

We can define the  $i$ -th Fibonacci matrix as

$$F_n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}.$$

We can get the value of the  $x^n$  term by noting that

$$f_n x^n = x \cdot f_{n-1} x^{n-1} + x^2 \cdot f_{n-2} x^{n-2}.$$

Note that this recurrence is linear, since  $x$  can be treated as a constant. We can express this linear recurrence using matrices, which can be evaluated the following expression in  $\mathcal{O}(\log n)$  using matrix exponentiation:

$$\begin{bmatrix} x & x^2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}.$$

To find the sums over these terms, we simply need to add another term to the transition matrix, which yields

$$\begin{bmatrix} x & x^2 & 0 \\ 1 & 0 & 0 \\ x & x^2 & 1 \end{bmatrix}^n \begin{bmatrix} f_1 \\ f_0 \\ f_0 + f_1 \end{bmatrix}.$$