Lemma 1. *Every Panaitopol Prime p can be represented as $n^2 + (n+1)^2$ for some positive integers n.*

Proof. Suppose $p = \frac{x^4-y^4}{x^3+y^3}$. Expanding and simplifying, we get that $p(x^2 - xy + y^2) = (x-y)(x^2+y^2)$. Because p is prime, it must be a factor of either $(x-y)$ or (x^2+y^2). If $p \backslash x-y$, x^2+y^2 must divide x^2-xy+y^2, which is not possible since xy is positive. Thus, $p \backslash x^2+y^2$ and $x-y \backslash x^2-xy+y^2$.

Because $x-y \backslash x^2-xy$ and $x-y \backslash y^2-xy$, we know that $x-y$ divides x^2, y^2, and xy. Let $a = x^2/(x-y)$, $b = y^2/(x-y)$, and $c = xy/(x-y)$. From these definitions we know that $ab = c^2$ and $a+b-2c = x-y$.

Let $d = \gcd(a,b,c)$ and a_0, b_0, and c_0 equal a/d, b/d, and c/d. Plugging in for the initial expression for p, we get that

$$\begin{aligned} p &= \frac{(a+b-2c)(a+b)}{a+b-c} \\ &= \frac{(a_0+b_0-2c_0)(a_0+b_0)d}{a_0+b_0-c_0}. \end{aligned}$$

Suppose there was some prime q that divided $a_0+b_0-2c_0$ and $a_0+b_0-c_0$. Then, $q \backslash c$, and because $xy = c(x-y)$ we know that either x or y must have a factor of q. Without loss of generality, assume this is x. Because $x-y \backslash x^2$, we know that y must also have a factor of q. It follows that a_0 and b_0 must consequently be divisible by q, violating the maximality of d. Thus, $\gcd(a_0+b_0-2c_0, a_0+b_0-c_0) = 1$.

Thus $a_0+b_0-c_0 \backslash d(a_0+b_0)$, and furthermore this quotient must be greater than 1 because c_0 is positive. As p is prime, we know that $a_0+b_0-2c_0 = 1$. By definition, $c_0 = \sqrt{a_0 b_0}$, so we get $\sqrt{a_0} - \sqrt{b_0} = 1$. Clearly, the only integer solutions are $a_0 = (n+1)^2$ and $b_0 = n^2$. All that remains is to solve for x and y and plug back into the initial formula for p.

□

It suffices to check every n from 1 to $\sqrt{5 \cdot 10^5}$.