Theorem 1. *Let N be a random variable defined on the non-negative integers, and let X_i be a sequence of random variables independent of N and* $\mathrm{E}[X_i] = \mathrm{E}[X]$ *for all i. Then,*

$$\mathrm{E}\left[\sum_{i=1}^{N} X_i\right] = \mathrm{E}[N]\mathrm{E}[X].$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[\sum_{i=1}^{N} X_i\right] &= \mathrm{E}\left[\mathrm{E}\left[\sum_{i=1}^{n} |N = n\right]\right] \\
&= \sum_{i=1}^{\infty} \mathrm{E}\left[\sum_{j=1}^{n} X_j\right] \cdot \Pr[N = n] \\
&= \sum_{i=1}^{\infty} n\mathrm{E}[X] \cdot \Pr[N = n] \\
&= \mathrm{E}[X] \cdot \sum_{i=1}^{\infty} n\Pr[N = n] \\
&= \mathrm{E}[N]\mathrm{E}[X]
\end{aligned}$$

□

Theorem 2. *Let N be a random variable defined on the non-negative integers, and let X_i be a sequence of random variables independent of N and* $\mathrm{Var}[X_i]$ *same for all i. Then,*

$$\mathrm{Var}\left[\sum_{i=1}^{N} X_i\right] = \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{E}[X]^2\mathrm{Var}[N]$$

Proof.

$$
\begin{aligned}
\mathrm{Var}\left[\sum_{i=1}^{N} X_i\right] &= \mathrm{E}\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}\left[\mathrm{E}\left[\left(\sum_{i=1}^{N} X_i\right)^2 | N = n\right]\right] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}\left[E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right]\right] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}\left[\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] + \mathrm{E}\left[\sum_{i=1}^{n} X_i\right]^2\right] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}\left[n\mathrm{Var}[X_i] + (n\mathrm{E}[X])^2\right] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \sum_{i=1}^{\infty}(n\mathrm{Var}[X_i] + (n\mathrm{E}[X])^2)\mathrm{Pr}[N = n] - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{E}[N^2]\mathrm{E}[X]^2 - \mathrm{E}\left[\sum_{i=1}^{N} X_i\right]^2 \\
&= \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{E}[N^2]\mathrm{E}[X]^2 - (\mathrm{E}[N]\mathrm{E}[X])^2 \\
&= \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{E}[X]^2(\mathrm{E}[N^2] - \mathrm{E}[N]^2) \\
&= \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{E}[X]^2\mathrm{Var}[N]
\end{aligned}
$$

□