Fall 2012 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

1. On Black Friday a store reduced its price on a camera by 30%. Two weeks later, the item still not having sold, the store reduced the Black Friday sale price by 50%. The final price on the camera is what per cent of its original price?

   (A) 20 **(B)** 35 **(C)** 50 **(D)** 65 **(E)** 80

   Solution:B $(.7)(.5) = .35$

2. If one defines
$$(a, b) \wedge (c, d) = ad - bc$$
   solve this equation for x: $(2, x) \wedge (7, -4) = 3$

   (A) $-\frac{7}{11}$ **(B)** $\frac{11}{7}$ **(C)** $\frac{7}{11}$ **(D)** 11 **(E)** $-\frac{11}{7}$

   Solution: E $-8 - 7x = 3$ leads to $x = -11/7$

3. In the sequence of numbers
$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$
   each number after the second is the sum of the previous two numbers. Find the value of a.

   (A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

   Solution: B So $b + 1 = -1$, thus $b = -2$. Then $a + (-2) = 1$ and $a = 3$.

4. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

   (A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

   Solution: C One way: in the classic shoestring method to find the area each product will be doubled. So the overall area will be doubled.

5. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) 1/34 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** 14

   Solution: C The product of all the fractions is both a/e and 1.

6. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

   Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

7. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

   (A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

   Solution: A Note that $-8 < -2 < 4$. The others do not work.

8. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

   (A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

   Solution: E By Euclidean Algorithm or otherwise, $11 \cdot 17 - 6 \cdot 31 = 1$. Here $n = 11$ and $m = 6$. To make n smaller it would have to decrease in multiples of 31 which would force it be negative. So 11 is the smallest positive value of n where m is positive too.

9. In how many ways can 9 students be divided into 3 groups of 3 students each?

   (A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

   Solution: C $(\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3})/3! = 280$ because the groups are unordered.

10. How many solutions does the equation $|x - 2| = |4 - x|$ have?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

    Solution:B Only $x = 3$ is equidistant from 2 and 4.

11. Which of the integers below can be expressed in the form $p^2 + q^2 + r^2 + s^2 + t^2$ where p, q, r, s and t are all odd integers?

(A) 2012 **(B)** 2013 **(C)** 2014 **(D)** 2015 **(E)** 2016

Solution: B The square of an odd is one greater than a multiple of 4. So the sum of 5 such is also one greater than a multiple of 4. This eliminates each answer except 2013 and 2013 can be written in such a manner.

12. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

Solution: C Add all the divisors of 60 except for the two largest as they do not correspond to actual polygons.

13. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: E Cheryl should take 5 chips on her first move to reduce the pile to 96 chips. a multiple of 6. No matter what Matthew takes on his turn, Cheryl takes the complement in terms of six, leaving Matthew with a pile of 90 chips. In this way she guarantees that Matthew will eventually face a pile of 6 chips. Whatever he takes, she can take the last remaining chips, thereby winning. If she does not take 5 chips, Matthew will turn the table on her and leave her with 96 chips! So she must take 5 on her first turn in order to win.

14. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

Solution: A Let $x =$ the distance from the center of the fountain to the base of the higher tower. Then $50 - x =$ the distance from the center of the fountain to the base of the lower tower. By the Pythagorean Theorem $x^2 + 40^2 = (50 - x)^2 + 30^2$ or $x^2 + 1600 = x^2 - 100x + 2500 + 900$ or $100x = 1800$ or $x = 18$.

15. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

Solution: C Let r =rate of boat in still water and c =rate of the current. Take the distance from A to B as unit distance. Then $1/(r + c) = 3$ and $1/(r - c) = 4$. Then $r + c = 1/3$ and $r - c = 1/4$. Subtracting these equations yields $2c = 1/12$ or $c = 1/24$. Thus it would take 24 days for the tube to float from A to B.

16. Find the value of x if

$$\begin{aligned} 3x + 2y - z &= 1 \\ -x + y - 3z &= 7 \\ x + 2y + 9z &= -1 \end{aligned}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution:A Take -3 times the first equation and add it to 4 times the second, then add that result to the last equation to get $-12x = 24$ or $x = -2$.

17. A frog is on a number line and can jump either one unit to the left or one unit to the right. If it starts at the origin and jumps randomly 6 times, what is the probability it is back at the origin at the end of those 6 jumps?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{5}{16}$

Solution: E $\binom{6}{3}/2^6 = 5/16$

18. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.