# ECA CLASSIFICATION USING CYCLES

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Computational Discrete Math

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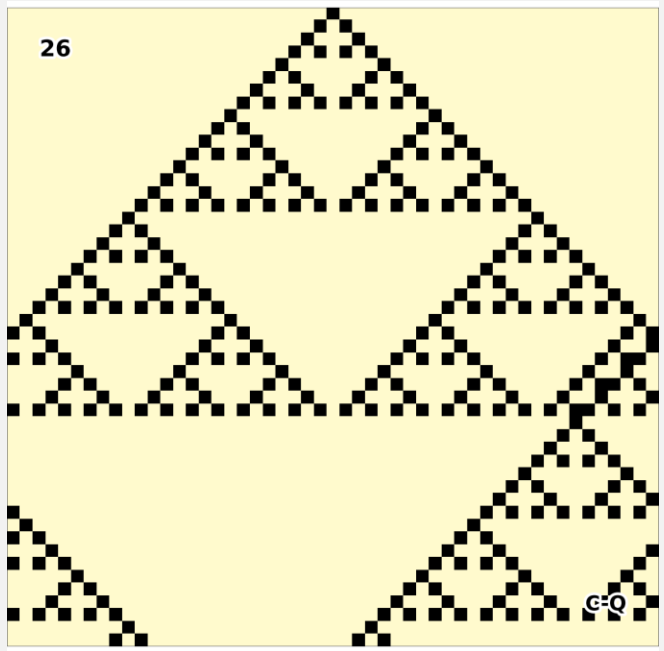
# PART I

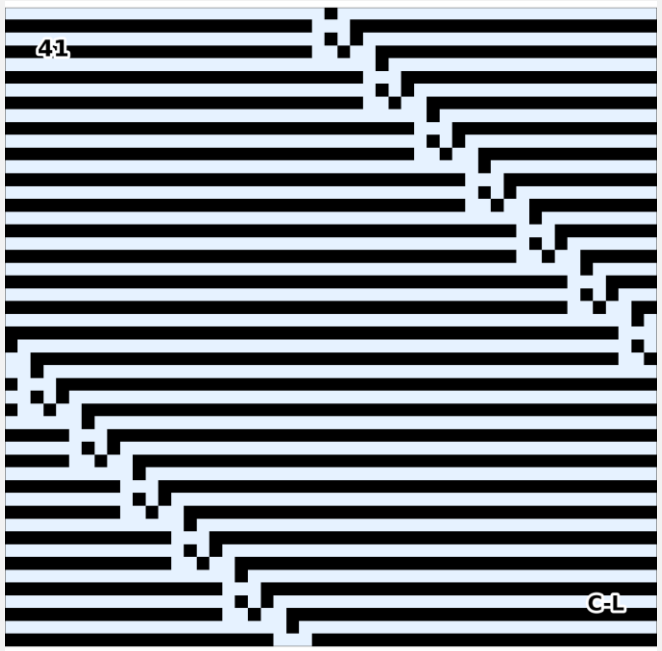
CYCLE DETECTION

# WHAT IS A CYCLE?

- We can define the function f as the global map of the ECA rule for a fixed-width string with wraparound. For some length n, we use the initial point 0…010…0 of length n.
- Since the state space is finite, a cycle is guaranteed when iterating over this initial point. We can use a few different methods to obtain it.

# EXAMPLE: N=51



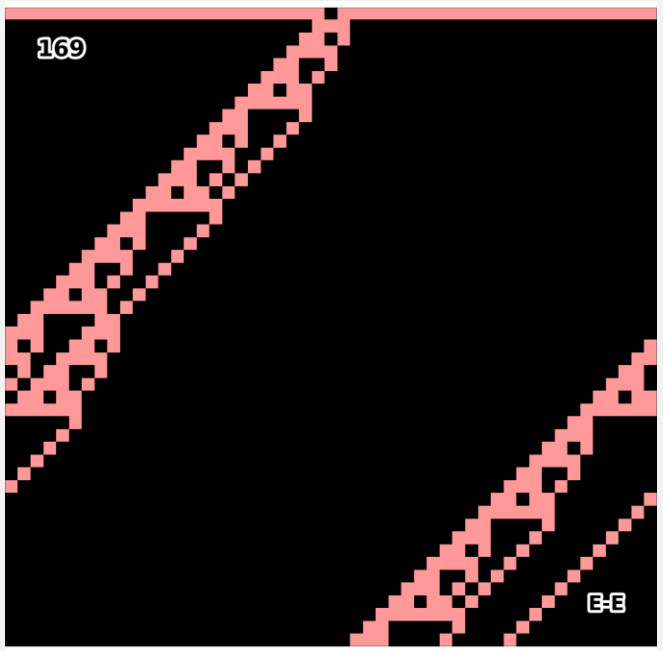




Illustration of wrap-around behavior of certain ECAs.
Top Left indicates the rule number. Bottom right shows the classification- we'll get to that later.

# WHAT ALGORITHMS ARE THERE?

## HASH-MAP

- Stores every generated state in a lookup table to check for duplicates.
- Simplest implementation, uses the least amount of functions applications.
- Cons: Memory intensive. Will struggle to detect large cycles.

## FLOYD'S TORTOISE & HARE

- Two pointers move at different speeds. A collision is guaranteed and indicated a loop.
- Uses constant memory but performs ~3x as many function evaluations as hash-map.

# WHAT ALGORITHMS ARE THERE?

## BRENT'S

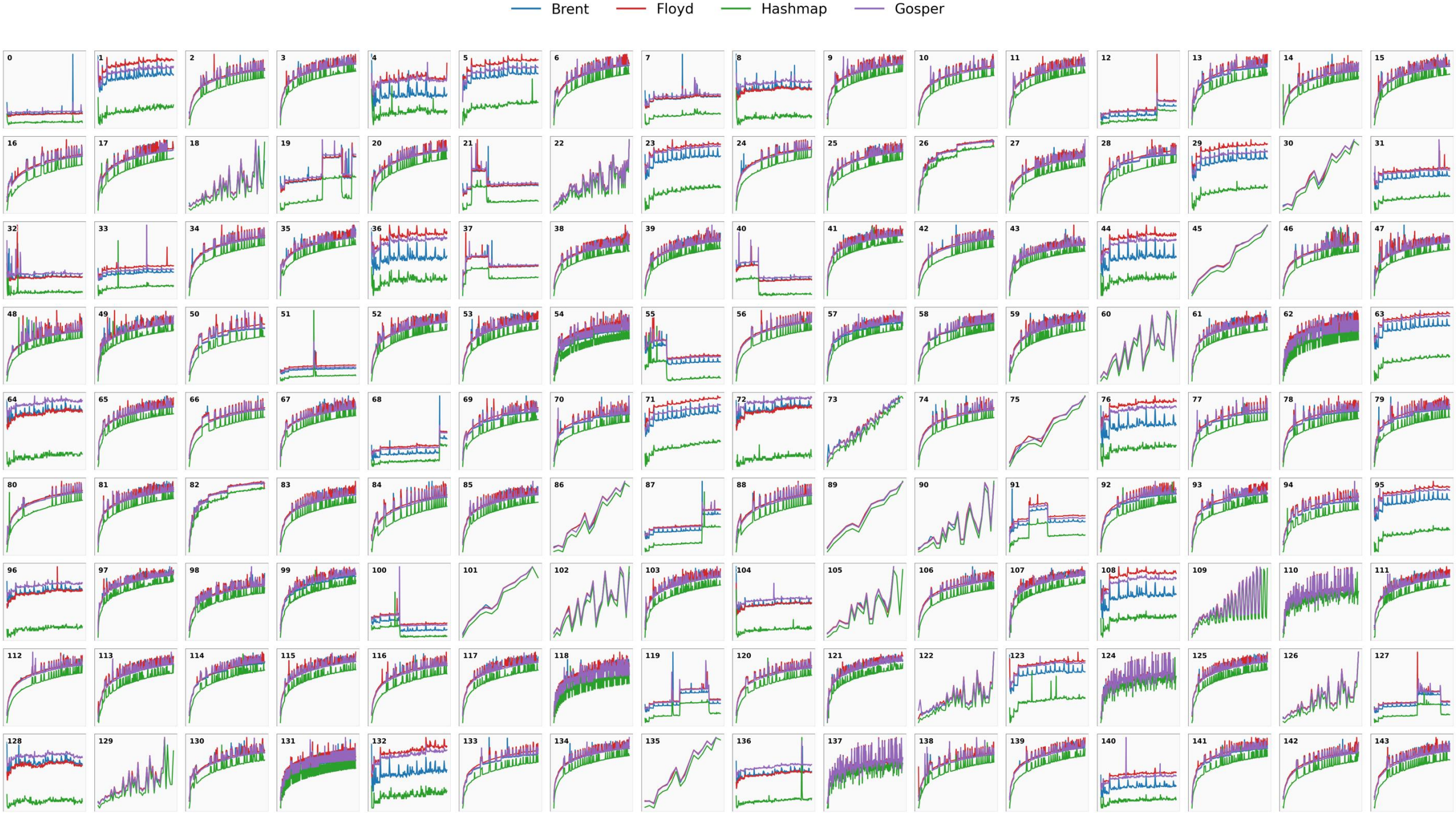
- The "slow" pointer waits stationary for exponential intervals before jumping to the "fast" pointer's location.
- Evaluates much fewer steps that Floyd with no memory needed.
- Cons: Requires secondary calculation to find transient

## GOSPER'S

- Saves specific states (at power-of-2 intervals) in a small table for comparison.
- Detects transient and period in single pass
- Uses log(L) memory

# LETS TEST IT

- For each ECA, start from n=3 and increment n+=2 to keep the initial config symmetrical. Determine the transient and cycle using each algorithm.
- Repeat until:
  - We hit a limit (default n=500, to prevent constant or linear from eating up all my time)
  - It takes longer than TIMEOUT (default t=60s) to compute the cycle
- Plot it as **Time (log scale) vs N (linear scale)**


Brent
Floyd
Hashmap
Gosper
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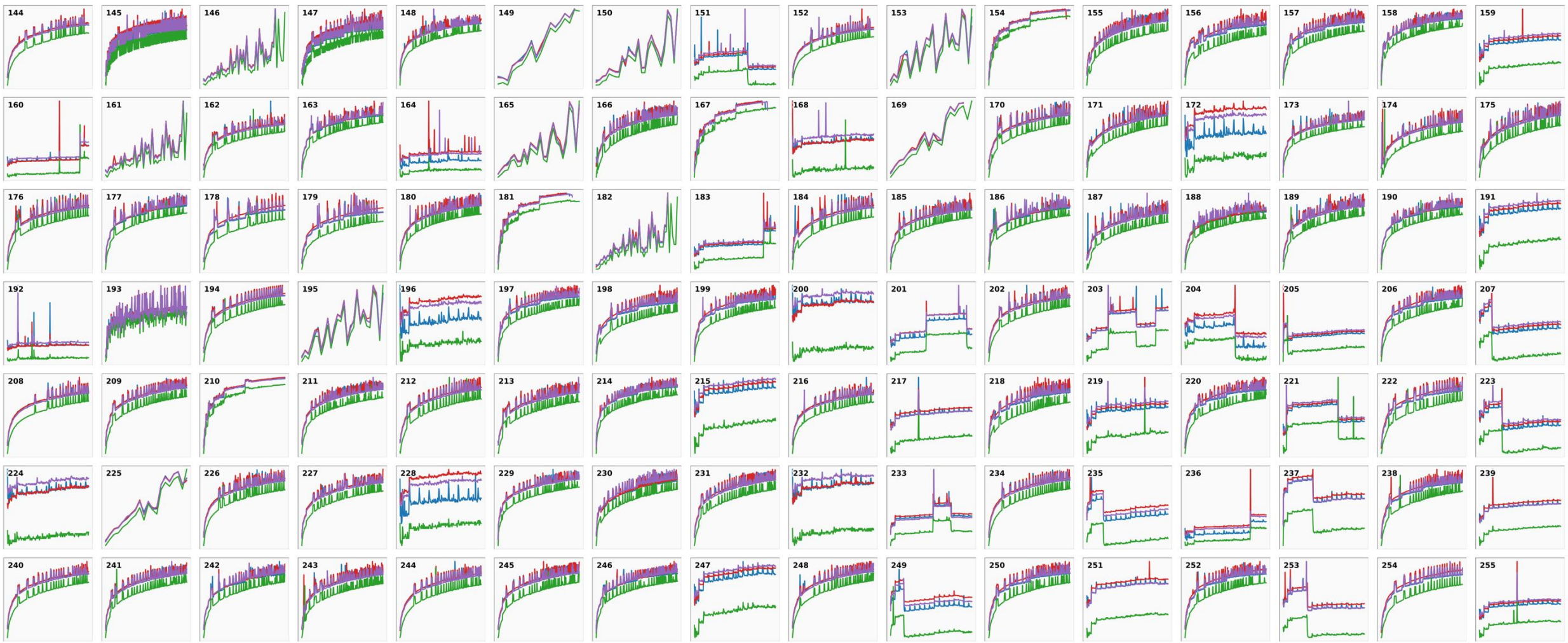
Ponder deeply.



# RESULT

- Hash-map wins for small n as expected
- Hash-map wins for large n, unexpectedly, as I fully expected my computer to run out of memory- some rules can blow up to cycle lengths of millions within that allotted time.
- Brent's comes in second place. I use brent's for future simulations to prevent any memory related issues, even though none seem to come up.
  - In the end, hash-map can only computer one or two more values compared to the other approaches.

# PART 2

ECA CLASSIFICATION

# PERIOD LENGTH

- Before crunching the numbers, here's some intuition behind why period length might matter and how we might classify them.
- A ruleset that consistently reaches a fixed point (length = 1) will have rules that cause it to 'die out'.
- A ruleset whose period increases linearly with n probably has rules that depend on the time it takes to reach the boundary condition, but not much else.
- A ruleset whose period increase exponentially with n probably has randomness or chaoticity involved, as it near almost the entire state space because forming a cycle.

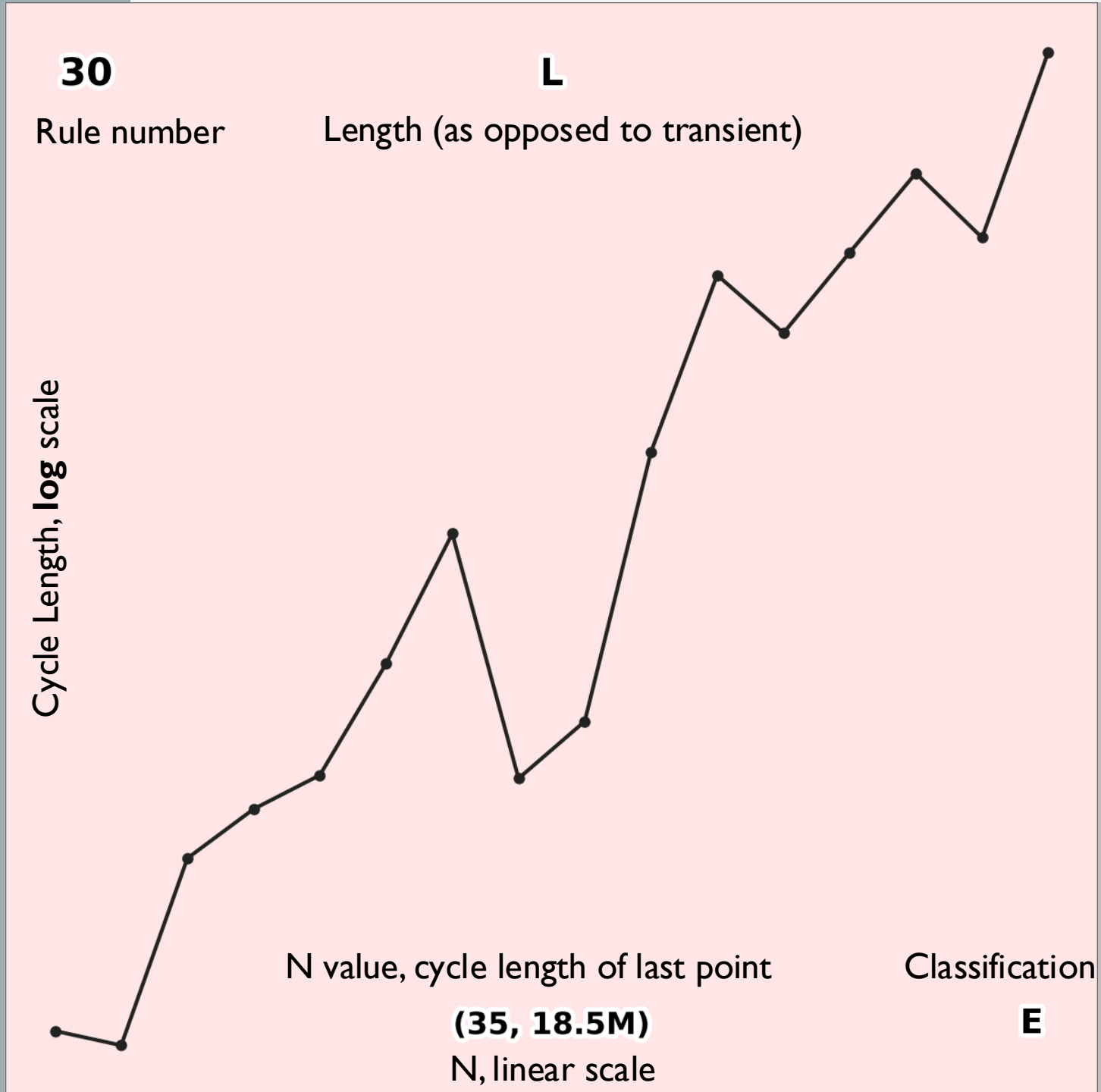
# SIMPLE CLASSIFICATION

- C – graph is about-constant.
- L – graph is about-linear
- Q – graph is about-quadratic
- E – graph is about-exponential

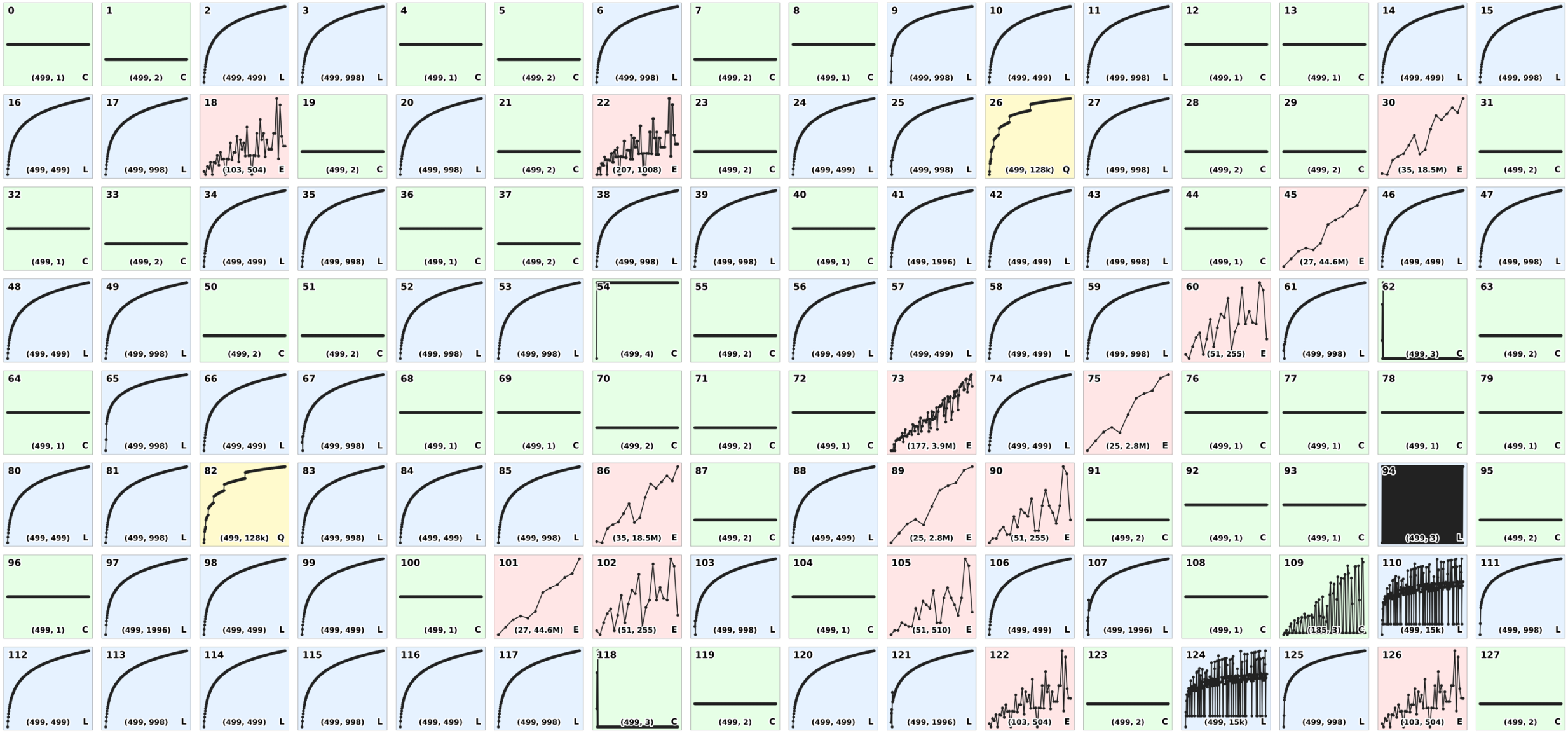
- I used sketchy heuristics to code these up. The hope is that as I run the simulations for longer periods of time, the more likely it is for each rule to fall under its ‘correct’ complexity class.

# INTERPRETING RESULTS

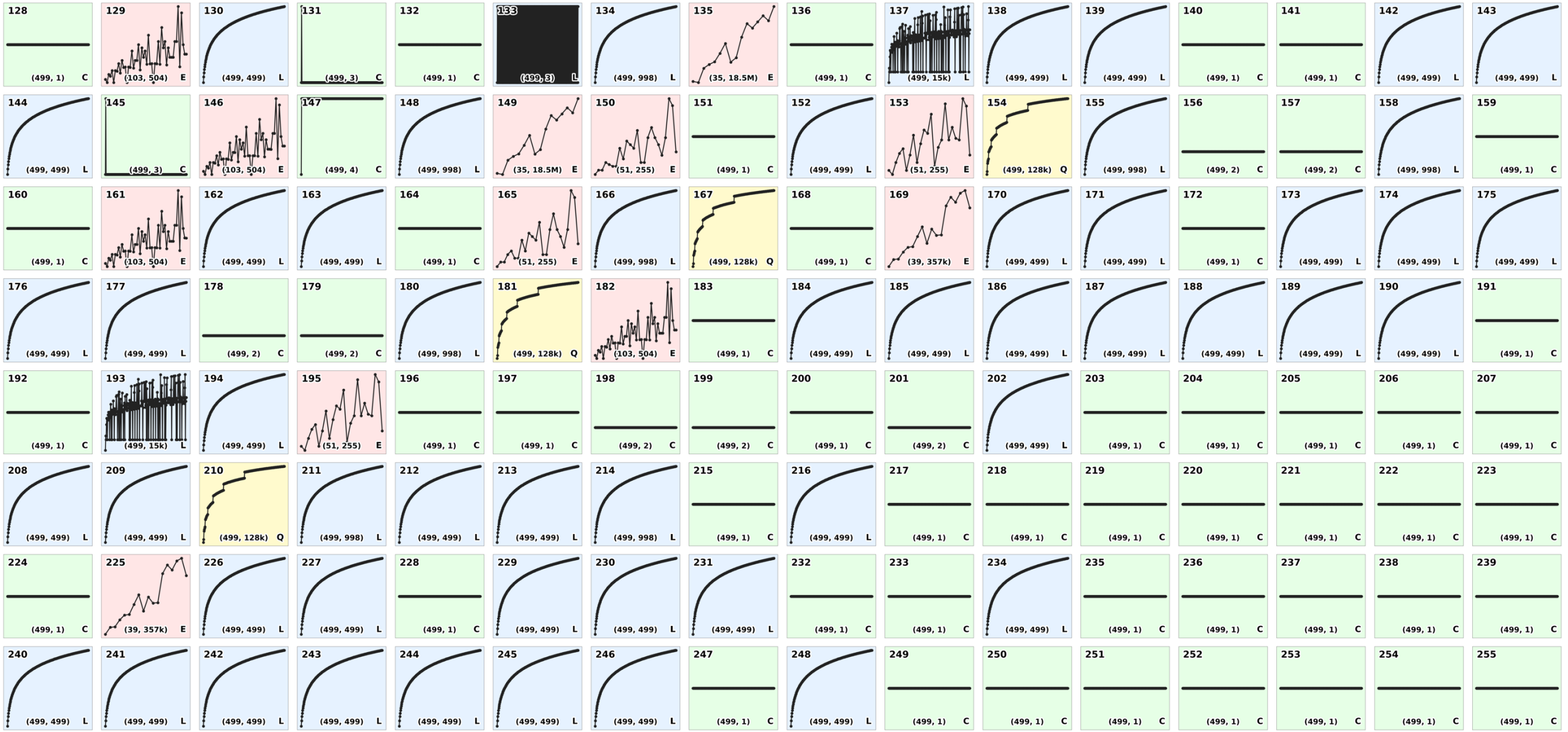
- Each N run using Brent's for up to 3 minutes
- Up to N = 499



# Cycles Growth Analysis



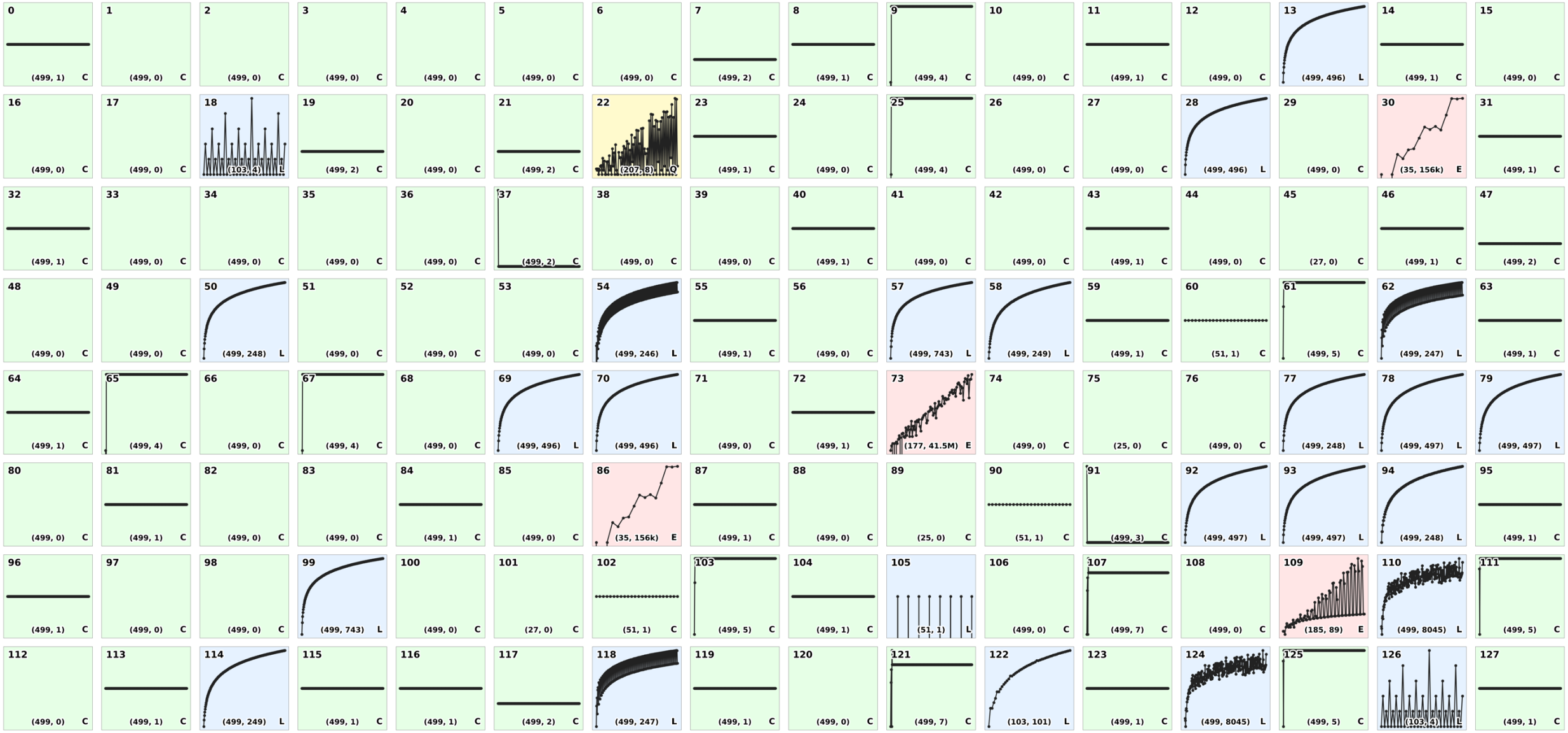
# Ponder even more deeply.

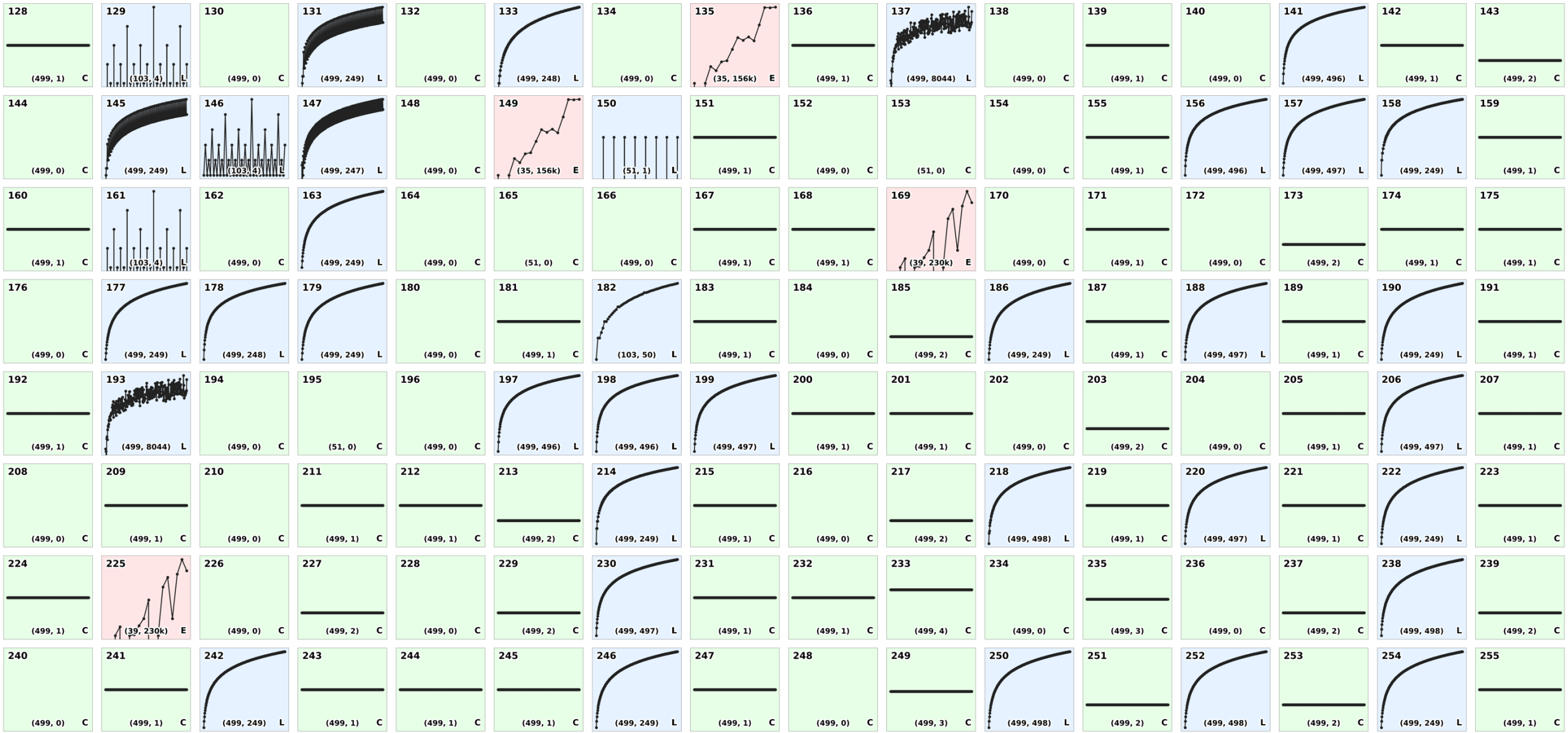


# TRANSIENT LENGTH

- One can similarly gather intuition from the transient. It is certainly less clear to me, but we can still infer order and chaos based on the constant/exponential growth.
- A linear transient implies that it takes some time to ‘saturate’ the grid- that is, to reach the wraparound zone. Afterwards, it never goes back. This is commonly seen in automata that branch out left and right initially.

# Transients Growth Analysis




128 (499, 1) C
129 (103, 4) L
130 (499, 0) C
131 (499, 249) L
132 (499, 0) C
133 (499, 248) L
134 (499, 0) C
135 (35, 156k) E
136 (499, 1) C
137 (499, 8044) L
138 (499, 0) C
139 (499, 1) C
140 (499, 0) C
141 (499, 496) L
142 (499, 1) C
143 (499, 2) C
144 (499, 0) C
145 (499, 249) L
146 (103, 4) L
147 (499, 247) L
148 (499, 0) C
149 (35, 156k) E
150 (51, 1) L
151 (499, 1) C
152 (499, 0) C
153 (51, 0) C
154 (499, 0) C
155 (499, 1) C
156 (499, 496) L
157 (499, 497) L
158 (499, 249) L
159 (499, 1) C
160 (499, 1) C
161 (103, 4) L
162 (499, 0) C
163 (499, 249) L
164 (499, 0) C
165 (51, 0) C
166 (499, 0) C
167 (499, 1) C
168 (499, 1) C
169 (39, 230k) E
170 (499, 0) C
171 (499, 1) C
172 (499, 0) C
173 (499, 2) C
174 (499, 1) C
175 (499, 1) C
176 (499, 0) C
177 (499, 249) L
178 (499, 248) L
179 (499, 249) L
180 (499, 0) C
181 (499, 1) C
182 (103, 50) L
183 (499, 1) C
184 (499, 0) C
185 (499, 2) C
186 (499, 249) L
187 (499, 1) C
188 (499, 497) L
189 (499, 1) C
190 (499, 249) L
191 (499, 1) C
192 (499, 1) C
193 (499, 8044) L
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195 (51, 0) C
196 (499, 0) C
197 (499, 496) L
198 (499, 496) L
199 (499, 497) L
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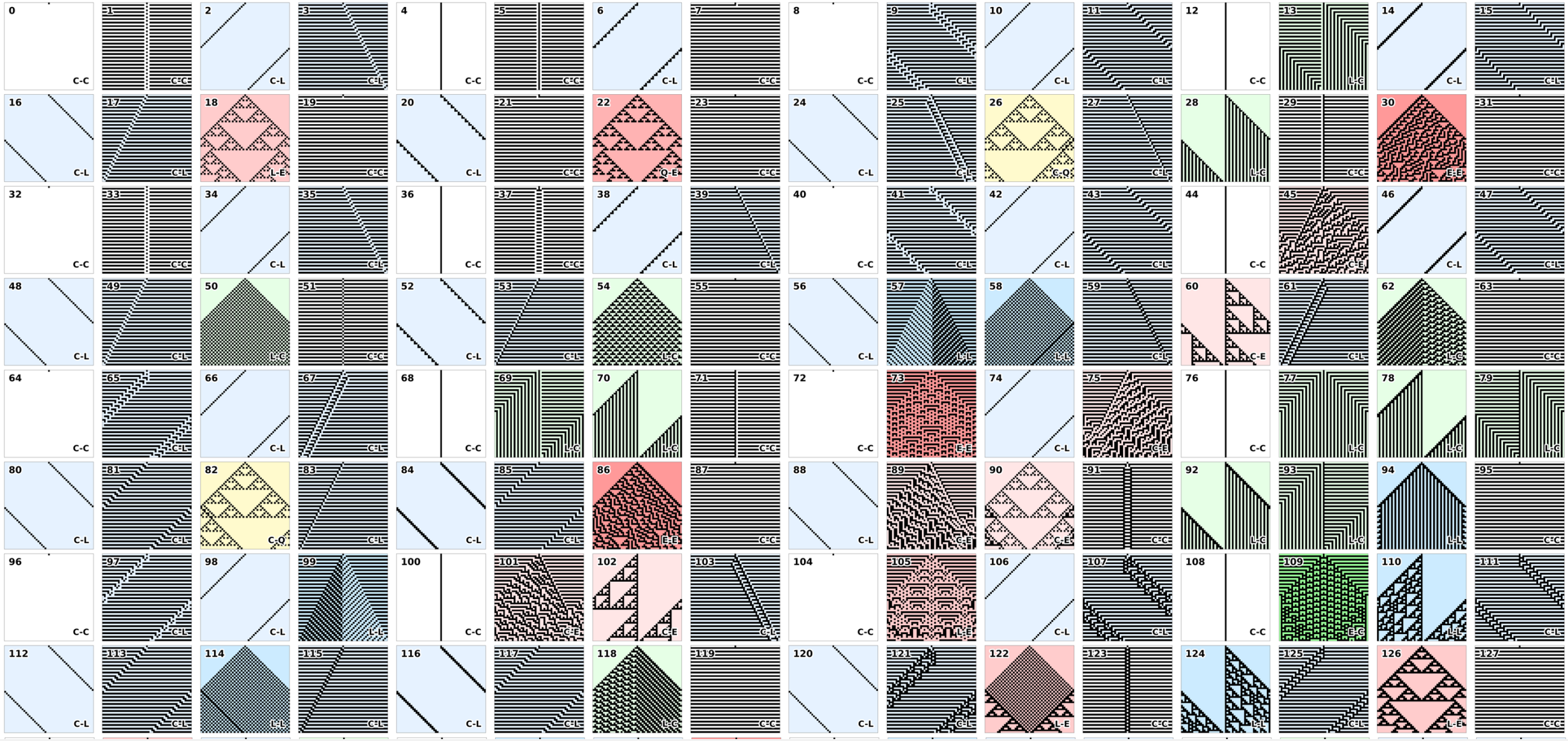
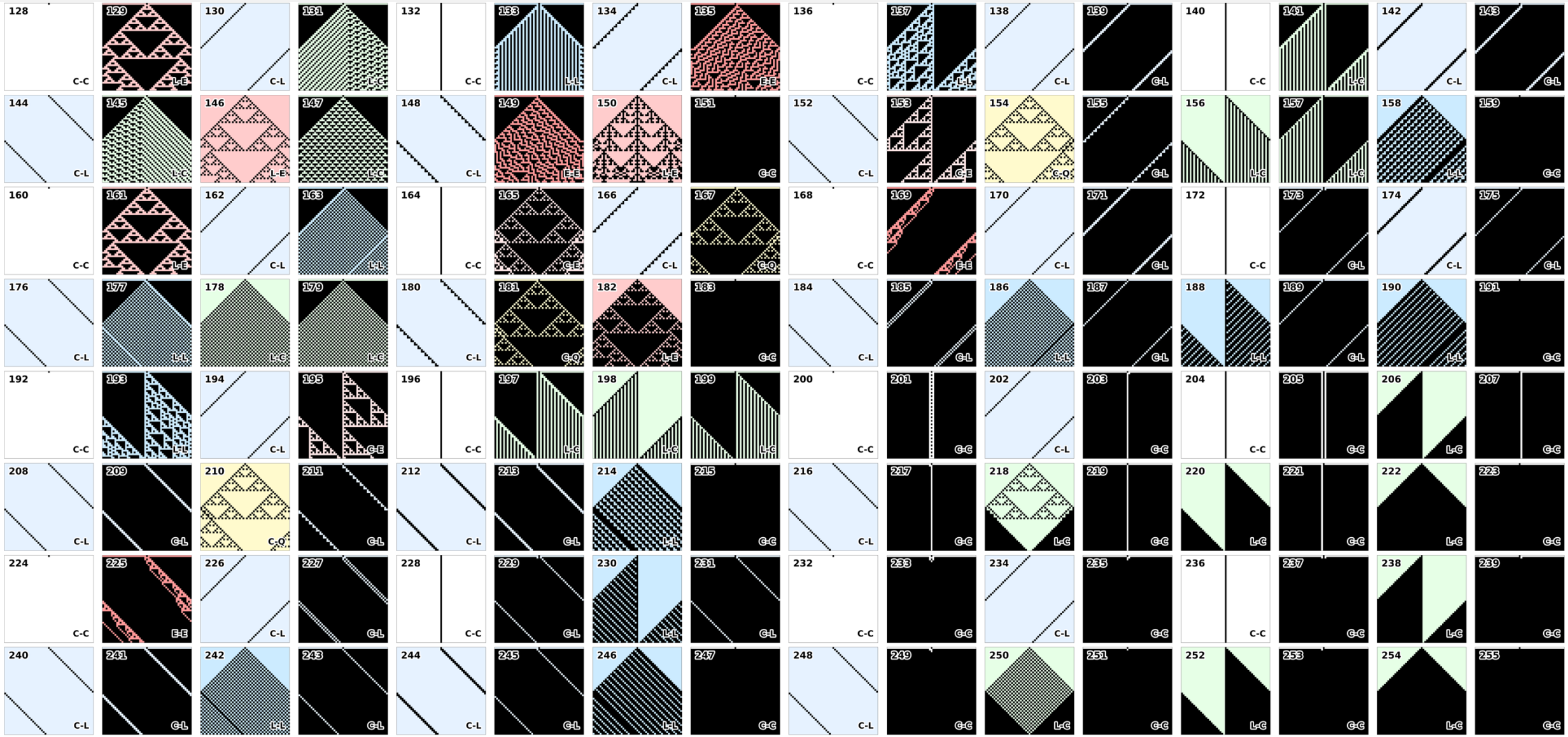
202 (499, 0) C
203 (499, 2) C
204 (499, 0) C
205 (499, 1) C
206 (499, 497) L
207 (499, 1) C
208 (499, 0) C
209 (499, 1) C
210 (499, 0) C
211 (499, 1) C
212 (499, 1) C
213 (499, 2) C
214 (499, 249) L
215 (499, 1) C
216 (499, 0) C
217 (499, 2) C
218 (499, 498) L
219 (499, 1) C
220 (499, 497) L
221 (499, 1) C
222 (499, 249) L
223 (499, 1) C
224 (499, 1) C
225 (39, 230k) E
226 (499, 0) C
227 (499, 2) C
228 (499, 0) C
229 (499, 2) C
230 (499, 497) L
231 (499, 1) C
232 (499, 1) C
233 (499, 4) C
234 (499, 0) C
235 (499, 3) C
236 (499, 0) C
237 (499, 2) C
238 (499, 498) L
239 (499, 2) C
240 (499, 0) C
241 (499, 1) C
242 (499, 249) L
243 (499, 1) C
244 (499, 1) C
245 (499, 1) C
246 (499, 249) L
247 (499, 1) C
248 (499, 0) C
249 (499, 3) C
250 (499, 498) L
251 (499, 2) C
252 (499, 498) L
253 (499, 2) C
254 (499, 249) L
255 (499, 1) C

# DUAL CLASSIFICATION

- And finally, we can combine the classifications of the transient and cycles to produce a new classification

# Rule Complexity Visuals




128 C-C
129 L-E
130 C-L
131 L-C
132 C-C
133 L-L
134 C-L
135 E-E
136 C-C
137 L-L
138 C-L
139 C-L
140 C-C
141 L-C
142 C-L
143 C-L
144 C-L
145 L-C
146 L-E
147 L-C
148 C-L
149 E-E
150 L-E
151 C-C
152 C-L
153 C-E
154 C-O
155 C-L
156 L-C
157 L-C
158 L-L
159 C-C
160 C-C
161 L-E
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163 L-L
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169 E-E
170 C-L
171 C-L
172 C-C
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174 C-L
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187 C-L
188 L-L
189 C-L
190 L-L
191 C-C
192 C-C
193 L-E
194 C-L
195 L-E
196 C-C
197 L-C
198 L-C
199 L-C
200 C-C
201 C-C
202 C-L
203 C-C
204 C-C
205 C-C
206 L-C
207 C-C
208 C-L
209 C-L
210 C-O
211 C-L
212 C-L
213 C-L
214 L-L
215 C-C
216 C-L
217 C-C
218 L-C
219 C-C
220 L-C
221 C-C
222 L-C
223 C-C
224 C-C
225 E-E
226 C-L
227 C-L
228 C-C
229 C-L
230 L-L
231 C-L
232 C-C
233 C-C
234 C-L
235 C-C
236 C-C
237 C-C
238 L-C
239 C-C
240 C-L
241 C-L
242 L-L
243 C-L
244 C-L
245 C-L
246 L-L
247 C-C
248 C-L
249 C-C
250 L-C
251 C-C
252 L-C
253 C-C
254 L-C
255 C-C

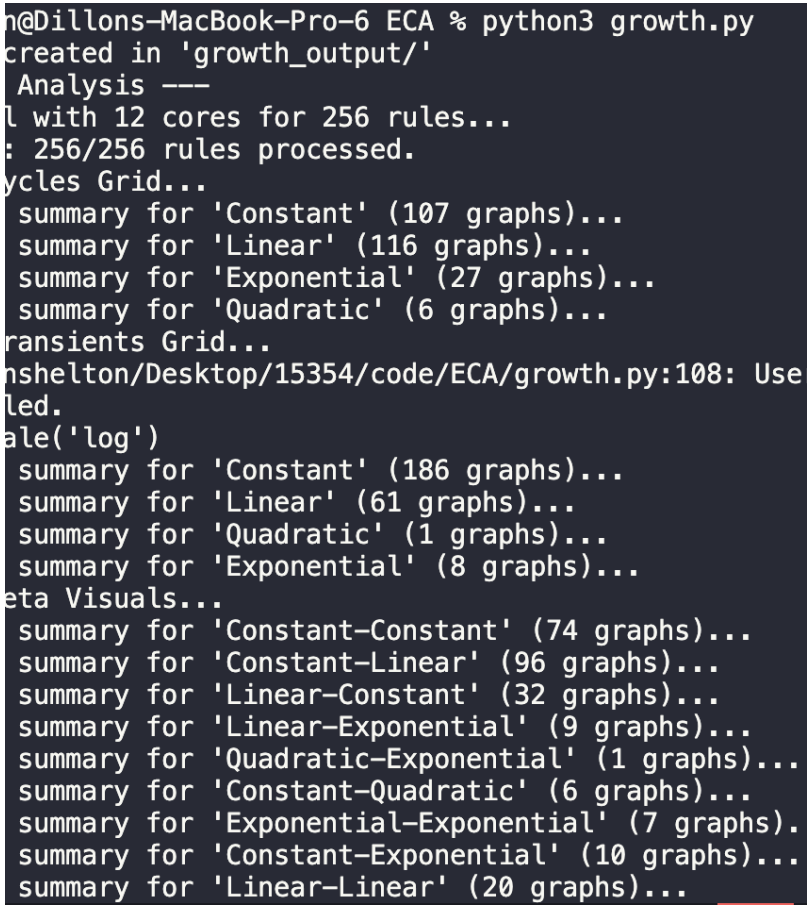
# PART 3

Analysis

# OKAY, THAT WAS A LOT

I'll give a quick summary of each of the meta-classifications.

I don't bother removing mirror/inverse rules

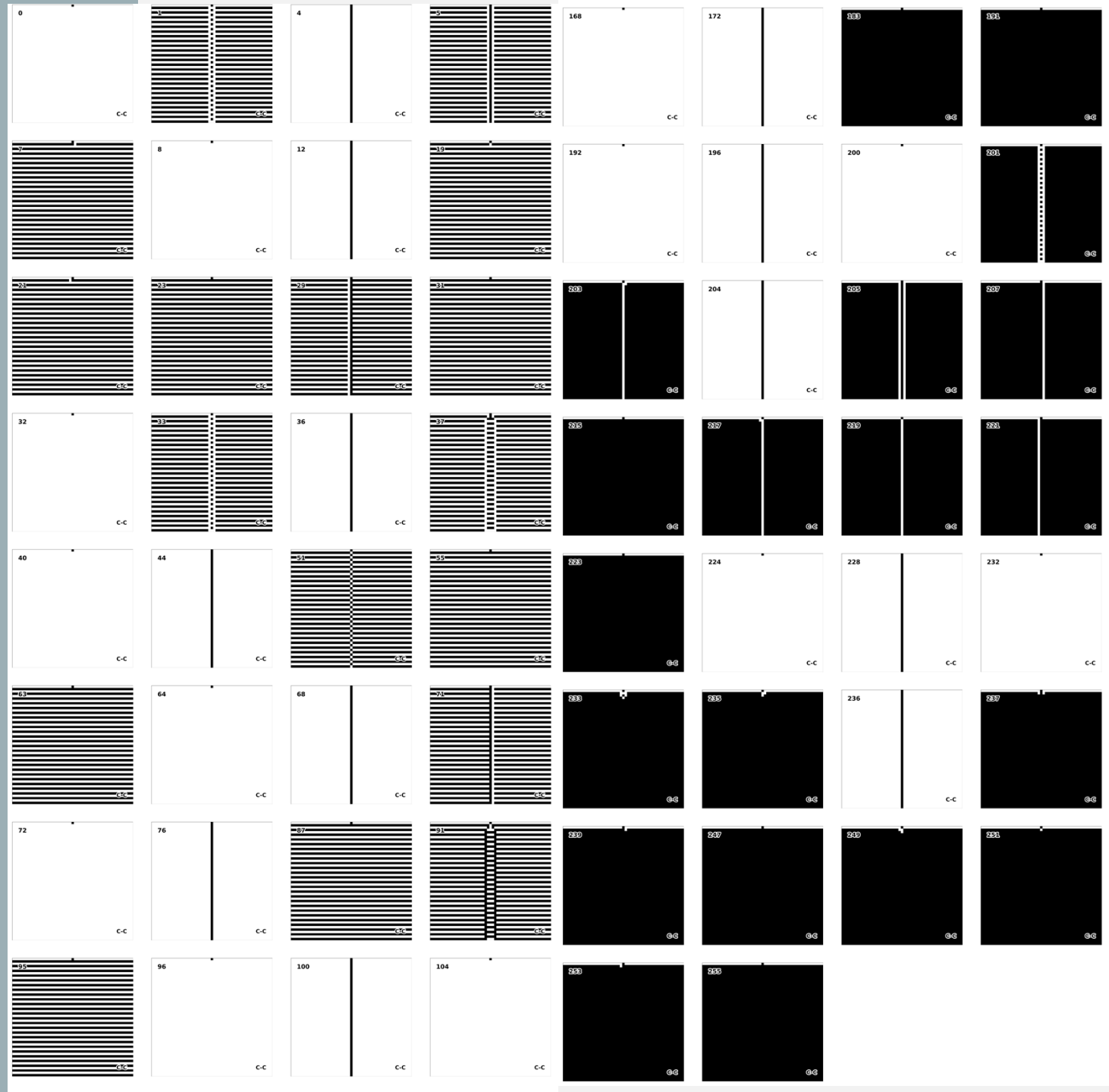


(transient class) - (period class)

# CONSTANT CONSTANT

74 Rules

Characterized by all-white, all-black, or alternating all-black all-white rows, with constant behavior down the center.

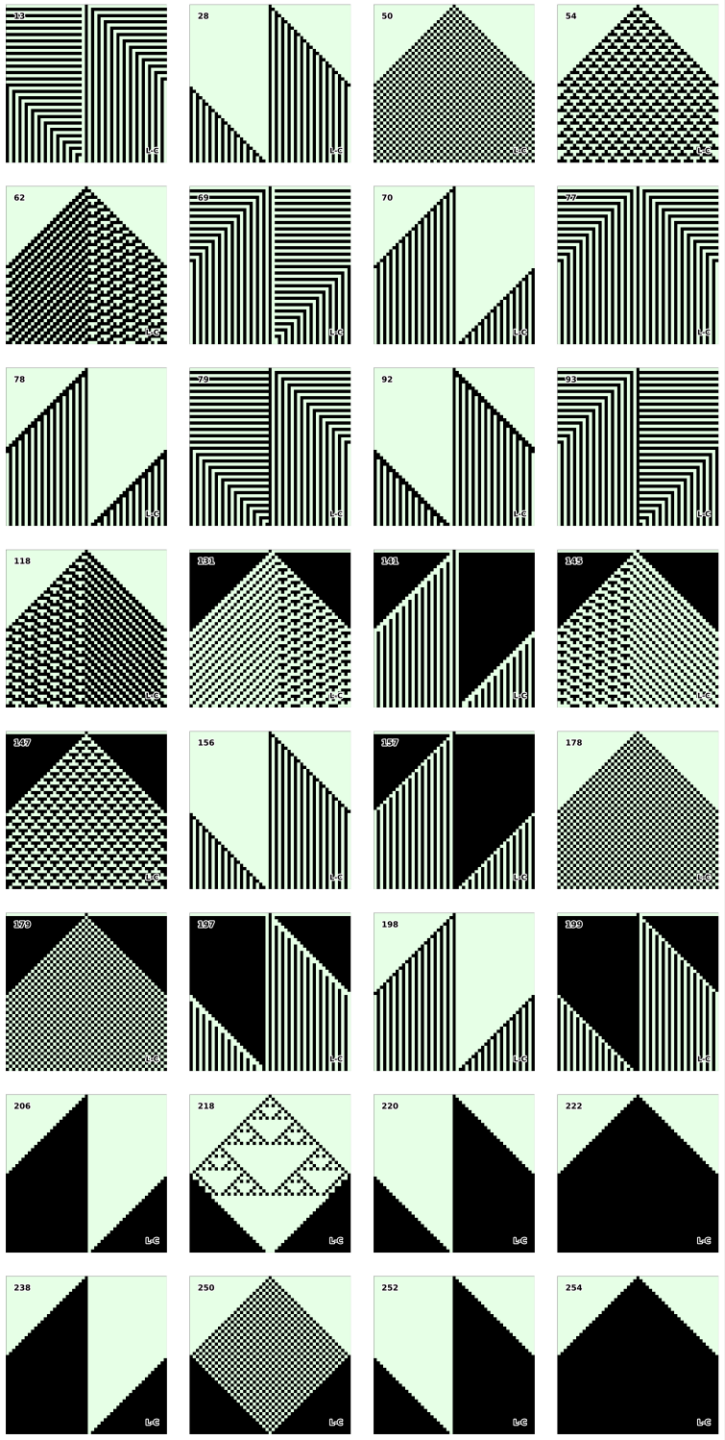


(transient class) - (period class)

# LINEAR CONSTANT

32 Rules

Characterized by grids which propagate out one cell at a time. Once both sides hit the edge, the behavior forms a cycle.

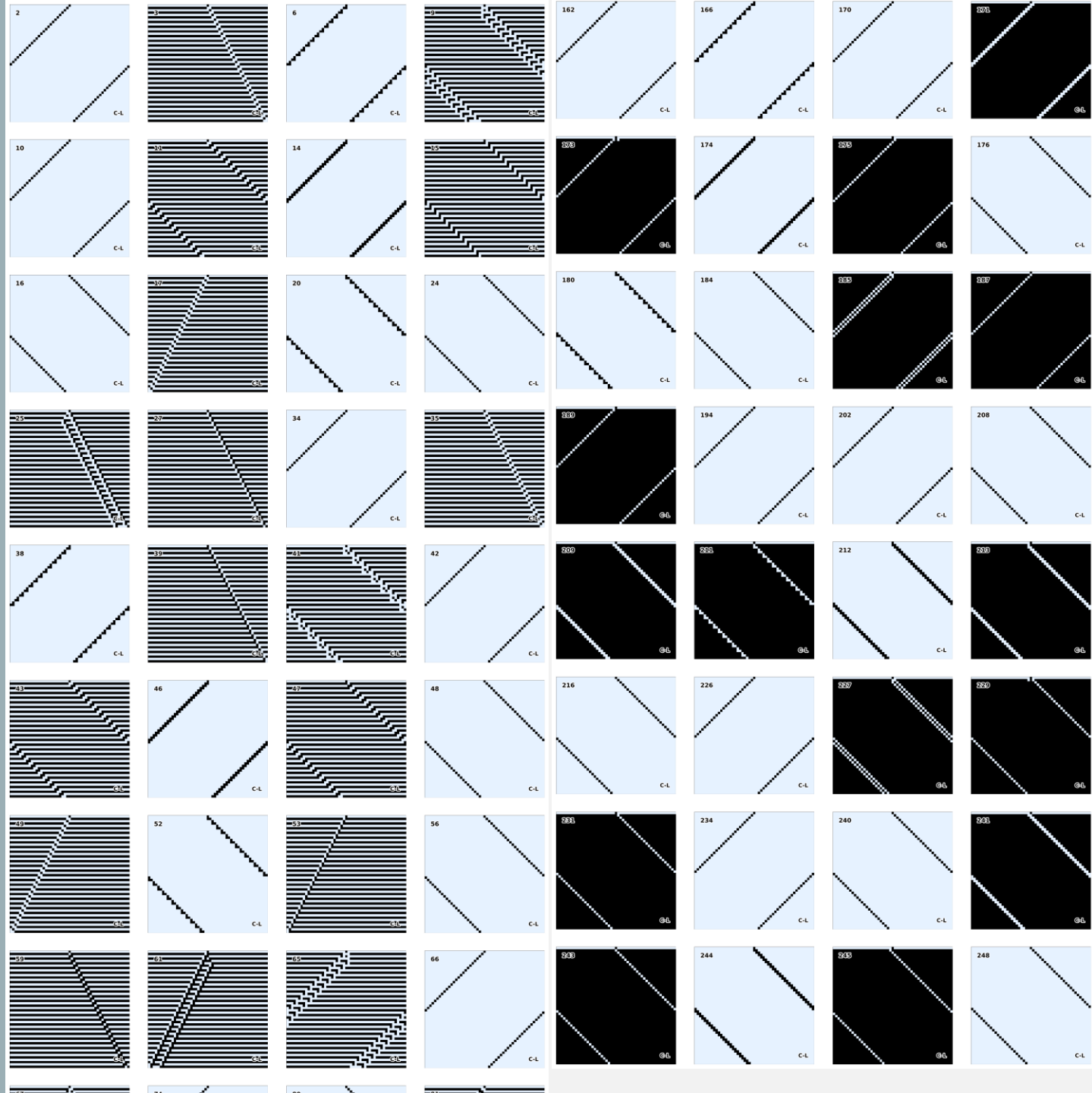


(transient class) - (period class)

# CONSTANT LINEAR

96 Rules

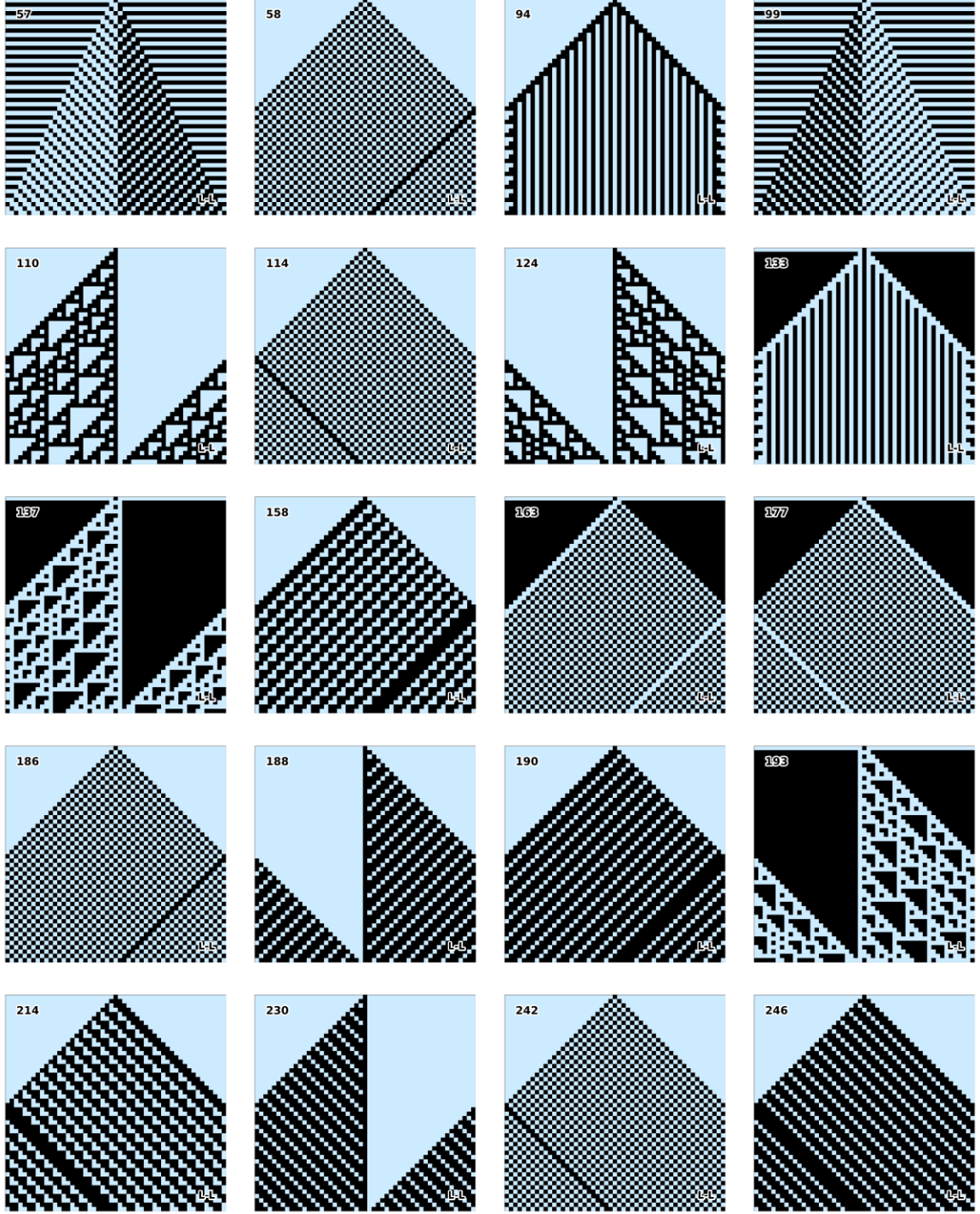
Characterized lines which shoot in one direction. Graphs repeat when the ‘line’ wraps around and returns to the center.



(transient class) - (period class)

# LINEAR LINEAR

20 Rules



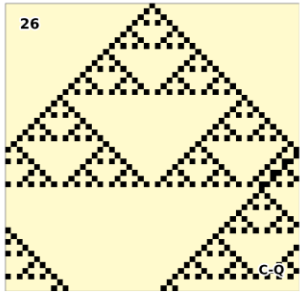
(transient class) - (period class)

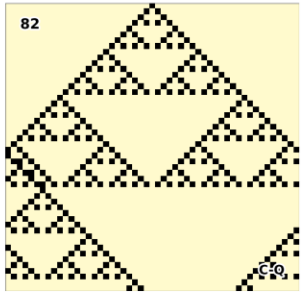
# CONSTANT QUADRATIC

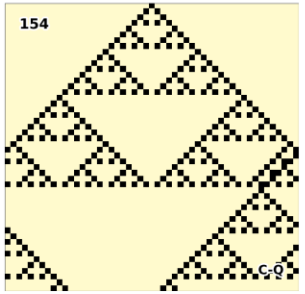
6 Rules

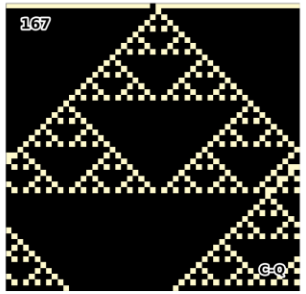
Quadratic periods are very interesting in that they are not actually quadratic. Take a look at the right- It doubles at powers of two, then increases linearly.

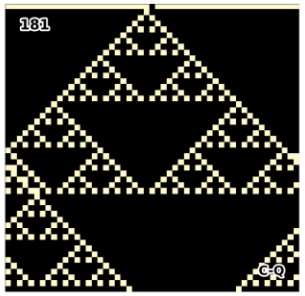
They are certainly worth looking at in the future.

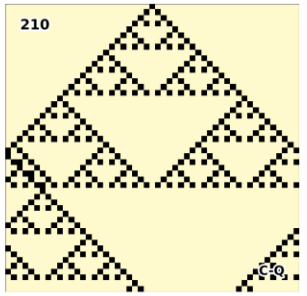


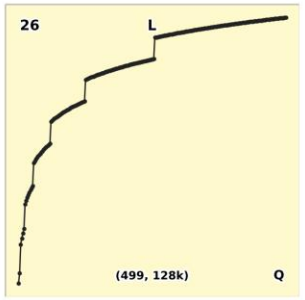



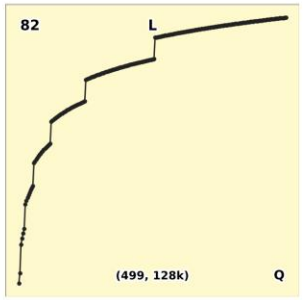



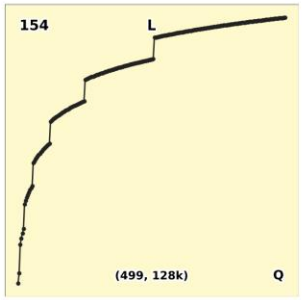



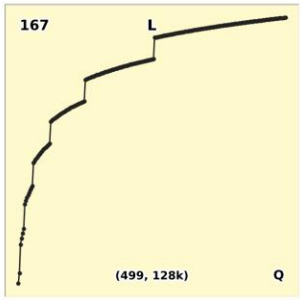



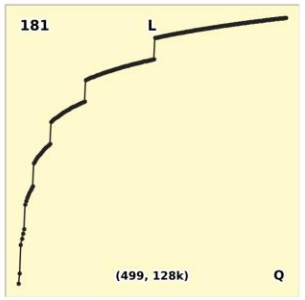



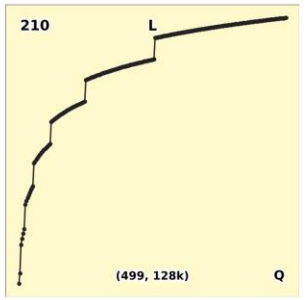



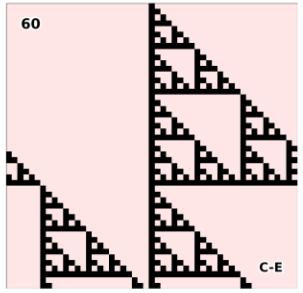



(transient class) - (period class)

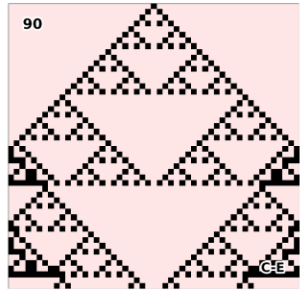
# CONSTANT EXPONENTIAL

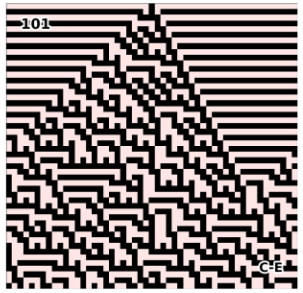
10 Rules

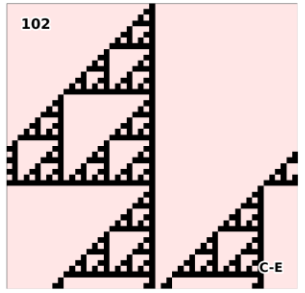
These hit the same transient every time, but take exponentially longer to get back to it.

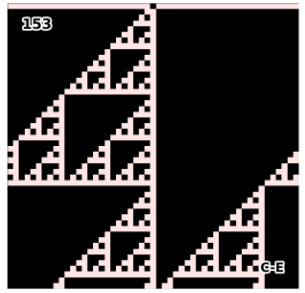



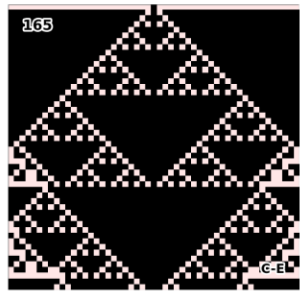


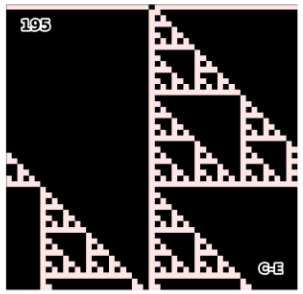



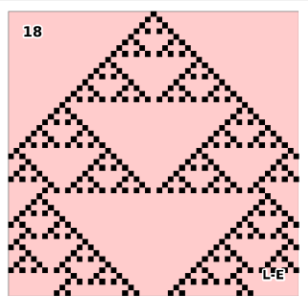



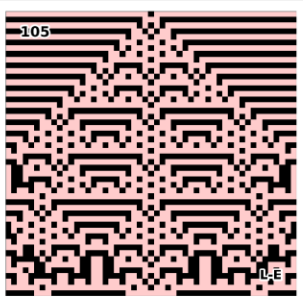
(transient class) - (period class)

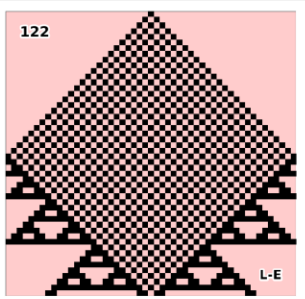
# LINEAR EXPONENTIAL

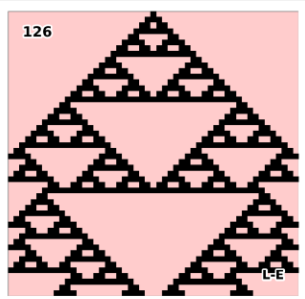
9 Rules

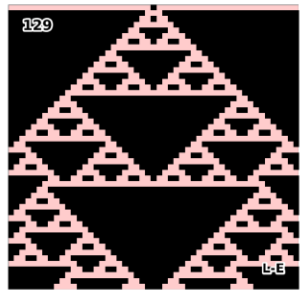
Again, the ECA ‘branches out’ until it hits the wraparound, but this time explores almost every other state before forming a cycle.

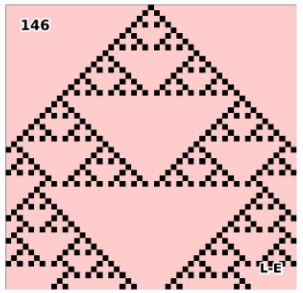


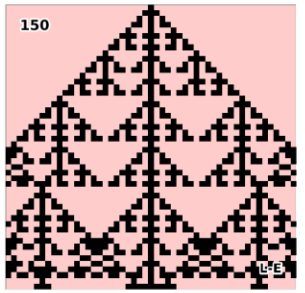



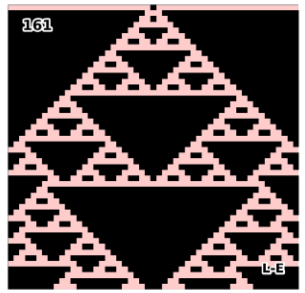



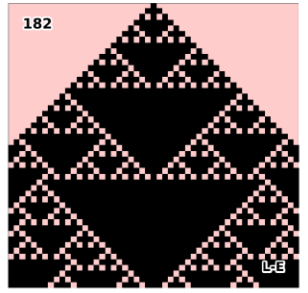



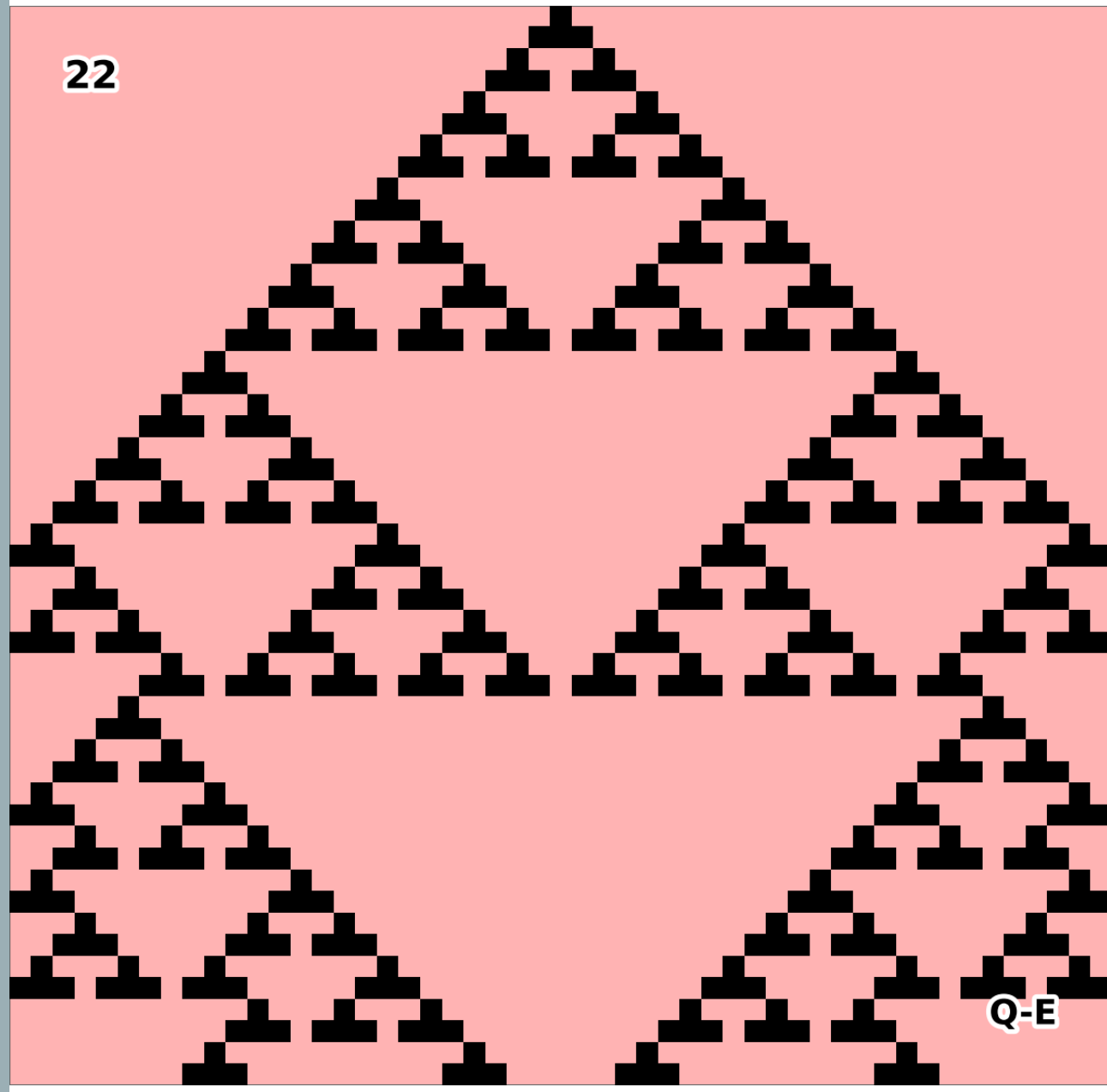



(transient class) - (period class)

# QUADRATIC EXPONENTIAL

1 Rules

I'm not sure how to interpret quadratic transients.

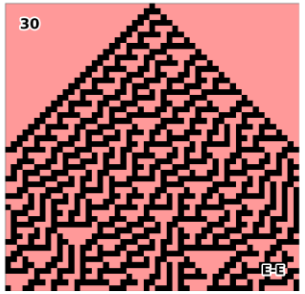


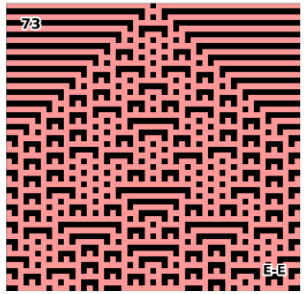
(transient class) - (period class)

# EXPONENTIAL EXPONENTIAL

7 Rules

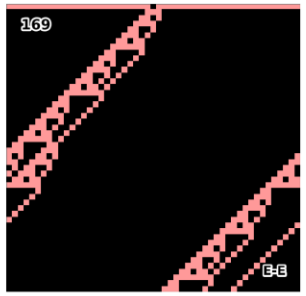
These are generally seem as the ‘chaotic’ rules.

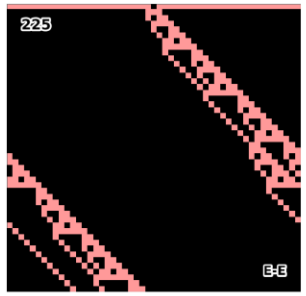








# LIMITATIONS

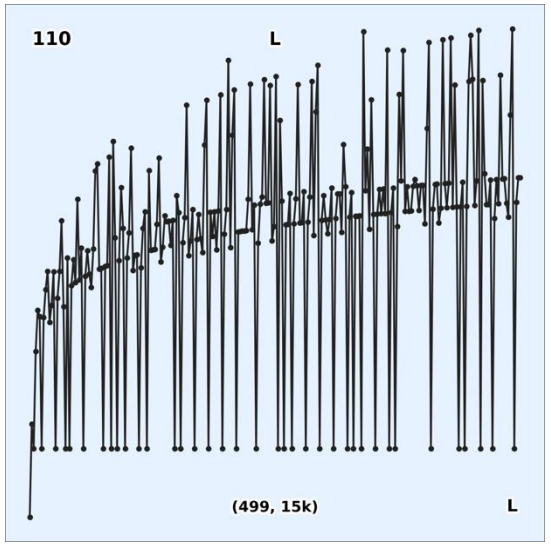
# CONCLUSION

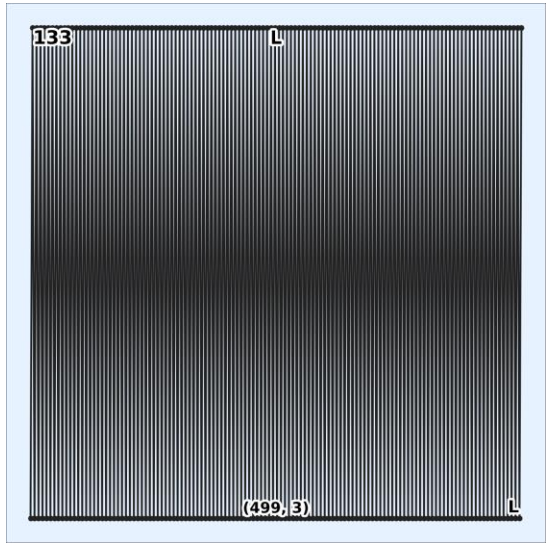
- Classifying ECA based on their cycle length and transient growth can provide valuable intuitions on how the ECAs behave.
- However, the complexity classes are not descriptive enough to describe certain automata.
- Nonetheless, investigating these patterns can provide new insights on ECA, and the graphs generated here are a first step in doing so.

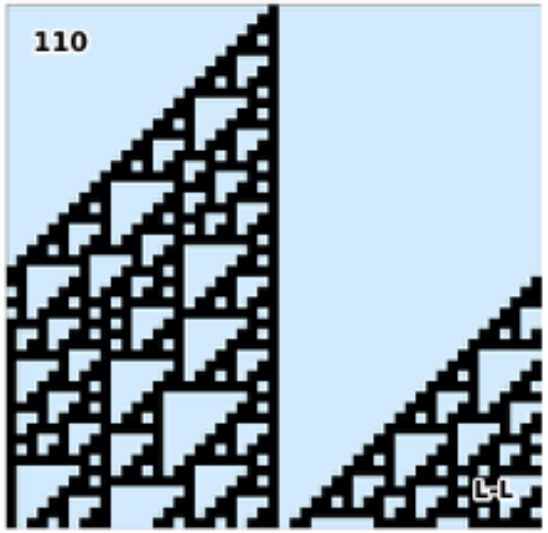
# OSCILLATING LENGTHS

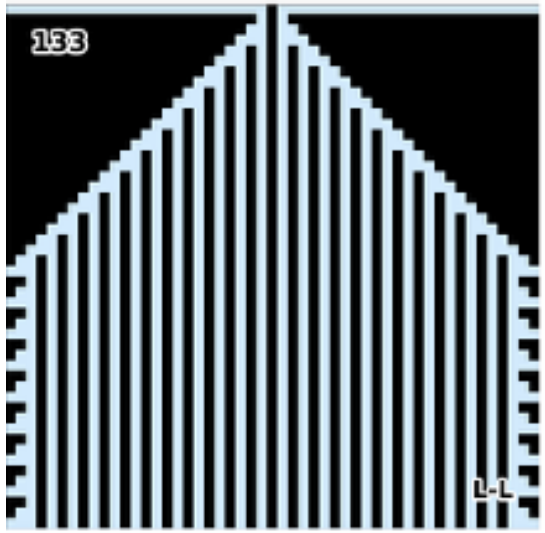
Rule 110, which is universal, has wild jumps.

Rule 133 is stuck between two values, Due to parity.



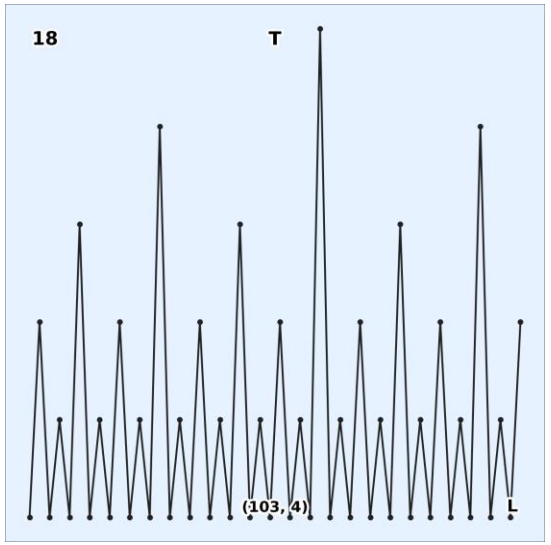



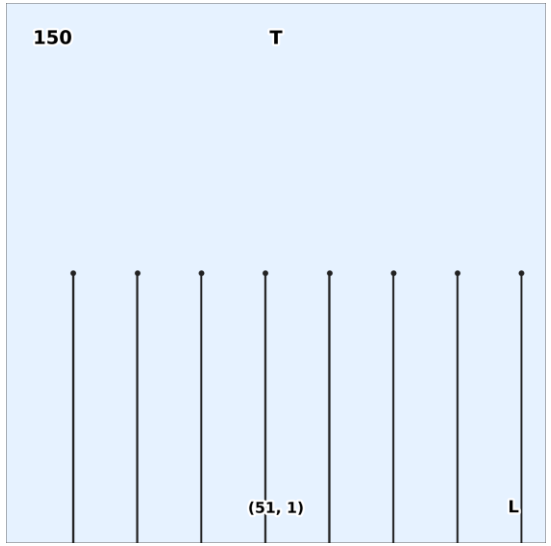



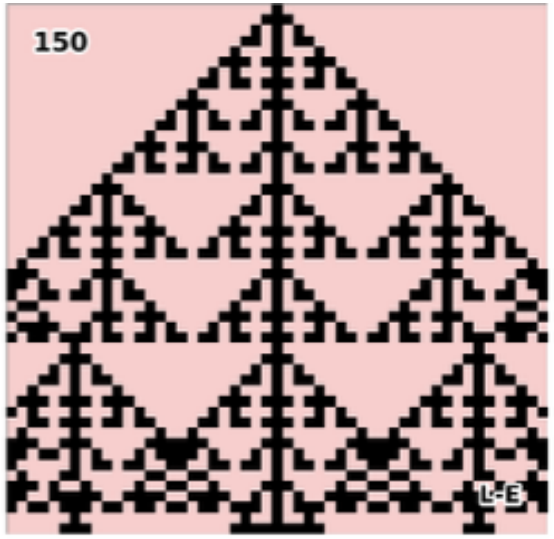
# OSCILLATING TRANSIENTS

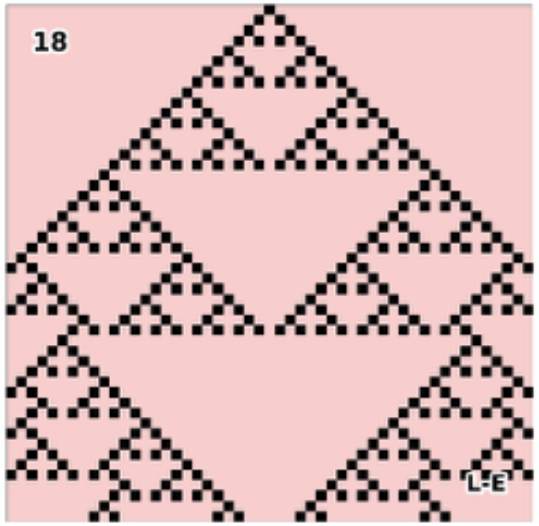
Rule 18, has interesting behavior that Probably depends on the binary representation of n.

Rule 133 has transients probably depending on the factors of n.



# OSCILLATING BOTH- RULE 109

These classifications are due to the atrocity
That is my classify function.

But either way they would not really fit
Any of the complexity classes I proposed.