# Foundations of probability theory

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## 1 • Introduction

In the usual measure-theoretical formulation of probability theory, the following result is a corollary of the Law of Large Numbers:

THEOREM 1.1: *The frequency interpretation of probability*

*Let* $\mathsf{X}, \mathsf{X}_1, \mathsf{X}_2, \ldots$ *be i.i.d. real-valued random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For every* $B \in \mathcal{B}(\mathbb{R})$ *we have*

$$\mathbb{P}(\mathsf{X} \in B) = \lim_{n \to \infty} \frac{|\{j \in \{1, \ldots, n\} \mid \mathsf{X}_j(\omega) \in B\}|}{n}$$

*for* $\mathbb{P}$*-almost all* $\omega \in \Omega$.

PROOF. Thorbjørnsen Korollar 13.6.2. [Maybe reproduce the proof given LLN for completeness?] □

That is, given a sequence $(\mathsf{X}_n)_{n \in \mathbb{N}}$, and one extra $\mathsf{X}$, of i.i.d. random variables, the probability that $\mathsf{X}$ lies in some Borel set $B$ can be thought of as the proportion of the $\mathsf{X}_n$ that lie in $B$, as $n$ tends to infinity. In other words, probability is a measure of the *frequency* with which an outcome of a random experiment obtains, if we repeat the experiment many times.

Whether or not this is the correct interpretation of probability as it occurs in the natural world we will not discuss here. Nonetheless the above result is an uncontroversial consequence of the theory, and it certainly aligns with our intuitive understanding of probability.

In this note we turn this result on its head and attempt to use it to motivate the formalisation of probability theory in terms of measure spaces. As we shall see, this is not entirely successful and will require some leaps that are not entirely justified by our conceptual grasp of probability.

# 2 • Event spaces as Boolean algebras

If the probability of an event is supposed to be a measure of how often this event occurs, then it seems reasonable to assume that we are, in principle, able to distinguish when this event occurs. For example, rolling a six-sided die the state of affairs 'the result of the die roll is three' is an event, since we can determine the outcome of the roll just by looking at the die. To take another example, throwing a ball the state of affairs 'the ball was thrown more than 50 metres' is also an event: That is, we can determine whether or not the length of the throw was strictly greater than 50 metres.

One might take a different view: One might agree that it is possible to *affirm* that the length of the throw, measures in metres, lies in the interval $(50,\infty)$. If the length is $L$, we can simply take a ruler whose subdivisions are smaller than $L-50$ in metres. However, one might disagree that it is possible to *refute* that $L \in (50,\infty)$. For if $L$ is exactly 50 metres, then since any measurement of $L$ carries some error, it is in practice impossible to determine whether $L$ is 50 (or slightly smatter), or whether it is slightly larger than 50. We will not pursue this line further but refer the reader to Vickers (1989) for more on this *logic of affirmative assertions*.

To be precise, after performing the relevant random experiment, we will assume that we are always able to decide whether or not the event has occured or not. In particular, if $E$ is an event, then the state of affairs '$E$ does not obtain' is also an event, denoted $E'$: If $E$ obtains, then $E'$ does not. And conversely, if $E$ does not obtain, then $E'$ does obtain. We call $E'$ the *complement of* or the *complementary event to $E$*.[1]

## References

Kolmogorov, A. N. (1956). *Foundations of the Theory of Probability*. 2nd ed. Chelsea Publishing Company. 84 pp.

Vickers, Steven (1989). *Topology via Logic*. 1st ed. Cambridge University Press. 200 pp. ISBN: 0-521-36062-5.

[1] In contrast, in the logic of affirmative assertions we do not allow complementation (i.e. negation). Hence it may not be surprising that this logic ends up being closely tied to topology.