Foundations of probability theory

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1 • Introduction

In the usual measure-theoretical formulation of probability theory, the following result is a corollary of the Law of Large Numbers:

THEOREM 1.1: *The frequency interpretation of probability*

Let $\mathsf{X}, \mathsf{X}_1, \mathsf{X}_2, \ldots$ *be i.i.d. real-valued random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For every* $B \in \mathcal{B}(\mathbb{R})$ *we have*

$$\mathbb{P}(\mathsf{X} \in B) = \lim_{n \to \infty} \frac{|\{j \in \{1, \ldots, n\} \mid \mathsf{X}_j(\omega) \in B\}|}{n}$$

for $\mathbb{P}$*-almost all* $\omega \in \Omega$.

PROOF. Thorbjørnsen Korollar 13.6.2. [Maybe reproduce the proof given LLN for completeness?] □

That is, given a sequence $(\mathsf{X}_n)_{n \in \mathbb{N}}$, and one extra X, of i.i.d. random variables, the probability that X lies in some Borel set B can be thought of as the proportion of the X_n that lie in B, as n tends to infinity. In other words, probability is a measure of the *frequency* with which an outcome of a random experiment obtains, if we repeat the experiment many times.

Whether or not this is the correct interpretation of probability as it occurs in the natural world we will not discuss here. Nonetheless the above result is an uncontroversial consequence of the theory, and it certainly aligns with our intuitive understanding of probability.

In this note we turn this result on its head and attempt to use it to motivate the formalisation of probability theory in terms of measure spaces. As we shall see, this is not entirely successful and will require some leaps that are not entirely justified by our conceptual grasp of probability.

2 • Metric Boolean algebras

We begin by reviewing some of the purely algebraic properties of Boolean algebras.

Definition 2.1: *Boolean algebras*

A *Boolean algebra* is a structure $\langle B; \vee, \wedge, ', 0, 1\rangle$ such that

(i) $\langle B; \vee, \wedge\rangle$ is a distributive lattice,

(ii) 0 and 1 are elements of B such that $x \vee 0 = x$ and $x \wedge 1$ for all $x \in B$, and

(iii) $'$ is a unary operation such that $x \vee x' = 1$ and $x \wedge x' = 0$ for all $x \in B$.

The binary operations $\vee$ and $\wedge$ are called *join* and *meet*, respectively. For $x \in B$ the element x' is called the *complement* of x. In a general bounded lattice L, an element $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$ is called a complement of $x \in L$. If L is distributive, complements are unique. Recall also that the lattice structure on B induces a partial order $\leq$ such that $x \leq y$ if and only if $x \vee y = y$ for $x, y \in B$.

Let B be a Boolean algebra. For $x, y \in B$ we define the *symmetric difference* between x and y by

$$x \triangle y = (x \wedge y') \vee (y \wedge x').$$

If $x \triangle y = 0$, then it is easy to show that $x = y$.

Before proceeding we note the following technical result that we shall need later:

Lemma 2.2

Let $\langle B; \vee, \wedge, ', 0, 1\rangle$ be a Boolean algebra. Let $(a_i)_{i\in I}$ be a collection of elements in B such that $a_i \wedge a_j = 0$ when $i \neq j$. If $\bigvee_{i\in I} a_i \in B$, then $\bigvee_{i\in J} a_i \in B$ for any $J \subseteq I$ such that $I \setminus J$ is finite (i.e. J is cofinite).

Proof. Let $(a_i)_{i\in I}$ be such a collection of elements, and let $J \subseteq I$ be cofinite. It suffices to prove the lemma in the case $I \setminus J = \{i_0\}$, since the general case then follows by induction. We claim that

$$\bigvee_{i\in J} a_i = a'_{i_0} \wedge \bigvee_{i\in I} a_i.$$

Let $j \in J$ and notice that, since $a_{i_0} \wedge a_j = 0$,

$$a'_{i_0} \wedge a_j = (a'_{i_0} \wedge a_j) \vee (a_{i_0} \wedge a_j) = (a'_{i_0} \vee a_{i_0}) \wedge a_j = 1 \wedge a_j = a_j.$$

Now because $a_j \leq \bigvee_{i\in I} a_i$ we get

$$a_j = a'_{i_0} \wedge a_j \leq a'_{i_0} \wedge \bigvee_{i\in I} a_i.$$

Conversely, suppose that $a_j \leq s$ for all $j \in J$. Then $a_i \leq a_{i_0} \vee s$ for all $i \in I$, so

$$\bigvee_{i \in I} a_i \leq a_{i_0} \vee s.$$

It follows that

$$a'_{i_0} \wedge \bigvee_{i \in I} a_i \leq a'_{i_0} \wedge (a_{i_0} \vee s) = 0 \vee (a_{i_0} \wedge s) \leq s,$$

as desired. □

Definition 2.3: *Abstract measure spaces*

A *measure* on a Boolean algebra B is a map $\mu\colon B \to [0,\infty)$ such that $x \wedge y = 0$ implies

$$\mu(x \vee y) = \mu(x) + \mu(y) \tag{2.1}$$

for all $x, y \in B$. If μ is a measure on a Boolean algebra B, then we call the pair (B, μ) a *generalised abstract measure space*. If $x \neq 0$ implies that $\mu(x) > 0$, then μ is called *positive definite*. If $\mu(1) = 1$, then we call μ a *probability measure*.

It is clear that $\mu(\emptyset) = 0$ and that μ is increasing. It follows that $\mu(x) \leq \mu(1)$ for all $x \in B$. Notice that we require that μ is finite, but this is no restriction since we are ultimately interested in the case where μ is a probability measure.

The property (2.1) is called *(finite) additivity* of μ. We will later define more restrictive structures and measures upon them, hence the adjective 'generalised'.

Proposition 2.4: *Boole's inequality*

Let (B, μ) be a generalised abstract measure space B. Then for any $x, y \in B$ we have

$$\mu(x \vee y) \leq \mu(x) + \mu(y).$$

Proof. Notice that

$$(x \wedge y') \wedge y = x \wedge (y' \wedge y) = 0,$$

and that

$$(x \wedge y') \vee y = (x \vee y) \wedge (y' \vee y) = x \vee y.$$

It follows by additivity of μ that

$$\mu(x \vee y) = \mu\big((x \wedge y') \vee y\big) = \mu(x \wedge y') + \mu(y) \leq \mu(x) + \mu(y),$$

as desired. □

Definition 2.5: *Metric Boolean algebras*

A *pseudometric Boolean algebra* is a tuple (B,ρ), where B is a Boolean algebra and ρ is a pseudometric on B such that the maps $x \mapsto x'$, $(x,y) \mapsto x \vee y$, and $(x,y) \mapsto x \wedge y$ are continuous.

If ρ is a metric, then (B,ρ) is called a *metric Boolean algebra*.

Next we equip abstract measure spaces with a canonical pseudometric. If (B,μ) is an abstract measure space, define a map $\rho_\mu \colon B \times B \to [0,\infty)$ by

$$\rho_\mu(x,y) = \mu(x \triangle y) \tag{2.2}$$

for $x,y \in B$. The next proposition shows that ρ_μ is in fact a pseudometric. We will always equip an abstract measure space with this pseudometric.

Proposition 2.6

Given an abstract measure space (B,μ), the map ρ_μ defined in (2.2) makes (B,ρ_μ) into a pseudometric Boolean algebra. Furthermore, ρ_μ is a metric if and only if μ is positive definite.

Notice that the measure μ can be written in terms of ρ_μ, since $\mu(x) = \rho_\mu(x,0)$. Furthermore, since (pseudo)metrics are continuous, it follows that $\mu(x_n) \to \mu(x)$ whenever $x_n \to x$ in B.

Proof. *ρ_μ is a pseudometric*: We only need to prove the triangle inequality. To this end, let $x,y,z \in B$ and notice that

$$\begin{aligned} x \wedge z' &= (x \wedge z) \wedge (y' \vee y) \\ &= (x \wedge z' \wedge y') \vee (x \wedge z' \wedge y) \\ &\leq (x \wedge y') \vee (y \wedge z'). \end{aligned}$$

Similarly we have $z \wedge x' \leq (z \wedge y') \vee (y \wedge x')$. It follows that

$$\begin{aligned} x \triangle z &= (x \wedge z') \vee (z \wedge x') \\ &\leq (x \wedge y') \vee (y \wedge x') \vee (y \wedge z') \vee (z \wedge y') \\ &= (x \triangle y) \vee (y \triangle z). \end{aligned}$$

Now Boole's inequality implies that

$$\rho_\mu(x,z) = \mu(x \triangle y) \leq \mu(x \triangle y) + \mu(y \triangle z) = \rho_\mu(x,y) + \rho_\mu(y,z),$$

as desired.

Continuity of lattice operations: Let $x \in B$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B that converges to x. Notice that $x_n' \triangle x' = x_n \triangle x$, so $\rho_\mu(x_n', x') = \rho_\mu(x_n, x)$. Hence the complementation map $x \mapsto x'$ is continuous.

Let further $(y_n)_{n\in\mathbb{N}}$ be a sequence converging to a point $y \in B$. A short calculation shows that

$$\begin{aligned}(x_n \vee y_n) \triangle (x \vee y) &= (x_n \wedge x' \wedge y') \vee (y_n \wedge x' \wedge y') \vee (x \wedge x_n' \wedge y_n') \vee (y \wedge x_n' \wedge y_n') \\ &\leq (x_n \wedge x') \vee (x \wedge x_n') \vee (y_n \wedge y') \vee (y \wedge y_n') \\ &= (x_n \triangle x) \vee (y_n \triangle y).\end{aligned}$$

Thus Boole's inequality shows that

$$\rho_\mu(x_n \vee y_n, x \vee y) \leq \rho_\mu(x_n, x) + \rho_\mu(y_n, y), \tag{2.3}$$

which implies continuity of the join map $(x, y) \mapsto x \vee y$.

Finally, continuity of the meet map $(x, y) \mapsto x \wedge y$ follows since

$$x \wedge y = \big(x' \vee y'\big)',$$

so it is a composition of continuous functions.

Positive definiteness: The last claim follows directly from the fact that $x \triangle y = 0$ if and only if $x = y$ for all $x, y \in B$. □

It is well-known that any (pseudo)metric space has a completion, i.e. can be isometrically embedded as a dense subset of a complete (pseudo)metric space. See for instance Willard Corollary 24.5. A natural question is then: If (B, ρ) is a (pseudo)metric Boolean algebra with metric completion $(\overline{B}, \overline{\rho})$, does $\overline{B}$ also carry the structure of a Boolean algebra?

This is indeed the case, and we sketch the construction: Let $x, y \in \overline{B}$, and let (x_n) and (y_n) be sequences in B that converge to x and y, respectively. Then these are Cauchy sequences in B, and the calculation leading to (2.3) show that $(x_n \vee y_n)$ is also a Cauchy sequence. Thus it converges to some element of $\overline{B}$. Denote it $x \vee y$. We define x' and $x \wedge y$ similarly. It is easy to check that these operations satisfy the conditions in Definition 2.1. Furthermore, the completion $\overline{\rho}$ of the pseudometric ρ makes $(\overline{B}, \overline{\rho})$ into a pseudometric Boolean algebra.

This takes care of the metric structure. The next proposition shows that we can also extend the measure on an abstract measure space to its completion.

PROPOSITION 2.7

Let (B, μ) be an abstract measure space, and let $(\overline{B}, \overline{\rho}_\mu)$ be the completion of (B, ρ_μ).

Define a map $\overline{\mu}\colon \overline{B} \to [0,\infty)$ by

$$\overline{\mu}(x) = \lim_{n\to\infty} \mu(x_n), \tag{2.4}$$

where $(x_n)_{n\in\mathbb{N}}$ is any sequence in B that converges to x. Then $\overline{\mu}$ is a well-defined measure on $\overline{B}$.

PROOF. First notice that for any $x \in \overline{B}$ there does in fact exist a sequence in B converging to x. If $(x_n)_{n\in\mathbb{N}}$ is such a sequence, it is a Cauchy sequence in B, and the reverse triangle inequality shows that $(\mu(x_n))$ is a Cauchy sequence in $\mathbb{R}$, hence convergent. Thus the limit on the right-hand side of (2.4) exists.

Now let (y_n) be another sequence in B that approximates x. Another application of the reverse triangle inequality then shows that

$$|\mu(x_n) - \mu(y_n)| \leq \rho_\mu(x_n, y_n) \leq \rho_\mu(x_n, x) + \rho_\mu(y_n, x) \to 0.$$

Hence $\mu(x_n)$ and $\mu(y_n)$ converge to the same value, and thus $\overline{\mu}$ is well-defined.

Next we show that $\overline{\mu}$ is finitely additive. Let $x, y \in \overline{B}$ with $x \wedge y = 0$ and choose approximating sequences (x_n) and (y_n) in B. Then

$$(x_n \vee y_n) \wedge (x_n \wedge y_n)' = \big(x_n \wedge (x_n \wedge y_n)'\big) \vee \big(y_n \wedge (x_n \wedge y_n)'\big)$$

is the join of disjoint elements of B, so

$$\mu\big((x_n \vee y_n) \wedge (x_n \wedge y_n)'\big) = \mu\big(x_n \wedge (x_n \wedge y_n)'\big) + \mu\big(y_n \wedge (x_n \wedge y_n)'\big).$$

By continuity of the lattice operations we have $(x_n \wedge y_n)' \to 1$, so the three elements given as arguments to μ above are elements in approximating sequences for $x \vee y$, x and y respectively. By definition of $\overline{\mu}$ it follows that

$$\overline{\mu}(x \vee y) = \overline{\mu}(x) + \overline{\mu}(y)$$

as desired. □

LEMMA 2.8

Let (B,μ) be an abstract measure space. Every monotonic sequence in B is a Cauchy sequence.

PROOF. Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence in B. Then since $x_n \leq 1$ we also have $\mu(x_n) \leq \mu(1) < \infty$ for all $n \in \mathbb{N}$. Thus the sequence $(\mu(x_n))$ is a bounded increasing sequence in $\mathbb{R}$, hence it converges to some $\alpha \geq 0$. For $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\mu(x_n) \in (\alpha - \varepsilon, \alpha]$ for all $n \geq N$.

If then $m, n \geq N$ with $m \leq n$, then $x_m \leq x_n$ and so

$$x_m \wedge x_n' \leq x_n \wedge x_n' = 0.$$

Hence $x_m \triangle x_n = x_n \wedge x_m'$, so it follows that

$$\rho_\mu(x_m, x_n) = \mu(x_n \wedge x_m') = \mu(x_n) - \mu(x_m) < \varepsilon.$$

Thus (x_n) is indeed a Cauchy sequence. The case where (x_n) is decreasing is similar. □

Proposition 2.9

Let (B, ρ) be a metric Boolean algebra with completion $(\overline{B}, \overline{\rho})$. Any sequence $(x_n)_{n\in\mathbb{N}}$ in B has a join in $\overline{B}$, and

$$\bigvee_{n\in\mathbb{N}} x_n = \lim_{n\to\infty} \bigvee_{i\leq n} x_i.$$

Similarly for meets.

As far as I know, it is not possible to generalise this result to the case where ρ is only a pseudometric. [We also need the measure, don't we? Kolmogorov only looks at this when there is a positive definite measure.]

Proof. The sequence $(\bigvee_{i\leq n} x_i)_{n\in\mathbb{N}}$ is increasing, so by Lemma 2.8 it has a limit $s \in \overline{B}$. For $k \in \mathbb{N}$ and $n \geq k$ we have $x_k \leq \bigvee_{i\leq n} x_i$, i.e.

$$x_k \vee \bigvee_{i\leq n} x_i = \bigvee_{i\leq n} x_i.$$

Taking the limit as $n \to \infty$, continuity of (binary) joins implies that $x_k \vee s = s$, or $x_k \leq s$. Thus s is an upper bound of the sequence (x_n).

On the other hand, if $t \in \overline{B}$ is an upper bound of (x_n), then $x_k \leq t$. We have just seen that taking limits preserves inequalities, so this implies that $s \leq t$ as desired.

The corresponding result for meets follows similarly, or from the fact that complementation is continuous. □

The previous proposition motivates the following definition:

Definition 2.10: *Abstract σ-algebra*

An *abstract σ-algebra* is a Boolean algebra $\langle B; \vee, \wedge, ', 0, 1\rangle$ with countable joins. That is, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements in B, then their join $\bigvee_{n\in\mathbb{N}} a_n$ exists.

If the join $\bigvee_{n\in\mathbb{N}} a_n$ exists, then it follows by taking complements that the meet $\bigwedge_{n\in\mathbb{N}} a_n$ also exists. In the context of abstract measure spaces we obtain the following:

DEFINITION 2.11: *Abstract measure spaces*

Let (B, ρ) be

3 • Event spaces as Boolean algebras

If the probability of an event is supposed to be a measure of how often this event occurs, then it seems reasonable to assume that we are, in principle, able to distinguish when this event occurs. For example, rolling a six-sided die the state of affairs 'the result of the die roll is three' is an event, since we can determine the outcome of the roll just by looking at the die. To take another example, throwing a ball the state of affairs 'the ball was thrown more than 50 metres' is also an event: That is, we can determine whether or not the length of the throw was strictly greater than 50 metres.

One might take a different view: One might agree that it is possible to *affirm* that the length of the throw, measured in metres, lies in the interval $(50, \infty)$. If the length L does in fact lie in the above interval, we can simply take a ruler whose subdivisions are smaller than $L - 50$ in metres. However, one might disagree that it is possible to *refute* that $L \in (50, \infty)$. For if L is exactly 50 metres, then since any measurement of L carries some error, it is in practice impossible to determine whether L is 50 (or slightly smatter), or whether it is slightly larger than 50. We will not pursue this line further but refer the reader to Vickers (1989) for more on this *logic of affirmative assertions*.

To be precise, after performing the relevant random experiment, we will assume that we are always able to decide whether or not the event has occured or not. In particular, if E is an event, then the state of affairs 'E does not obtain' is also an event, denoted E': If E obtains, then E' does not. And conversely, if E does not obtain, then E' does obtain. We call E' the *complement of* or the *complementary event to* E.[1]

Next consider two events E_1 and E_2. Since we are able to decide whether each of them have obtained, the same is true for the event 'both E_1 and E_2 have obtained' and the event 'at least one of E_1 and E_2 has obtained'. The first is called the *conjunction* of E_1 and E_2 and is denoted $E_1 \wedge E_2$, and the second is the *disjunction* $E_1 \vee E_2$ of E_1 and E_2.

Finally it seems natural to allow an 'empty event' 0 which never occurs, as well as a 'universal event' 1 that always occurs.

With these operations in place, we claim that this logic of events constitute a *Boolean algebra*. We recall the definition of such a structure:

[1] In contrast, in the logic of affirmative assertions we do not allow complementation (i.e. negation). Hence it may not be surprising that this logic ends up being closely tied to topology.

Definition 3.1: *Boolean algebras*

A *Boolean algebra* is a structure $\langle B; \vee, \wedge, ', 0, 1\rangle$ such that

(i) $\langle B; \vee, \wedge\rangle$ is a distributive lattice,

(ii) $a \vee 0 = a$ and $a \wedge 1$ for all $a \in B$, and

(iii) $a \vee a' = 1$ and $a \wedge a' = 0$ for all $a \in B$.

We leave it to the reader to reflect on whether or not Boolean algebras offer a reasonable way to formalise our intuition of events (in the opinion of the author, they do). This leads naturally to the following definition:

Definition 3.2: *Generalised abstract probability spaces*

A *generalised abstract probability space* is a pair $(\mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a set and $\mathbb{P}\colon \mathcal{F} \to [0,1]$, such that

(i) $\langle \mathcal{F}; \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra,

(ii) $\mathbb{P}(1) = 1$, and

(iii) $E \wedge F = 0$ implies $\mathbb{P}(E \vee F) = \mathbb{P}(E) + \mathbb{P}(F)$ for all $E, F \in \mathcal{F}$.

In [ref] we consider more restrictive abstract probability spaces, hence the adjective ‘generalised’. We refer to the property (iii) as *(finite) additivity* of $\mathbb{P}$. If E and F are events such that $E \wedge F = 0$, then we call E and F *disjoint* or *incompatible*.

We define an ordering on events in the usual way: If E and F are events such that $E \vee F = F$, then we write $E \leq F$ and say that E *implies* F. The intuition is that, if F obtains just when *either* E or F obtains, then there is no way for E to obtain without F also doing so.

Lemma 3.3

Let $(\mathcal{F}, \mathbb{P})$ be a generalised abstract probability space.

(i) $\mathbb{P}(0) = 0$.

(ii) *$\mathbb{P}$ is increasing, i.e. $E \leq F$ implies $\mathbb{P}(E) \leq \mathbb{P}(F)$ for all $E, F \in \mathcal{F}$.*

Proof. The first claim follows since $0 \vee 0 = 0 \wedge 0 = 0$, so by additivity of $\mathbb{P}$,

$$\mathbb{P}(0) = \mathbb{P}(0 \vee 0) = \mathbb{P}(0) + \mathbb{P}(0),$$

hence $\mathbb{P}(0) = 0$.

To prove the second claim, first notice that if $E \le F$ then $F = E \vee (F \wedge E')$. But $E \wedge (F \wedge E') = 0$, so

$$\mathbb{P}(F) = \mathbb{P}\big(E \vee (F \wedge E')\big) = \mathbb{P}(E) + \mathbb{P}(F \wedge E') \ge \mathbb{P}(E),$$

as claimed. □

4 • Abstract σ-algebras and continuity

Of course, the map $\mathbb{P}$ above is supposed to be a probability measure. But measures are *countably* additive, not just finitely so. It is however difficult to justify this on conceptual groups. In fact, according to Kolmogorov himself:

> Since the new axiom [countable additivity] is essential for infinite fields of probability only, it is almost impossible to elucidate its empirical meaning. (...) For, in describing any observable random process we can obtain only finite fields of probability. Infinite fields of probability occur only as idealized models of real random processes. *We limit ourselves, arbitrarily, to only those models which satisfy Axiom VI* [countable additivity]. This limitation has been found expedient in researches of the most diverse sort. (Kolmogorov 1956)

And furthermore:

> [S]omewhat more complicated problems require, if the theory is to be simple and tractable, that probability be subject to the *axiom of denumerable additivity*. However, the justification of that axiom remains purely empirical, in that we have not yet encountered any interesting problem for which we have not been able to construct a probability field conforming to the axiom in question. (Kolmogorov and Jeffrey 1995)

Kolmogorov (1956) introduces the axiom in the following form:

Definition 4.1: *Axiom of continuity*

A generalised abstract probability space $(\mathcal{F}, \mathbb{P})$ is said to satisfy the *axiom of continuity* if it has the following property: If $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of events in $\mathcal{F}$ such that $\bigwedge_{n\in\mathbb{N}} E_n$ exists and equals 0, then $\lim_{n\to\infty} \mathbb{P}(E_n) = 0$.

This axiom implies that $\mathbb{P}$ is countably additive when the join of a sequence of elements in $\mathcal{F}$ also lies in $\mathcal{F}$. To prove this fact we need a lemma, which is obvious if $\mathcal{F}$ is a set algebra:

LEMMA 4.2

Let $\langle B; \vee, \wedge, ', 0, 1\rangle$ be a Boolean algebra. Let $(a_i)_{i\in I}$ be a collection of elements in B such that $a_i \wedge a_j = 0$ when $i \neq j$. If $\bigvee_{i\in I} a_i \in B$, then $\bigvee_{i\in J} a_i \in B$ for any $J \subseteq I$ such that $I \setminus J$ is finite (i.e. J is cofinite).

PROOF. Let $(a_i)_{i\in I}$ be such a collection of elements, and let $J \subseteq I$ be cofinite. It suffices to prove the lemma in the case $I \setminus J = \{i_0\}$, since the general case then follows by induction. We claim that

$$\bigvee_{i\in J} a_i = a'_{i_0} \wedge \bigvee_{i\in I} a_i.$$

Let $j \in J$ and notice that, since $a_{i_0} \wedge a_j = 0$,

$$a'_{i_0} \wedge a_j = (a'_{i_0} \wedge a_j) \vee (a_{i_0} \wedge a_j) = (a'_{i_0} \vee a_{i_0}) \wedge a_j = 1 \wedge a_j = a_j.$$

Now because $a_j \leq \bigvee_{i\in I} a_i$ we get

$$a_j = a'_{i_0} \wedge a_j \leq a'_{i_0} \wedge \bigvee_{i\in I} a_i.$$

Conversely, suppose that $a_j \leq s$ for all $j \in J$. Then $a_i \leq a_{i_0} \vee s$ for all $i \in I$, so

$$\bigvee_{i\in I} a_i \leq a_{i_0} \vee s.$$

It follows that

$$a'_{i_0} \wedge \bigvee_{i\in I} a_i \leq a'_{i_0} \wedge (a_{i_0} \vee s) = 0 \vee (a_{i_0} \wedge s) \leq s,$$

as desired. □

PROPOSITION 4.3: *The generalised addition theorem*

Let $(\mathcal{F}, \mathbb{P})$ be a generalised abstract probability space satisfying the axiom of continuity. If $(E_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint events in $\mathcal{F}$ such that $E = \bigvee_{n\in\mathbb{N}} E_n \in \mathcal{F}$, then

$$\mathbb{P}(E) = \sum_{n=1}^{\infty} \mathbb{P}(E_n).$$

PROOF. By Lemma 2.2, $R_n = \bigvee_{i>n} E_i$ exists in $\mathcal{F}$, and we claim that $\bigwedge_{n\in\mathbb{N}} R_n = 0$. Let $L \in \mathcal{F}$ be a lower bound of R_n for $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$ we have

$$L \vee \bigvee_{i>n} E_i = \bigvee_{i>n} E_i,$$

and taking the meet of each side with E_n' yields $L \wedge E_n' = 0$. Hence $\bigvee_{n\in\mathbb{N}} L \wedge E_n = 0$. Now notice that $\bigwedge_{n\in\mathbb{N}} E_n = 0$, since $E_n \wedge E_m = 0$ when $n \neq m$. It follows that $\bigvee_{n\in\mathbb{N}} E_n' = 1$, and so

$$L = L \wedge \bigvee_{n\in\mathbb{N}} E_n' = \bigvee_{n\in\mathbb{N}} L \wedge E_n' = 0.$$

Thus $\bigwedge_{n\in\mathbb{N}} R_n = 0$ as claimed. It now follows from finite additivity of $\mathbb{P}$ and the axiom of continuity that

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E_i) + \mathbb{P}(R_n) \to \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

as $n \to \infty$ as desired. □

Of course, we are not always guaranteed that a generalised abstract probability space is closed under countable joins.

Definition 4.4: *Abstract σ-algebra*

An *abstract σ-algebra* is a Boolean algebra $\langle B; \vee, \wedge, ', 0, 1\rangle$ with countable joins. That is, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements in B, then their join $\bigvee_{n\in\mathbb{N}} a_n$ exists.

If the join $\bigvee_{n\in\mathbb{N}} a_n$ exists, then it follows by taking complements that the meet $\bigwedge_{n\in\mathbb{N}} a_n$ also exists.

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