

# Semantic foundations of equality saturation

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## Abstract

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Equality saturation is an emerging technique for program and query optimization developed in the programming language community. It performs term rewriting over an E-graph, a data structure that compactly represents a program space. Despite its popularity, the theory of equality saturation lags behind the practice. In this paper, we define a fixpoint semantics of equality saturation based on tree automata and uncover deep connections between equality saturation and the chase. We characterize the class of chase sequences that correspond to equality saturation. We study the complexities of terminations of equality saturation in three cases: single-instance, all-term-instance, and all-E-graph-instance. Finally, we define a syntactic criterion based on acyclicity that implies equality saturation termination.

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## 1 Introduction

Given a set of identities between terms, the word problem asks whether the identities imply two ground terms  $t_1, t_2$  are equivalent, i.e.  $t_1 \approx t_2$ . This fundamental problem has applications including automated theorem proving, program verification, and query equivalence checking. In his Ph.D. thesis, Nelson [23] introduced a data structure called *E-graph* for efficiently answering the word problem. At the core, an E-graph is a compact representation of an equivalence relation over a possibly infinite set of ground terms. During the 2000s, researchers applied E-graphs to program optimization [15, 28]. The compiler populates an E-graph with many equivalent programs, using axiomatic rewrites, then extracts the best program from the equivalent ones. In particular, Tate et.al. [28] coined the term *equality saturation (EqSat)* and gave a procedural description of the algorithm. In 2021, Willsey et al. [32] proposed egg, which introduced important algorithmic improvements and made EqSat practical. Since 2021, EqSat has been applied to a wide range of topics in domain-specific program optimization, including floating-point computation [25] computational fabrication [21], machine learning systems [33], and hardware design [29, 4]. There is also a growing interest in using EqSat for query optimization in data management. For example, EqSat is used to optimize queries in OLAP [6], linear algebra [30], tensor algebra [26], and Datalog [31].

The equality saturation procedure consists of repeatedly selecting an identity  $u \approx v$  from the given set, matching the term  $u$  with the E-graph, then adding the term  $v$  to the E-graph, if it wasn't already there. Equality saturation terminates when no new terms can be added. There are striking connections between equality saturation and database concepts. Zhang et al. [35] observed that the *matching* step is the same as conjunctive query evaluation, and

described significant speedups in `egg` by using a Worst Case Optimal Join algorithm [24] for matching. A recent system, `egglog` [34], unified EqSat and Datalog to improve `egg`'s support for program optimization and program analysis.

In this paper we study another deep connection between equality saturation and the chase procedure for Tuple Generating Dependencies (TGDs) and Equality Generating Dependencies (EGDs) [8]. Our hope is that these results will help solve some of the open problems in equality saturation by using techniques and results for the chase procedure. Before describing our results we give a gentle introduction to EqSat and describe some of its open problems.

► **Example 1.** Consider a simple language with two binary operators  $f, g$  and constant  $a$ . We want to optimize the following term  $t$  (the “8th power” of  $f$  on  $a$ ):

$$t = f(f(f(a, a), f(a, a)), f(f(a, a), f(a, a))) \quad (1)$$

We are given a single identity,  $f(x, x) \approx g(x, x)$ , which says two terms  $f(t_1, t_2)$  and  $g(t_1, t_2)$  are equivalent, provided that  $t_1, t_2$  are equivalent. Starting with the initial term  $t$ , EqSat constructs an E-graph  $G$  and grows it to represent all terms equivalent to  $t$ . The literature defines an E-graph as a set of E-classes, where each E-class is a set of E-nodes, and each E-node is labeled with a function symbol and has a number of E-class children equal to the arity of the symbol. EqSat starts by constructing an E-graph  $G$  representing  $t$ , shown on the left in Figure 1. There are 4 E-classes, each consisting of one single E-node; the E-class  $c_4$  represents precisely the term  $t$ . Next, EqSat repeatedly applies the identity  $f(x, x) \approx g(x, x)$ , by matching the left-hand side  $f(x, x)$  to the E-graph, then adding the right-hand side  $g(x, x)$  to the E-graph: we formalize this in Sec. 3. The resulting E-graph  $H$  is on right of Figure 1. There are 4 E-classes,  $c_1, \dots, c_4$ , each consisting of 1 or 2 E-nodes. For example,  $c_4$  has two E-nodes, and represents two equivalent terms,  $f(t_1, t_2) \approx g(t_1, t_2)$ , where  $t_1, t_2$  are any terms represented by  $c_3$ . By continuing this reasoning, we observe that  $c_4$ , represents a total of  $2^7$  possible terms, namely all terms of the form:  $h(h(h(a, a), h(a, a)), h(h(a, a), h(a, a)))$  where each  $h$  can be either  $f$  or  $g$ .

**Open problems about EqSat** We still understand very little about equality saturation. Most descriptions of EqSat focus on an imperative understanding<sup>1</sup> of equality saturation and E-graphs. E-graphs are described by their individual components (e.g., a hashconsing data structure, a union-find, etc.), and EqSat is commonly defined in pseudocode as a sequence of operations. In other words, the semantics of EqSat is the output of the specific algorithm, if it terminates; if the algorithm diverges, the semantics is undefined.

We also do not know much about when EqSat terminates. The termination problem asks, given a set of rewrite rules,  $G, H$ , before and after EqSat, whether EqSat terminates on a given input E-graph, or whether it terminates on all input E-graphs. This is a fundamental problem of EqSat and has applications in program/query optimization and equivalence checking: If EqSat terminates on the symmetric closure of a

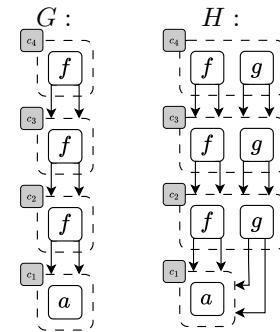


Figure 1 Two E-graphs

<sup>1</sup> A notable exception is `egglog` [34], whose semantics is based on fixpoints instead of implementation details. Some early works also define E-graphs (under a different name like abstract congruence closure) as tree automata similar to ours [27, 2, 13].

set of (variable-preserving) rewrite rules  $\mathcal{R}$ , it decides the word problem of the equational theory defined by  $\mathcal{R}$  (Lemma 22). With an appropriate cost model, EqSat can further pick the optimal program among all programs equivalent to the input, e.g. by using Knuth’s algorithm [16].

**Our contribution** After a review of some background material in Sec. 2, we introduce E-graphs and EqSat in Sec. 3. Our definition, in Sec. 3.1, applies even to the cases when equality saturation does not terminate and, for that purpose, we define the E-graph to be a reachable, deterministic tree automaton, with possibly infinitely many states. By explicitly allowing infinite E-graphs we can define a formal semantics even when EqSat does not terminate. We show that concepts in tree automata are in 1-1 correspondence with those in E-graphs: the automaton states correspond to E-classes, and the transitions correspond to E-nodes. A term is represented by an E-graph iff it is accepted by the E-graph viewed as a standard tree automaton. We prove that, for any two E-graphs there exists at most one homomorphism between them, and, therefore, E-graphs are *rigid* tree automata. Next, in Sec. 3.2, we define a few basic operations on E-graphs, such as E-matching, insertion, congruence closure, and least upper bounds, by relying on tree-automata concepts. Using these operations, we define in Sec. 3.3 an *immediate consequence operator* (ICO), and define EqSat formally as the least fixpoint of the ICO. The least fixpoint always exists and is unique, even if the fixpoint procedure does not terminate, in which case the least fixpoint may be infinite. Finally, we prove an important lemma, called the *Finite Convergence Lemma*, stating that, if the least fixpoint is finite, then equality saturation procedure converges in finitely many steps. This is not immediately obvious because, while E-matching and insertion strictly increase the size of the E-graph, congruence closure may decrease it. A similar proposition fails for TGDs and EGDs: there exists an infinite chase where all instances have bounded size, hence its “limit” is finite.

Next, in Sec. 4 we describe the connection between EqSat and the chase. After a brief review of the chase in Sec. 4.1, we start by presenting a reduction from the Skolem chase to equality saturation, denoted  $\text{SKLCH} \Rightarrow \text{EQSAT}$  (Sec. 4.2), then from equality saturation to the standard chase, denoted  $\text{EQSAT} \Rightarrow \text{STDCH}$  (Sec. 4.3). For  $\text{SKLCH} \Rightarrow \text{EQSAT}$ , given a set of dependencies, we show there exists a set of rewrite rules where EqSat produces an encoded result of the Skolem chase and has the same termination behavior. For  $\text{EQSAT} \Rightarrow \text{STDCH}$ , we show that, given a set of rewrite rules, there exists a set of dependencies where the standard chase produces an encoded result of EqSat (whether it terminates). Since the standard chase is a non-deterministic process, we characterize the type of chase sequences that terminate when EqSat terminates (Theorem 30). We call them *EGD-fair* chase sequences. Intuitively, a chase sequence is called EGD-fair if it applies EGDs to a fixpoint frequently enough. We show in Theorem 30 that,

$$\begin{aligned} \text{EqSat terminates} &\Leftrightarrow \text{one chase sequence terminates} \\ &\Leftrightarrow \text{all EGD-fair chase sequences terminate.} \end{aligned}$$

The notion of EGD-fair chase sequences is of independent interest.

Finally, we present our main decidability results for EqSat in Sec. 5: we show that the single-instance termination problem of EqSat, denoted as  $\mathcal{T}_G^{\text{EQSAT}}$ , is R.E.-complete, and the all-term-instance termination problem of EqSat, denoted as  $\mathcal{T}_{\forall t}^{\text{EQSAT}}$ , is  $\Pi_2$ -complete. Our proof is based on a non-trivial reduction from the Turing machine, first presented in the undecidability proof of the finiteness of congruence classes defined by string rewriting systems [22]. While the single-instance case easily follows from the undecidability of Skolem chase termination, our approach allows us to also prove the  $\Pi_2$ -completeness of the all-term-instance termination case by a reduction from the universal halting problem. We also show

the all-E-graph-instance termination problem of EqSat, denoted as  $\mathcal{T}_{\forall G}^{\text{EQSAT}}$ , is undecidable, although the exact upper bound is open.

We contrast the termination problems of EqSat with those of the Skolem chase and the standard chase. The single-instance termination problems are R.E.-complete in all three cases [19, 5], and the all-instance termination of the Skolem chase ( $\mathcal{T}_{\forall}^{\text{SKLC}\text{H}}$ ) is R.E.-complete as well [19, 11]. This shows that  $\mathcal{T}_{\forall}^{\text{SKLC}\text{H}}$  is easier than  $\mathcal{T}_{\forall t}^{\text{EqSAT}}$ . The case for the standard chase is more interesting. There are two all-instance termination problems of the standard chase: for all database instances, whether all chase sequences terminate in finitely many steps ( $\mathcal{T}_{\forall, \forall}^{\text{STDCH}}$ ), and whether there exists at least one chase sequence that terminate ( $\mathcal{T}_{\forall, \exists}^{\text{STDCH}}$ ). It has been shown  $\mathcal{T}_{\forall, \exists}^{\text{STDCH}}$  is  $\Pi_2$ -complete [12], but the exact complexity of  $\mathcal{T}_{\forall, \forall}^{\text{STDCH}}$  is open. Grahne and Onet showed if we allow one *denial constraint*,  $\mathcal{T}_{\forall, \forall}^{\text{STDCH}}$  is  $\Pi_2$ -complete [12], although Gogacz and Marcinkowski [11] conjectured that this problem is indeed in R.E.

In Sec. 6 we propose a sufficient syntactic criterion that guarantees EqSat termination, called *weak term acyclicity*, which is based on the classic notion of *weak acyclicity* [8]. If a set of rewrite rules is weakly term acyclic, then EqSat terminates for all input E-graphs.

## 2 Background

### 2.1 Term Rewriting Systems

We review briefly the standard definition of a term rewriting system from [1]. A *signature* is a finite set  $\Sigma$  of function symbols with given arities. If  $V$  is a set of variables, then  $T(\Sigma, V)$  denotes the set of terms constructed inductively using symbols from  $\Sigma$  and variables from  $V$ . Members of  $T(\Sigma, V)$  are called *patterns*, and members of  $T(\Sigma) \stackrel{\text{def}}{=} T(\Sigma, \emptyset)$  are called *ground terms*, or simply *terms* thereafter. A *substitution* is a function  $\sigma : V \rightarrow T(\Sigma)$ ; if  $u$  is a pattern, then we denote by  $u[\sigma]$  the term obtained by applying the substitution  $\sigma$  to  $u$ . A *rewrite rule*  $r$  has the form  $lhs \rightarrow rhs$  where  $lhs$  and  $rhs$  are patterns and the variables in  $rhs$  are a subset of those  $lhs$ ,  $\text{Var}(rhs) \subseteq \text{Var}(lhs)$ . A *term rewriting system* (TRS),  $\mathcal{R}$ , is a set of rewrite rules.  $\mathcal{R}$  defines a *rewrite relation*  $\rightarrow_{\mathcal{R}}$  as follows:  $lhs[\sigma] \rightarrow_{\mathcal{R}} rhs[\sigma]$  for any substitution  $\sigma$  and rule  $lhs \rightarrow rhs$  in  $\mathcal{R}$ , and, if  $u \rightarrow_{\mathcal{R}} v$  then  $f(w_1, \dots, w_{i-1}, u, w_{i+1}, \dots, w_k) \rightarrow_{\mathcal{R}} f(w_1, \dots, w_{i-1}, v, w_{i+1}, \dots, w_k)$  for any function symbol  $f \in \Sigma$  of arity  $k$ , and any terms  $w_j$ ,  $j = 1, k; j \neq i$ . Let  $\rightarrow_{\mathcal{R}}^*$  be the reflexive and transitive closure of  $\rightarrow_{\mathcal{R}}$ . We define  $(\leftarrow_{\mathcal{R}}) \stackrel{\text{def}}{=} (\rightarrow_{\mathcal{R}})^{-1}$ ,  $(\leftrightarrow_{\mathcal{R}}) \stackrel{\text{def}}{=} (\rightarrow_{\mathcal{R}}) \cup (\leftarrow_{\mathcal{R}})$ , and  $(\approx_{\mathcal{R}}) \stackrel{\text{def}}{=} (\leftrightarrow_{\mathcal{R}})^*$ .  $\approx_{\mathcal{R}}$  is a congruence relation. We define the set of reachable terms  $R^*(t) = \{t' \mid t \rightarrow_{\mathcal{R}}^* t'\}$ . If a term rewriting system  $\mathcal{R}$  is variable-preserving (i.e.,  $\text{Var}(lhs) = \text{Var}(rhs)$  for all rules), we define  $\mathcal{R}^{-1} = \{rhs \rightarrow lhs \mid (lhs \rightarrow rhs) \in \mathcal{R}\}$ . It follows that  $(\rightarrow_{(\mathcal{R}^{-1})}) = (\leftarrow_{\mathcal{R}})$ .

### 2.2 Tree automata

Let  $\Sigma$  be a signature. A (bottom-up) tree automaton is a tuple  $\mathcal{A} = \langle Q, \Sigma, Q_{\text{final}}, \Delta \rangle$ , where  $Q$  is a (potentially infinite)<sup>2</sup> set of states,  $Q_{\text{final}} \subseteq Q$  is a set of final states, and  $\Delta$  is a set of transitions of the form  $f(q_1, \dots, q_n) \rightarrow q$  where  $q, q_1, \dots, q_n \in Q$ , and  $f \in \Sigma$ . Denote by  $\Sigma \cup Q$  the signature  $\Sigma$  extended with  $Q$  where each state is viewed as a symbol of arity 0. Then  $\Delta$  is a term rewriting system for  $\Sigma \cup Q$ , and we will denote by  $\rightarrow_{\mathcal{A}}^*$  (rather than  $\rightarrow_{\Delta}^*$ ) the rewrite relation defined by  $\Delta$ . A term  $t \in T(\Sigma)$  is accepted by a state  $q$  if  $t \rightarrow_{\mathcal{A}}^* q$ ,

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<sup>2</sup> In this paper, we allow tree automata (and thus E-graphs) to have an infinite number of states and transitions. Talking about infinite E-graphs allow us to define the semantics of equality saturation even when the algorithm does not terminate.

and we write  $\mathcal{L}(q)$  for the set of terms accepted by  $q$ . The language accepted by  $\mathcal{A}$  is  $\mathcal{L}(\mathcal{A}) \stackrel{\text{def}}{=} \{t \in T(\Sigma) \mid \exists q_f \in Q_{\text{final}}, t \rightarrow_{\mathcal{A}}^* q_f\}$ . A tree language  $L \subseteq T(\Sigma)$  is called regular if it is accepted by some *finite* tree automaton.

Fix two tree automata  $\mathcal{A} = \langle Q, \Sigma, Q_{\text{final}}, \Delta \rangle, \mathcal{B} = \langle Q', \Sigma, Q'_{\text{final}}, \Delta' \rangle$ . A *homomorphism*,  $h : \mathcal{A} \rightarrow \mathcal{B}$ , is a function  $h : Q \rightarrow Q'$  that maps final states to final states, and, for every transition  $f(c_1, \dots, c_k) \rightarrow c$  in  $\mathcal{A}$  there exists a transition  $f(h(c_1), \dots, h(c_k)) \rightarrow h(c)$  in  $\mathcal{B}$ . An *isomorphism*<sup>3</sup> is a homomorphism  $h : \mathcal{A} \rightarrow \mathcal{B}$  for which there exists an inverse homomorphism  $h^{-1} : \mathcal{B} \rightarrow \mathcal{A}$  such that  $h^{-1} \circ h = id_{\mathcal{A}}$  and  $h \circ h^{-1} = id_{\mathcal{B}}$ . The following holds:

► **Lemma 2.** *Let  $h : \mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism,  $t \in T(\Sigma)$ , and  $c$  be a state of  $\mathcal{A}$ . If  $t \rightarrow_{\mathcal{A}}^* c$ , then  $t \rightarrow_{\mathcal{B}}^* h(c)$ . In particular,  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ .*

**Proof.** We prove the statement by induction on the structure of the term  $t \in T(\Sigma)$ . Assuming  $t = f(t_1, \dots, t_k)$  for  $k \geq 0$ <sup>4</sup> and  $t \rightarrow_{\mathcal{A}}^* c$ , then there exists states  $c_1, \dots, c_k$  such that  $t_i \rightarrow_{\mathcal{A}}^* c_i$  and  $\mathcal{A}$  contains the transition  $f(c_1, \dots, c_k) \rightarrow c$ . By induction hypothesis  $t_i \rightarrow_{\mathcal{B}}^* h(c_i)$  for  $i = 1, \dots, k$ , and since  $h$  is a homomorphism, there exists a transition  $f(h(c_1), \dots, h(c_k)) \rightarrow h(c)$  in  $\mathcal{B}$ , proving that  $t \rightarrow_{\mathcal{B}}^* h(c)$ . ◀

We write  $\mathcal{A} \sqsubseteq \mathcal{B}$  whenever there exists a homomorphism  $\mathcal{A} \rightarrow \mathcal{B}$ . Observe that  $\sqsubseteq$  is a preorder relation. In the next section, we show that this preorder relation  $\sqsubseteq$  becomes a partial order when restricted to E-graphs (Lemma 12).

We call an automaton  $\mathcal{A}$  *deterministic* if  $t \rightarrow_{\mathcal{A}}^* q_1$  and  $t \rightarrow_{\mathcal{A}}^* q_2$  implies  $q_1 = q_2$  for states  $q_1, q_2$ . We call  $\mathcal{A}$  *reachable* if every state  $q$  accepts some ground term:  $\exists t \in T(\Sigma), t \rightarrow_{\mathcal{A}}^* q$ .

### 3 E-graphs and Equality Saturation

Most papers discussing E-graphs use an operational definition not suitable for a theoretical analysis. We introduce an equivalent definition of E-graphs in terms of tree automata, similar to Kozen's partial tree automata [18]. Throughout this section we fix the signature  $\Sigma$ .

#### 3.1 E-graphs

► **Definition 3.** *An E-graph is a deterministic and reachable tree automaton  $G = \langle Q, \Sigma, \Delta \rangle$  (without a set of final state  $Q_{\text{final}}$ ).*

Our definition maps one-to-one to the classical definition of E-graphs: An E-class is a state  $c \in Q$  of the tree automaton, and an E-node is a transition  $f(c_1, \dots, c_k) \rightarrow c$ . A term  $t$  is *represented* by the E-class  $c$  if  $t$  is accepted by  $c$ , i.e.  $t \rightarrow_G^* c$ . In the literature, the sets of E-classes and E-nodes are denoted  $C$  and  $N$  respectively. We will use states/E-classes and transitions/E-nodes interchangeably in this paper. E-graphs do not define a set of “final” E-classes, and for that reason we omit the final states  $Q_{\text{final}}$  from Definition 3<sup>5</sup>, similarly to [18].

<sup>3</sup> Notice that a bijective homomorphism is not necessarily an isomorphism.

<sup>4</sup> The base case is covered by the case  $k = 0$ .

<sup>5</sup> Alternatively, consider  $Q_{\text{final}} = Q$ .

► **Example 4.** The E-graph  $H$  in Figure 1 is the automaton  $\langle Q, \Sigma, \Delta \rangle$ , where  $\Sigma = \{a, f(\cdot, \cdot), g(\cdot, \cdot)\}$ , there are four states  $Q = \{c_1, \dots, c_4\}$ , and  $\Delta$  consists of seven transitions:

$$a() \rightarrow c_1 \quad f(c_1, c_1) \rightarrow c_2 \quad g(c_1, c_1) \rightarrow c_2 \quad \dots \quad f(c_3, c_3) \rightarrow c_4 \quad g(c_3, c_3) \rightarrow c_4$$

An example of rewritings is  $f(a, a) \rightarrow_H f(c_1, a) \rightarrow_H f(c_1, c_1) \rightarrow_H c_2$ , showing that the term  $f(a, a)$  is represented by the E-class  $c_2$ .

It is folklore that E-graphs represent equivalences of terms. We make this observation formal, by defining the semantics of an E-graph to be a certain partial congruence. A *partial equivalence relation*, or PER, on a set  $A$  is a binary relation  $\approx$  that is symmetric and transitive. Its *support* is the set  $\text{supp}(\approx) \stackrel{\text{def}}{=} \{x \mid x \approx x\} \subseteq A$ . Equivalently, a PER can be described by its support and an equivalence relation on the support. A PER on the set of terms  $T(\Sigma)$  is *congruent* if  $s_i \approx t_i$  for  $i = 1, \dots, n$  and  $f(s_1, \dots, s_n) \in \text{supp}(\approx)$  implies  $f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)$ . A PER is *reachable* if  $f(s_1, \dots, s_n) \in \text{supp}(\approx)$  implies  $s_i \in \text{supp}(\approx)$ , for  $i = 1, \dots, n$ . A *Partial Congruence Relation* (PCR)<sup>6</sup> on  $T(\Sigma)$  is a congruent and reachable PER.

An E-graph  $G$  induces a PCR  $\approx_G$  defined as follows:  $t_1 \approx t_2$  if there exists some E-class  $c$  in  $G$  that accepts both  $t_1$  and  $t_2$ , i.e.  $t_1 \rightarrow_G^* c \underset{G}{\leftarrow} t_2$ . We check that  $\approx_G$  is a PCR:  $\approx_G$  is symmetric by definition, and transitivity follows from determinacy, because  $t_1 \rightarrow_G^* c \underset{G}{\leftarrow} t_2$  and  $t_2 \rightarrow_G^* c' \underset{G}{\leftarrow} t_3$  implies  $t_1 \rightarrow_G^* c = c' \underset{G}{\leftarrow} t_3$ . Suppose  $f(s_1, \dots, s_n) \rightarrow_G^* c$ : then there exists states  $c_i$  s.t.  $s_i \rightarrow_G^* c_i$ , and a transition  $f(c_1, \dots, c_n) \rightarrow c$ , proving reachability; if, in addition,  $s_i \approx_G t_i$  for  $i = 1, \dots, n$ , then  $t_i \rightarrow_G^* c_i$ , which implies  $f(t_1, \dots, t_n) \rightarrow_G^* c$ , proving congruence,  $f(s_1, \dots, s_n) \approx_G f(t_1, \dots, t_n)$ .

► **Definition 5.** The semantics of an E-graph  $G$  is the PCR  $\approx_G$ .

► **Theorem 6.** For any PCR  $\approx$  over  $T(\Sigma)$  there exists a unique  $G$  such that  $(\approx_G) = (\approx)$ .

**Proof sketch.** The states of  $G$  are the equivalence classes of  $\approx$ , denoted as  $[t]$  for  $t \in \text{supp}(\approx)$ , and the transitions are  $f([t_1], \dots, [t_n]) \rightarrow [f(t_1, \dots, t_n)]$  for all  $t_1, \dots, t_n, f(t_1, \dots, t_n)$  in the support of  $\approx$ . One can check by induction on the size of  $t$  that  $t \in \text{supp}(\approx)$  iff  $t \in \text{supp}(\approx_G)$ , and  $t \rightarrow_G^* [s]$  iff  $t \approx s$ , proving that  $(\approx_G) = (\approx)$ . ◀

Thus, the semantics of an E-graph  $G$  is a PCR  $\approx_G$ , which is a congruence on  $\mathcal{L}(G) \stackrel{\text{def}}{=} \text{supp}(\approx_G)$ . We say that  $G$  represents the set of terms  $\mathcal{L}(G)$ .

► **Example 7.** Continuing Example 1, the semantics of the E-graph  $H$  in Figure 1 is the PCR  $\approx_H$  that equates  $a \approx_H a$  (witnessed by state  $c_1$ ),  $f(a, a) \approx_H g(a, a)$  (by state  $c_2$ ),  $f(f(a, a), g(a, a)) \approx_H g(f(a, a), f(a, a))$  (by state  $c_3$ ), etc.

► **Example 8.** Let  $\Sigma = \{a, f(\cdot)\}$ . Consider the E-graph  $G$  with a single state  $c$  and transitions  $a() \rightarrow c$ ,  $f(c) \rightarrow c$ . It represents infinitely many terms,  $f^{(k)}(a)$ , for  $k \geq 0$ , and its semantics is the PCR  $a \approx_G f(a) \approx_G f(f(a)) \approx_G \dots$

► **Example 9.** Let  $\Sigma = \{a, f(\cdot), g(\cdot)\}$  and consider the infinite E-graph  $G$  with states  $c, c_0, c_1, c_2, \dots$  and transitions

$$a \rightarrow c_0 \quad f(c_i) \rightarrow c_{i+1} \quad g(c_i) \rightarrow c \quad i = 0, 1, 2, \dots$$

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<sup>6</sup> PCRs are studied in the literature as congruences on partial algebras (e.g., [18]).

The PCR consists of  $g(a) \approx_G g(f(a)) \approx_G g(f(f(a))) \approx_G \dots$ , defined by the state  $c$ . No other distinct terms are in  $\approx_G$ , for example  $f(a) \not\approx_G f(f(a))$  because they are represented by the distinct states  $c_1$  and  $c_2$  respectively. Although  $G$  represents a regular language,  $\{f^{(k)}(a) \mid k \geq 0\} \cup \{g(f^{(k)}(a)) \mid k \geq 0\}$ , its semantics  $\approx_G$  cannot be captured by a finite E-graph. This example shows that  $\approx_G$  differs from the Myhill-Nerode equivalence relation [17], under which all terms  $f^{(k)}(a)$  would be equivalent. It also illustrates the subtle distinction between tree automata and E-graphs. An optimizer that wants to use the identity  $g(x) = g(f(x))$ , but not  $x = f(x)$ , needs this E-graph to represent all terms equivalent to  $g(a)$ , and cannot use the finite tree automaton accepting the regular language  $\mathcal{L}(c)$  because that would incorrectly equate all terms  $f^{(k)}(a)$ .

Recall the definitions of tree automata homomorphisms in Sec. 2.2. When restricted to E-graphs, homomorphisms have some interesting properties:

► **Lemma 10.** *If  $h : G \rightarrow H$  is a homomorphism, then  $(\approx_G) \subseteq (\approx_H)$ .*

**Proof.** Assume  $t_1 \approx_G t_2$ . Then there exists some E-class  $c$  where  $t_1 \rightarrow_G^* c \xrightarrow{G} t_2$ . By Lemma 2,  $t_1 \rightarrow_H^* h(c) \xleftarrow{H} t_2$ , implying  $t_1 \approx_H t_2$ . ◀

► **Lemma 11.** *There exists at most one homomorphism  $h : G \rightarrow H$ .*

**Proof.** Call the *weight* of a state  $c$  in  $G$  the size of the smallest term  $t$  such that  $t \rightarrow_G^* c$ . Since  $G$  is reachable, every state has a finite weight. Given two homomorphisms  $h_1, h_2 : G \rightarrow H$ , we prove by induction on the weight of  $c$  that  $h_1(c) = h_2(c)$ . Let  $t$  be a term of minimal size such that  $t \rightarrow_G^* c$ , and assume  $t = f(t_1, \dots, t_k)$ , for  $k \geq 0$ . Then there exists states  $c_1, \dots, c_k$  such that  $t_i \rightarrow_G^* c_i$ ,  $i = 1, k$ , and a transition  $f(c_1, \dots, c_k) \rightarrow c$  in  $G$ . By induction hypothesis  $h_1(c_i) = h_2(c_i)$  for  $i = 1, k$ . By the definition of a homomorphism,  $H$  contains both transitions  $f(h_1(c_1), \dots, h_1(c_k)) \rightarrow h_1(c)$  and  $f(h_2(c_1), \dots, h_2(c_k)) \rightarrow h_2(c)$ , and we conclude  $h_1(c) = h_2(c)$  because  $H$  is deterministic. ◀

We call a tree automaton  $\mathcal{A}$  rigid [14] if the identity mapping is the only homomorphism  $\mathcal{A} \rightarrow \mathcal{A}$ . It follows from Lemma 11 that every E-graph is a rigid tree automaton.

► **Lemma 12.**  *$\sqsubseteq$  over E-graphs forms a partial order up to isomorphism.*

**Proof.** Obviously  $\sqsubseteq$  is reflexive and transitive. To prove anti-symmetry, assume two homomorphisms  $h : G \rightarrow H$ ,  $h' : H \rightarrow G$ . The composition  $h' \circ h$  is a homomorphism  $G \rightarrow G$ , and, by uniqueness, it must be the identity on  $G$ ; similarly,  $h \circ h'$  is the identity on  $H$ , proving that  $h$  is an isomorphism, thus  $G, H$  are isomorphic. ◀

Next, we define models for term rewriting systems.

► **Definition 13.** *We say that an E-graph  $H = \langle Q, \Sigma, \Delta \rangle$  is a model of a TRS  $\mathcal{R}$  if, for every rule  $\text{lhs} \rightarrow \text{rhs}$  in  $\mathcal{R}$  and any substitution  $\sigma : \text{Var}(\text{lhs}) \rightarrow Q$ , if  $\text{lhs}[\sigma] \rightarrow_G^* c$  then  $\text{rhs}[\sigma] \rightarrow_G^* c$ . If  $G$  is another E-graph, then we say that  $H$  is a model for the pair  $\mathcal{R}, G$  if  $G \sqsubseteq H$  and  $H$  is a model of  $\mathcal{R}$ .  $H$  is a universal model if for any other model  $H'$ , it holds that  $H \sqsubseteq H'$ .*

When it exists, the universal model is unique up to isomorphism, because  $H \sqsubseteq H'$  and  $H' \sqsubseteq H$  implies  $H, H'$  are isomorphic.

Continuing Example 7, let  $\mathcal{R}$  consists of the rule  $f(x, x) \rightarrow g(x, x)$ , and let  $G, H$  be the E-graphs in Figure 1.  $G$  is not a model of  $\mathcal{R}$ , because for the substitution  $\sigma(x) = c_1$  we have  $\text{lhs}[\sigma] = f(c_1, c_1) \rightarrow_G c_2$ , but  $\text{rhs}[\sigma] = g(c_1, c_1) \not\rightarrow_G^* c_2$ . On the other hand, one can check that  $H$  is a model of  $\mathcal{R}$ ; in fact it is a model of  $\mathcal{R}, G$ , because  $G \sqsubseteq H$ .

Given an E-graph  $G$  and a TRS  $\mathcal{R}$ , equality saturation constructs a universal model  $H$  of  $\mathcal{R}, G$ , by repeatedly applying some simple operations on  $G$ , which we define next.

### 3.2 Operations over E-graphs

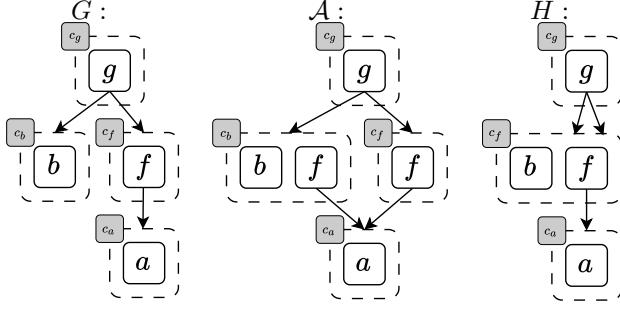
E-matching, Insertion, and Rebuilding are the building blocks of equality saturation. They are defined in the literature operationally [20]. We provide here a formal definition, using tree automata terminology. Throughout this section we fix an E-graph  $G = \langle Q, \Sigma, \Delta \rangle$ .

**E-matching** a rule  $\text{lhs} \rightarrow \text{rhs} \in \mathcal{R}$  in  $G$  returns the set of pairs  $(\sigma, c)$ , where  $\sigma : \text{Var}(\text{lhs}) \rightarrow Q$  is a substitution such that  $\text{lhs}[\sigma] \rightarrow_G^* c$ . For example, considering the E-graph on the left of Figure 1 and the rule  $f(x, x) \rightarrow g(x, x)$ , E-matching returns  $(\{x \mapsto c_i, y \mapsto c_i\}, c_{i+1})$  for  $i = 1, 2, 3$ . E-matching is analogous to computing the triggers for a TGD or EGD (Sec. 4.1).

The **Insertion** of a pair  $(t, c)$  into  $G$ , where  $t \in T(\Sigma \cup Q)$  and  $c \in Q$ , returns an automaton  $\mathcal{A}$  such that  $G \sqsubseteq \mathcal{A}$  and  $t \rightarrow_{\mathcal{A}}^* c$ . To define  $\mathcal{A}$ , we need the following:

► **Definition 14.** Fix a term  $t \in T(\Sigma \cup Q)$  and a state  $c \in Q$ . The flattening for  $t$  with root  $c$ , in notation  $\text{FL}(t \rightarrow^* c)$ , or just  $\text{FL}$  when  $t, c$  are clear from the context, is an E-graph that has one distinct state  $q_u$  for each subterm  $u$  of  $t$ , and has a transition  $f(q_{u_1}, \dots, q_{u_k}) \rightarrow q_u$ , for all subterms  $u$  of the form  $u = f(u_1, \dots, u_k)$  and  $f \in \Sigma$ . Moreover, it is enforced that the state of the root node is  $c$  (i.e.,  $q_t = c$ ). One can check that  $t \rightarrow_{\text{FL}}^* c$ , and that  $\approx_{\text{FL}}$  is the identity on all subterms of  $t$ . Flattening is also called normalization [9].

The result of *inserting*  $(t, c)$  in  $G$  is  $\mathcal{A} \stackrel{\text{def}}{=} G \cup \text{FL}(t \rightarrow^* c)$  (i.e. we take the set-union of all states and all transitions). The result  $\mathcal{A}$  is a reachable tree automaton, but it is non-deterministic in general, thus it is not an E-graph; the next operation, rebuilding, converts it back into an E-graph.  $G \sqsubseteq \mathcal{A}$  holds, because the inclusion  $G \rightarrow \mathcal{A}$  is a homomorphism.



► **Figure 2** Example E-graphs on insertion and rebuilding.

As an example, if we insert the pair  $(f(c_a), c_b)$  in the E-graph  $G$  in Figure 2, the result is  $\mathcal{A}$  in the center of the figure; flattening  $\text{FL}(f(c_a) \rightarrow c_b)$  has a single transition  $f(c_a) \rightarrow c_b$ .

**Rebuilding** converts  $\mathcal{A}$  into a deterministic automaton. Formally, let  $\mathcal{A}$  be any reachable tree automaton, and recall that  $\mathcal{A}$  may be infinite. The *congruence closure*  $\text{CC}(\mathcal{A})$  is an E-graph (i.e. deterministic, reachable automaton) such that  $\mathcal{A} \sqsubseteq \text{CC}(\mathcal{A})$  and, for any other E-graph  $G'$ , if  $\mathcal{A} \sqsubseteq G'$ , then  $\text{CC}(\mathcal{A}) \sqsubseteq G'$ . We prove the following in Appendix A.1:

► **Lemma 15.** For any reachable tree automaton  $\mathcal{A}$ ,  $\text{CC}(\mathcal{A})$  exists and is unique.

The procedure of computing  $\text{CC}(\mathcal{A})$ , also known as *rebuilding* in EqSat literature [32], can be done efficiently in the finite case, for instance with Tarjan's algorithm [7]. The idea is to repeatedly find violating transitions  $f(c_1, \dots, c_k) \rightarrow c$  and  $f(c_1, \dots, c_k) \rightarrow c'$  with  $c \neq c'$ , and replace every occurrence of  $c$  with  $c'$ , until fixpoint. This is similar to determinizing  $\mathcal{A}$ , but instead of constructing powerset states like  $\{c_1, c_4, c_7\}$ , we equate states  $c_1 = c_4 = c_7$ ; thus,  $\text{CC}(\mathcal{A})$  has at most as many states as  $\mathcal{A}$ , and the procedure always terminates for finite  $\mathcal{A}$ , as merging shrinks the number of states.

► **Example 16.** The tree automaton  $\mathcal{A}$  in Figure 2 is non-deterministic, because  $f(a) \rightarrow^* c_b$  and  $f(a) \rightarrow^* c_f$ . The congruence closure algorithm merges  $c_b$  and  $c_f$ , and produces the E-graph  $H$  in Figure 2. Notice that  $H$  represents strictly more terms than  $\mathcal{A}$ . For example,  $H$  represents  $g(b, b)$ , because  $g(b, b) \rightarrow_H^* c_g$ , but  $\mathcal{A}$  does not represent  $g(b, b)$ .

**Least upper bound of E-graphs** Let  $(G_i)_{i \in I}$  be a (possibly infinite) family of E-graphs. Recall that their least upper bound  $G$  is an E-graph such that  $G$  is an upper bound for every E-graph in the set,  $G_i \sqsubseteq G$  for all  $i \in I$ , and for any other upper bound  $G'$ , it holds that  $G \sqsubseteq G'$ . We prove the following in Appendix A.2:

► **Lemma 17** (Least upper bound). *The least upper bound exists and is given by  $\text{CC}(\mathcal{A})$ , where  $\mathcal{A}$  is the automaton consisting of the disjoint union of the states and the disjoint union of the transitions of all E-graphs  $G_i$ .*

We will denote the least upper bound by  $\bigsqcup_{i \in I} G_i$ . It is also possible to show that every family of E-graphs admits a greatest lower bound, by using a product construction [13], but we do not need it in this paper.

### 3.3 Equality saturation

The standard definition of equality saturation in the literature is procedural: given an E-graph  $G = \langle Q, \Sigma, \Delta \rangle$  and TRS  $\mathcal{R}$ , equality saturation repeatedly applies matching/insertion/rebuilding. EqSat is undefined when this process does not terminate. We provide here an alternative definition, as the least fixpoint of an *immediate consequence operator* (ICO), and prove that it always exists. We start by introducing the ICO:

$$\text{ICO}_{\mathcal{R}} \stackrel{\text{def}}{=} \text{CC} \circ T_{\mathcal{R}} \quad (2)$$

$T_{\mathcal{R}}$  is the match/apply operator: it computes all E-matches then inserts the *rhs*'s into  $G$ :

$$T_{\mathcal{R}}(G) \stackrel{\text{def}}{=} G \cup \bigcup \{ \text{FL}(rhs[\sigma] \rightarrow_G^* c) \mid (lhs \rightarrow rhs) \in \mathcal{R}, \sigma : \text{Var}(lhs) \rightarrow Q, lhs[\sigma] \rightarrow_G^* c \}$$

$\text{CC}$  is the rebuilding operator of Lemma 15.

► **Lemma 18.**  *$\text{ICO}_{\mathcal{R}}$  is inflationary ( $G \sqsubseteq \text{ICO}_{\mathcal{R}}(G)$  for all  $G$ ) and monotone.*

**Proof.** That  $\text{ICO}_{\mathcal{R}}$  is inflationary follows from  $G \sqsubseteq T_{\mathcal{R}}(G) \sqsubseteq \text{CC}(T_{\mathcal{R}}(G))$ . We prove that both  $T_{\mathcal{R}}$  and  $\text{CC}$  are monotone. Let  $H \stackrel{\text{def}}{=} \bigcup \{ \text{FL}(rhs[\sigma] \rightarrow^* c) \mid (lhs \rightarrow rhs) \in \mathcal{R}, \sigma : \text{Var}(lhs) \rightarrow Q, lhs[\sigma] \rightarrow_G^* c \}$ , thus  $T_{\mathcal{R}}(G) = G \cup H$ . Any homomorphism  $G \rightarrow G'$  can be extended to a homomorphism  $G \cup H \rightarrow G' \cup H$ , which proves that  $T_{\mathcal{R}}$  is monotone. Consider two automata  $\mathcal{A}, \mathcal{A}'$  and assume  $\mathcal{A} \sqsubseteq \mathcal{A}'$ , i.e. there exists a homomorphism from  $\mathcal{A}$  to  $\mathcal{A}'$ . Denote  $G \stackrel{\text{def}}{=} \text{CC}(\mathcal{A})$ ,  $G' \stackrel{\text{def}}{=} \text{CC}(\mathcal{A}')$ . Then,  $\mathcal{A} \sqsubseteq \mathcal{A}' \sqsubseteq G'$ , which implies  $\mathcal{A} \sqsubseteq G'$ . By the definition of  $G = \text{CC}(\mathcal{A})$ , we have  $G \sqsubseteq G'$ , proving that  $\text{CC}$  is monotone. ◀

With Lemmas 17 and 18, we show the following in Appendix A.3:

► **Theorem 19.** *Fix an E-graph  $G$ , and consider the class  $\mathcal{C}_G$  of E-graphs  $G' \sqsupseteq G$ . Then  $\text{ICO}_{\mathcal{R}} : \mathcal{C}_G \rightarrow \mathcal{C}_G$  has a least fixpoint, given by*

$$\text{EQSAT}(\mathcal{R}, G) \stackrel{\text{def}}{=} \bigsqcup_{i \geq 0} \text{ICO}_{\mathcal{R}}^{(i)}(G) \quad (3)$$

Furthermore,  $\text{EQSAT}(\mathcal{R}, G)$  is a universal model of  $\mathcal{R}, G$ ; we call it equality saturation.

Given  $G, \mathcal{R}$ , our semantics of EqSat is the least fixpoint in (3), which is also the unique universal model of  $G, \mathcal{R}$ . When  $\text{EQSAT}(\mathcal{R}, G)$  is finite, then this coincides with the standard procedural definition in the literature. A common case (e.g., in program and query optimization settings) is when  $G$  represents a single term  $t$ , more precisely  $G = \text{FL}(t \rightarrow^* c)$  with fresh state  $c$ ; in that case we denote  $\text{EQSAT}(\mathcal{R}, G)$  as  $\text{EQSAT}(\mathcal{R}, t)$ .

**Properties of equality saturation** We establish several basic facts of EQSAT.

► **Lemma 20** (Inflationary).  $G \sqsubseteq \text{EQSAT}(\mathcal{R}, G)$ .

► **Corollary 21.**  $\mathcal{L}(G) \subseteq \mathcal{L}(\text{EQSAT}(\mathcal{R}, G))$  and  $(\approx_G) \subseteq (\approx_{\text{EQSAT}(\mathcal{R}, G)})$ .

Lemma 20 follows from Lemma 18, while Corollary 21 follows from Lemma 2 and Lemma 10. Thus,  $\text{EQSAT}(\mathcal{R}, G)$  represents more terms than  $G$ , and identifies more pairs of terms than  $G$ . Next, we examine the relationship between the PCR defined by  $\text{EQSAT}(\mathcal{R}, G)$  and the relations  $\mathcal{R}^*$  and  $\approx_{\mathcal{R}}$  defined by the TRS  $\mathcal{R}$  (see Sec. 2.1). If  $\approx_1, \approx_2$  are two PCRs, then we denote by  $\approx_1 \vee \approx_2$  the smallest PCR that contains both. We prove:

► **Lemma 22** (Representation). *Let  $w \in T(\Sigma)$  be a term represented by some state of the E-graph  $H \stackrel{\text{def}}{=} \text{EQSAT}(\mathcal{R}, G)$ . The following hold:  $\mathcal{R}^*(w) \subseteq [w]_{\approx_H} \subseteq [w]_{\approx_{\mathcal{R}} \vee \approx_G}$ .*

**Proof.** The definitions of E-matching and insertion imply that, if  $u \rightarrow_{\mathcal{R}} v$  and  $u$  is represented by some state  $c$  of some E-graph  $K$ , then  $v$  is represented by the same state of the E-graph  $\text{ICO}(K) = \text{CC}(T_{\mathcal{R}}(K))$ . Therefore, if  $u \rightarrow_{\mathcal{R}} v$  and  $u$  is represented by some state of  $H$ , then  $v$  is represented by the same state of  $\text{ICO}(H) = H$  (because  $H$  is a fixpoint of  $\text{ICO}$ ). This implies that  $\mathcal{R}^*(w) \subseteq [w]_{\approx_H}$ .

For the second part, we denote by  $G_k = \text{ICO}^{(k)}(G)$ , and check by induction on  $k$  that  $[w]_{\approx_{G_k}} \subseteq [w]_{\approx_{\mathcal{R}} \vee \approx_G}$ . When  $k = 0$  then  $G_0 = G$  and the claim is obvious. For the inductive step we observe that the only new identities introduced by  $G_{k+1}$  are justified by  $\mathcal{R}$ . ◀

In other words, if  $w \rightarrow_{\mathcal{R}}^* v$ , then  $\text{EQSAT}(\mathcal{R}, G)$  will equate  $w$  with  $v$ ; and if  $\text{EQSAT}(\mathcal{R}, G)$  equates  $w$  with  $v$  then this can be derived from  $\approx_{\mathcal{R}}$  and  $\approx_G$ . In general,  $w \approx_{\mathcal{R}} v$  does not imply  $w \approx_H v$ . For a simple example, let  $\mathcal{R} = \{a \rightarrow b\}$ , thus  $a \approx_{\mathcal{R}} b$ , and let  $G$  represent only the term  $b$ . Then  $H = \text{EQSAT}(\mathcal{R}, G) = G$  represents only the term  $b$ , thus  $a \not\approx_H b$ .

► **Example 23.** For some TRS, the starting E-graph  $G$  determines whether EqSat terminates in a finite number of steps. For a simple example, consider  $\Sigma = \{f(\cdot), g(\cdot), a\}$ ,  $\mathcal{R} = \{f(g(x)) \rightarrow g(f(x))\}$ . If the initial E-graph  $G$  represents only the term  $f(g(a))$  (and its subterms), then EqSat terminates, and the resulting  $H \stackrel{\text{def}}{=} \text{EQSAT}(\mathcal{R}, G)$  represents a PCR where  $f(g(a)) \approx_H g(f(a))$ . On the other hand, if  $G$  is the E-graph with states  $c_f, c_g$  and transitions  $\{g(c_f) \rightarrow c_g, f(c_g) \rightarrow c_f, a \rightarrow c_f\}$ , then  $G$  already represents infinitely many terms  $\mathcal{L}(c_f) \cup \mathcal{L}(c_g)$ , where  $\mathcal{L}(c_f) = \{a, f(g(a)), f(g(f(g(a))))\ldots\}$  and  $\mathcal{L}(c_g) = \{g(a), g(f(g(a))), \ldots\}$ . After equality saturation,  $H = \text{EQSAT}(\mathcal{R}, G)$  represents all terms in  $T(\Sigma)$  where the numbers of occurrences  $\#f, \#g$  of  $f, g$  satisfy  $\#f \leq \#g \leq \#f + 1$ . This is not a regular language, hence  $H$  is infinite, and EqSat will not terminate in a finite number of steps.

► **Example 24.** We show that both inclusions in Lemma 22 can be strict. Let  $\mathcal{R} = \{a \rightarrow b, c \rightarrow b\}$  and let  $G$  be the E-graph representing a single term  $f(a, b)$  with transitions  $\{a \rightarrow c_a, b \rightarrow c_b, f(c_a, c_b) \rightarrow c_f\}$ . Then  $H = \text{EQSAT}(\mathcal{R}, G)$  has transitions:  $\{a \rightarrow c_a, b \rightarrow c_b, f(c_a, c_b) \rightarrow c_f\}$ . We have:<sup>7</sup>  $\mathcal{R}^*(f(a, b)) = f(a|b, b)$ ,  $[f(a, b)]_{\approx_H} = f(a|b, a|b)$ , and  $[f(a, b)]_{\approx_{\mathcal{R}} \vee \approx_G} = [f(a, b)]_{\approx_{trs}} = f(a|b|c, a|b|c)$ . All three sets are different.

However, the three expressions in Lemma 22 are equal in an important special case:

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<sup>7</sup> We use  $f(a|b, b)$  as a shorthand for  $\{f(t, b) \mid t = a \vee t = b\}$  (as in regular expressions) and similarly for other terms.

► **Corollary 25.** Suppose  $\mathcal{R}$  is a variable-preserving term rewriting system. Let  $\text{Sym}(\mathcal{R}) = \mathcal{R} \cup \mathcal{R}^{-1}$ , and let  $w \in T(\Sigma)$  be a term represented by some state of the E-graph  $H^{\leftrightarrow} \stackrel{\text{def}}{=} \text{EQSAT}(\text{Sym}(\mathcal{R}), G)$ . The following hold:  $(\text{Sym}(\mathcal{R}))^*(w) = [w]_{\approx_{H^{\leftrightarrow}}} = [w]_{\approx_{\mathcal{R}} \vee \approx_G}$ .

Let  $G$  be a finite E-graph. If  $\text{EQSAT}(\mathcal{R}, G)$  is infinite, then EqSat does not terminate in a finite number of steps. Somewhat surprisingly, the converse does hold: if  $\text{EQSAT}(\mathcal{R}, G)$  is finite, then EqSat terminates in a finite number of steps. This follows from the next lemma, whose proof is deferred to Appendix B.

► **Lemma 26** (Finite convergence). Let  $\mathcal{G} : G_1 \sqsubset G_2 \sqsubset \dots$  be an ascending sequence of finite E-graphs. If  $G_\infty = \bigsqcup_i G_i$  is finite, then the sequence  $\mathcal{G}$  is finite.

► **Corollary 27** (Finite convergence of EqSat). Let  $\mathcal{R}$  be a term rewriting system and  $G$  be a finite E-graph. If  $\text{EQSAT}(\mathcal{R}, G)$  is finite, EqSat converges in a finite number of steps.

## 4 Equality saturation and the chase

In this section, we briefly review necessary background on databases and the chase. Then, we will show the fundamental connections between equality saturation and the chase.

### 4.1 The Chase Procedure

**Databases and conjunctive queries** A relational database *schema* is a tuple of relation names  $\mathcal{S} = (R_1, \dots, R_m)$  with associated arities  $\text{ar}(R_i)$ . A database instance is a tuple of relation instances  $I = (R_1^I, \dots, R_m^I)$ , where  $R_i^I \subseteq \text{Dom}^{\text{ar}(R_i)}$  for some domain  $\text{Dom}$ . We allow an instance to be infinite. We often view a tuple  $\vec{a}$  in  $R_i^I$  as an atom  $R_i(\vec{a})$ , and view the instance  $I$  as a set of atoms. The domain  $\text{Dom}$  is the disjoint union of set of *constants* and a set of *marked nulls*.

A *conjunctive query*  $\lambda(\vec{x})$  is a formula with free variables  $\vec{x}$  of the form  $R_1(\vec{x}_1) \wedge \dots \wedge R_k(\vec{x}_k)$ , where each  $\vec{x}_i$  is a tuple of variables from  $\vec{x}$ . The *canonical database* of a conjunctive query consists of all the tuples  $R_i(\vec{x}_i)$ , where the variables  $\vec{x}$  are considered marked nulls.

Let  $I, J$  be two database instances. A *homomorphism* from  $I$  to  $J$ , in notation  $h : I \rightarrow J$ , is a function  $h : \text{Dom}(I) \rightarrow \text{Dom}(J)$  that is the identity on the set of constants, and maps each atom  $R(\vec{a}) \in I$  to an atom  $R(h(\vec{a})) \in J$ . The notion of homomorphism immediately extends to conjunctive queries and/or database instances. The output of a conjunctive query  $\lambda(\vec{x})$  on a database  $I$  is defined as the set of homomorphisms from  $\lambda(\vec{x})$  to  $I$ . We say that a database instance  $I$  *satisfies* a conjunctive query  $\lambda(\vec{x})$ , denoted by  $I \models \exists \vec{x} \lambda(\vec{x})$ , if there exists a homomorphism  $\lambda(\vec{x}) \rightarrow I$ .

**Dependencies** TGDs and EGDs describe semantic constraints between relations. A TGD is a first-order formula of the form  $\lambda(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}. \rho(\vec{x}, \vec{z})$  where  $\lambda(\vec{x}, \vec{z})$  and  $\rho(\vec{x}, \vec{y})$  are conjunctive queries with free variables in  $\vec{x} \cup \vec{y}$  and  $\vec{x} \cup \vec{z}$ . An EGD is a first-order formula of the form  $\lambda(\vec{x}) \rightarrow x_i = x_j$  where  $\lambda(\vec{x})$  is a conjunctive query with free variables in  $\vec{x}$  and  $\{x_i, x_j\} \subseteq \vec{x}$ .

Fix a set of TGDs and EGDs  $\Gamma$ . If  $I$  is a database instance and  $d \in \Gamma$ , then a *trigger* for  $d$  in  $I$  is a homomorphism from  $\lambda(\vec{x}, \vec{y})$  (resp.  $\lambda(\vec{x})$ ) to  $I$ . An *active trigger* is a trigger  $h$  such that, if  $d$  is a TGD, then no extension  $h$  to a homomorphism  $h' : \rho(\vec{x}, \vec{z}) \rightarrow I$  exists, and, if  $d$  is an EGD, then  $h(x_i) \neq h(x_j)$ . We say that  $I$  is model for  $\Gamma$ , and write  $I \models \Gamma$ , if it has no active triggers.

Given  $\Gamma$  and  $I$  we say that some database instance  $J$  is a *model* for  $\Gamma, I$ , if  $J \models \Gamma$  and there exists a homomorphism  $I \rightarrow J$ .  $J$  is called *universal model* if there is a homomorphism from  $J$  to every model of  $\Gamma$  and  $I$ . Universal models are unique up to homomorphisms.

**The chase** The chase is a fixpoint algorithm for computing universal models. We consider two variants of the chase here: the *standard chase* and *Skolem chase*. Both the standard chase and the Skolem chase produce a universal model of  $\Gamma, I$  [8, 3, 19]. The standard chase computes answers by deriving a sequence of *chase steps* until all dependencies are satisfied. A chase step, denoted as  $I \xrightarrow{d,h} J$ , takes as inputs an instance  $I$ , a homomorphism  $h$ , and a dependency  $d$ , where  $h$  is an active trigger of  $d$  in  $I$ , and produces an output instance  $J$  by adding some tuples (for TGDs) or collapsing some elements (for EGDs). Specifically, if  $d$  is a TGD, the chase step extends  $I$  with the tuple  $h'(\rho(\vec{x}, \vec{z}))$ , where  $h'$  is an extension of  $h$  that maps the variables  $\vec{z}$  on which  $h$  is undefined to fresh marked nulls. If  $d$  is an EGD, if  $h(x_i)$  (or  $h(x_j)$ ) is a marked null, a chase step replaces in  $I$  every occurrence of  $h(x_i)$  with  $h(x_j)$  (or  $h(x_j)$  with  $h(x_i)$ ). If neither  $h(x_i)$  nor  $h(x_j)$  is a marked null and  $h(x_i) \neq h(x_j)$ , then the chase fails.

A standard chase sequence starting at  $I_0$  is a sequence of successful chase steps  $I_0 \xrightarrow{d_1,h_1} I_1 \xrightarrow{d_2,h_2} \dots$  that is *fair*: for all  $i \geq 0$ , for each dependency  $d$  and active trigger  $h$  of  $d$  in  $I_i$ , some  $j \geq i$  must exist such that  $h$  is no longer an active trigger of  $d$  in  $I_j$ . The *result* of a (possibly infinite) chase sequence is  $\bigcup_{i \geq 0} \bigcap_{j \geq i} I_j$  [3]. A chase sequence is *terminating* if it ends with  $I_n$  and  $I_n \models \Gamma$ , in which case  $I_n$  is the result of the chase sequence. The standard chase is non-deterministic: depending on the order of firing, the chase sequence can be different. Different chase sequences can even differ on whether they terminate.

The Skolem chase, discussed in [19], differs from the standard chase in several ways. It first *skolemizes* each TGD  $d : \lambda(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}. \rho(\vec{x}, z_1, \dots, z_k)$  to  $\lambda(\vec{x}, \vec{y}) \rightarrow \rho(\vec{x}, f_{z_1}^d(\vec{x}), \dots, f_{z_k}^d(\vec{x}))$ , where each  $f_{z_j}^d$  is an uninterpreted function from  $\text{Dom}^{|\vec{x}|}$  to  $\text{Dom}$ . The result of the Skolem chase, denoted as  $\text{SKLCH}(\Gamma, I)$ , is the least fixpoint of the immediate consequence operator (ICO) of the Skolemized TGDs. Note that the Skolem chase does not directly handle EGDs but uses a technique called *singularization* [19] to simulate EGDs with TGDs.

## 4.2 Reducing the Skolem chase to equality saturation

In this section, we show how to reduce the Skolem chase to EqSat. We only consider TGDs, since in the Skolem chase, EGDs are modeled as TGDs using singularization [19].

We show an encoding where there exists a simple mapping from E-graphs to database instances, defined by

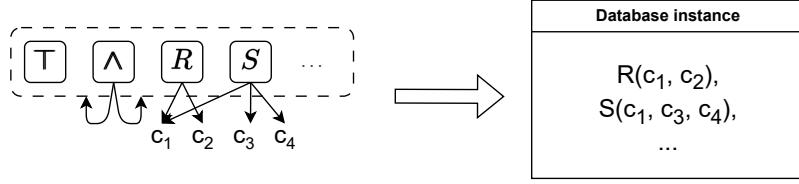
$$\xi(G) = \{R(c_1, \dots, c_k) \in \mathcal{L}(G) \mid R \text{ is a relation symbol in } \mathcal{S}\}$$

such that, given a set of dependencies  $\Gamma$ , running EqSat on an encoded term rewriting system from  $\Gamma$  corresponds to running the Skolem chase on the set of dependencies via  $\xi$ . Intuitively, given an E-graph,  $\xi$  collects every term that corresponds to a tuple from the language of  $G$ . An illustration of  $\xi$  is shown in Figure 3.

► **Theorem 28.** *Given a database schema  $\mathcal{S} = (R_1, \dots, R_m)$ , a set of TGDs  $\Gamma$ , and an initial database  $I$ , it is possible to define a signature  $\Sigma$ , a term rewriting system  $\mathcal{R}$  over  $\Sigma$ , and an initial term  $t$  such that*

$$\xi(\text{EQSAT}(\mathcal{R}, t)) = \text{SKLCH}(\Gamma, I).$$

Moreover, the Skolem chase terminates if and only if equality saturation terminates.



**Figure 3** Mapping results of encoded EqSat back to database instances.

The intuition for the construction is that we can uniformly treat relational atoms as E-nodes contained in a special E-class, and Skolem functions naturally correspond to terms in EqSat. More specifically,

- Add symbols  $\{\top, \wedge(\cdot, \cdot)\}$  to the signature  $\Sigma$ . Add rewrite  $r_{\top} : \top \rightarrow \wedge(\top, \top)$  to  $\mathcal{R}$ . Let the initial term  $t$  be  $\top$ .
- Add every Skolem function symbol to  $\Sigma$ , and for every  $n$ -ary relational symbol  $R \in \mathcal{S}$ , add a  $n$ -ary function symbol to  $\Sigma$ , and add rewrite rule  $r_R : R(x_1, \dots, x_n) \rightarrow \top$ .
- For every Skolemized TGD

$$d : R_1(\vec{x}_1, \vec{y}_1) \wedge \dots \wedge R_n(\vec{x}_n, \vec{y}_n) \rightarrow R'_1(\vec{x}'_1, \vec{f}_{z1}^d) \wedge \dots \wedge R'_m(\vec{x}'_m, \vec{f}_{zm}^d),$$

replace the conjunctions in the head and body with nested applications of  $\wedge$  and  $\top$ :

$$r_d : \wedge(R_1(\vec{x}_1, \vec{y}_1), \wedge(\dots \wedge(R_n(\vec{x}_n, \vec{y}_n), \top))) \rightarrow \wedge(R'_1(\vec{x}'_1, \vec{f}_{z1}^d), \wedge(\dots \wedge(R'_m(\vec{x}'_m, \vec{f}_{zm}^d), \top)))$$

and add  $r_d$  to  $\mathcal{R}$ .

- For each constant  $c$  in the input database  $I$ , add a nullary function symbol  $c$  to  $\Sigma$ . For each tuple  $t = R(c_1, \dots, c_n)$  in the input database  $I$ , add rewrite  $r_t : \top \rightarrow R(c_1, \dots, c_n)$ .

**Proof of Theorem 28.** See Appendix C.1. ◀

### 4.3 Reducing equality saturation to the standard chase

We show how to reduce equality saturation to the standard chase. The encoding itself is straightforward. However, the standard chase is non-deterministic and can have different chase sequences, so a natural question is what kind of the chase sequence will converge finitely, given that EqSat terminates, and vice versa. We show that as long as the chase sequence applies EGDs frequent enough, the chase sequence will always converge. We capture this notion as EGD-fairness.

► **Definition 29.** Given a database schema  $\mathcal{S}$ , a set of dependencies  $\Gamma$  over  $\mathcal{S}$ , and an initial database  $I_0$ . We call a chase sequence  $I_0, I_1, \dots$  of  $\Gamma$  and  $I$  EGD-fair if for every  $i$ , either  $I_i$  is a model of  $\Gamma$  and the chase terminates, or there exists some  $j > i$  such that  $I_j$  is a model of the EGD subset of  $\Gamma$ .

Given that EqSat terminates, what can we say about chase sequences that are not EGD-fair? In fact, such chase sequences may not terminate. Despite this, it can be shown that the result of such chase sequences, terminating or not, is isomorphic to the result of equality saturation (when encoded as a database). On the other hand, to show that equality saturation terminates, it is sufficient to show an arbitrary chase sequence terminates.

The following theorem shows the connection between EqSat and the standard chase.

► **Theorem 30.** *Given signature  $\Sigma$ , a set of rewrite rules  $\mathcal{R}$  over  $\Sigma$ , and an initial E-graph  $G$ , it is possible to define a relational schema  $\mathcal{S}$ , a set of dependencies  $\Gamma$  over  $\mathcal{S}$ , and an initial database  $I$  over  $\mathcal{S}$ . The following three statements are equivalent:*

1. Equality saturation terminates for  $\mathcal{R}$  and  $t$ .
2. There exists a terminating chase sequence of the standard chase for  $\Gamma$  and  $I$ .
3. All EGD-fair chase sequences of the standard chase terminate for  $\Gamma$  and  $I$ .

Moreover, denote the result of an arbitrary chase sequence as  $I_\infty$ . If equality saturation terminates,  $I_\infty$  is isomorphic to the database encoding the resulting E-graph of  $\text{EqSAT}(\mathcal{R}, G)$ .

The encoding consists of two steps. First, we can encode an E-graph as a database. Second, we encode the match/apply operator and congruence closure operator as a set of TGDs and EGDs. To encode an E-graph as a database:

- Take the domain  $\text{Dom}$  to be the set of all E-classes, which are treated as marked nulls.
- For every function symbol  $f$  of arity  $n$ , add relation symbol  $R_f$  of arity  $n + 1$  to  $\mathcal{S}$ .
- For every E-node  $f(c_1, \dots, c_n) \rightarrow c$ , add a tuple  $R_f(c_1, \dots, c_n, c)$  to the database  $I$ .

Under this encoding, each E-class is treated as a marked null, and each E-node is treated as a tuple.

The encoding of the match/apply operator and congruence closure operator is plain:

- For every function symbol  $f$  of arity  $n$ , add a functional dependency  $R_f(x_1, \dots, x_n, x) \wedge R_f(x_1, \dots, x_n, x') \rightarrow x = x'$  to  $\Gamma$ .
- For every rewrite rule  $\text{lhs} \rightarrow \text{rhs}$  in  $\mathcal{R}$ , flatten the left- and right-hand side into conjunctions of relational atoms, unify the variable denoting the root node of  $\text{lhs}$  with that of  $\text{rhs}$ , and add existential quantifiers to the head accordingly. For example, rule  $f(f(x, y), z) \rightarrow f(x, f(y, z))$  is flattened into  $R_f(x, y, w_1) \wedge R_f(w_1, z, r) \rightarrow \exists w_2, R_f(x, y, w_2) \wedge R_f(w_2, v, r)$ . There are two corner cases to the above translations. First, if  $\text{lhs}$  is a single variable  $x$ , we need to introduce  $n$  rules of the form  $R_f(y_1, \dots, y_k, x) \rightarrow \dots$ , one for each function symbol, to “ground”  $x$ . For instance, suppose  $\Sigma = \{f(\cdot, \cdot), g(\cdot)\}$ , rewrite rule  $x \rightarrow g(x)$  is flattened into two dependencies:  $R_f(y_1, y_2, x) \rightarrow R_g(x, x)$  and  $R_g(y_1, x) \rightarrow R_g(x, x)$ . Second, in the case that the right-hand side is a single variable  $x$ , we need to add an EGD instead of a TGD. For example, rule  $f(x, y) \rightarrow x$  is encoded as an EGD  $R_f(x, y, r) \rightarrow x = r$ .

**Proof of Theorem 30.** See Appendix C.2. ◀

## 5 The termination theorems of equality saturation

Finally, we present our main results here.

► **Theorem 31** (Single-instance termination). *The following problem is R.E.-complete:*

- *Instance:* A term rewriting system  $R$ , a term  $t$ .
- *Question:* Does  $\text{EqSat}$  terminate with  $R$  and  $t$ ?

► **Theorem 32** (All-term-instance termination). *The following problem is  $\Pi_2$ -complete:*

- *Instance:* A term rewriting system  $R$ .
- *Question:* Does  $\text{EqSat}$  terminate with  $R$  and  $t$  for all terms  $t$ ?

While Theorem 31 follows immediately from the fact the the Skolem chase is undecidable, our proof in Appendix D.1 is based on Narendran et al. [22], which allows us to also show Theorem 32. We encode a Turing machine as a term rewriting system with the property that the congruence classes of initial configurations corresponds to traces of running such configurations, and that  $\text{EqSat}$  terminates if and only if congruence class is finite. For the

all-term-instance case, we then show that the congruence class of an arbitrary term is infinite if and only if the congruence class of an initial configuration is. The actual proof is slightly more involved so we refer the reader to Appendix D.1 for more details.

The technique above does not apply to the all-E-graph-instance case, however. The all-E-graph-instance termination can be thought of as having inputs both a term and a set of ground identities, and we have no control over the latter. Still, we are able to prove that this problem is undecidable by a reduction from the Post correspondence problem (Appendix D.2), while the exact upper bound is unknown.

► **Theorem 33** (All-E-graph-instance termination). *The following problem is undecidable:*

- *Instance:* A term rewriting system  $R$ .
- *Question:* Does EqSat terminate with  $R$  and  $G$  for all E-graphs  $G$ ?

## 6 Weak term acyclicity for equality saturation termination

We can adapt the classic weak acyclicity criterion [8], which is used to show the termination of the chase algorithm, to equality saturation. The adapted criterion, which we call *weak term acyclicity*, is more powerful than simply translating EqSat rules to TG-Ds/EGDs and applying weak acyclicity. We demonstrate weak term acyclicity with two examples, and the full definition can be found at Appendix E.

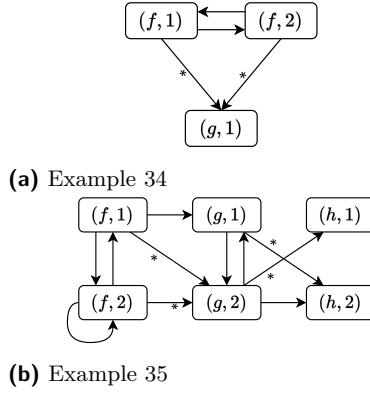
► **Example 34.** Consider  $\mathcal{R} = \{f(f(x, y), z) \rightarrow g(f(z, x))\}$ . This ruleset is weakly term acyclic, with the weak term dependency graph shown in Figure 4a. Note however if we derive the dependencies  $\Gamma$  using the method in Sec. 4.3 from  $\mathcal{R}$ ,  $\Gamma$  is not weakly acyclic.

► **Example 35.** Consider  $\mathcal{R} = \{g(f(x_1, y_1), f(z_1, x_1)) \rightarrow g(z_1, f(y_1, x_1)), g(x_2, y_2) \rightarrow h(y_2, g(y_2, x_2))\}$ . This ruleset is weakly term acyclic. Its weak term dependency graph is shown in Figure 4b.

## 7 Conclusion

We have presented a semantic foundation for E-graphs and EqSat: We identified E-graphs as reachable and deterministic tree automata and defined the result of EqSat as the least fixpoint according to E-graph homomorphisms. We defined the universal model of E-graphs and showed the fixpoint EqSat produces is the universal model (Theorem 19). We showed several basic properties about E-graphs, including a finite convergence lemma (Lemma 26). We then established connections between EqSat and the chase in both directions (Sec. 4) and characterize chase sequences that correspond to EqSat with EGD-fairness (Definition 29). Our main results are on the terminations of EqSat in three cases: single-instance, all-term-instance, and all-E-graph-instance. Finally, adapting ideas from weak acyclicity for the chase, we defined weak term acyclicity which implies EqSat termination.

The correspondence between EqSat and the chase established in this paper may help further port the rich results of database theory to EqSat, as the current paper only scratches the surface of the deep literatures of the chase. Another direction is to use our better understanding of EqSat to design more efficient and expressive EqSat tools and better support



■ **Figure 4** Example weak term dependency graphs. Special edges are marked with \*.

downstream applications of EqSat. Finally, many problems about EqSat are still open. For example, the exact upper bound of the all-E-graph-instance termination is not known. Other problems include rule scheduling, evaluation algorithm, and E-graph extraction.

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## A Proof basic properties of E-graphs and EqSat

### A.1 Proof for Lemma 15

► **Lemma 15.** *For any reachable tree automaton  $\mathcal{A}$ ,  $CC(\mathcal{A})$  exists and is unique.*

**Proof sketch.** Suppose  $\mathcal{A} = \langle Q, \Sigma, Q_{final}, \Delta \rangle$  and let  $L = \bigcup \{\mathcal{L}(c) \mid c \in \mathcal{A}\}$ , the set of terms represented by any state of  $\mathcal{A}$ . Define  $(\approx_L) \subseteq L \times L$  as the smallest equivalence relation satisfying  $t_1 \approx_L t_2$  if some state  $c$  accepts both  $t_1$  and  $t_2$ . Let  $C$  be the set of equivalence classes of  $\approx_L$ . Let

$$N = \{f([t_1]_{\approx_L}, \dots, [t_k]_{\approx_L} \rightarrow [f(t_1, \dots, t_k)]_{\approx_L}) \mid f(t_1, \dots, t_k) \in L\}.$$

We claim that  $G = \langle C, \Sigma, N \rangle$  is such an E-graph.  $G$  is an E-graph as we can show  $G$  is deterministic and reachable. We can also show that if two terms  $t_1$  and  $t_2$  are accepted by the same state of  $\mathcal{A}$ , it holds that  $t_1 \approx_L t_2$ , and that if two terms  $t_1 \approx_L t_2$ ,  $t_1$  and  $t_2$  are accepted by the same E-class of  $G$ . Therefore, terms accepted by the same state of  $\mathcal{A}$  are accepted by the same E-class of  $G$ . Denote this mapping  $h$ . It can be shown that  $h$  is a homomorphism from  $\mathcal{A}$  and  $G$ .

Let  $G'$  be another E-graph and  $u$  be a homomorphism from  $\mathcal{A}$  to  $G'$ . We need to show there is a homomorphism from  $G$  to  $G'$ . This can be done by noticing that if  $t_1 \approx_L t_2$ ,  $t_1$  and  $t_2$  are accepted by the same E-class of  $G'$ . We can then define  $h'([t]_{\approx_L}) = c$  where  $t$  is accepted by  $c$  in  $G'$  and show  $h'$  is a homomorphism from  $G$  to  $G'$ .

The uniqueness of  $G$  follows from the anti-symmetry of  $\sqsubseteq$ . ◀

### A.2 Proof for Lemma 17

► **Lemma 17 (Least upper bound).** *The least upper bound exists and is given by  $CC(\mathcal{A})$ , where  $\mathcal{A}$  is the automaton consisting of the disjoint union of the states and the disjoint union of the transitions of all E-graphs  $G_i$ .*

**Proof.**  $G_i \sqsubseteq G$  since there is an identity homomorphism from  $G_i$  to  $\mathcal{A}_{\sqsubseteq}$  and a homomorphism, denoted as  $h_{CC}$ , from  $\mathcal{A}_{\sqsubseteq}$  to  $G$  by Lemma 15.

Next, we show if  $G'$  is an E-graph such that  $\forall i. G_i \sqsubseteq G'$ ,  $G \sqsubseteq G'$ . Denote the homomorphism from  $G_i$  to  $G'$  as  $h_i$ . There is a homomorphism from  $\mathcal{A}_{\sqsubseteq}$  to  $G'$ , defined by  $h(c) = h_i(c)$

for  $c$  is an E-class of  $G_i$ . By Lemma 15, there is a homomorphism  $h'$  from  $G$  to  $G'$ .

$$\begin{array}{ccc} \text{htc} & & G' \\ \downarrow & & \downarrow \\ \mathcal{A}_{\sqsubseteq}(\bigcup_i Q_i, \bigcup_i \Delta_{\sigma_i}) & & \mathcal{A}_{\sqsubseteq}(Q'_i, \Delta'_{\sigma'_i}) \end{array}$$

◀

### A.3 Proof for Theorem 19

► **Theorem 19.** Fix an E-graph  $G$ , and consider the class  $\mathcal{C}_G$  of E-graphs  $G' \sqsupseteq G$ . Then  $\text{ICO}_{\mathcal{R}} : \mathcal{C}_G \rightarrow \mathcal{C}_G$  has a least fixpoint, given by

$$\text{EQSAT}(\mathcal{R}, G) \stackrel{\text{def}}{=} \bigsqcup_{i \geq 0} \text{ICO}_{\mathcal{R}}^{(i)}(G) \quad (3)$$

Furthermore,  $\text{EQSAT}(\mathcal{R}, G)$  is a universal model of  $\mathcal{R}, G$ ; we call it equality saturation.

**Proof sketch.** By definition,  $H$  is a model of  $\mathcal{R}, G$  iff  $G \sqsubseteq H$  and  $H$  is a fixpoint of  $\text{ICO}_{\mathcal{R}}$ . Therefore, it suffices to prove that  $H = \bigsqcup_{i \geq 0} \text{ICO}_{\mathcal{R}}^{(i)}(G)$  is the least fixpoint. We first prove that  $H$  is a fixpoint.  $H \sqsubseteq \text{ICO}_{\mathcal{R}}(H)$  since  $\text{ICO}_{\mathcal{R}}$  is inflationary, so we prove the opposite. Apply definition of  $T_{\mathcal{R}}$ ,  $T_{\mathcal{R}}(H) = H \cup (\bigcup \text{FL})$ , where  $\bigcup \text{FL}$  abbreviates the union of E-matches into  $H$ . Since  $\mathcal{R}$  is finite and every E-match  $\sigma : \text{Var}(\text{lhs}) \rightarrow Q$  uses a finite number of states, there exists an  $i$  such that all E-matches use only the states of  $\text{ICO}_{\mathcal{R}}^{(i)}(G)$ . This implies  $(\bigcup \text{FL}) \sqsubseteq \text{ICO}_{\mathcal{R}}^{(i)}(G) \sqsubseteq H$ , proving that  $T_{\mathcal{R}}(H) \sqsubseteq H$ . It follows that  $\text{ICO}_{\mathcal{R}}(H) = \text{CC}(T_{\mathcal{R}}(H)) \sqsubseteq \text{CC}(H) = H$  because  $H$  is deterministic. Thus,  $H$  is a fixpoint. We prove that it is the least: let  $H'$  be another fixpoint. We use induction on  $i$  to prove  $\text{ICO}_{\mathcal{R}}^{(i)}(G) \sqsubseteq H'$ : assuming this holds for  $i$ , we derive  $\text{ICO}_{\mathcal{R}}(\text{ICO}_{\mathcal{R}}^{(i)}(G)) \sqsubseteq \text{ICO}_{\mathcal{R}}(H') = H'$  thus it holds for  $i+1$ . Therefore  $H = \bigsqcup_{i \geq 0} \text{ICO}_{\mathcal{R}}^{(i)}(G) \sqsubseteq H'$ , completing the proof. ◀

## B Proof for the finite convergence lemma

► **Lemma 26** (Finite convergence). Let  $\mathcal{G} : G_1 \sqsubset G_2 \sqsubset \dots$  be an ascending sequence of finite E-graphs. If  $G_{\infty} = \bigsqcup_i G_i$  is finite, then the sequence  $\mathcal{G}$  is finite.

**Proof.** For the sake of contradiction we assume  $\mathcal{G}$  is infinite. Let us first define the *rank* of an E-class as follows.

$$\begin{aligned} \text{rank}_G(c) &= \min_{(f(c_1, \dots, c_n) \rightarrow c) \in N} \text{rank}_G(f(c_1, \dots, c_n)) \\ \text{rank}_G(f(c_1, \dots, c_n)) &= 1 + \max\{0, \text{rank}_G(c_1), \dots, \text{rank}_G(c_n)\} \end{aligned}$$

Intuitively, the rank of an E-class is the smallest depth of the terms represented by this E-class. By E-graph's reachability, every E-class has a finite rank. We define the rank of an E-graph as the greatest rank of its E-classes. Some observations about E-graph ranks:

► **Fact 36.** Every finite E-graph has a finite rank.

► **Fact 37.** The set of E-graphs with a bounded rank is finite.<sup>8</sup>

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<sup>8</sup> To see this, E-graph rank bounds the number of E-classes an E-graph can have.

► **Fact 38.** *Ranks only decrease: Let  $G$  and  $H$  be two E-graphs and  $h : G \rightarrow H$  a homomorphism.  $\text{rank}_G(c) \geq \text{rank}_H(h(c))$ .*

Let us denote the homomorphism between  $G_i$  and  $G_j$  as  $h_{i,j}$  for all  $i \leq j$ , and the homomorphism between  $G_i$  and  $G_\infty$  as  $h_{i,\infty}$ . Since  $G_\infty$  is finite, denote its rank as  $N$ .

We say a homomorphism  $h : G \rightarrow H$  is *Union<sub>N</sub>* if there exist two E-classes  $c_1$  and  $c_2$  such that  $\text{rank}_G(c_1) \leq N$ ,  $\text{rank}_G(c_2) \leq N$ , and  $h(c_1) = h(c_2)$ . In other words,  $H$  “unions” two E-classes with rank  $\leq N$  of  $G$ . We claim there exists some lower bound  $M$  such that for all  $M \leq i \leq j$ ,  $h_{i,j}$  is not *Union<sub>N</sub>*. This is because every *Union<sub>N</sub>* homomorphism necessarily merges two equivalence classes with rank  $\leq N$ , and this cannot happen indefinitely because each such equivalence class need to be backed by a term with depth  $\leq N$ , which is finite.

Find  $a_1 > M$  such that  $G_{a_1}$  contains an E-class  $c_1$  with rank  $N + 1$ . This is always possible by Fact 37 and the fact that any E-graph with rank  $> N$  must contain some E-classes with rank  $N + 1$ . Let  $n_1 = f_1(\vec{c}_1) \rightarrow c_1$  be an E-node of E-class  $c_1$  such that  $\max_i \text{rank}_G((\vec{c}_1)_i) \leq N$ .

Since all E-classes of  $G_\infty$  have rank  $\leq N$ , there must exist some  $b_1 > a_1$  such that  $h_{a_1,b_1}(c)$  in  $G_{b_1}$  has rank  $\leq N$ . Next, find an  $a_2 > b_1$  such that  $G_{a_2}$  contains an E-class  $c_2$  with rank  $N + 1$ . Repeat the same process to obtain a sequence  $\mathcal{N}$  of E-nodes

$$f_1(\vec{c}_1) \rightarrow c_1 \in G_{a_1}$$

$$f_2(\vec{c}_2) \rightarrow c_2 \in G_{a_2}$$

...

that satisfies  $\text{rank}_{G_{a_i}}(c_i) = N + 1$ ,  $\max_j (\vec{c}_i)_j \leq N$ , and  $\text{rank}_{G_{b_i}}(h_{a_i,b_i}(c_i)) \leq N$ . Note by Fact 38, it also holds that for all  $k > b_i$ ,  $h_{a_i,k}(c_i)$  also has rank  $\leq N$ .

For the rest of the proof, we claim that for all  $i$  and  $j$ ,  $h_{a_i,\infty}(f_i(\vec{c}_i)) \neq h_{a_j,\infty}(f_j(\vec{c}_j))$ , which implies  $G_\infty$  has infinitely many distinct E-classes as  $\mathcal{G}$  is infinite. This is a contradiction to the fact that  $G_\infty$  is finite.

To prove this claim, we assume  $i < j$  without loss of generality. Observe that  $h_{a_i,a_j}(c_i)$  has rank  $\leq N$  and  $c_j$  has rank  $N + 1$  in  $G_{a_j}$ , so

$$h_{a_i,a_j}(c_i) \neq c_j.$$

By determinacy of E-graphs,

$$h_{a_i,a_j}(f_i(\vec{c}_i)) \neq f_j(\vec{c}_j).$$

Because E-classes in  $\vec{c}_i$  and  $\vec{c}_j$  has rank  $\leq N$ , it holds that for all  $k \geq a_j$ ,

$$\begin{aligned} h_{a_j,k}(h_{a_i,a_j}(f_i(\vec{c}_i))) &\neq h_{a_j,k}(f_j(\vec{c}_j)) \\ h_{a_i,k}(f_i(\vec{c}_i)) &\neq h_{a_j,k}(f_j(\vec{c}_j)) \end{aligned}$$

since  $h_{a_j,k}$  is not *Union<sub>N</sub>*. This inequality is preserved *ad infinitum*. By virtue of  $G_\infty$  being the least upper bound,

$$h_{a_i,\infty}(f_i(\vec{c}_i)) \neq h_{a_j,\infty}(f_j(\vec{c}_j)).$$



$$\begin{aligned}
r_{\top} : \quad & \top \rightarrow \wedge(\top, \top) \\
r_R : \quad & R(x_1, \dots, x_n) \rightarrow \top \quad \text{for relation symbol } R \in \sigma \\
r_d : \quad & \wedge(R_1(\vec{x}_1, \vec{y}_1), \dots) \rightarrow R_1(\vec{x}_1, \vec{y}_1) \wedge \dots \rightarrow \\
& \wedge(R'_1(\vec{x}'_1, \vec{f}'_{z1}), \dots) \quad \text{for dep. } d : R'_1(\vec{x}'_1, \vec{f}'_{z1}) \wedge \dots \in \Gamma \\
r_t : \quad & \top \rightarrow R(c_1, \dots, c_n) \quad \text{for initial tuple } t = R(c_1, \dots, c_n) \in I
\end{aligned}$$

**Figure 5** A summary of encoded rewrite rules in Theorem 28 for the given  $\Gamma$  and  $I$ . The initial term to equality saturation is  $\top$ .

## C Proof for the reductions between EqSat and the chase

### C.1 Proof for Theorem 28

► **Theorem 28.** *Given a database schema  $\mathcal{S} = (R_1, \dots, R_m)$ , a set of TGDs  $\Gamma$ , and an initial database  $I$ , it is possible to define a signature  $\Sigma$ , a term rewriting system  $\mathcal{R}$  over  $\Sigma$ , and an initial term  $t$  such that*

$$\xi(EQSAT(\mathcal{R}, t)) = SKLCH(\Gamma, I).$$

Moreover, the Skolem chase terminates if and only if equality saturation terminates.

**Proof.** Denote  $G = EQSAT(\mathcal{R}, t)$  and  $I = SKLCH(\Gamma, I)$ . Denote the E-class of  $G$  that represents  $\top$  as  $c_{\top}$ .

We show the following holds between  $G$  and  $I$ :

A tuple  $R(c_1, \dots, c_n)$  is in  $I$  if and only if  $R(c_1, \dots, c_n)$  is represented by  $G$ , where  $R$  is a function symbol in  $\Sigma$  and  $c_1, \dots, c_n$  are Skolem terms or constants.

- $\Rightarrow$ : We prove by induction on the iteration of applying match/apply operator  $T_{\mathcal{R}}$  when a tuple is first derived in  $I$ .
  - Base case: this is clear from Figure 3. Denote the E-graph after applying the first iteration of equality saturation as  $G_1$ . In the first iteration of equality saturation, rules that are applied are  $\top \rightarrow \wedge(\top, \top)$  and  $\top \rightarrow R(c_1, \dots, c_n)$  for each tuple in the initial database, so every tuple  $R(c_1, \dots, c_n)$  in the input database is represented  $G_1$ . Since  $G_1 \sqsubseteq G$ , every tuple is also represented by  $G$ .
  - Inductive case: Assume a tuple  $R_i(h(\vec{x}'_i), h(\vec{f}'_{zi}))$  is produced at the  $n$ -th iteration of  $T_{\mathcal{R}}$  by dependency

$$d : R_1(\vec{x}_1, \vec{y}_1) \wedge \dots \wedge R_n(\vec{x}_n, \vec{y}_n) \rightarrow R'_1(\vec{x}'_1, \vec{f}'_{z1}) \wedge \dots \wedge R'_m(\vec{x}'_m, \vec{f}'_{zm})$$

and substitution  $h$ . By inductive hypothesis, every tuple  $R_i(h(\vec{x}'_i), h(\vec{y}'_i))$  in the substituted body of  $d$  is also represented by  $c_{\top}$  of  $G$ .

It is straightforward to see that  $G$  contains E-nodes  $\top \rightarrow c_{\top}$  and  $\wedge(\top, \top) \rightarrow c_{\top}$ . As a result,

$$\begin{aligned}
& \wedge(R_1(h(\vec{x}_1), h(\vec{y}_1)), \dots, \wedge(R_n(h(\vec{x}_n), h(\vec{y}_n)), \top)) \rightarrow^* \wedge(c_{\top}, \dots, \wedge(c_{\top}, c_{\top})) \\
& \qquad \qquad \qquad \rightarrow^* c_{\top}
\end{aligned}$$

Therefore, to reach fixpoint, equality saturation would also add the substituted right-hand side

$$\wedge(R'_1(\vec{x}'_1, \vec{f}'_{z1}), \dots, \wedge(R'_m(\vec{x}'_m, \vec{f}'_{zm}), \top))$$

- to  $c_{\top}$ . Similarly, equality saturation would fire series of rules of the form  $r_R : R(x_1, \dots, x_n) \rightarrow \top$ , so  $R_i(h(\vec{x}_i), h(\vec{f}_{z_i}^d))$  would be represented by  $c_{\top}$  of  $G$ .
- $\Leftarrow$ : It can be shown that every term  $R(\vec{c})$  that is inserted into the E-graph at iteration  $n$  will be unioned with the  $c_{\top}$  at iteration  $n + 1$ . Therefore, we only need to show that for every term  $R(\vec{c})$  represented by some E-class of  $G$ ,  $R(\vec{c})$  is in  $I$ . Similar to the first case, let us prove by inducting on the first time a  $R$ -term  $R(\vec{c})$  is inserted into the E-graph.
  - Base case: At iteration 1, all terms  $R(c_1, \dots, c_n)$  inserted correspond to tuples from the initial database.
  - Inductive case: Suppose at iteration  $i > 1$ , a  $R$ -term  $R(\vec{c})[\sigma]$  is inserted into the E-graph because of some rewrite rule  $r$  and E-class substitution  $\sigma$ . By definition, the rewrite rule  $r$  has to be of the form

$$r_d : \wedge(R_1(\vec{x}_1, \vec{y}_1), \dots) \rightarrow \wedge(R'_1(\vec{x}'_1, \vec{f}'_{z_1}), \dots).$$

By inductive hypothesis, all tuples  $R_i(\vec{c}_i)$  represented by  $R_i(\vec{x}_i, \vec{y}_i)[\sigma]$  are in  $I$ , so the dependency that  $r$  is mapped from would be fired and tuples represented by  $R(\vec{c})[\sigma]$  are added to  $I$ .

Next, we show that the Skolem chase terminates if and only if equality saturation terminates: if equality saturation terminates, an E-graph with a finite number of transitions is produced, so the mapped database is finite. Since the Skolem chase monotonically adds tuples to the database, the Skolem chase has to terminate as well. On the other hand, if the Skolem chase is finite, the produced E-graph needs to be finite as well, since it only contains  $\top \rightarrow c_{\top}$  and  $\wedge(\top, \top) \rightarrow c_{\top}$  besides E-nodes that are in one-to-one correspondence with the databases. By Lemma 26, equality saturation terminates as well. ◀

## C.2 Proof for Theorem 30

► **Theorem 30.** *Given signature  $\Sigma$ , a set of rewrite rules  $\mathcal{R}$  over  $\Sigma$ , and an initial E-graph  $G$ , it is possible to define a relational schema  $\mathcal{S}$ , a set of dependencies  $\Gamma$  over  $\mathcal{S}$ , and an initial database  $I$  over  $\mathcal{S}$ . The following three statements are equivalent:*

1. Equality saturation terminates for  $\mathcal{R}$  and  $t$ .
2. There exists a terminating chase sequence of the standard chase for  $\Gamma$  and  $I$ .
3. All EGD-fair chase sequences of the standard chase terminate for  $\Gamma$  and  $I$ .

Moreover, denote the result of an arbitrary chase sequence as  $I_{\infty}$ . If equality saturation terminates,  $I_{\infty}$  is isomorphic to the database encoding the resulting E-graph of EqSAT( $\mathcal{R}, G$ ).

To prove this theorem, we show the following lemma. Recall that a database is a core if every homomorphism from the database to itself is an isomorphism. Two homomorphically equivalent cores are isomorphic.

► **Lemma 39.** *Given a set of rewrite rules  $\Sigma$ , let  $I$  be a database instance that encodes an E-graph and  $\Gamma$  be a set of dependencies encoding EqSat of  $\Sigma$ , both using the encoding in Sec. 4.3. Suppose  $I_k$  is a database obtained after a finite number of chase steps of  $\Gamma$  and  $I$ . If  $I_k$  is closed under the EGD subset of  $\Gamma$ ,  $I_k$  is a core database instance.*

Consider the tree automaton  $\mathcal{A}$  encoded by  $I_k$ . Because  $I_k$  is closed under EGDs,  $\mathcal{A}$  is deterministic. Since every dependency of  $\Sigma$  preserves reachability of the encoded E-graph,  $\mathcal{A}$  is reachable, so  $\mathcal{A}$  is an E-graph. Therefore,  $\mathcal{A}$  is a core tree automaton. The definition of homomorphisms is preserved under our encoding, so the only homomorphism from  $I_k$  to  $I_k$  is the identity mapping. Therefore  $I_k$  is a core.

► **Corollary 40.** Fix  $\Sigma, I, \Gamma$  as above. Let  $I_\infty$  be the result of a (potentially non-terminating) chase sequence of  $\Sigma$  and  $I$ .  $I_\infty$  is a core database instance.

$I_\infty$  is closed under the EGD subset of  $\Gamma$ . By the above lemma we only need to consider the case where  $I_\infty$  is the result of a non-terminating chase. Since  $I_\infty$  is closed under the EGDs, it encodes a deterministic tree automaton  $\mathcal{A}$ . Suppose for the sake of contradiction that  $\mathcal{A}$  is not reachable, with some state  $q$  that does not accept any term. There must exist a finite number  $m$  where  $q$  stays unreachable for all  $\geq m$  iterations. However, this is impossible, since the head of each dependency of  $\Gamma$  preserves reachability.

**Proof of Theorem 30.** We use the encoding in Sec. 4.3 and treat every tuple  $R_f(c_1, \dots, c_n, c)$  as an E-node  $f(c_1, \dots, c_n) \rightarrow c$  in an E-graph. It is straightforward to see that every insertion of the match/apply operator corresponds to a chase step of some TGD (EGD if  $rhs$  is a single-variable) of the above the encoding, and every merge by CC corresponds to a chase step of some FD. Therefore, a terminating run of equality saturation corresponds to a terminating chase sequence  $((1) \implies (2))$ . Denote this chase sequence as the EqSat-encoding chase sequence  $\mathcal{I}_{\text{EQSAT}}$ , and the resulting database is isomorphic to the E-graph (when encoded as a database).

Suppose there is a terminating chase sequence that produces some database instance  $I$ . By our lemma  $I$  is core. Let  $J$  be the database instance produced by  $\mathcal{I}_{\text{EQSAT}}$ .  $J$  is also a core, since it encodes an E-graph. Because  $I$  and  $J$  are both universal models, they are homomorphically equivalent. Homomorphically equivalent cores are isomorphic,  $I$  and  $J$  has to be isomorphic. Since  $I$  is finite,  $J$  is also finite. By Lemma 26, equality saturation also terminates  $((2) \implies (1))$ .

The fact that  $(3) \implies (1)$  is obvious, since  $\mathcal{I}_{\text{EQSAT}}$  is EGD-fair. To show  $(1) \implies (3)$ , let  $\mathcal{J} : J_0, J_1, \dots$  be an arbitrary EGD-fair chase sequence with result  $J_\infty$ , and consider the EqSat-encoding chase sequence  $\mathcal{I}_{\text{EQSAT}} : I_0, I_1, \dots, I_n$ . We claim that for all  $i$ , there exists a finite  $j$  such that there is a homomorphism from  $I_i$  to  $J_j$  and  $J_j$  is closed under EGDs. This can be proved by induction on  $i$ . The case for  $i = 0$  holds trivially. Suppose  $I_i \sqsubseteq J_j$  for some  $j$ . The fact that some  $j'$  exists such that there is a homomorphism from  $I_{i+1}$  to  $J_{j'}$  follows from fairness of chase sequences. With EGD-fairness, it is then possible to find another  $j'' > j'$  such that  $J_{j''}$  is closed under EGDs and  $J_j \sqsubseteq J_{j'} \sqsubseteq J_{j''}$ .

As a result, some  $m$  exists such that there is a homomorphism from  $I_n$  to  $J_m$ . There is also a homomorphism from  $J_m$  to  $J_\infty$  from the chase sequence and a homomorphism from  $J_\infty$  to  $I_n$  since  $J_\infty$  is a universal model. By our lemma  $I_n, J_m, J_\infty$  are all core models. Since they are homomorphically equivalent, they have to be isomorphic. so  $J_m$  and  $J_\infty$  coincide. Therefore, The chase sequence  $\mathcal{J} : J_0, J_1, \dots, J_m$  stops in  $m$  steps  $((1) \implies (3))$ .

$$\begin{array}{ccccccc} I_0 & & \dots & & I_n \\ & & & & & & \\ J_0 & & \dots & & J_m & & \dots & & J_\infty \end{array}$$

It is left to show that if EqSat terminates, any arbitrary chase sequence, terminating or not, has result isomorphic to the output of equality saturation. This can be shown similarly by noticing that  $J_\infty$  is a core model and homomorphically equivalent to the result of EqSat, so they have to be isomorphic.



## D Proof for the termination theorems

Before we proceed to the proofs, let us complement the backgrounds on term rewriting system. Given a term rewriting system  $\mathcal{R}$ , a normal form is a term that cannot be rewritten any further. We say  $n$  is a normal form of  $t$  if  $t$  reduces to  $n$  and  $n$  is a normal form. A TRS  $\mathcal{R}$  is *terminating* if there is no infinite rewriting chain  $t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow \dots$ . A TRS  $\mathcal{R}$  is *confluent* if for all  $t, t_1, t_2, t_1 \leftarrow_{\mathcal{R}}^* t \rightarrow_{\mathcal{R}}^* t_2$  implies there exists a  $t'$  such that  $t_1 \rightarrow_{\mathcal{R}}^* t' \leftarrow_{\mathcal{R}}^* t_2$ . We call a confluent and terminating TRS *convergent*. Every term in a terminating TRS has at least one normal form, every term in a confluent TRS has at most one normal form, and every term in a convergent TRS has exactly one normal form.

### D.1 Proof for Theorem 31 and Theorem 32

- ▶ **Theorem 31** (Single-instance termination). *The following problem is R.E.-complete:*
  - *Instance:* A term rewriting system  $R$ , a term  $t$ .
  - *Question:* Does EqSat terminate with  $R$  and  $t$ ?

This problem is in R.E. since we can simply run EqSat with  $R$  and  $t$  to test whether it terminates. To show this problem is R.E.-hard, we reduce the halting problem of Turing machines to the termination of EqSat. The proof follows that of Narendran et al. [22].

In this proof, we consider a degenerate form of EqSat that works with *string* rewriting systems instead of TRS. A string can be viewed as a degenerate term, and a string rewriting rule can be viewed as a degenerate term rewriting rule. For example, the string  $uvw$  corresponds to a term  $u(v(w(\epsilon)))$ , where  $u, v, w$  are unary functions and  $\epsilon$  is a special constant marking the end of a string. A string rewriting rule  $uvw \rightarrow vuw$  corresponds to a (linear) term rewriting rule  $u(v(w(x))) \rightarrow v(u(w(x)))$  where  $x$  is a variable.

For each Turing machine  $\mathcal{M}$ , we produce a string rewriting system  $\mathcal{R}$  such that the congruence closure of  $R$ ,  $(\approx_{\mathcal{R}})$ , satisfies that each congruent class of  $\approx_{\mathcal{R}}$  corresponds to a trace of the Turing machine. As a result, informally, the following statements are equivalent:

1. the Turing machine halts;
2. the trace of the Turing machine is finite;
3. the congruent class in  $\approx_{\mathcal{R}}$  is finite;
4. EqSat terminates.

A first simple idea is to encode the transition relation of a Turing machine directly as a string rewriting system, so that the congruence class of the initial configuration  $w_0$  contains the trace. For example, consider transition  $q_i abRq_j$ , which says if the current state is  $q_i$  and the symbol being scanned is  $a$ , then write  $b$  to the tape, move right, and change the state to  $q_j$ . It is tempting to encode this transition as a string rewriting rule  $q_i a \rightarrow_{\mathcal{R}} b q_j$ . The issue, however, is that two different initial configurations can lead to the same configuration. For example, consider a Turing machine that clears its input and then halts. Every input string leads to the same configuration, so its termination on an input does not imply the finiteness of its congruent class.

To address this issue, following Narendran et al. [22], we require the string rewrite rule not only encode the transition relation, but also stores the history of the computation. As a result, even if two different initial configurations lead to the same configuration, the rewritten strings that correspond to the same configuration are different, so different initial configurations lead to different congruent classes. More specifically, we introduce dummy symbols that stores states and symbols before transitions.

**Turing machine** A Turing machine  $\mathcal{M} = (Q, \Sigma, \Pi, \Delta, q_0, \beta)$  consists of a set of states  $Q$ , the input and the tape alphabet  $\Sigma$  and  $\Pi$  (with  $\Sigma \subseteq \Pi$ ), a set of transitions  $\Delta$ , an initial state  $q_0 \in Q$ , and a special blank symbol  $\beta \in \Pi$ . Each transition in  $\Delta$  is a quintuple in  $Q \times \Pi \times \Pi \times \{L, R\} \times Q$ . For example, transition  $q_i a b R q_j$  means if the current state is  $q_i$  and the symbol being scanned is  $a$ , then replace  $a$  with  $b$ , move the head to the right, and transit to state  $q_j$ . We assume the Turing machine is two-way infinite (so that the head can move in both directions indefinitely) and deterministic. Each configuration of  $\mathcal{M}$  can be represented as  $\triangleright u q_i v \triangleleft$ , where  $\triangleright, \triangleleft$  are left and right end markers,  $u$  is the string to the left of the read/write head  $q_i$  is the current state,  $v$  is the string to the right. If  $v$  is non-empty, the first character of  $v$  is the symbol being scanned. Otherwise, the symbol being scanned is the blank symbol  $\beta$ . We say  $w_1 \vdash_{\mathcal{M}} w_2$  if configuration  $w_1$  can transition to configuration  $w_2$  in a Turing machine  $\mathcal{M}$ , and we omit  $\mathcal{M}$  when it's clear from the context.

We say a configuration  $w$  is *halting* configuration if it cannot be transited further. We say a configuration  $w$  is *mortal* if there exists a finite sequence of configurations  $w = w_0 \vdash w_1 \vdash \dots \vdash w_n$  such that  $w_n$  is a halting configuration.

**Alphabet of the constructed string rewriting system** Compared to a direct encoding of the transition relation, the string rewriting system  $\mathcal{R}$  we construct has the following characteristics:

- It distinguishes between symbols to the left and to the right of the current state: Define  $\bar{\Pi} = \{\bar{a} \mid a \in \Pi\}$  as the alphabet used exclusively to the left of the state (and  $\Pi$  to the right).
- We introduce new dummy symbol to store information about the states and symbols. Define  $D_L = \{L_z \mid z \in \bar{\Pi} \cup \{\triangleleft\} \text{ or } z \in \{\triangleright\} \cup \bar{\Pi}\}$  and  $D_R$  similarly. Similar to  $\bar{\Pi}$  and  $\Pi$ ,  $D_L$  (resp.  $D_R$ ) is used exclusively for dummy symbols to the left (resp. right) of the state symbol.
- Similar to  $q_i \in Q$ , of which the symbol in  $\Pi$  to the immediate right is being scanned, we also define the “left” counterpart  $\bar{Q}_i = \{\bar{q}_i \mid q_i \in Q\}$ , where the symbol being scanned is to the immediate left of  $\bar{q}_i$ . For instance, in the string representation  $\triangleright u \bar{a} \bar{q}_i v \triangleleft$ , the current state is  $q_i$  and the symbol being scanned  $a$ .

In summary, the rewriting system we construct works over the following regular string language

$$CONFIG = \triangleright (\bar{\Pi} \cup D_L)^* (Q \cup \bar{Q}) (\Pi \cup D_R)^* \triangleleft$$

Strings in  $CONFIG$  are in a many-to-one mapping to configurations of a Turing machine. We denote this mapping as  $\pi$ :  $\pi(w)$  converts each  $\bar{a} \bar{q}_i$  to  $q_i a$  (and  $\triangleright \bar{q}_i$  to  $\triangleright q_i \beta$ ), removes dummy symbols  $L_z$  and  $R_z$ , and replace  $\bar{a}$  with  $a$ . For example,

$$\pi(\triangleright L_{q_0, a} \bar{b} L_{q_1, b} \bar{c} \bar{q}_3 d R_{q_i, \triangleleft} \triangleleft) = \triangleright b q_3 c d \triangleleft$$

Now, for each transitions in  $\mathcal{M}$ , our string rewriting system  $\mathcal{R}$  is defined in Figure 6. It consists of two parts. The first part encodes each transition rule of  $\mathcal{M}$  as two rewrite rules, one for the case when the state symbol is  $q_i \in Q$ , and one for the case when the state symbol is  $\bar{q}_i \in \bar{Q}$ . It also introduces dummy symbols to store the source configuration of each transition. Moreover, for each  $z$ , we have the two additional sets of rewrite rules

$$\begin{aligned} q_i R_z &\rightarrow_{\mathcal{R}} L_z L_z q_i \\ L_z \bar{q}_i &\rightarrow_{\mathcal{R}} \bar{q}_i R_z R_z. \end{aligned}$$

transitions in $\mathcal{M}$	rewrites in $R$
$q_i ab R q_j$	$q_i a \rightarrow_{\mathcal{R}} L_{q_i, a} \bar{b} q_j$ $\bar{a} \bar{q}_i \rightarrow_{\mathcal{R}} L_{\bar{a}, \bar{q}_i} \bar{b} q_j$
$q_i \beta b R q_j$	$q_i \triangleleft \rightarrow_{\mathcal{R}} L_{q_i, \triangleleft} \bar{b} q_j \triangleleft$ $\triangleright \bar{q}_i \rightarrow_{\mathcal{R}} \triangleright L_{\triangleright, \bar{q}_i} \bar{b} q_j$
$q_i ab L q_j$	$q_i a \rightarrow_{\mathcal{R}} \bar{q}_j b R_{q_i, a}$ $\bar{a} \bar{q}_i \rightarrow_{\mathcal{R}} \bar{q}_j b R_{\bar{a}, \bar{q}_i}$
$q_i \beta b L q_j$	$q_i \triangleleft \rightarrow_{\mathcal{R}} \bar{q}_j b R_{q_i, \triangleleft}$ $\triangleright \bar{q}_i \rightarrow_{\mathcal{R}} \triangleright \bar{q}_j b R_{\triangleright, \bar{q}_i}$

U

for each $z$	$q_i R_z \rightarrow_{\mathcal{R}} L_z L_z q_i$ $L_z \bar{q}_i \rightarrow_{\mathcal{R}} \bar{q}_i R_z R_z$
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Figure 6 The string rewriting system  $R$  derived from a Turing machine  $\mathcal{M}$ .

They are used to ensure that the state symbol is adjacent to the symbol being scanned and shuffling dummy symbols around.

To explain what these dummy symbol–shuffling rules do more precisely, let us define two types of strings of  $CONFIG$ .

► **Definition 41.** *Type-A strings are strings in  $CONFIG$  where the symbol being scanned is to the immediate right of  $q_i$  or to the immediate left of  $\bar{q}_i$ . In other words, we call a string  $s$  a type-A string if  $s$  contains  $q_i a$  or  $\bar{a} \bar{q}_i$ . Type-B strings are strings in  $CONFIG$  that are not type-A.*

The rewrite rules above convert any type-B strings into type-A in a finite number of steps.

Now, we observe that the string rewriting system  $R$  we constructed above has several properties:

1. Reverse convergence: the critical pair lemma states that if a rewriting system is terminating and all its critical pairs are convergent, it is convergent. Define  $R^{-1}$  to be a string rewriting system derived from  $R$  by swapping left- and right-hand side of each rewrite rule.  $R^{-1}$  is terminating since rewrite rules in  $R^{-1}$  decreases the sizes of terms, and  $R^{-1}$  has no critical pairs. Therefore,  $R^{-1}$  is convergent.
2. For each type-A string  $w$ , then either
  - there exists no  $w'$  with  $w \rightarrow_{\mathcal{R}} w'$  and  $\pi(w)$  is a halting configuration;
  - there exists a unique  $w'$  such that  $w \rightarrow_{\mathcal{R}} w'$ . Moreover, it holds that  $w' \in CONFIG$  and  $\pi(w) \vdash \pi(w')$ .
3. For each type-B string  $w$ , there exists a unique  $w'$  such that  $w \rightarrow_{\mathcal{R}} w'$ . It holds that  $w'$  is in  $CONFIG$  and  $\pi(w) = \pi(w')$ . Moreover, if  $w_0 \rightarrow_{\mathcal{R}} w_1 \rightarrow_{\mathcal{R}} \dots$  is a sequence of type-B strings, the sequence must be bounded in length, since the state symbols  $q_i$  and  $\bar{q}_i$  move towards one end according to the auxillary rules above.
4. From 2 and 3, it follows that  $\rightarrow_{\mathcal{R}}$  closed under  $CONFIG$  (i.e.,  $w \in CONFIG$  and  $w \rightarrow_{\mathcal{R}} w'$  implies  $w' \in CONFIG$ ) and is deterministic over  $CONFIG$  (i.e.,  $w \rightarrow_{\mathcal{R}} w_1$  and  $w \rightarrow_{\mathcal{R}} w_2$  implies  $w_1 = w_2$ ).

These observations allow us to prove the following lemma

► **Lemma 42.** *Given a Turing machine  $\mathcal{M}$ , construct a string rewriting system  $R$  as above. Let  $w_0$  be a string in  $CONFIG$ .  $\pi(w_0)$  is a mortal configuration of  $\mathcal{M}$  if and only if  $[w_0]_{\mathcal{R}}$ , the equivalence class of  $w_0$  in  $R$ , is finite.*

Rw	$\underbrace{w_0 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} w_{a_0-1}}_{\text{finite}}$	$\rightarrow_{\mathcal{R}}$	$w_{a_0}$	$\underbrace{w_{a_1+1} \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} w_{a_1-1}}_{\text{finite}}$	$\rightarrow_{\mathcal{R}}$	$w_{a_1}$	$\dots$
Type	B ... B		A	B ... B		A	
Config	$\pi(w_0) = \dots = \pi(w_{a_0-1})$	$\vdash_{\mathcal{M}}$	$\pi(w_{a_0})$	$\pi(w_{a_0+1}) = \dots = \pi(w_{a_1-1})$	$\vdash_{\mathcal{M}}$	$\pi(w_{a_1})$	$\dots$

■ **Figure 7** Rewriting sequence for a string  $w_0$  over  $\mathcal{R}$ .

**Proof.** Without loss of generality, we assume  $w_0$  is a normal form with respect to  $R^{-1}$ . If this is not the case, since  $R^{-1}$  is convergent,  $w_0$  has a unique normal form  $w'_0$ . Moreover,  $[w_0] = [w'_0]$  and  $\pi(w'_0) \vdash^*_{\mathcal{M}} \pi(w'_0)$ , so it suffices to consider  $w'_0$ .

- $\Leftarrow$ : Suppose  $[w_0]_{\mathcal{R}}$  is finite. Since the size of each rewrite rule is strictly increasing, no cycle in the rewriting sequence is possible. Therefore, there must exist a finite sequence of  $W : w_0 \rightarrow_{\mathcal{R}} w_1 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} w_n$  such that  $w_n$  is a normal form of  $\mathcal{R}$ . By our observation above, since  $w_n$  cannot be rewritten further,  $w_n$  is a type-A string, and  $\pi(w_n)$  is a halting configuration.

Take the subsequence of  $S$  consisting of all type-A strings:

$$w_{a_0} \rightarrow_{\mathcal{R}}^* w_{a_1} \rightarrow_{\mathcal{R}}^* \dots \rightarrow_{\mathcal{R}}^* w_{a_k} = w_n.$$

We have  $\pi(w_{a_i}) \vdash \pi(w_{a_{i+1}})$  for all  $i$  and  $\pi(w_{a_k})$  is a halting configuration. This implies a finite trace of the Turing machine:

$$\pi(w_0) = \pi(w_{a_0}) \vdash \pi(w_{a_1}) \vdash \dots \vdash \pi(w_{a_n}),$$

which implies  $w_0$  is a mortal configuration of  $\mathcal{M}$ .

- $\Rightarrow$ : Suppose otherwise  $\pi(w_0)$  is a mortal configuration of  $\mathcal{M}$  and  $[w_0]_{\mathcal{R}}$  is infinite. Since  $w_0$  is a normal form with respect to  $R^{-1}$  and  $R^{-1}$  is convergent, for any  $w$  with  $w \approx_{\mathcal{R}} w_0$ ,  $w \rightarrow_{\mathcal{R}-1}^* w_0$ , or equivalently  $w_0 \rightarrow_{\mathcal{R}}^* w$ . Since  $[w_0]_{\mathcal{R}}$  is infinite, there are infinitely many  $w$  satisfying  $w_0 \rightarrow_{\mathcal{R}}^* w$ . By König's lemma, there exists an infinite rewriting sequence:  $W : w_{a_0} \rightarrow_{\mathcal{R}} w_1 \rightarrow_{\mathcal{R}} \dots$ . Again, take the subsequence of  $S$  consisting of every type-A string:

$$W' : w_{a_0} \rightarrow_{\mathcal{R}}^* w_{a_1} \rightarrow_{\mathcal{R}}^* \dots$$

Since every type-B subsequence of  $W$  is bounded in length,  $W'$  is necessarily infinite. This implies an infinite trace of the Turing machine:

$$\pi(w_0) = \pi(w_{a_0}) \vdash \pi(w_{a_1}) \vdash \dots,$$

which is a contradiction. ◀

An overview of the rewriting sequences starting at  $w_0$  of  $\mathcal{R}$  is shown in Figure 7.

We are ready to prove the undecidability of the termination problem of EqSat.

**Proof of Theorem 31.** Given a Turing machine  $\mathcal{M}$ . We construct the following two-tape Turing machine  $\mathcal{M}'$ :

► **Definition 43.**  $\mathcal{M}'$  alternates between the following two steps:

1. Simulate one transition of  $\mathcal{M}$  on its first tape.

2. Read the string on its second tape as a number, compute the next prime number, and write it to the second tape.

$\mathcal{M}'$  halts when the simulation of  $\mathcal{M}$  reaches an accepting state.

It is known that a two-tape Turing machine can be simulated using a standard Turing machine, so we assume  $\mathcal{M}'$  is a standard Turing machine and takes input string  $(s_1, s_2)$ , where  $s_1$  is the input to its first tape and  $s_2$  is the input to its second tape. Let  $\mathcal{R}'$  be the string rewriting system derived from  $\mathcal{M}'$  using the encoding we introduced in the lemma.

Given a string  $s$ , let  $w$  be the initial configuration  $\triangleright q_0(s, 2) \triangleleft$ . The following conditions are equivalent:

1.  $\mathcal{M}$  halts on input  $s$ .
2.  $\mathcal{M}'$  halts on input  $(s, 2)$ .
3.  $[w]_{\mathcal{R}'}$  is finite.
4.  $[w]_{\mathcal{R}'}$  is regular.

(1) and (2) are equivalent by our construction, and (2) and (3) are equivalent by Lemma 42. (3) implies (4) trivially, and (4) implies (3) because if  $[w]_{\mathcal{R}'}$  is infinite, the trace of  $\mathcal{M}'$  will compute every prime number, which is not regular.

Run EqSat with initial string  $w$  and string rewriting system  $\mathcal{R}' \cup \mathcal{R}'^{-1}$ . We claim EqSat terminates if and only if  $\mathcal{M}$  halts on  $s$ :

- $\Rightarrow$ : Suppose EqSat terminates with output E-graph  $G$ . By Corollary 25,  $[w]_G = [w]_{\mathcal{R}'}$ . Since every finite E-graph represents a regular language,  $[w]_{\mathcal{R}'}$  is regular. Therefore,  $\mathcal{M}$  halts on  $s$ .
- $\Leftarrow$ : Suppose  $\mathcal{M}$  halts on  $s$ . Let  $G$  be the E-graph output by EqSat. By Corollary 25,  $[w]_G = [w]_{\mathcal{R}'}$ . By the equivalences above,  $[w]_{\mathcal{R}'}$  is finite. Because the set of represented terms increases in every iteration of EqSat, EqSat has to stop in a finite number of iterations.

By the undecidability of the halting problem, the termination problem of EqSat is undecidable. Therefore, the termination problem of EqSat is R.E.-complete.  $\blacktriangleleft$

► **Theorem 32** (All-term-instance termination). *The following problem is  $\Pi_2$ -complete:*

- *Instance: A term rewriting system  $R$ .*
- *Question: Does EqSat terminate with  $R$  and  $t$  for all terms  $t$ ?*

It does not suffice to just use Lemma 42 to prove this theorem, since Lemma 42 only states properties of strings in  $CONFIG$ , while all-instance termination considers all possible strings (i.e. strings in  $\Sigma^*$ ). To address this mismatch, the following lemma by Narendran et al. bridges the gap between  $\Sigma^*$  and  $CONFIG$ .

► **Lemma 44** ([22]). *Given a Turing machine  $\mathcal{M}$ , let  $\mathcal{R}$  be the term rewriting system constructed using the encoding in the proof of Theorem 31. If there exists a string  $w \in \Sigma^*$  such that  $[w]_{\mathcal{R}}$  is infinite, then there exists a string  $s \in CONFIG$  such that  $[s]_{\mathcal{R}}$  is infinite.*

**Proof.** Suppose such a string  $w_0$  exists in  $\Sigma^*$ . Since  $\mathcal{R}$  is reverse convergent, we can assume that  $w_0$  is a normal form of  $\mathcal{R}^{-1}$  and has infinitely many reachable terms  $w'$  ( $w_0 \rightarrow_{\mathcal{R}} w'$ ). By König's lemma, there exists an infinite rewrite sequence  $w_0 \rightarrow_{\mathcal{R}} w_1 \rightarrow_{\mathcal{R}} \dots$ .

Note that the left-hand side and the right-hand side of each rule of  $\mathcal{R}$  each contain exactly one state symbol, so all  $w_i$  has the same number of state symbols, denoted as  $k$ . Moreover, the new state symbols in  $w_{i+1}$  and the old state symbol in  $w_i$  have the same relative positions to other state symbols.

For each  $r$  such that  $1 \leq r \leq k$ , we define the  $r$ -th segment of  $w_i$ , denoted as  $w_i^r$ , as the longest substring of  $w_i$  that contains  $r$ th state symbol that is in the regular language  $\triangleright^{\{0,1\}}(\bar{\Pi} \cup D_L)^*(Q \cup \bar{Q})(\Pi \cup D_R)^* \triangleleft^{\{0,1\}}$ . The only difference between this language and  $CONFIG$  is that the left and right end marker is optional.

It follows that if the reduction  $w_i \rightarrow_{\mathcal{R}} w_{i+1}$  involves the  $r$ th state symbol, then for  $1 \leq j \leq r$ , if  $j \neq r$ , then  $w_i^j = w_{i+1}^j$ , and  $w_i^r \rightarrow_{\mathcal{R}} w_{i+1}^r$ .

Therefore, it is possible to find some index  $r$  such that there is an infinite rewrite sequence  $w_{a_1}^r \rightarrow_{\mathcal{R}} w_{a_2}^r \rightarrow_{\mathcal{R}} \dots$ . Since rewrite rules of  $\mathcal{R}$  preserve the number of endmarkers, it is possible to uniformly add endmarkers to the left and right of each  $w_{a_i}^r$ . Denote the result as  $w'_{a_i}$ , which is in  $CONFIG$ .  $w'_{a_1} \rightarrow_{\mathcal{R}} w'_{a_2} \rightarrow_{\mathcal{R}} \dots$  is an infinite rewrite sequence in  $CONFIG$ , and  $[w'_{a_1}]_{\mathcal{R}}$  is infinite.  $\blacktriangleleft$

The other direction of Lemma 44 is trivial. With this lemma, we can prove Theorem 32.

**Proof of Theorem 32.** This problem is in  $\Pi_2$  since we can formulate this problem as a  $\forall\exists$ -sentence in first-order logic.

Given a Turing machine  $\mathcal{M}$ , construct another Turing machine  $\mathcal{M}'$  and term rewriting system  $\mathcal{R}'$  derived from  $\mathcal{M}'$  same as in the proof of Theorem 31. The  $\Pi_2$ -hardness follows from the following equivalences:

1. Every configuration of  $\mathcal{M}$  is mortal.
2.  $[w]_{\mathcal{R}'}$  is finite for all string  $w \in CONFIG$ .
3.  $[w]_{\mathcal{R}'}$  is finite for all string  $w$ .
4. Equality saturation terminates on  $\mathcal{R}' \cup \mathcal{R}'^{-1}$  for all string  $w$ .

The equivalence between (1) and (2) follows from Lemma 42. The equivalence between (2) and (3) follows from Lemma 44. For any string  $w$ , the equivalence between (3) and (4) can be proved similarly to the proof of Theorem 31. Since the universal halting problem is  $\Pi_2$ -hard, the all-instance termination problem of EqSat is  $\Pi_2$ -hard.  $\blacktriangleleft$

## D.2 Proof for Theorem 33

► **Theorem 33** (All-E-graph-instance termination). *The following problem is undecidable:*

- *Instance:* A term rewriting system  $R$ .
- *Question:* Does EqSat terminate with  $R$  and  $G$  for all E-graphs  $G$ ?

**Proof.** Our proof is inspired by Gilleron and Tison [10]. We reduce the Post correspondence problem (PCP) to this problem. Let  $A$  be an alphabet. The input to the problem is two finite lists of  $A$ -words  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$ . PCP asks if there exists a non-empty sequence of indices  $i_1, \dots, i_k$  such that  $\alpha_{i_1} \dots \alpha_{i_k} = \beta_{i_1} \dots \beta_{i_k}$ .

We use unary functions and a special nullary function  $\epsilon$  to represent strings as terms. A word is represented as consecutive applications of unary symbols. The ruleset we construct simulates runs of PCP. Let  $\mathcal{R}_{PCP}$  be

$$\begin{aligned} k(x, y) &\rightarrow k(i(x), \alpha_i(y)) \quad \text{for } i = 1, \dots, n \\ k(i(x), y) &\rightarrow r(i(x), y) \quad \text{for } i = 1, \dots, n \\ r(i(x), \beta_i(z)) &\rightarrow r(x, z) \quad \text{for } i = 1, \dots, n \\ r(\epsilon, \epsilon) &\rightarrow goal \end{aligned}$$

Intuitively, we think of the term rewriting starts with term  $k(\epsilon, \epsilon)$ . The first rule explores different choices of indices and stores the corresponding  $\alpha$  sequence, and the third rule verifies

if the given choice of indices is an acceptable solution. The second rule is to make sure the solution is non-empty.  $\mathcal{R}_{PCP}$  involves a special nullary function  $goal$  which, once populated in the E-graph, will be rewritten to every other term. Let  $\mathcal{R}_{goal}$  be

$$\begin{aligned} goal &\rightarrow \epsilon \\ goal &\rightarrow i(goal) \quad \text{for } i = 1, \dots, n \\ goal &\rightarrow s(goal) \quad \text{for } s \in A \\ goal &\rightarrow k(goal, goal) \\ goal &\rightarrow r(goal, goal). \end{aligned}$$

To handle inputs that do not represent  $k(\epsilon, \epsilon)$ , let  $\mathcal{R}_{st}$  be

$$\begin{aligned} k(x, y) &\rightarrow k(\epsilon) \\ r(x, y) &\rightarrow k(\epsilon). \end{aligned}$$

Let  $\mathcal{R} = \mathcal{R}_{PCP} \cup \mathcal{R}_{goal} \cup \mathcal{R}_{st}$ . We claim that equality saturation terminates on  $\mathcal{R}$  and  $G$  for all E-graph  $G$  if and only if PCP has a solution.

- $\Rightarrow$ : Let  $G$  be an E-graph representing a single term  $k(\epsilon, \epsilon)$ . It is easy to see that in this case EqSat terminates if and only if  $goal$  is populated if and only if there is a solution to PCP.
- $\Leftarrow$ : Suppose PCP has a solution. It follows that if an E-graph contains  $k(\epsilon, \epsilon)$ ,  $goal$  will be populated by Lemma 22 and EqSat will terminate. Note that for EqSat to not terminate, the input E-graph has to represent at least some  $k$ - or  $r$ - term. By  $\mathcal{R}_{st}$ , however, if an E-graph represents any  $k$ - or  $r$ - term it will be rewritten to  $k(\epsilon, \epsilon)$ , so EqSat terminates for arbitrary E-graphs.



## E Weak term acyclicity and EqSat termination

Given a signature  $\Sigma$ , we define a position as a pair  $(f, i)$ , where  $f$  is a  $n$ -ary function symbol in  $\Sigma$  and  $1 \leq i \leq n$ . For  $u, v \in T(\Sigma, V)$ , we define  $Pos_u(v)$  as the set of positions  $(f, i)$  satisfying  $f(p_1, \dots, p_{i-1}, v, p_{i+1}, \dots, p_n)$  is a sub-pattern of  $u$ . For instance,  $Pos_{g(f(x), x)}(x) = \{(f, 1), (g, 2)\}$ . A rule is called *non-degenerate* if its left-hand side is not solely variables. Any degenerate rule  $x \rightarrow rhs$  can be made into a set of non-degenerate rules by substituting  $x$  with  $f(x_1, \dots, x_n)$  for each function symbol  $f \in \Sigma$ .

► **Definition 45** (Weak term acyclicity). *Let  $\Sigma$  be a signature. Let  $\mathcal{R}$  be a set of non-degenerate rewrite rules over  $\Sigma$ . The weak term dependency graph of  $\mathcal{R}$  consists of positions as nodes. Moreover, for each rewrite rule  $lhs \rightarrow rhs$  in  $\mathcal{R}$ ,*

1. *for each variable  $x \in \text{Var}(rhs)$ , add an edge from  $u$  to  $v$  for every combination of  $u \in Pos_{lhs}(x)$  and  $v \in Pos_{rhs}(x)$ .*
2. *for each proper, non-variable sub-pattern  $p$  of  $rhs$ , if  $p$  does not occur in  $lhs$ , for each variable  $x \in \text{Var}(p)$ , add a special edge from  $u$  to  $v$  for every combination of  $u \in Pos_{rhs}(x)$  and  $v \in Pos_{rhs}(p)$ .*

$\mathcal{R}$  is called weakly term acyclic if no cycle of  $\mathcal{R}$ 's weak term dependency graph contain a special edge.

The definition of weak term acyclicity follows the structure of the definition of weak acyclicity. For example, proper, non-leaf sub-patterns of  $\text{rhs}$  in our case act like existentially quantified variables. However, some key differences allow weak term acyclicity to capture termination of more rules than applying weak acyclicity on TGDs/EGDs directly derived from EqSat rules. If a sub-pattern in  $\text{rhs}$  already occurs in  $\text{lhs}$ , it will not introduce new E-classes. Moreover, in the chase, for a TGD  $\lambda(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}. \rho(\vec{x}, \vec{z})$ , values assigned to existential variables  $\vec{z}$  depend on  $\vec{x}$ . In EqSat, because of functional dependencies, the E-classes that a pattern  $u = f(p_1, \dots, p_n)$  can be instantiated to are fully determined by  $\text{Var}(u)$ , a subset of all free variables of  $\text{lhs}$ .

► **Theorem 46.** *If a term rewriting system  $\mathcal{R}$  is weakly term acyclic, then equality saturation (defined in (3)) converges in steps polynomial to the size of the input E-graph.*

**Proof sketch.** The proof is essentially the same as the proof of weak acyclicity from Fagin et al. [8]. We provide a sketch here.

Define an incoming path of position  $(f, i)$  to be any path of the weak term dependency graph ending at  $(f, i)$ , and define the rank<sup>9</sup> of a position  $(f, i)$  as the maximum number of special edges on any incoming path to  $(f, i)$ . Since the weak term dependency graph is weakly term acyclic, the rank of any position is finite. We can prove by induction on the ranks of positions that there exists a polynomial  $P_i$  that bounds the total number of distinct E-classes at all positions of rank  $k$  at any intermediary E-graph produced by EqSat at some iteration. There are three kinds of E-classes at positions of rank  $i$ : E-classes that are already present at such positions in the input E-graph, E-classes that are copied over from positions of ranks  $< i$ , and new E-classes created at positions of rank  $i$ . The last kind is bound by the number of special edges, multiplied by the number of distinct E-classes at positions of ranks  $< i$ .

Denote by  $P$  the sum of all  $P_i$ .  $P$  is a polynomial that bounds the number of distinct E-classes of any E-graph produced by EqSat at any iteration. Let  $ar$  be the maximum arity of any function symbol in  $\Sigma$ .  $Q^{ar+1}$  bounds the number of distinct E-nodes. Since any E-graph  $G = \langle Q, \Sigma, \Delta \rangle$  is fully determined by its E-classes and E-nodes, at most  $Q \cdot Q^{ar+1} = Q^{ar+2}$  distinct E-graphs can occur during EqSat. Since EqSat is inflationary, it terminates in at most  $Q^{ar+2}$  iterations. ◀

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<sup>9</sup> Ranks in a weak term dependency graph are not related to ranks in an E-graph.