Recursive Algorithms coded in Python

- Version 1.0 -

Thomas Peters

Contents

Introduction and Acknowledgement

This text is about recursive algorithms. So, at first we must define what a recursive algorithm is. *An algorithm is called recursive, if it contains at least one function which calls itself.* A well-known and simple example is the computation of *factorials*. $n!$ stands for the product of the integer numbers from 1 to n. But usually *factorials* are defined recursively, and $0! = 1$ is also defined.

$$n! = \begin{cases} n \cdot (n-1)! & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases} .$$

This definition can easily be converted into a Python function:

```
def fakrek(n):
    if n == 0:  return 1
    else:  return n*fakrek(n-1)
```

However, it is not necessary to write your own function. Predefined functions for factorials are available in modules *math*, *numpy* and *scipy*.

Some basic knowledge of Python is provided in this text. If you are an absolute beginner you should study a tutorial on Python first. There you will learn how to code loops with `for` or `while` and how to deal with functions. It is not necessary to study the programming of classes; they will not occur in this text because the codes are as simple as possible.
There are some examples in the following text which use *turtlegraphics*. Originally this kind of graphics was developed for the programming language Logo [2]. The turtle was a small robot that could be directed to move around by typing commands on the keyboard. But soon it appeared on the screen. Turtlegraphics is also available in Python's module *turtle* [3]. This kind of graphics allows numerous nice examples which use recursions, and you can watch how the graphics develops on the screen.

Chapter 1 starts with some examples of simple recursive algorithms. In chapter 2 some mathematical applications of recursions are described. Chapter 3 is about recursive algorithms with more than one call. In that chapter we shall see that recursive algorithms are not always advantageous, and it may be better to use iterative algorithms instead. Finally, in chapter 4 some algorithms with mutual recursive calls are discussed.

There are some general programming concepts which are related with recursive algorithms. 'Divide and Conquer' divides the input into smaller and smaller sub-problems until a trivial solution can be found for each one. Then the partial solutions are put together to yield a complete solution of the original problem. Well-known examples are *binary searching*, see Section 1.1.3, and *merge sort*, see Section 3.4. Another concept is 'Branch and Bound'. There the aim is to minimize the set of possible solutions. This is done by branching the set of feasible solutions into subsets and installing suitable bounds. A typical example is solving integer linear problems, see Section 3.7. 'Backtracking' is a concept which is especially useful for constraint satisfaction problems e.g, for the knapsack problem, see Section 3.6.

The programs to be discussed are coded in a rather simple manner. Certainly experts would code some of them shorter and in a more elegant way. But in this text comprehensibility was emphasized most. Modules for diverse purposes have been created for Python during the last years. So many of the problems discussed in this text can easily be solved applying one of these modules. But here only such modules are used that are included in the Python package except *turtle*.
The codes were written in Python 3.9.13 [1]. Meanwhile there will be more recent versions, and it is possible that some parts of the codes will not work any longer. But in most cases it will not be difficult to update the codes.

Many thanks to my friend Horst Oehr for reading the text and testing the codes. Horst gave me many useful hints to improve the text and the figures.

Chapter 1

Simple Recursions

1.1 Some Examples for Beginners

1.1.1 Harmonic Series

The harmonic series is defined by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \sum_{n=1}^{\infty} \frac{1}{n}$$

Every added fraction is smaller than the preceding one. But the value of partial sums

$$H_N = \sum_{n=1}^{N} \frac{1}{n} \tag{1.1}$$

grows slowly with increasing N. In the 14th century the french philosopher Nicole Oresme showed that the partial sums diverge [4]. That means they grow infinitely with increasing N. But actually N must be large to surpass even a relatively small number such as 8. We develop a short program which computes partial sums (Eq. (1.1)) of the harmonic series to explore the minimal N such that $\sum_{n=1}^{N} \frac{1}{n}$ yields more than 8.

A suitable recursive function is very simple:

```
def harmrek(n):
    if n == 1:  return 1
    else:  return harmrek(n-1) + 1/n
```

The computation of the partial sum with upper bound n is reduced to computing it with the upper bound n-1 and adding $\frac{1}{n}$. If $n = 1$, 1 is returned.

The main program essentially consists of just two lines:

```
n = int(input('n = '))
print('H(',n,') = ',harmrek(n))
```

The function *input* interprets any input as a string. We have to convert it into an integer number. That is why we must surround the call of `input` by `int(...)`. The second line shows the upper limit n and the resulting partial sum.

The harmonic partial sums can be approximated by $\ln n + \gamma$, where γ is the Euler-Mascheroni constant which has the value $\gamma \approx 0.5772$ [5]. In order to compare our results with the approximation we must import the module *math* at the beginning. The constant γ must be defined, and one more line is added to the main program:

```
print('approximated by',math.log(n)+ gamma)
```

The function *log* of the module *math* computes the natural logarithm of the argument.

The approximation is the better the greater n is. Here are two examples:

```
H( 10 ) = 2.9289682539682538, approximated by 2.879800757896
H( 1000 ) = 7.485470860550343, approximated by 7.48497094388367
```

Applying the approximation yields that to surpass the partial sum 8 you must set $n \approx 1673$. Using this program you can easily find the exact solution. Do not try much larger numbers. Otherwise the program will not work and you'll get the error message: `RecursionError: maximum recursion depth exceeded in comparison`. There are some internet sites on the limit of recursions in Python. Most of them state that it is 1000. You can find out what the maximum recursion depth on your system really is. Just type this small program and run it:

```
import sys
print(sys.getrecursionlimit())
```

It is even possible to change the limit of recursions on your system. However, according to experts this is not recommended.

1.1.2 Hexagons

This is a little program showing some features of turtlegraphics. An overview with all possibilities of turtlegraphics in Python can be found here [3]. We should like to plot some regular hexagons which all have the same center. The edges of each succeeding hexagon shall be shorter than the edges of the

current one. Primarily we want to plot a single hexagon with turtlegraphics. The first thing we always have to do when dealing with turtlegraphics is to import the module:

```
import turtle as tu
```

That means the module is available, and whenever we want to use its functions we may e.g. write `tu.forward(100)` instead of `turtle.forward(100)`. The turtle is adjusted in such a way that it initially moves "eastward". To obtain a hexagon it has to turn left, but by which angle? Every two edges of a hexagon include the angle 120°. The turtle would move straight ahead if we do not tell it to turn. Figure 1.1 shows that the correct angle to turn left is 60°.

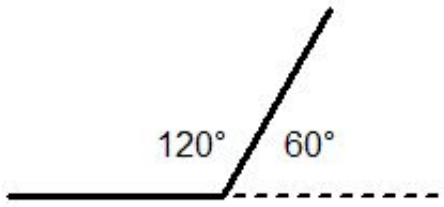


Figure 1.1: The turtle must turn left by 60° to draw the next edge of a hexagon.

With that we can construct a hexagon with edge length a just by these two lines:

```
for i in range(6):
    tu.forward(a); tu.left(60)
```

That is all we need to draw one single hexagon. But we want to have many concentric hexagons. The turtle is initially at position (0,0), that is the middle of its panel. It would be nice if the hexagons would be drawn around this center. So we must first move the turtle to the position where drawing of the hexagon begins. Figure 1.2 shows that the turtle must move down by $\frac{\sqrt{3}}{2} \cdot a$, approximated by $0.866 \cdot a$, and it has to move to the left by $\frac{1}{2}a$. Only when these motions have been performed the turtle can start to draw. In Python it is coded this way:

```
tu.up()
tu.setpos(-0.5*a,-0.866*a)
tu.down()
```

When the hexagon is finished the turtle has to move back to the center without drawing:

```
tu.up()
tu.setpos(0,0)
```

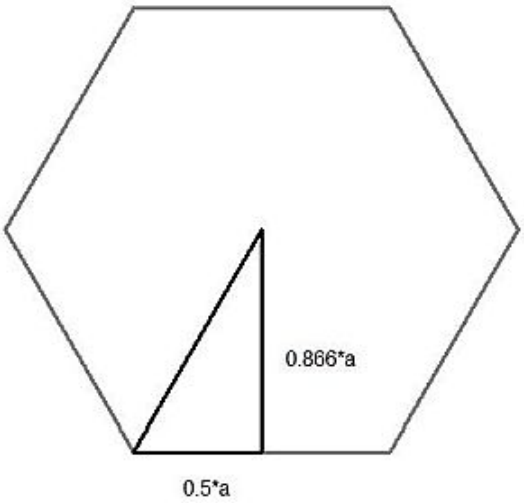


Figure 1.2: The turtle has to move without drawing down by $0.866 \cdot a$ and to the left by $\frac{1}{2}a$.

After all this is done a recursive call may follow like this one:
`if a > 25:  hexagon(a*0.8)`
The `if`-clause is very important. If it was not there the program would not terminate and run forever. Every recursive call needs a so-called *termination condition*. The values "edge length" $a = 25$ and "reduction factor" 0.8 are arbitrary.
The hexagons look a bit nicer with colors. We want the outer hexagon to be red. Then the colors of the hexagons change continuously to blue. We can achieve this by the following three lines:

```
red *= 0.8; blue *= 1.25
if blue > 1:  blue = 1
tu.pencolor(red,0,blue)
```

The variables for colors red, green and blue range from 0 to 1. The current values for `red` and `blue` are multiplied by 0.8 and 1.25 respectively. So the contribution of red decreases while the blue color intensifies. The next line ensures that `blue` cannot be greater than 1. The last line tells the turtle to use the color composed of `red` and `blue`. There is no green used here. This is the code of the complete function *hexagon*:

```
def hexagon(a, red, blue):
    red *= 0.8; blue *= 1.25
    if blue > 1:  blue = 1
    tu.pencolor(red,0,blue)
    tu.up()
```

```
tu.setpos(-0.5*a,-0.866*a)
tu.down()
for i in range(6):
  tu.forward(a); tu.left(60)
tu.up(); tu.setpos(0,0)
if a > 25:  hexagon(a*0.8, red, blue)
```

Now let us take a glance at the main program. The colors have to be initialized: `red = 1; blue = 0.2`, and the width of the pen is fixed such that is not too small: `tu.pensize(width = 3)`. Then the essential function is called: `hexagon(200,red, blue)`. At last there are some lines to announce to the user that the drawing is finished. Finally we have to care that the panel of turtlegraphics can be closed properly. The last two lines may be different on your system. Here is the complete main program:

```
red = 1; blue = 0.2
tu.pensize(width = 3)
hexagon(200,red, blue)
tu.up(); tu.hideturtle()
tu.setpos(-300,-250); tu.pencolor((0,0,0))
tu.write('finished!',font = ("Arial",12,"normal"))
tu.mainloop()
tu.done()
```

The following figure 1.3 shows the first three steps of the concentric hexagons described above.

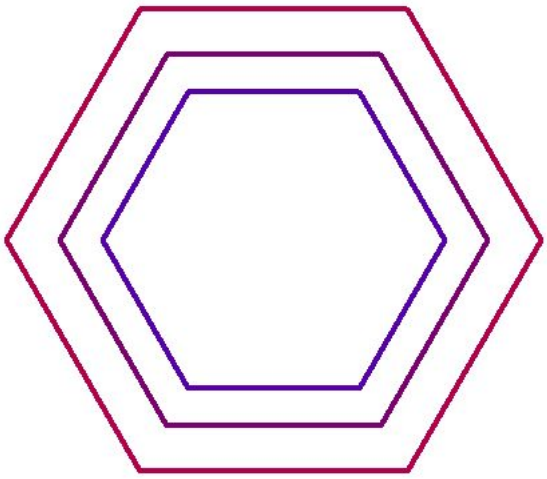


Figure 1.3: First three hexagons, colors intensified

1.1.3 Binary Search

A frequent task in data processing is searching for certain data. The input is a keyword or a number. The data processing program has to find the corresponding data set or it has to tell the user that there is none. If the keywords are not sorted the program must check each stored keyword. This is very time-consuming. However, if the keywords are sorted much better methods for searching are available. One of them is "binary search" which we demonstrate in this section.
Therefore a file with 1024 first names is loaded into the memory. We do not discuss this part of the program because it has nothing to do with binary search itself. The loaded names are stored in `namesList`.

The important function is `search`. It needs the two parameters `l,r`. These are the left and right bound between which the search is performed. In function `search` the variable `found` must be initialized. Then the middle `m` of `l` and `r` is determined: `m = round((l+r)/2)`. The current name `namesList[m]` is displayed such that we can see what the function does. There are three possibilities: (1) `na == namesList[m]` which means that name `na` was found. In this case searching ends, and `found = True` and `m` are returned. (2) The name `na` precedes `namesList[m]` in alphabetical order. In this case also the condition `l < m` must be checked. If both conditions are fulfilled searching continues in the interval [*l,m-1*]:
`if (na < namesList[m]) & (l < m):  found, m = search(l,m-1)`
However, if the first condition is true but the second is not, the name `na` is not in `namesList`. (3) The name `na` succeeds `namesList[m]` in alphabetical order. This case is quite analogous to the second case. If `m < r` searching continues in the right half of `namesList`, otherwise `na` cannot be found in the list of names.
We must still explain why `search` has to *return* `m` and `found`. The reason is that all variables in Python are local. Let us assume that `na` is found in a certain recursive call of `search`. For example `m` = 47 and `found = True`. These values are only "known" to the current call of `search`. The copy of `search` which called the version of `search` where `na` was found doesn't "know" anything about the successful search. That's why we must return parameters `m` and `found` to the calling copy of `search`.
Here is the complete listing of function `search`:

```
def search(l,r):
    found = False
    m = round((l+r)/2)
    print(' Name[',m,'] = ',namesList[m])
```

```
    if na == namesList[m]:  found = True
    if (na < namesList[m]) & (l < m):found,m = search(l,m-1)
    if (na > namesList[m]) & (m < r):found,m = search(m+1,r)
    return found, m
```

Now, some words about the main program. Besides the header there is just the input of the name `na` which is looked for. The call of function `search` follows: `found, m = search(0,n-1)` The left and right limits are `0` and `n-1`, these are the lowest and highest index of `namesList`. Finally the result is displayed on the screen.

Why is "binary search" advantageous? If a linear search in an unsorted list of 1024 entries is carried out, the mean number of comparisons is 512. In binary search there are initially also 1024 possibilities. After the first comparison this number reduces to 512, after the second to 256, and so on. Thus at most $10 = log_2 1024$ comparisons are necessary to find an entry or to find out that it is not in the list of keys. The multiple splitting of the list which serves as input is a good example for the general programming concept "Divide and Conquer".

1.2 Top-of-Stack Recursions

1.2.1 The Small Difference

Let us take a look at these two small functions:

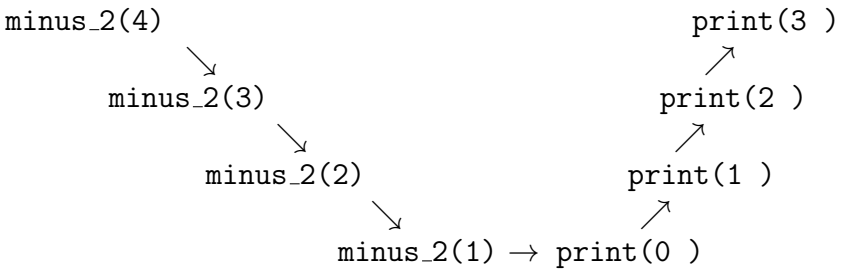
```
def minus(i):
    i -= 1
    print(i,end= ' ')
    if i > 0:  minus(i)

def minus_2(i):
    i -= 1
    if i > 0:  minus_2(i)
    print(i,end= ' ')
```

The functions are called by this tiny main program:

```
n = int(input('n = '))
    minus(n); print()
    minus_2(n); print()
```

Let us assume `n = 4`. Function `minus` will first display `3`, then the recursive

call follows. So 2 appears on the screen, after that 1 and finally 0. It is a countdown. Function minus_2 just does the reverse. The output starts with 0 and ends with 3. What is the reason? Before anything is displayed on the screen a recursive call is carried out. Multiple calls of function minus_2 are processed in the following sequence:



Recursions use a part of the computer memory which is called *stack*. It is comparable to a dead end street into which only a certain number of cars can drive and only backward out again. A stack follows the principle "LIFO" which stands for "Last In First Out". The above sketch shows that each recursive call pushes some information on the stack: the name of a function and one or more parameters. In function minus_2 information is only taken from the stack, when the base condition is reached that means `if i > 0` is no longer `True`. Then 0 is displayed, and the call `minus_2(1)` is finished. Consequently 1 is displayed, and `minus_2(2)` is done. So the processing moves up from the bottom to the top. That is the way recursive functions normally work.

But function minus is different. First printing is done and the recursive call comes afterwards. So displays and recursive calls alternate with each other until `i > 0` is `False`.
`minus(4)` → `print(3 )` → `minus(3)` → `print(2 )` → `minus(2)`
→ `print(1 )` → `minus(1)` → `print(0 )`
Functions where the recursive call is the last statement are called *top-of-stack recursions*. There are programming languages where the information of the recursive function call is removed, 'popped' from the stack when the call is finished. In such programming languages a top-of-stack recursion can run indefinitely. Python, however, does not belong to these languages, and the number of recursive calls is limited.

1.2.2 Rosettes

Though the possibilities of top-of-stack recursions are limited there are some nice examples. Here is one that applies turtlegraphics for drawing rosettes

by means of polygons with three, four, five or six edges. The main program is not very exciting. The input `n` denotes the number of edges, and `angle` is a variable set to `360/n`. It is necessary for drawing polygons. The width of the turtle's pen for drawing is set to `2`, such that lines are not too thin. Then the function call follows which draws the rosette: `shape(0,600/n)`. When finished the code ends with a passage described in Section 1.1.2.

Let us see how function `shape(count,x)` works. First the turtle's pencolor is determined. This is done by choosing randomly values for `red`, `green` and `blue`, for example `red = rd.uniform(0,1)`. Therefore the module `random` must be imported at the beginning: `import random as rd`. Then the polygon is drawn. Only two lines of code are needed:

```
for i in range(n):
  tu.forward(x); tu.right(angle)
```

`x` is the length of an edge. In the main program we set `x = 360/n`. `count` denotes the number of calls needed to draw a complete rosette. After having drawn a single polygon the turtle turns at a small angle: `tu.right(7)`. The last line is the recursive call:

```
if count < 360/7:  shape(count+1,x)
```

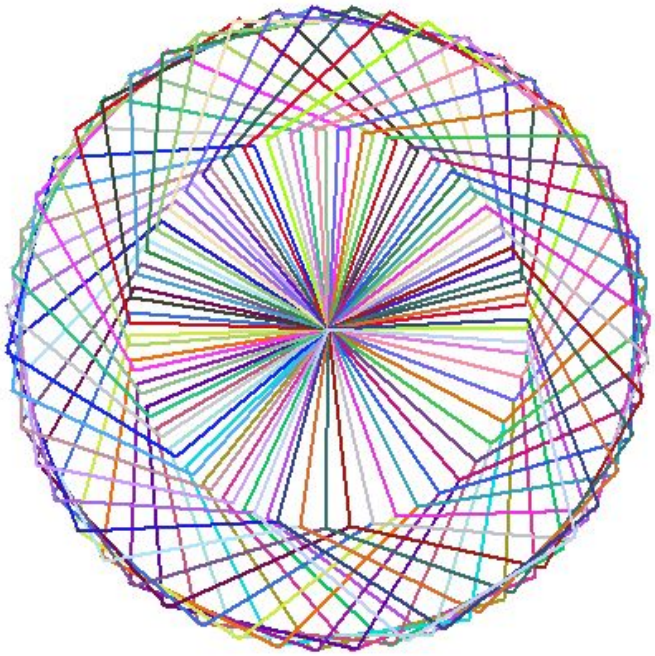


Figure 1.4: This rosette consists of pentagons.

That means that the length of an edge keeps its value, but variable `count` is increased by one. So many polygons are drawn rotated by small angles, and a rosette results. In total the code is very short and easy to understand. Nevertheless it yields a result like figure 1.4.

1.2.3 Greatest Common Divisor

The *greatest common divisor gcd* of two (positive) numbers is the greatest number by which both numbers can be divided without leaving a remainder. Here we discuss a possibility to obtain the gcd of more than two numbers. It applies the well-known Euclidean algorithm, see [6].
The main program is very short. The user is asked to enter at least two numbers. The `while`-loops ensure that `a` or `b` cannot be 0. Then function `gcd` is called, and that's it. Here is the listing:

```
a, b = 0, 0
while a == 0:  a = int(input('first number:  '))
while b == 0:  b = int(input('second number:  '))
gcd(a,b)
```

In the function itself we take care that negative numbers can also be processed. They are converted into positives if necessary.

```
if x < 0:  x = -x; if y < 0:  y = -y
```

Now four lines follow containing the Euclidean algorithm.

```
r = 1
while r > 0:
    r = x % y; x = y; y = r
print('The GCD is ',x)
```

`r` denotes the remainder and is initially set to 1. The `while`-loop runs until `r` is zero. The operation `x % y` computes the remainder obtained by dividing x by y. x need not be greater than y. For example 5 % 7 = 5. Afterwards the variables are shifted: `x` gets the value of `y`, and `y` is set to `r`. The proof of correctness of the algorithm can be found in [6].
Next the user is asked to enter another number to continue the calculation of the gcd.

```
    z = int(input('next number:  '))
    if z != 0:  gcd(x,z)
```

If `0` is entered the calculation is finished. Otherwise the gcd of the gcd obtained so far and the new number is calculated. Here is an example. We want to have the gcd of 525, 1365, 1015 and 749. Function `gcd` first returns 105 = gcd(525,1365). Then 1015 is entered, and gcd(105,1015) = 35 is displayed. After entering 749 the final result is gcd(35,749) = 7.

Since the recursive call is contained in the last line of the code of function `gcd` it is a top-of-stack recursion.

1.3 Exercises

1. **Alternating Harmonic Series**
   (i) Develop a program for computing partial sums of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + -... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

   Your program should contain a recursive function. Additionally compute the difference from ln 2 (that is `np.log(2)` in Python) which is the limit of this series [7].
   (ii) The alternating series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + -... = \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2n+1}$$

   has the limit $\frac{\pi}{4}$. Write a program that computes an approximation for π using partial sums of the above series.

2. **Snail-shell**
   Write a program which creates a drawing similar to figure 1.5. The snail-shell consists of squares with decreasing edge lengths. There are only two functions necessary to start and to stop filling a square: `tu.begin_fill()` and `tu.end_fill()` before and after the code of the drawing you want to fill. Hint: Red is coded by the triple (1,0,0), cyan by (0,1,1).

3. **Program X**
   Describe what functions `work` and `work2` display on the screen and why they do so. Which function is a top-of-stack recursion?

```
def work(k):
    print(st[k],end = '')
    if st[k] != ' ':  work(k+1)
def work2(k):
    if st[k] != ' ':  work2(k+1)
    print(st[k],end = '')
```

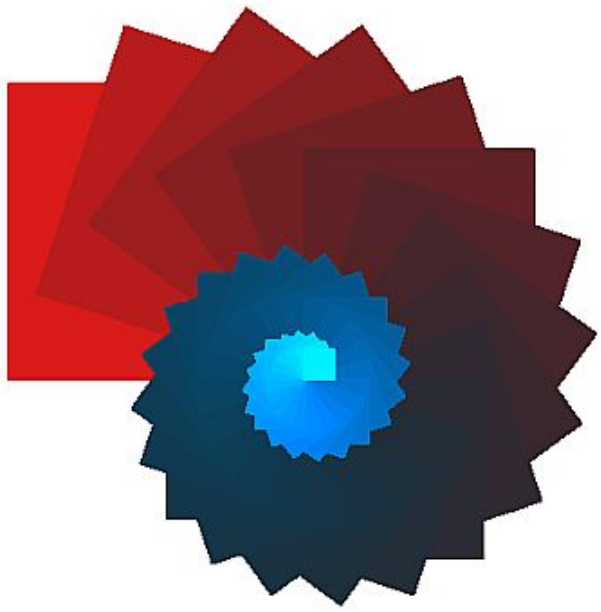


Figure 1.5: Snail-shell with colors changing from red to cyan

4. **Snowflakes**
   A snowflake consists of a crystal containing several smaller sub-crystals of the same shape as the first one. An elementary crystal is a star with a certain number of rays. Create a program for drawing snowflakes with at least 3 and at most 6 rays per star. Code a recursive function applying turtlegraphics. The drawing may look like 1.6, see also [8].

   Hints: From the center of the current star the turtle moves forward at length `x`. Then function `flake` is called with parameter `0.4*x` or so. When the function is finished the turtle has to move back at `x`. To draw the next ray the turtle must turn at angle $360°/n$ where n is the number of rays.

5. **Prime Factors**
   Write a program which computes the prime factors of a given (not too large) number. Your program should contain a function with a top-of-stack recursion.

6. **Street**
   Write a program with a top-of-stack recursion for drawing a "street" with houses of different sizes similar to figure 1.7.
   Hints: Shades of gray are obtained by `gray = rd.uniform(0.4,0.9)`.

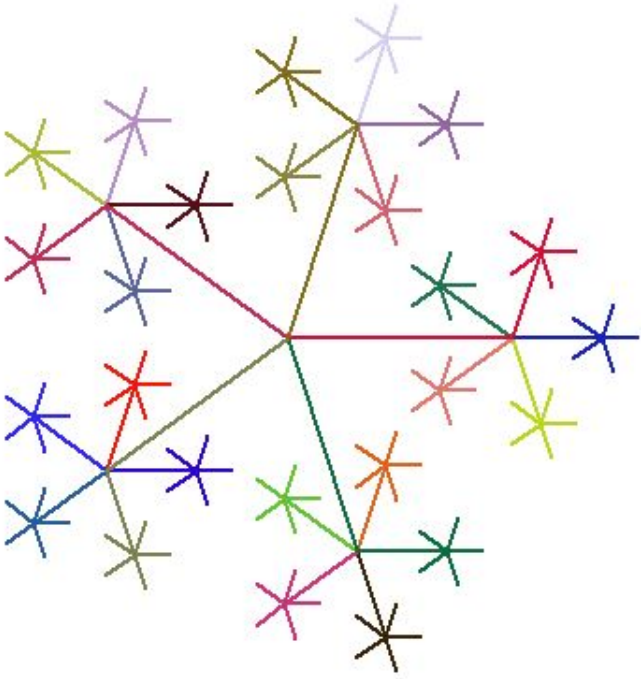


Figure 1.6: Snowflake - stars have five rays

Don't forget to import module *random*. The termination condition is obtained by checking the x-coordinate of the lower left corner, for example: `if tu.xcor() < 200:`.

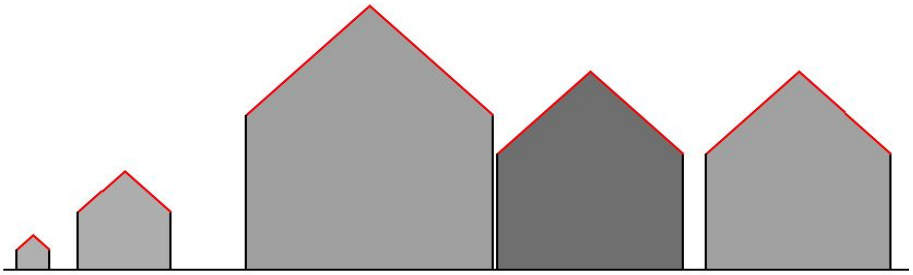


Figure 1.7: Street with houses of different sizes

Chapter 2

Some Recursive Algorithms in Mathematics

There are a lot of recursive algorithms in mathematics. Only few of them will be discussed here and coded in Python. An often represented example in this context is the computation of Fibonacci numbers. But there are a lot of websites on this item [9]. More applications of recursions in mathematics can be found in the next chapter because they are algorithms with more than one recursive call.

2.1 Reducing Fractions

A fraction can be reduced by dividing the numerator and the denominator by the greatest common divisor. However, normally reducing is not done this way. Instead one tries to divide numerator and denominator by small numbers as long as this is possible without leaving a remainder. Why not try to code this method in Python?

We will not discuss the main program in detail. The numerator and the denominator are entered, and they are expected to be positive. One more variable is needed: `prime = 2`. And then the recursive call comes:
`reduce(prime, num, den)`
where `num` and `den` stand for *numerator* and *denominator*. `prime` denotes a prime number. Initially `prime` is set to `2` which is the smallest prime. If possible, `num` and `den` shall be divided by `prime`.
From function `reduce` first function `search_prime` is called. It returns `prime` and `success`. If the latter is `True` reducing of the fraction takes place: `num = num // prime` and `den = den // prime`. The double slash stands for in-

teger division. That means decimals are cut off, and the result is integer. For example 22 // 5 = 4. Then the (intermediate) result is displayed: `print(' = ',num,'/',den,end = ' ')`. Thereby `end = ' '` effects that no new line begins when the `print(...)` command is finished. Finally `reduce` is called with possibly altered parameters. Here is the listing of the whole function:

```
def reduce(prime, num, den):
    prime, success = search_prime(prime, num, den)
    if success:
       num = num // prime; den = den // prime
       print(' = ',num,'/',den,end = ' ')
       reduce(prime, num, den)
```

Now, let's have a look at what function `search_prime` does. Its task is to provide a prime by which both, `num` and `den`, can be divided without remainder. Additionally it is checked whether or not reducing may continue.

There are two boolean variables `possible` and `success`. `possible` is `True` as long as reducing the fraction may be possible. That means `num` and `den` are both greater or equal compared to `prime`. `success` is set to `False`. But it will become `True` if `num` and `den` can be divided by `prime`. So, initially `possible = True; success = False`.
What is to happen, if `num` or `den` cannot be divided by `prime` without remainder? Surely `prime` must be increased: `prime += 1`. Then it has to be checked whether or not reducing is still possible: `if (num < prime) or (den < prime): possible = False`. If reducing is still possible (but `num` or `den` could not be divided by `prime`), function `search_prime` is called recursively: `prime, success = search_prime(prime, num, den)`. Note that `prime` has changed meanwhile.
However, if `num` or `den` can be divided by `prime`, the `else`-branch is active, and `success = True`. In the end `prime` and `success` are returned. Here is the listing of `search_prime`:

```
def search_prime(prime, num, den):
    possible = True; success = False
    if (num % prime != 0) or (den % prime != 0):
       prime += 1
       if (num < prime) or (den < prime):  possible = False
       if possible:
          prime, success = search_prime(prime, num, den)
    else:  success = True
    return prime, success
```

Now the program to reduce fractions in a special way is complete, and this is an example for the output: `48/84 = 24/42 = 12/21 = 4/7`.

2.2 Heron's Method

HERON lived about 50 after Christ in Alexandria, Egypt [10]. He invented a method for determining square roots approximately [11]. Let a be the *radicand*. That is the number to determine the square root x of. Then $x = \sqrt{a}$. x_0 denotes the initial approximation, e.g. $x_0 = 1$. Then we build

$$x_1 = \frac{1}{2} \cdot (x_0 + \frac{a}{x_0})$$

Evidently, one of the numbers x_0, $\frac{a}{x_0}$ is greater and the other one is smaller than $\sqrt{a}$. So x_1 is a better approximation of $\sqrt{a}$ than x_0 because it is the mean value of x_0 and $\frac{a}{x_0}$. Again one of the numbers x_1 and $\frac{a}{x_1}$ is greater than $\sqrt{a}$, and the other one is smaller. Thus we can continue and build $x_2 = \frac{1}{2} \cdot (x_1 + \frac{a}{x_1})$ and so on, in general

$$x_{i+1} = \frac{1}{2} \cdot (x_i + \frac{a}{x_i}) \tag{2.1}$$

But why do we know that the sequence $< x_i >$ really converges against $\sqrt{a}$? It can easily be shown by solving a quadratic equation that for all natural numbers $i \in \mathbb{N}$:

$$x_i > \sqrt{a} \text{ and } \frac{a}{x_i} < \sqrt{a} \tag{2.2}$$

Since $x_{i+1} < x_i$ for all i the sequence $< x_i >$ is decreasing monotonously, and the sequence $< \frac{a}{x_i} >$ is increasing monotonously. Because of eq. (2.2) the sequence of intervals $[\frac{a}{x_i}, x_i]$ is an interval nesting and converges against $\sqrt{a}$. Now we are ready to implement Heron's method. This is in fact not very difficult. The main program is short and looks like this:

```
# main program
print("Computing Square Roots using Heron's Method")
a = float(input('radicant = '))
d = int(input('number of decimals:  '))
epsilon = 10**(-d)
Heron(1,0)
```

Variable d denotes the accuracy of the desired approximation, and it is converted to *epsilon*. At last the function *Heron* is invoked with parameters 1 and 0. 1 is the zeroth approximation x_0, and 0 the number of approximations.

Also the function *Heron* looks very simple. First we increase `i` by one which indicates the number of steps carried out. Then we apply eq. (2.2): `x_new = 0.5*(x + a/x)` and the newly computed interval $[\frac{a}{x_i}, x_i]$ is displayed. Finally the recursive call has to be executed: `if x_new - a/x_new > epsilon: Heron(x_new,i)`. That's all. Here is the listing of the function:

```
def Heron(x,i):
    i += 1
    x_new = 0.5*(x + a/x)
    print('step ',i,'[',a/x_new,',',x_new,']')
    if x_new - a/x_new > epsilon:  Heron(x_new,i)
```

Heron's method can easily be extended for an approximation of the kth root of a given number. By applying the newton procedure for zeros on the function $f(x) = x^k - a$ it can be shown that eq. (2.2) has to be replaced by

$$x_{i+1} = (1 - \frac{1}{k}) \cdot x_i + \frac{a}{k \cdot x_i^{k-1}} \tag{2.3}$$

The reader should implement a program using eq. (2.3) for approximating kth roots. However, only approximations x_i can be obtained, not intervals.

2.3 Catalan Numbers

Many a reader may not know what Catalan numbers are. Nevertheless they play an important role in combinatorics. They are called after the Belgian mathematician Eugène Catalan (1814-1894). There are two equivalent definitions [12]:

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n} = \frac{(2n)!}{(n+1)! \cdot n!} \tag{2.4}$$

But what are they good for? Here are some examples.
(1) Parentheses: A term like $18 - 9 - 4 - 1$ makes no sense if there are no parentheses set. There are some possibilities to set parentheses. Here they are:

$$((18 - 9) - 4) - 1 = 4$$

$(18 - (9 - 4)) - 1 = 12$
$(18 - 9) - (4 - 1) = 6$
$18 - ((9 - 4) - 1) = 14$
$18 - (9 - (4 - 1)) = 12$

Since there are three operations there are five possibilities to set parentheses, and using eq. (2.4) it can easily be verified that $C_3 = 5$. It can be shown that the number of possible sets of parentheses in a term of n operations is C_n, the Catalan number of n.
(2) The number of different binary trees with n nodes is C_n. Figure 2.1 shows the non-isomorphic binary trees with three nodes. Notice that a branch or leaf with a key smaller than the parent key is a "left child"; if the key is greater than the parent key it is a "right child". Therefore 1-2-3 and 3-2-1 are not isomorphic. The proof using eq. (2.5) is done in [13].

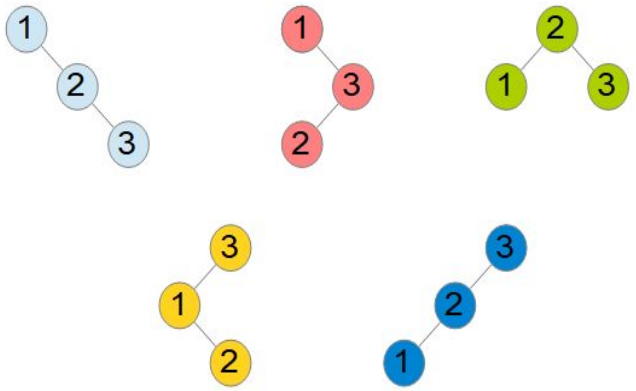


Figure 2.1: There are five non-isomorphic binary trees with three nodes.

(3) The number of non-crossing partitions of a set with n elements is C_n [14]. In Figure 2.2 $C_4 = 14$ non-crossing partitions of the set consisting of four elements is shown.

Indeed there are many more applications of Catalan numbers, but you can see from the above examples how useful theses numbers are. But now we will discuss three small programs for computing Catalan numbers in Python. According to definition (2.4) we can implement a program which calls the following recursive function for factorials three times:

```
def fakrek(n):
  if n == 0:  return 1
  else:  return n*fakrek(n-1)
```

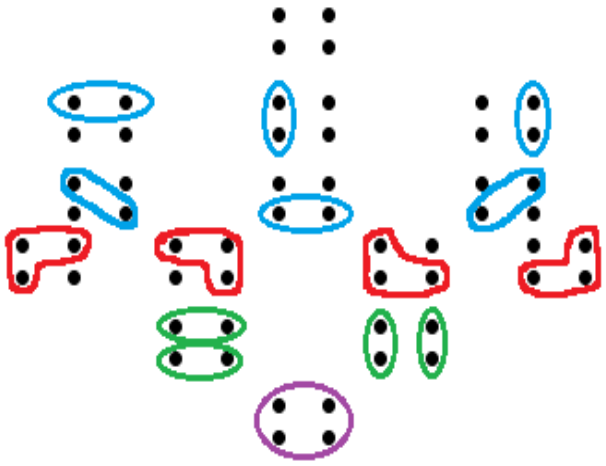


Figure 2.2: There are 14 non-isomorphic non-crossing partitions of a set with four elements.

The main program which computes Catalan numbers up to a limit n looks like this:

```
n = int(input('limit = '))
for k in range(n+1):
     C = fakrek(2*k)/(fakrek(k+1)*fakrek(k))
     print('C(',k,') = ',round(C))
```

This is a simple method to compute Catalan numbers. But there is another recursive formula which doesn't need factorials:

$$C_n = \sum_{j=0}^{n-1} C_j \cdot C_{n-j-1} \text{ where } C_0 = 1 \tag{2.5}$$

Thus it is easy to implement eq. (2.5) as a Python function:

```
def Cat(n):
     if n == 0:  return 1
     else:
       su = 0
       for j in range(n):
           su += Cat(j)*Cat(n-1-j)
        return su
```

And this is the corresponding very short main program:

```
n = int(input('limit = '))
```

```
for i in range(n+1):
    print('C(',i,') = ',round(Cat(i)))
```

For tests you should enter small numbers such as 5 or 10. You will see that the program works fine. You'll see the Catalan numbers up to 5 or 10 on the screen, but if you enter somewhat larger numbers such as $n = 18$ you will notice that the program gets slower and slower. The reason is that the number of recursive calls increases exponentially. If you choose a still larger n it may happen that the maximum depth for recursions does not suffice for the required computation. The program is canceled and an error message is displayed. That is not what we want.
How can we do better? We avoid the lots of recursive calls by introducing a list, called *CatList*. The list must be initialized in the main program by $CatList = [\ ]$. The modified main program is not very complicated either. We have just to append the current value of the function *Cat* to our *CatList*. Then the result can be displayed on the screen.

```
n = int(input('limit = '))
for i in range(n+1):
    CatList.append(Cat(i,CatList))
    print('C(',i,') = ',round(CatList[i]))
```

However, the function itself has to be modified. Instead of many recursive calls the required Catalan numbers are obtained from *CatList* which is already filled as far as necessary. Thus the implementation of the function looks like this:

```
def Cat(n,CatList):
  if n == 0:  return 1
  else:
    su = 0
    for j in range(n):
      su += CatList[j]*CatList[n-1-j]
  return su
```

This program seems not to differ very much from the latter, but the most important issue: There is *no* recursive call. Programs like these are called *iterative*. The Catalan numbers obtained one after the other are stored in a list or in an array. So they are at once available when needed.
Testing this iterative algorithm shows the enormous gain of speed. It costs only a moment to get results like this $C(\ 40\ ) = 262212704227649210882 0$.
So if you can convert a recursive algorithm into an iterative one the latter is preferable. But this conversion is not always possible.

2.4 Binomial Distribution

Assume we have a box with w white and b black balls. A ball is randomly drawn from the box n times, the color is checked, and then the ball is thrown back into the box. We consider drawing of a white ball as "success". The probability of success is $p := w/(w+b)$. If a certain sequence of white and black balls is fixed, the probability to draw k white balls is $p^k \cdot (1-p)^{n-k}$.

Let us consider an example with three white and five black balls. The probability to draw the sequence *bbwbwwbb* is $(\frac{3}{8})^3 \cdot (\frac{5}{8})^5 \approx 0.005$. But if we are not interested in the sequence of drawing white and black balls, and we only consider the probability of obtaining k white balls, i.e. $P(X = k)$, we have to multiply the obtained result with the number of possibilities to choose k white balls of n. This number is the *binomial coefficient* $\binom{n}{k}$ [15]. So the number of successes is given by the so-called *binomial distribution* [16]:

$$p(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \tag{2.6}$$

Let us look at the above example again. There are eight balls in a box, three of them are white, and we draw eight times. The probability to draw three white balls is according to eq. (2.6)

$$p(X = 3) = \binom{8}{3} \cdot 0.375^3 \cdot 0.625^5 \approx 0.2816$$

Eq. (2.6) can easily be implemented as a Python code because module `scipy` provides a function which computes the binomial coefficients. The whole listing is this:

```
import scipy.special as sp
     <<< headline and inputs >>>
p = w/(w + b)
for k in range(n+1):
     binom = sp.comb(n,k,exact = True)
     binom = binom*(p**k)*((1-p)**(n-k))
     print('Probability for ',k,' white balls:  ',binom)
```

This is a short code which needs no recursion, but the predefined module `scipy` instead. That is not really what we want. Instead we wish to create a "handmade" code with a recursive function. The `for`-loop of the main program is shorter:

```
for k in range(n+1):
    print('Probability for ',k,' white balls:  ',prob(n,k))
```

We have to discuss function `prob`. It starts with an `if`-clause: `if (white > step) or (white < 0):return 0`. That is so because the number of white balls cannot be greater than the number of draws and a negative number of white balls doesn't make sense. The first part of the `else`-branch contains another `if`-clause: `if step == 1:...`. The `if`-branch consists of still one more `if`-clause: `if white == 1:return p`, `else:return 1-p`. This case covers the situation in which only one ball is drawn. The probability obtaining a white ball is p, and the probability to draw a black ball is $1-p$. The last `else`-branch is a bit complicated:
`prob(1,0)*prob(step-1,white)+prob(1,1)*prob(step-1,white-1)`.
`prob(1,0)` can be replaced by `1-p`, and `prob(1,1)` is the same as `p`. Additionally there are two recursive function calls: `prob(step-1,white)` and `prob(step-1,white-1)`. To show the correctness we convert that line of the Python code back into an algebraic formula.

$$P(X=k) = \binom{n}{k} \cdot p^k (1-p)^{n-k} =$$
$$(1-p) \cdot \binom{n-1}{k} \cdot p^k (1-p)^{n-1-k} + p \cdot \binom{n-1}{k-1} \cdot p^{k-1} (1-p)^{n-k}$$

We can rewrite this in the following way:

$$\binom{n}{k} \cdot p^k (1-p)^{n-k} = p^k (1-p)^{n-k} \cdot \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$$

This is equivalent to

$$\binom{n}{k} = \cdot \binom{n-1}{k} + \binom{n-1}{k-1} \tag{2.7}$$

The proof of eq. (2.7) can be found in [17]. Thus we see that the last line of function `prob` is indeed correct. So we show the whole listing.

```
def prob(step,white):
    if (white > step) or (white < 0):return 0
    else:
       if step == 1:
          if white == 1:return p
          else:return 1 - p
       else:
```

```
return prob(1,0)*prob(step-1,white)+
   prob(1,1)*prob(step-1,white-1)
```

Our "handmade" program works quite fine for small numbers. Try `w` = 3, `b` = 5, `n` = 8. But if we set `w` = 9, `b` = 15, `n` = 24, i.e. three times the former inputs, the program gets very slow. The reason is that the number of recursive calls increases very strongly with the number of draws `n`, see chapter 3. To see that also try `w` = 3, `b` = 5, `n` = 30.

2.5 Determinants

The determinant of a square matrix is a floating point number which depends on the entries of the matrix . There are many applications of determinants, some examples are given below. The *determinant of a 2 x 2 matrix* can easily be computed:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \tag{2.8}$$

The definition of the determinant of a 3 x 3 matrix is more complicated, for example

$$\begin{aligned} &\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \\ &a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21} \cdot \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31} \cdot \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \end{aligned} \tag{2.9}$$

This looks rather complicated but eq. 2.9 is only a special case of what is called *development of a determinant by column*. Code *determinant_3* demonstrates the application of eq. 2.9. The general formula for n x n determinants is

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det A_{ij} \tag{2.10}$$

A is the matrix of which the determinant is to be computed, A_{ij} is the matrix which arises from A by leaving out the i-th row and the j-th column. The determinant is then developed by the j-th column. It is also possible to develop a determinant by a row. But this is not discussed here, it is an exercise instead.

As it can be seen from eq. 2.9 the computation of a determinant with three

rows can be recursively reduced to computing three determinants with two rows each. Eq. 2.10 shows that the computation of a determinant with n rows is done by computing n determinants with n-1 rows. So it makes sense to develop a recursive function for computing determinants. Before we start it should be mentioned that there is a function doing that in the module `numpy`. The built-in function `np.linalg.det` requires a two dimensional `numpy` array. It returns the value of a determinant, but you cannot see the path of computing.

Our code needs no extra module. The matrix `A` is entered row by row, so we get a list of lists. This is done in function `enterMatrix` which we do not discuss in detail. Additionally we need the set of indices {0, 1, ..., n-1}, in Python: `indices = {j for j in range(n)}`. The main program is short:

```
A,n = enterMatrix()
indices = {j for j in range(n)}
print('value of the determinant:',det(A,n,0,indices))
```

The essential function is `det` which depends on parameters `A, rank, column, ind`. Clearly `A` denotes the current matrix of which the determinant is to be computed. `rank` is the number of rows and columns of the current matrix `A`. `column` is the current column by which the determinant is developed. `ind` denotes the current set of indices. The word *current* must be emphasized because all these parameters will probably change during recursive calls.
Here is an example: The matrix A is defined in 2.11. We want to develop det(A) by column 0.

$$A = \begin{pmatrix} 1 & 0 & 3 & 4 \\ -2 & 1 & 1 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 4 & 1 & 5 \end{pmatrix} \tag{2.11}$$

Function `det` is called with parameters `rank = 4`, `column = 0`, ind = {0,1,2,3}. In the first recursive call the following subdeterminant has to be computed:

$$\det \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 0 \\ 4 & 1 & 5 \end{pmatrix}$$

The corresponding parameters in function `det` are `rank = 3`, `column = 1`, ind = {1,2,3}. The second recursive step has to compute

$$\det \begin{pmatrix} 0 & 0 \\ 1 & 5 \end{pmatrix}$$

Here the parameters are are `rank = 2`, `column = 2`, ind = {2,3}. At this step recursive calls end because a determinant with two rows can be computed directly. This is covered by the following part of `det`:

```
if rank == 2:
   # find correct indices
   k = list(ind)
   if k[0] > k[1]:  x = k[0]; k.pop(0); k.append(x)
   res = A[k[0]][column]*A[k[1]][column+1]- \
         A[k[1]][column]*A[k[0]][column+1]
```

Finding the two correct indices looks a bit strange. Since we have to care for the right order of the indices we need a list `k` with only two members. In order to achieve that at first the *set* `ind` is converted to a *list* `k`. If the first index is greater than the second, the first index is copied to `x`, then removed from the list and `x` is appended at the end. Then the formula for two-rowed determinants is applied.
The `else`-branch contains recursive calls. At first two local variables are needed. `res` is reserved for the value of the current determinant. `count` denotes a counter initialized with 0. It serves to ensure that the exponent in 2.10 is correct. Remember that the development of the determinant is done by (current) column 0. So `(-1)**count` yields the correct sign. `count` is not the same as `i` in the following `for` loop. Look at matrix 2.11 again. If `i = 1`, subdeterminant

$$\det \begin{pmatrix} 0 & 3 & 4 \\ 2 & 0 & 0 \\ 4 & 1 & 5 \end{pmatrix}$$

has to be computed. Therefore `i` obtains the values 0, 2, 3 one after the other. But since `count` is local, it takes the values 0, 1 and 2.
The `for`-loop runs from `0` to `n-1` and doesn't care about the current `rank`. If index `i` is in the current set `ind` of indices it must be removed from `ind` for the following recursive call: `ind=ind-{i}`. The call itself just applies 2.10:
`res += (-1)**count*A[i][column]*det(A,rank-1,column+1,ind)`
Thus `res` obtains the value of the current determinant during the runs of the `for`-loop. Afterwards `count` is increased, and `i` is added to `ind` again. This is the whole listing of the `else`-branch:

```
else:
   res = 0;count = 0
   for i in range(n):
    if i in ind:
      # remove row i
```

```
ind = ind - {i}
res+=(-1)**count*A[i][column]* \
      det(A,rank-1,column+1,ind)
count += 1
ind = ind.union({i})
```

There are some basic rules about determinants, see [18], [19]. From these it can easily be derived that a determinant is zero if one column is a linear combination of the other columns. The determinant 2.11 is -58, not zero; thus the vectors consisting of the columns are *linearly independent*. None of them is a linear combination of the other three vectors.
The absolute value of the determinant of three vectors as columns yields the volume of the *parallelepiped* generated by these vectors [20]. An example for a parallelepiped is a feldspar crystal in figure 2.3 However, if a determinant of three vectors is zero the vectors lie in a plane, they are *coplanar*.

Figure 2.3: feldspar crystal - example of a parallelepiped

Determinants are also good for solving linear systems of equations applying *Cramer's rule* [21]. If the system has a unique solution it is necessary to compute one more determinant than the number of variables to obtain it. However, if the determinant of the matrix of coefficients is zero, the system has no unique solution.

These are only some examples for the application of determinants, see [19] for more. Finally we have to remark that the development of a determinant by column or row is not a very efficient algorithm. The runtime is proportional to the factorial of the number of variables n that is $O(n!)$ [22]. More efficient algorithms are available. Moreover there are better algorithms for solving systems of equations and to prove linear independence. But determinants

are convenient to deal with, for example coding Cramer's rule is relatively simple.

2.6 Interval Halving Method

This section is about a method for determining the zeros of a polynomial function. There is no exact algorithm which allows the computation of zeros of polynomial functions with a degree greater than 4 [23]. But the methods for polynomials of degrees 3 and 4 are complicated [24], [25]. So we are going to perform a numerical approximation. A common method is the *interval halving method*, and we will first line out the structure of a program which applies this method.

Let M be the sum of the absolute values of the coefficients. Then all zeros lie within the interval [-M,M]. We build a table of function values with integer steps starting with -M and ending with M. If there is a change of signs between two function values there must be a zero (of odd multiplicity) between them, and the function `bisection` is called. Evidently the zero must lie either in the lower or in the upper half of the interval, and the change of signs occurs either in the lower half or in the upper. This fact provides a recursive algorithm which is stopped if the lengths of the intervals get smaller than a limit which we call `epsilon`.
The main program consists of two parts: the input and the processing part. Here is the listing:

```
max = 6 # the maximal degree
epsilon = 1e-5
print(); print('Interval Halving Method')
ok = False
while not(ok):
     coeff(); polynom()
     print(); ans = input('everything ok?  (y/n) ')
     if ans in ['Y','y']:  ok = True
limit(n,a); tov(n,a)
```

`max = 6` restricts the degree of the polynomial. The meaning of `epsilon` has already been explained. The `while`-loop allows to correct the input of the coefficients, which is done in function `coeff`. `polynomial` displays the current polynomial. If the input is correct, function `limit` is called to fix the interval [-M,M] where the table of values is to be computed. These three functions are not discussed in detail because they are very simple and no recursion is needed in them.

Function `tov` (which stands for "table of values") does not contain a recursion either but we will nevertheless discuss it. We denote the essentials of the listing:

```
def tov(n,a):
     for i in range(round(-M),round(M)):
        print(' ',i,' -> ',round(f(i),3))
        if f(i)*f(i+1) < -epsilon:
           bisection(i,i+1)
        if abs(f(i)) < epsilon:
           print(); print('A zero detected at ',i)
```

Function `f` builds a polynomial of the coefficients such that values can be computed, see below. `bisection` is called if `f(i)*f(i+1)< -epsilon`. This means that a change of signs occurs in the interval [`i`,`i+1`]. If the product `f(i)*f(i+1)` is not smaller than `-epsilon` there is no change of signs or an integer zero was found. The latter case is captured by the last two lines. Unfortunately the program will not detect every zero of polynomials. It may happen that two zeros of odd orders are very close together, for example $\frac{1}{4}$ and $\frac{1}{2}$. So the program will not detect the zeros of $p(x) = x^2 - \frac{3}{4}x + \frac{1}{8}$. In this case the increment of the table of values must be refined, see `zeros_ihm_ref.py`. The program will also fail detecting zeros of even multiplicity, see [26]. If you enter the polynomial $p(x) = x^3 - \frac{5}{2}x^2 + \frac{3}{4}x + \frac{9}{8}$ the program will only find $-\frac{1}{2}$ but not the zero $\frac{3}{2}$ which is also a minimum.

Function `f` looks very simple, but it isn't. It applies *Horner's method* [27]. The variable to be returned is `y`, and it is initially set to the coefficient of the maximal exponent: `y = a[n]`. A `for`-loop follows the index of which *decreases* to zero. The old value of `y` is multiplied by argument `x`, then the next coefficient is added: `a[j-1]`. Finally `y` is returned.

```
def f(x):
     y = a[n]
     for j in range(n,0,-1):  y = y*x + a[j-1]
     return y
```

At last the recursive function `bisection` is to be presented. Parameters are the left and right border `l` and `r` of the interval in which the change of signs occurs. `fl` and `fr` are the corresponding function values. They are displayed, so you can see how the function works. If `r-l < epsilon` the terminal condition is reached; otherwise a recursive call is processed depending on where the change of signs occurs. And this is the listing:

```
def bisection(l,r):
    fl = f(l); fr = f(r)
    print('[',round(l,5),',',round(r,5),'])
    if r - l < epsilon:
       print(); print('A zero lies in the interval')
       print('[',l,',',r,'].'); print()
    else:
       if fl*f((l+r)/2) <= 0:  bisection(l,(l+r)/2)
       else:  bisection((l+r)/2,r)
```

2.7 Exercises

1. **Heron's Method**
   Write a program to compute the kth root of a positive floating point number. See eq. (2.3).

2. **Pascal's Triangle**
   Implement a program to compute the numbers of Pascal's triangle. It contains the binomial coefficients, see [28].

3. **Catalan numbers**
   There are $C_4 = 14$ different binary trees with four nodes. Write them all down.

4. **Regula falsi**
   A well-known method to determine zeros of odd multiplicity is the *regula falsi* [29]. This method is very old, mentioned first in papyrus Rhind (about 1550 before Christ) and developed in more detail in the medieval Muslim mathematics. Let $P_0 = (x_0, f(x_0))$ and $P_1 = (x_1, f(x_1))$ be two points with $f(x_0) \cdot f(x_1) < 0$, where w.l.o.g. $x_0 < x_1$. Then $(x_2, 0)$ is the intersection of the line connecting P_0 and P_1 and the x-axis, see figure 2.4. It can be shown that

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 \cdot f(x_0)}{f(x_1) - f(x_0)} \tag{2.12}$$

   It is an approximation of the zero between x_0 and x_1. As in the interval halving method either $f(x_0) \cdot f(x_2) < 0$ or $f(x_2) \cdot f(x_1) < 0$. So this procedure can be continued using recursive calls until a sufficient precision has been reached.

Develop a program for approximating zeros of a given function applying eq. (2.12). If you use $f(x) = \sin(2 \cdot x)$ and your interval is [2, 4] you obtain an approximation for π.

Hint: If you want to enter floating point numbers as interval borders, you must convert your input into "float":
`border = float(input('enter border '))`.

5. **Determinants**
We mentioned that developing a determinant by column or by row is not a very efficient algorithm. Transformation into an upper triangular matrix is better. A determinant does not change its value, if a multiple of one row is added to another row. Adding $-\frac{a_{k1}}{a_{11}}$ times row 1 to row k (k = 2,...,n) yields a determinant which has the same value as the determinant given, and the first column consists of zeros except a_{11}. This procedure can be continued by recursive calls until there is no longer an element below a diagonal element [30]. At last an upper triangular matrix is received. The determinant is just the product of the diagonal elements. Unfortunately some rounding errors may occur caused by divisions.
Write a program that computes a determinant by triangulation provided that no diagonal element is zero!

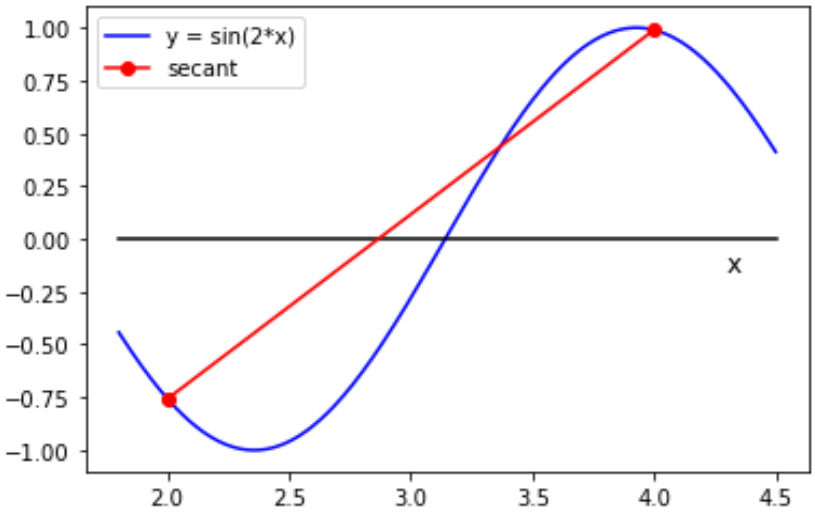


Figure 2.4: A change of signs occurs between $x_0 = 2$ and $x_1 = 4$. According to eq. (2.12) the intersection of the red secant and the x-axis is (2.87,0) . The blue graph intersects the x-axis in $(\pi, 0)$.

Chapter 3

Recursive Algorithms with Multiple Calls

There are many functions in which recursive calls occur twice or more. In the following only a small selection of these is presented. An important application is the computing of permutations. Extraordinary functions are the ACKERMANN function, HOFSTADTER'S Q-function, and the algorithm for Fibonacci numbers of higher order. But we start with a graphical application.

3.1 Dragon Curves

A dragon curve of step n [31] emerges from a dragon curve of step n-1 by replacing each line of the curve of step n-1 by the cathetes of a rectangular triangle such that the above mentioned line is the hypotenuse of the triangle. The right angle is directed outside as seen from the middle of the screen. A dragon curve of step 0 is just a straight line. Figure 3.1 shows dragon curves of steps 3 and 4 and the superposition of both curves. These curves are dragon curves because the right angles stand out like the scales of the armoured skin of a dragon.
Since the dragon curve is defined recursively it is obvious that we implement the drawing of the curve using turtlegraphics by a recursive function. It depends on two parameters: `size` and `step`. If `step` = 0, the function consists of only one statement: `tu.forward(size)`. Otherwise we have to care that the problem of drawing the dragon curve of *step n* is reduced to plotting the dragon curve of *step n-1*.

In a right angled isosceles triangle the proportion of the length of a cathete and the hypotenuse is $\frac{1}{\sqrt{2}}$, which is approximately 0.707. Thus in the function

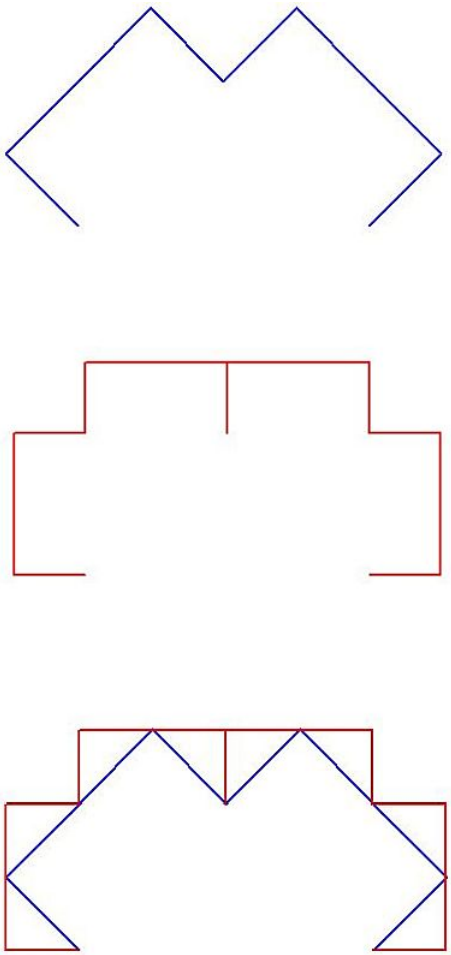


Figure 3.1: Dragon curves: step 3, step 4, and the superposition of both

`dragon` we redefine `size = round(0.707*size)`. Then the turtle rotates by 45° to the left, and `dragon` is called with *step*-1. Afterwards the turtle has to rotate by 90° to the right, and `dragon(size,step-1)` is called again. But in the end of this small section the turtle shall have the same direction as at the beginning. So the turtle has to rotate by 45° to the left again. Now the whole function can be given.

```
def dragon(size,step):
    if step == 0:  tu.forward(size)
    else:
        size = round(size*0.707)
        tu.left(45)
        dragon(size,step-1)
```

```
tu.right(90)
dragon(size,step-1)
tu.left(45)
```

The reader might think that the function is wrong for the following reason: If `step` is 2 at the beginning, `size` will be replaced by the 0.707-fold of `size`. Next the function calls `dragon(0.707*size,1)`. Thereby the original `size` will be replaced by the 0.707^2-fold of `size`, the half of the original `size`. What happens when the function `dragon` processes `step` = 0? Will `size` get smaller and smaller during further function calls?

Actually this will not happen. Python differs between variables used in different function calls although all of them are called `size`. The `size` of `step 2` is not the same as `size` of `step` 1 or `step` 0. Only during the current call `size` is changed: `size = round(size*0.707)`. As an example all recursive calls of `dragon`(100,2) are shown in the following table.

1	dragon(100,2)
2	dragon(71,1)
3	dragon(50,0)
4	dragon(50,0)
5	dragon(71,1)
6	dragon(50,0)
7	dragon(50,0)

The main program shall overlay the dragon curves of orders 0 to 5. It would be possible to choose a higher limit for the number of curves. But the errors caused by rounding would spoil the image. If you want to have a dragon curve of high order it is recommended to modify the program a bit such that only the curve you want is drawn and no one else.

Perhaps it is convenient to set the width of the turtle to more than one: `tu.width(2)`. The main part of the program consists only of a loop in which the color is set with which the turtle has to draw. The small function `chooseColor()` just determines randomly the so-called pencolor. The last part of the main program ensures that the panel for turtlegraphics can close correctly and that the program returns to the editor. The main program is denoted here, `chooseColor()` afterwards.

```
tu.width(2)
for step in range(6):
     tu.pencolor(chooseColor())
     tu.up()
     tu.setpos(-160,-50)
     tu.down()
     dragon(270,step)
tu.up(); tu.hideturtle()
tu.setpos(-300,-150)
tu.pencolor((0,0,0))
tu.write('finished!',font = ("Arial",12,"normal"))
tu.mainloop()
tu.done()
```

For using `chooseColor()`, the module `random` must be imported, for example this way: `import random as rd`

```
def chooseColor():
       red = rd.uniform(0,0.9)
       green = rd.uniform(0,0.9)
       blue = rd.uniform(0,0.9)
       return (red, green, blue)
```

3.2 Permutations

We want to develop a code which displays all permutations of numbers 1,...,n, where $2 \leq n \leq 6$. Generally a *permutation* of a finite set of elements is an arrangement of the elements in a row. For example the elements of the set {1,2,3} can be arranged in six different ways: (123), (132), (213), (231), (312), (321). It is easy to show that the number of permutations of n objects is $n!$, the factorial of n [32].

The main program contains the input of the variable `pmax`, the permutations of `1,...,pmax` will be computed. Then list `p` is filled with `1,...,pmax`. Finally function `perm` is called, that's all. By `perm` the numbers will be rearranged so that all permutations are obtained.

Before we discuss function `perm` we look at the auxiliary function `swap`. It processes three parameters `p`, `i` and `j`. `p` is the list in which `p[i]` and `p[j]` are to be swapped. To do this we need a variable `x`. First `p[i]` is assigned

to x, then p[j] is assigned to p[i]. At last p[j] is replaced by x. Convince yourself that p[i] and p[j] would have the same value if no auxiliary variable were used.

The essential function with recursive calls is perm. Let us first consider the special case n = 2.

```
if n == 2:
   print(p)
   swap(p,pmax-2,pmax-1)
   print(p)
   swap(p,pmax-2,pmax-1)
```

The current permutation p is displayed, then the last two numbers are swapped. The obtained new permutation is displayed. Afterwards the exchange of the two numbers is canceled by one more call of swap.
Example: p=[1,3,2,4]. First p=[1,3,2,4] is displayed, then the last two numbers of p are swapped, and p=[1,3,4,2] is displayed. Finally swapping is canceled, and p=[1,3,2,4] again.
The else-branch is not so easy to understand:

```
else:
   perm(n-1)
   for i in range(pmax-n+1,pmax):
      swap(p,pmax-n,i)
      perm(n-1)
   for i in range(pmax-n+1,pmax):swap(p,i-1,i)
```

Let us consider the case pmax=3. We know that p = [1,2,3], and perm(n-1) = perm(2) is called. So [1,2,3] and [1,3,2] are displayed. Then a for-loop is starts. It ranges from i = pmax-n+1 to pmax-1, that means i will assume the values 1 and 2. If i=1 the following happens: p[pmax-n]=p[0] and p[1] are swapped, and perm(n-1) = perm(2) is called again. [2,1,3] and [2,3,1] are displayed. Now consider i=2. p[pmax-n]=p[0] and p[2] are swapped, and perm(2) is executed. [3,1,2] and [3,2,1] are displayed. The effect of the second for-loop is to reestablish the initial order of the numbers. Before starting the order is [3,1,2]. When i=1, 3 and 1 are swapped, and we get [1,3,2]. And when i=2, 3 and 2 are swapped. The original order is back.
But what if pmax>3, for example pmax=4? perm(n-1)=perm(3) computes the permutations of three numbers. But now pmax-n+1=4-3+1=2. So the loop ranges from 2 to 3, and we obtain the permutations of [2,3,4], instead of [1,2,3]. The first for-loop changes 2,3,4 to the first position one after the other, and perm(n-1) creates the permutations of [1,3,4], [1,2,4] and [1,2,3]. So all 4! = 24 permutations are obtained.

3.3 Ackermann Function

In 1926 W. Ackermann invented a function which we present in the version of Rózsa Péter [33]. This is the definition:

$$A(m,n) = \begin{cases} n+1 \text{ if } m = 0 \\ A(m-1,1) \text{ if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m,n-1)) \text{ otherwise} \end{cases} \tag{3.1}$$

It is not difficult to code this recursive definition in Python:

```
def A(m,n):
    if m == 0:y = n+1
    else:
      if n == 0:y = A(m-1,1)
      else:y = A(m-1,A(m,n-1))
    return y
```

We do not discuss the main program. It just consists of the input and the call of function A. Try to fill the following table:

y/x	0	1	2	3	4
0					
1					?
2					X
3					X
4					X

Table 3.1: Fill in values of the Ackermann function

Possibly you fail to determine $A(4,1)$. Instead the following message is displayed: "maximum recursion depth exceeded in comparison". It is caused by the large number of recursive calls. So we ask: What is wrong about the Ackermann function? What is it good for?
These questions are topics of theoretical computer science. In 1926 D. Hilbert supposed that a function is computable if and only if it is composed of certain elementary functions. Such functions are called *primitive recursive* [34]. Values of these can be computed by programs which use only `for`-loops. But Ackermann invented his function as an example for a computable function that is not primitive recursive. It is not possible to code it using only `for`-loops. However, it is intuitively computable, and it can

be shown that it is *WHILE computable* [35]. That means: There exists a program using a `while`-loop instead of recursions which computes values of the Ackermann function [36]. It uses a stack which we emulate by a list in Python. Here is the listing:

```
print('Ackermann function with WHILE')
x = int(input('m = '))
y = int(input('n = '))
m = x; n = y
stack = []
stack.append(m) # push(m; stack)
stack.append(n) # push(n; stack)
count = 0
while len(stack) != 1:
     count += 1
     print('count = ',count,' stack = ',stack)
     n = stack[-1]; stack.pop(-1) # n := pop(stack)
     m = stack[-1]; stack.pop(-1) # m := pop(stack)
     if m == 0:
        stack.append(n+1) # push(n + 1; stack)
     else:
        if n == 0:
           stack.append(m-1) # push(m-1; stack)
           stack.append(1) # push(1; stack)
        else:
           stack.append(m-1) # push(m-1; stack)
           stack.append(m) # push(m; stack)
           stack.append(n-1) # push(n-1; stack)
result = stack[-1] # pop(stack)
print(); print('number of runs = ',count)
print(); print('A(',x,',',y,') = ',result)
```

This program does not need any function, and it is not recursive. Comments show the corresponding operations on the stack. Since the arguments `m` and `n` will change during the run of the program we use variables `x` and `y`, so that a correct output is displayed. Then the arguments `m` and `n` are pushed onto the stack. In the body of the `while`-loop the definition 3.1 is "translated" into stack operations. In addition the variable `count` counts the number of runs of the `while`-loop, and also the list `stack` is displayed.
Again try to fill table 3.1. Also try to determine $A(4,1)$. The author's computer obtained the correct value, but more than 2.86 billion runs of the `while`-loop were necessary.

3.4 Mergesort

Sorting data is one of the most important applications of computers. So it is not astonishing that there are many sorting algorithms, most of them are well documented, and we will not treat them here. Only if the data are sorted fast searching methods such as binary search (section 1.1.3) can be applied. We want to develop a small program which demonstrates the functionality of one of the oldest sorting algorithms. JOHN VON NEUMANN invented it in 1945. Mergesort is not an in-place algorithm. That means it needs more storage than the data to be sorted.
The principle is explained quickly [37]. Mergesort is a "divide and conquer" algorithm. In the first step the list (or array) of keys to be sorted is split into two halves. Then the first and the second half are split again. This process is continued until the obtained lists (or arrays) each consist of only one element, and nothing is to sort. When this is done the elements are composed to lists of two elements, then to lists of four elements and so on using the zipper principle. Finally the sorted list (or array) is obtained.

Our program will sort a list of 40 randomly chosen capital characters. We use turtlegraphics without any drawing to demonstrate mergesort. Thus the program begins with `import turtle as tu` and `import random as rd`. The main program contains preliminaries for turtlegraphics and it initializes the list of characters `a`. The essential recursive function `mergesort` is called, and the operations for termination of turtlegraphics are done, see section 1.1.2.

We will not discuss function `update` in detail. `update(kk,x,y)` moves the turtle to position $(x, y.)$. Any character that may be found at that position is deleted. Then the turtle writes `a[kk]`.

The most important function is `mergesort`. Its parameters are `l` and `r`, the left and right limitations for `mergesort`. Parameters `h` and `p` have nothing to do with sorting itself. `h` is a measure of the `y`-position on the panel of turtlegraphics, and with the help of `p` the horizontal position is determined. Usually `l` will be smaller than `r`. In this case the part of list `a` beginning with `l` and ending with `r` is displayed. The exact position of every character was found out by trial and error. The mean value `m` of `l` and `r` is computed using integer division `//`. Two recursive calls of `mergesort` follow. In the first call `r` is replaced by `m`, in the second `l` is substituted by `m+1`. Thus it is ensured that each character of `a` between `l` and `r` appears exactly once in the parts ranging from `l` to `m` and `m+1` to `r`. Parameters `h` and `p` are passed to function `mergesort` with necessary changes. Finally the zipper function

`merge` is called which will be discussed next. However, if `l=r`, there is only one character to be displayed. Here is the listing of `mergesort`:

```
def mergeSort(l, r, h, p):
    if l < r:
       for i in range(l,r+1):update(i,p+13*i,260-50*h)
       m = (l+r)//2
       mergeSort(l, m, h+1,p-65//(2**h))
       mergeSort(m+1, r, h+1,p+65//(2**h))
       merge(l, m, r, h, p)
    else:  update(l,p+13*l,260-50*h)
```

Function `merge` needs all the parameters used by `mergesort` too and also it needs `m`, the mean of `l` and `r`. Further three auxiliary lists are necessary: `L = a[l:m+1]` and `R = a[m+1:r+1]` are the parts of `a` to be merged. `A= []` is initially empty, in the end it will contain the sorted part of `a` from `l` to `r`. A `while`-loop follows which realizes the zipper principle.
If `L` is empty `R` is added to `A`, and correspondingly `L` is added to `A`, if `R` is empty. But if neither list is empty the lexicographically first of characters `L[0]` or `R[0]` is appended to `A`. Important is deleting the character just appended to `A`. When the `while`-loops are done the sorted list `A` is copied into `a` at the correct position and it is displayed. This is the somewhat confusing looking listing:

```
def merge(l, m, r, h, p):
    A = []; L = a[l:m+1]; R = a[m+1:r+1]
       while (L != []) or (R != []):
          if L == []:  A += R; R = []
          elif R == []:  A += L; L = []
          else:
             while (L != []) & (R != []):
                if L[0] < R[0]:  A.append(L[0]); L.pop(0)
                else:  A.append(R[0]); R.pop(0)

    for i in range(len(A)): a[l+i] = A[i]
    for i in range(l,r+1):  update(i,p+13*i,260-50*h)
```

Figure 3.2 shows the panel of turtlegraphics when sorting is partially done. There is another code in the appendix without turtlegraphics. It is easy to use it for sorting data sets where the keys are not just characters.

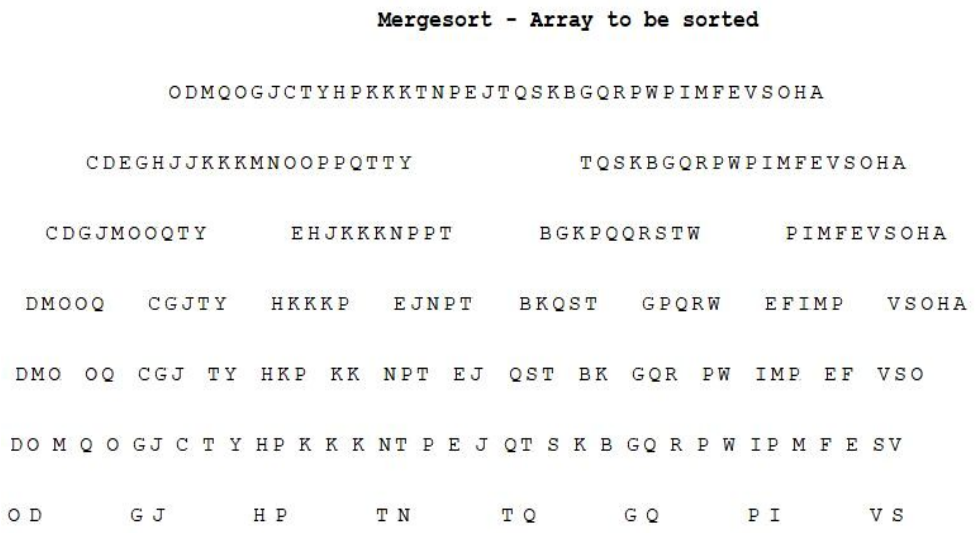


Figure 3.2: Nearly three quarters of the given list are sorted.

3.5 Tower of Hanoi

Tower of Hanoi (or Brahma) is a mathematical game invented by Édouard Lucas in 1883 [38]. The *source pile* consists of n disks with increasing diameters, the smallest at the top. Thus the pile has a conical shape. The disks shall be piled up at another position, called *target*. It is not allowed to place a disk with a larger diameter on a disk with a smaller size. The upper disk would break apart. Only one disk may be moved at a time. Further only one *auxiliary pile* is allowed for moving all disks from the source to the target, see figure 3.3. There are some legends about this game, see [38], [39].

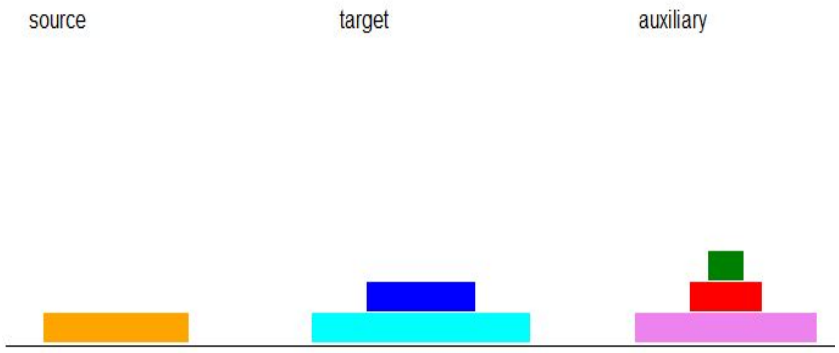


Figure 3.3: Tower of Hanoi

We develop a short program with a recursive function which solves the game. In the program part there is also the program `Tower_plot` which uses turtle-graphics. It demonstrates all the necessary moves, but the code is more elaborate.

As usual the main program of our code is very short. It just consists of the input of the number of disks and the call of function `shift` which returns the number of moves.

`shift` is the essential function. Its parameters are `x,y,z,k,count`, initialized in the main program with `x='source', y='target', z='auxiliary'`. `k` is the relevant number of disks; there may be more but only the upper `k` disks are considered. `count` counts the number of moves which is returned. It is initialized with 0. Notice that all these parameters will change during the recursive calls. Let us look at the listing:

```
def shift(x,y,z,k,count):
    if k == 1:
       count += 1
       print('disk ',k,' ',x,' -> ',y)
    else:
       count = shift(x,z,y,k-1,count)
       count += 1
       print('disk ',k,' ',x,' -> ',y)
       count = shift(z,y,x,k-1,count)
    return count
```

If `k=1` `count` is increased by one, and the disk is moved from `x` to `y`, i.e. from the current source pile to the current target. The `else`-branch contains recursive calls and it is more complex. For understanding imagine we could move the upper `k-1` disks at a time. In the first recursive call `count = shift(x,z,y,k-1,count)` the roles of the target and the auxiliary pile are reversed. So the pile of the upper `k-1` disks is moved to the auxiliary position. Then disk `k` moves to the target: `print('disk ',k,' ',x,'->',y)`. The following call `count = shift(z,y,x,k-1,count)` means that the upper `k-1` disks are moved from the auxiliary position to the target. But we cannot move more than one disks at a time. So many recursive calls may be necessary to move all disks to the target correctly. It can be shown that the minimum number of necessary moves is $2^n - 1$ where n is the number of disks.

Let us look at an example. We set `n` $= 3$. `'source', 'target', 'auxiliary'` are abbreviated by `s,t,a`. In the main program `shift(s,t,a,3,0)` is called.

Then the following happens:

```
shift(s,t,a,3,0)
   shift(s,a,t,2,0)
      shift(s,t,a,1,0):  count = 1; disk 1:s → t
   count = 2; disk 2:s → a
   shift(t,a,s,2):  count = 3; disk 1:t → a
count = 4; disk 3:s → t
   shift(a,t,s,2,4)
      shift(a,s,t,1,4):  count = 5; disk 1:a → s
   count = 6; disk 2:a → t
   shift(s,t,a,1,6):  count = 7; disk 1:s → t
```

It should be mentioned that there exists an iterative solution of the game too, see [40]. Anyhow the complexity of the algorithms (recursive or iterative) is high. It can be shown that it is proportional to 2^n, see [41]. That's why you should not choose the number of disks greater than 6.

3.6 The Knapsack Problem

It is not difficult to explain what the *knapsack problem* is [42], [43]. Assume there are several objects you want to put in your knapsack to transport them. Every object has a known value and a known weight. The total weight of objects carried in the knapsack may not exceed a certain limit `maxweight`. Otherwise the knapsack would be damaged. But `maxweight` is smaller than the sum of weights of all objects. So some of them must be left behind. The problem is to find objects such that the sum of their values is maximal and the sum of their weights is less or equal to `maxweight`. The set of objects determined in this way is optimal to be transported in the knapsack.

The knapsack problem is interesting because it is an NP-problem [44], [45]. That means there is no known algorithm to solve it in polynomial time. However, the knapsack problem can be solved using dynamic programming, see [43]. This method works fine but it needs a lot of computer storage. Here we prefer the method *backtracking* ([46], [47]) which needs no large storage. We show how it works by a small example.

The weights and the values are given by the lists `weight = [7,4,6,8,6]` and `value = [4,4,5,4,6]`. The limit of the weights is `maxweight` = 13. The current optimal value is named `optival`, the corresponding numbers of the objects (starting with 0) are stored in `optiList`. Initially `optival = 0`

and `optiList = []` of course.
In the following scheme we show the way the optimal solution is found in the example. Explanations see below.

$$
\begin{array}{l}
[](0) \to [0](4) \to [0,1](8) \to [0,1,2](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad [0,1](8) \to [0,1,3](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad [0,1](8) \to [0,1,4](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [0](8) \to [0,2](9) \to [0,2,3](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad [0,2](9) \to [0,2,4](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [0](9) \to [0,3](-) \\
\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [0](9) \to [0,4](10) \\
\qquad\qquad\qquad\quad \swarrow \\
\quad \swarrow \\
[](10) \to [1](10) \to [1,2](10) \to [1,2,3](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad [1,2](10) \to [1,2,4](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [1](10) \to [1,3](10) \to [1,3,4](-) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [1](10) \to [1,4](10) \\
\qquad\qquad\qquad\quad \swarrow \\
\quad \swarrow \\
[](10) \to [2](10) \to [2,3](-) \\
\qquad\qquad\qquad\quad \swarrow \\
\qquad\qquad\qquad [2](10) \to [2,4](11) \\
\qquad\qquad\qquad\quad \swarrow \\
[](11) \to [3](11) \to [3,4](-) \\
\qquad\qquad\qquad\quad \swarrow \\
\quad \swarrow \\
[](11) \to [4](11)
\end{array}
$$

The currently chosen numbers of the objects are stored in `indList`. It is part of list `[0,1,2,3,4]`. The number in parentheses is the current optimal value `optival`. If there is a dash instead the corresponding `indList` is not feasible, i.e. the sum of the associated weights exceeds `maxweight`. If an arrow points to the right the next possible number is added to `indList`. Then the feasibility must be checked. Diagonal arrows stand for backtracking, and the last number of `indList` is canceled.
The algorithm takes the first element of the list of weights and checks whether or not `maxweight` is exceeded. It is not, thus `optival` is set to `4` and `indList = optiList = [0]`. Also the sum of the first two weights is less than `maxweight`. So `optival = 8` and `indList = optiList = [0,1]`. However, the sum of the first three weights is greater than `maxweight`. `indList = [0,1,2]`, but `optival` and `optiList` remain unchanged, `2` must be removed from `indList`. Next the algorithm tries to add `3` to `indList`, but `[0,1,3]` is also infeasible etc.
The number of `indlists` which have to be checked is 23. Since there are five objects there are 32 possibilities in total. But it is unnecessary to check `[0,1,2,3]` because we already know that `[0,1,2]` is infeasible. The reduction of the number of checks does not seem to be very large in our example. But try the example in the exercises with eleven objects.

To code this method in Python we need some more variables. `valList` and `wtList` are lists that contain values and weights. But they are not the same as `value` and `weight`. `valList` is the list of values of the objects of `indList`. `wtList` is defined analogously. For example: If `indList = [0,2,4]`, `valList = [4,5,6]` and `wtList = [7,6,6]` in the above example. `marked` is a list of boolean variables which are initially all set `False`. `marked[index]` is turned to `True` if it is impossible that a new and better `optival` could be obtained by an `indList` containing `index`. We will discuss this later in detail. `n` is just the number of objects, and `count` counts the `indLists` to be considered. It is not necessary for the algorithm however. `base` on the other hand is very important. It tells the essential function `search` which number to begin with if `indList` is temporarily empty. You can see that this may happen when you look at the scheme above.
All lists except `marked` are initialized with `[]` and all numbers are initialized with `0`. Besides these settings the main program only consists of the call of function `search` and the display of the results. This is the listing:

```
n = len(weight)
indList, optiList, valList, wtList = [], [], [], []
marked = [False for i in range(n)]
```

```
base, count, optival = 0, 0, 0
search(-1)
print(); print('optimal value:  ',optival)
print('optimal objects:  ',optiList)
print('number of checks:  ',count)
```

The recursive function `search` does the essential work in this program. First variables `count, base, optival, optiList` are declared `global`. What does this mean, and why is this necessary? Normally all variables in Python are local. That means they are only defined in the function where they occur. But variables which are defined in the main program (not in any function) are global. They may be used in functions as well. But we want to change the variables mentioned above in the function. In this case we have to declare them `global`. This keyword is not needed für printing and accessing.
Then the auxiliary function `find` is called: `y=find(current)`. This function returns the first index not `marked` or `-1` if there is no such index from the argument `current+1` on. When the program starts `current = 0`, no index is marked, so `find` returns `0`. However, later `y` will be set to `-1`. In that case, backtracking is necessary.
But let's consider the case `if y >= 0` first. `current` is set to `y`, and this value is appended to `indList`, `wtList` and `valList`. This does not yet imply that we found a new solution of our problem. The constraint `sum(wtList) <= maxweight` and the condition `sum(valList) > optival` must be checked. If both conditions are fulfilled, a new and better solution was found:

```
if (sum(wtList) <= maxweight) & (sum(valList) > optival):
       optival = sum(valList); optiList = indList.copy()
```

To avoid unnecessary searching the following two lines are important:

```
if sum(wtList) > maxweight:
       for ii in range(current+1,n):marked[ii] = True
```

Consider `indList=[0,1,2]` in the scheme above. Then `sum(wtList) = 7+4+6 = 17 > maxweight`. Consequently indices `3` and `4` are marked, and `indLists [0,1,2,3]` , `[0,1,2,4]` and `[0,1,2,3,4]` are not checked. The last line of the `if`-branch is the recursive call `search(current)` without any terminal condition.
Now let us look at the `else`-branch. It is processed if `y=-1`, i.e. all further indices are marked or there is none. Then backtracking has to be performed. Thus `current` must be marked: `marked[current] = True`. `clean` is an auxiliary function called with parameter `indList[-1]`, the last item of `indList`. All markers from `indList[-1]+1` on are set to `False`. Why is this necessary? Look at our example again. `indList=[2,3]` is infeasible because

`weight[2]+weight[3]=6+8>maxweight`. So `3` and `4` are marked. But still `indList=[2,4]` has not been checked, so `4` must be unmarked again. In fact `indList=[2,4]` yields the optimal solution, and we don't want to miss it. Then the last items of `indList`, `wtList` and `valList` are removed. If `indList` is not empty, `current` is assigned to the new last item of `indList`: `if indList != []:current = indList[-1]`.
For an empty `indList` things get more complicated. In this case `current` must be set to `-1+base`. Remember that `base` was set to `0` in the main program. You can see from the scheme above that `indList` is empty for the first time if all possibilities for `0` in `indList` are depleted. Therefore `base` must be increased, and the next series starts with `indList = [1]`; no `0` occurs anymore. Finally the recursive call follows including the terminal condition: `if base<n:search(current)`. Here is the whole listing of `search`:

```
def search(current):
     global count, base, optival, optiList
     y = find(current)
     if y >= 0:
          count += 1; current = y
          indList.append(current)
          wtList.append(weight[current])
          valList.append(value[current])
          if (sum(wtList)<=maxweight) & (sum(valList)>optival):
             optival = sum(valList); optiList = indList.copy()
          if sum(wtList) > maxweight:
             for ii in range(current+1,n):marked[ii] = True
          print('count = ',count,)
          print('optival = ',optival,' indList = ',indList)
          search(current)
     else:  # backtracking
          marked[current] = True
          clean(indList[-1])
          indList.pop(-1); wtList.pop(-1); valList.pop(-1)
          if indList != []:current = indList[-1]
          else:
             current = -1 + base; base += 1
          if base < n:  search(current)
```

Finally we denote the listings of the small auxiliary functions:

```
def find(x):
     posi = -1
```

```
    for j in range(n-1,x,-1):
        if (not marked[j]):  posi = j
    return posi

def clean(x):
    y = x + 1
    while y <= n-1:
       marked[y] = False; y += 1
```

3.7 ILP with Branch and Bound

ILP stands for **I**nteger **L**inear **P**rogram. This is a linear optimization problem in which only integer solutions are to be determined. Such problems are important because often only integer solutions make sense. For example you can't buy 4.73 cars, and only integer numbers of sofas can be produced. It is not possible just to solve a linear optimization problem with real solutions and then obtain the solution for your ILP by rounding. The reason is that rounded solutions may be infeasible.

There are several methods for finding optimal solutions of a general linear optimization problem. A well-known method is the simplex method, see [48], [49]. This method usually works very well, but it can only be shown that it has exponential runtime. But in fact such problems can be solved in polynomial time (see [50]). This means that there exists an integer k such that the running time is limited by n^k where n is the number of variables. There is an efficient *scipy* function for solving general linear problems. So we shall concentrate on solving integer linear problems.

One strategy to solve them is *Branch and Bound*. Generally this is a pattern for certain algorithms. The problem is split into two or more problems which can be solved easier than the original one. In most cases this results in a tree. Bounding means that some branches may be canceled because of constraints included in the original problem.

Especially it is useful for integer optimization problems. One might think that finding only integer solutions is easier than solving the corresponding problem for real solutions. But the opposite is correct. No algorithm is known that solves ILPs in polynomial time. Branch and Bound may solve your ILP in less than exponential time if you are lucky.

The first step consists in a *relaxation*. The constraint that only integer solutions are to be determined is omitted. The obtained optimal solution may contain non integer values, for example $x_3 = 1.5$. Then the original problem is split into two new problems. The first one is the original problem with the additional constraint $x_3 \leq 1$ and the second is the original problem with the additional constraint $x_3 \geq 2$. When the first problem is solved it may happen that $x_4 = 7.3$. In this case another branching has to be performed. The branching is repeated until one of the following cases occurs: (1) An integer solution was found. (2) The current linear problem is infeasible. (3) A better solution has already been obtained. The next step consists in collecting all integer solutions, and finally the best of these is selected.

Now we will code an ILP in Python. In order to explain how the program works we use an example found in a text from the university of Siegen (Germany) [51]. The advantage is that this example is not very complicated but on the other hand it is not trivial. Here it is:

The *objective function* to be minimized is given by $c(x) := -x_1 - x_2$.
The *constraints* are

$$\begin{aligned} -2x_1 + x_2 &\leq -\frac{1}{2} \\ -x_2 &\leq -\frac{1}{2} \\ 2x_1 + x_2 &\leq \frac{11}{2} \end{aligned}$$

First the required data must be available for processing, i.e. the objective function and the constraints. It is rather tough to type all the needed data using input functions. So we rather store the data in a text file. It can be created by an editor for plain text. Whenever you want to create your own text file, remember that the entries have to be separated by exactly one space. For our example it looks like this:

$$\begin{array}{rrr} -1 & -1 & 0 \\ -2 & 1 & -0.5 \\ 0 & -1 & -0.5 \\ 2 & 1 & 5.5 \end{array}$$

The first line consists of the coefficients of the objective function. At the end of the line a zero is added because all lines must have equal lengths. All the other lines contain the coefficients of the constraints. Look at the third line.

The inequality does not contain x_1. But in the text file a zero was entered at the position of the missing coefficient of x_1. The function `read` serves for reading the text file. In the main program the variable `filename` must be replaced by the name of the file, in our example 'ILP1.txt'. We will not explain the details of this function. It uses function `loadtxt` of module *numpy*. This module must be imported at the beginning of the code. More important is the output. It returns `obj, A, B`. `obj` is a list which contains the coefficients of the objective function. `A` is a matrix and `B` is a vector such that the constraints are given in the form $A \cdot x \leq B$, where x is the vector of variables.

Function `solve_LP` contains the function `linprog` provided by the module *scipy*. Thus we must also import the module `scipy.optimize`. `linprog` returns a lot of outputs, but most of them are not important with regard to our problem. So `solve_LP` just returns `x,y,succ`. `x` is the vector of optimal variables corresponding to the given linear optimizing problem. Some of these values may be non-integers. `y` is the optimal objective value. `succ` announces whether or not the solution is feasible.

The next two functions check the input `x`. `isInt` returns `True` if `x` is nearly an integer. The mathematical equation $x = round(x)$ will always be wrong, if x is a floating point number because of the finite precision of calculation. Thus we use `(abs(x-xlow) < epsilon) or (abs(xup - x) < epsilon)`. `epsilon` is a value fixed in the main program below, and `xlow` $= \lfloor x \rfloor$, `xup` $= \lceil x \rceil$ are the biggest lower integer and the smallest greater integer with respect to `x`. Whereas `isInt` requires a single number as input, `allInt` demands a list. It is needed to check whether all variables of list `x` obtained from `solve_LP` are integer.

The functions discussed so far were only preliminaries for the important ones. The most essential function is `branch`. It is called whenever the preceding optimization yields at least one variable which is not integer. It returns a boolean variable which has the value `True` if a solution of the ILP was found.

The parameters are `ind, x, A, B, xList, yList`. `ind` is the first index of a non-integer variable. `x` is the current list of variables, `A` is the current matrix, and `B` is the current vector of the right hand side. We emphasize the word 'current' because the values of these variables will probably change during the recursive calls of `branch`. `xList` is the list of all lists of variables, and `yList` is the list of the corresponding optimal values obtained during preceding optimizations.

`branch` initializes the local variables `found_l` and `found_r` with `False`. They will turn to `True` as soon as an integer solution is found in the left or right branch. The line `xl,yl,Al,Bl,succ = left(ind,x,A,B)` follows. The function `left` adds one more constraint to the current system. The new constraint consists of a 1 inserted at the position of index `ind`. All other coefficients on the left hand side are set to 0. Vector `B` is supplemented by $= \lfloor x[ind] \rfloor$. Then the new LP is solved by calling `solve_LP` with parameters `obj`, the objective function, and `A0` and `B0`; these are the new built matrix of coefficients and the vector on the right hand side. If the solution is not feasible (that means `succ == False`) `y0` is set to `np.inf`, that is infinity. So it is ensured that this is certainly not the optimal solution of the problem. `left` returns the new optimal variables of the LP `x0`, the new optimal solution `y0` and the expanded matrix `A0`, vector `B0` and finally `succ` to decide whether or not the solution of the linear problem was successful.

The function `right` works in a quite similar way. We have just to remember that the new constraint $x[ind] \geq \lceil b[ind] \rceil$ has to be replaced by $-x[ind] \leq -\lceil b[ind] \rceil$. So we have to set `A_[ind] = -1; A1 = A + [A_]`. The vector `B` must be supplemented by `B1 = B + [-np.ceil(x[ind])]`.

Now let's return to function `branch`. If an integer and feasible solution was found the obtained new list `xl` and the corresponding optimal value `yl` are appended to the lists `xList` and `yList` respectively. `found_l` is then set to `True`. This is coded in these three lines:

```
if allInt(xl) & (yl != np.inf):
  xList.append(xl); yList.append(yl)
  found_l = True
```

This was outlined for the left branch. The corresponding part for the right branch looks quite analogously.
However, in most cases we shall not obtain integer solutions, and branching must go on. Whenever we have obtained a non-integer variable by the functions `left` and `right` we could perform another branching. This is done in the Python script `IPL_2`. It is obvious that the computing effort grows exponentially with the number of non-integer variables. But we can reduce the number of branches by bounding.

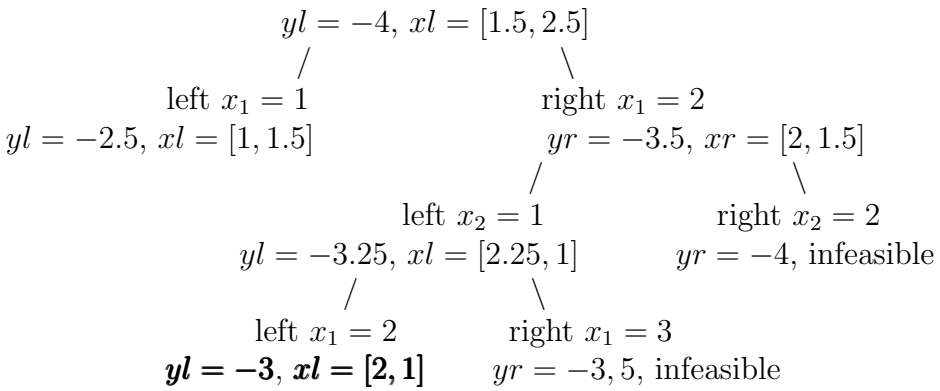
We return to the example from above to explain the idea. The LP relaxation yields the optimal value $y = -4$ where $x_1 = 1.5$, $x_2 = 2.5$. So branching has to be performed. The additional constraints are $x_1 \leq 1$ for the left branch and $x_1 \geq 2$ for the right one. Function `left` yields `yl = -2.5` and `xl = [1, 1.5]`, whereas function `right` gives `yr = -3.5` and `xr = [2, 1.5]`. The

minimum of `yl` and `yr` is a bound to the optimal value of the ILP. It cannot be smaller than -3.5. Since `yr` is less than `yl` we shall first try to find an integer solution using the right branch. If we obtain such a solution with an optimal value < -2.5 it is not necessary to consider the left branch. Only if the right branch yields no feasible integer solution it is necessary to try to get a solution from the left branch.

In our example the right branch seems to be the better choice, and we haven't got an integer solution yet. Thus the following part of function `branch` will be executed:

```
if (yr <= bound) & (yr != np.inf) & (found_r == False):
    i = 0
    while (i < len(xr)-1) & isInt(xr[i]) :  i += 1
    if i < len(xr):found_r=branch(i,xr,Ar,Br,xList,yList)
```

First we state that all the conditions of the `if`-clause are fulfilled. The next two lines detect the first index `i` where the variable `x[i]` has no integer value, if there is any. If such an index is found finally the recursive call of function `branch` is executed. The corresponding part for the left branch looks quite the same and need not be discussed here.
Our example now gives `yl = -3.25` and `xl = [2.25, 1]`. The right branch yields `yr = -4` and `xr = [2, 2]`. But this solution is not feasible, and it has to be canceled.
One more branching is necessary to obtain `yl=-3`, `xl = [2,1]` and `yr=-3.5`, `xl = [3,0.5]`. The latter solution is not feasible, so the optimal value is $y = -3$ with $x_1 = 2, x_2 = 1$. Since this optimal value is smaller than the former received left branch value -2.5 we need not consider the left branch of the first branching. The complete solution is shown in the sketch below.



Example for an ILP The complete tree according to our example; there is no need to pursue the upper left branch because the optimal objective value is -3 which is less than yl in the upper left branch.

Function `branch` is nearly complete, but there is still one case left. It is possible that the putative better branch (with the lower value of the objective function) gives no feasible solution. Then the remaining branch has to be considered.

```
y2 = np.inf; found2 = False
if yl > bound:  y2 = yl; x2 = xl; A2 = Al; B2 = Bl
if yr > bound:  y2 = yr; x2 = xr; A2 = Ar; B2 = Br
```

The preceding three lines effect that parameters of the remaining branch are assigned index 2. So we need only one more function call for branching. We also have to keep in mind that this call is only necessary if no integer solution has been found yet and if `y2` is greater than `bound`. The interior of the `if`-clause is quite analogous to the cases treated before.

```
if (y2 != np.inf) & (found_l == False) & (found_r == False):
   i = 0
   while (i < len(x2)-1) & isInt(x2[i]) :  i += 1
   if i < len(x2):  found2 = branch(i,x2,A2,B2,xList,yList)
```

There are three possibilities to find an integer solution during the course of function `branch`: `find_l` or `find_r` may turn to `True`, and finally `find2` may become `True`. Therefore the last line of the function is
`return found_l or found_r or found2`.
Here is the listing of function `branch`:

```
def branch(ind,x,A,B,xList,yList):
       found_l = False; found_r = False
       xl,yl,Al,Bl,succ = left(ind,x,A,B)
       if succ:
          if allInt(xl) & (yl != np.inf):
             xList.append(xl); yList.append(yl)
             found_l = True
       xr,yr,Ar,Br,succ = right(ind,x,A,B)
       if succ:
          if allInt(xr) & (yr != np.inf):
             xList.append(xr); yList.append(yr)
             found_r = True

       bound = min(yl,yr)
```

```
    if (yl <= bound) & (yl != np.inf) & (found_l == False):
       i = 0
       while (i < len(xl)-1) & isInt(xl[i]) :  i += 1
       if i < len(xl):  found_l = branch(i,xl,Al,Bl,xList,yList)
    if (yr <= bound) & (yr != np.inf) & (found_r == False):
       i = 0
       while (i < len(xr)-1) & isInt(xr[i]) :  i += 1
       if i < len(xr):  found_r = branch(i,xr,Ar,Br,xList,yList)
    y2 = np.inf; found2 = False
    if yl > bound:  y2 = yl; x2 = xl; A2 = Al; B2 = Bl
    if yr > bound:  y2 = yr; x2 = xr; A2 = Ar; B2 = Br
    if (y2 != np.inf) & (found_l == False) & (found_r == False):
       i = 0
       while (i < len(x2)-1) & isInt(x2[i]) :  i += 1
       if i < len(x2):  found2 = branch(i,x2,A2,B2,xList,yList)

    return found_l or found_r or found2
```

The *main program* is not so exciting. It starts with some initialization: `epsilon = 1e-6`, `xList` and `yList` are set to `[]`. A text file (see above) is read to obtain the objective function as well as the constraints. The LP is solved: `x,y, succ = solve_LP(obj,A,B)`. If the solution of the LP is already integer or if it is infeasible the result is displayed. Otherwise branching begins. In the end `xList`, `yList` and the result are displayed.

The program part contains three versions: `ILP` only displays the results whereas `ILP_3` also shows a lot of intermediate results so you can see how it works. There is also a version `ILP_2` where only branching is implemented but no bounding. The runtime difference between the latter two versions may seem unimportant, but take notice of exercise 6.

3.8 Exercises

1. **Curve of Hooks [52] and Snowflake Curve [53]**
   Develop programs for drawing these curves. Step one of them is depicted in figure 3.4. The recursive function `draw` belonging to the *curve of hooks* starts with `width = width // 4`. For the *snowflake curve* it is sufficient to divide `width` by 3. If `step == 0` the recursive call `draw(width,step-1)` simply means `tu.forward(width)`.
   The main program is the same for both curves. Here it is:

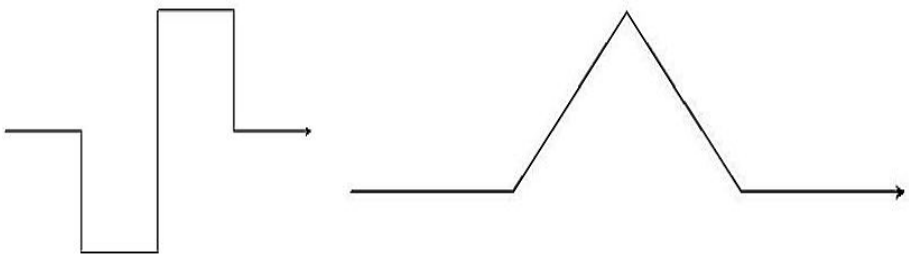


Figure 3.4: Curve of hooks and snowflake curve

```
for count in range(3):
    tu.up(); tu.setpos(-300,0); tu.down()
    tu.width(2)
    draw(800,count)
    time.sleep(3)
    tu.clearscreen()
tu.up(); tu.hideturtle()
tu.setpos(-300,-250)
tu.pencolor((0,0,0))
tu.write('finished!',font = ("Arial",12,"normal"))
tu.mainloop(); tu.done()
```

Hint: Do not forget to `import time` at the beginning of the programs.

2. **Fibonacci Function of higher Order**
   The *Fibonacci function fibop of order p* of a number n is defined by the sum of the $p+1$ predecessors of n [54]. If $n = p$, $fibop(p) = 1$, if $n < p : fibop(n) = 0$.
   Write a program for computing values of *fibop* applying a recursive or an iterative algorithm. The user may decide the kind of algorithm.

3. **Q function**
   The Q function invented by D. R. Hofstadter [55] is defined thus:

$$Q(k) = \begin{cases} 1, \text{ if } k \leq 2 \\ Q(k - Q(k-1)) + Q(k - Q(k-2)) \text{ otherwise} \end{cases}$$

   Write a program for computing values of this function applying a recursive or an iterative algorithm. The user may decide the kind of algorithm.
   Remark: It has not been proven that the Q function is well-defined. Possibly the recursion demands negative arguments for which Q does not exist. But this is unlikely [55], [56].

4. **Permutations**
   The following code corresponds to HEAP'S algorithm for computing permutations [57], [58]. Perform a desk test for this code for n=3, A = [1,2,3] and denote in which order the permutations are displayed.

```
def Perm(m):
    if m == 1:
      print(A)
    for i in range(m):
      Perm(m-1)
      if m % 2 == 1:
        x = A[0]; A[0] = A[m-1]; A[m-1] = x
      else:
        x = A[i]; A[i] = A[m-1]; A[m-1] = x
```

5. **Knapsack Problem**
   (i) Run program "Knapsack_backtracking" with `weight` = [6,8,7], `value` = [5,6,8] and `maxweight` = 13.
   (ii) Run it with `weight` = [5, 7, 6, 8, 10, 11, 12, 15, 17, 30, 18] , `value` = [8, 9, 6, 5, 10, 5, 10, 17, 19, 20, 13] and `maxweight` = 33.
   (iii) Alter the program such that it checks all possibilities for arranging the set of objects.

6. **ILP**
   (1) Test all text files using program versions `ILP_3` and `ILP_2`.
   (2) Find examples for every case that can occur and test your examples:
   (i) the LP is infeasible,
   (ii) the LP relaxation yields an integer solution,
   (iii) the LP is feasible but the ILP is not.
   (3) Solve the following ILP (adapted from [59]) .
   max $-2x_1 + x_2 - 8x_3$
   subject to:

$$\begin{aligned} 2x_1 - 4x_2 + 6x_3 &\leq 3 \\ -x_1 + 3x_2 + 4x_3 &\leq 2 \\ 2x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

   Use programs `ILP_3` and `ILP_2`. Count the different number of calls of function `branch`. Do you think that it is worthwhile to use bounding?

Chapter 4

Some Mutually Recursive Algorithms

4.1 Mondrian

Piet Mondrian (1872 - 1944) was a famous Dutch painter [60], [61]. Many museums especially in the Netherlands exhibit his paintings [62]. Works belonging to the artistic movement "new plasticism" are often composed of black horizontal and vertical bars which form a grid pattern. Some of the rectangular areas of the grid are filled with intensive colors red, blue and yellow [63], [64].
We want to develop a program which produces images modeled after this kind of paintings. Of course their quality is not comparable to original works. It is easy to see that the balance is missing. One reason is the use of random numbers in the program. We shall just apply the technique of the kind of paintings described above. A possible result is shown in Figure 4.2.

The essential functions are `horizontal` and `vertical`. They are *mutually recursive*, i.e. `vertical` calls `horizontal` and `horizontal` calls `vertical` until a terminal condition has been reached. Both functions need four parameters `left`, `right`, `bottom` and `top`. They mark the region where the next vertical or horizontal bar is to be painted.
First of all we need two global parameters: `minspan` and `half`. `minspan` is the minimal difference between `top` and `bottom` in function `vertical` and between `right` and `left` in function `horizontal`. `half` is the half of the width of a black bar. The values of these parameters remain constant throughout the program. Their amounts depend on the size of the panel for turtlegraphics. For example set `minspan = 120` and `half = 5`.

Now let us discuss function `vertical` in detail. We define `span` as the difference of `right` and `left`. The interesting part is the following `if`-clause of the listing below. There `le` and `ri` are set: `le = span//4+left` and `ri = -span//4+right`. Thus `le` and `ri` are a bit away from the left and the right border of the region where a vertical bar could be drawn. Then `middle` is initialized by `left` or `-1`, in any case it is outside the suitable region for a vertical bar. But the following `while`-loop takes care that `middle` gets a value between `le` and `ri`:
`while (middle<le) or (middle>ri):middle=left+rd.randint(1,span)`.
Now, function `fill` is called:
`fill(middle-half,middle+half,bottom,top,0)`.
It effects that a new bar is drawn. The left and right borders are `middle-half` and `middle+half`, whereas `bottom` and `top` remain unchanged. The last parameter `0` stands for the color black. At last function `horizontal` is called twice. In these function calls it is tried to draw a horizontal bar between `left` and `middle-half` and then between `middle+half` and `right` respectively. But this happens only if there is enough space.
If the condition `span >= minspan` is not fulfilled the rectangle with borders `left`, `right`, `bottom` and `top` may be colored with red, blue or yellow by calling function `dye`.
And here is the complete listing of function `vertical`:

```
def vertical(left,right,bottom,top):
    span = right - left
    if span >= minspan:
      le = span//4 + left
      ri = -span//4 + right
      if left < 0:  middle = 2*left
      else:middle = -1
      while (middle < le) or (middle > ri):
        middle = left + rd.randint(1,span)
      fill(middle-half,middle+half,bottom,top,0)
      horizontal(left,middle-half,bottom,top)
      horizontal(middle+half,right,bottom,top)
    else:dye(left,right,bottom,top)
```

Function `horizontal` works quite analogously to function `vertical`. Figure 4.1 demonstrates the way both functions call each other.

Now let us look at the main program. First we define three global constants. Above we already mentioned `minspan = 120` and `half = 5`. The third constant is `colorset = {1,2,3}`. It is needed for dying an area. The

small function `dye` produces random numbers from 1 to 18. If the number is at most 3, the area bordered by `left`, `right`, `bottom` and `top` will be filled with a color. Otherwise the area will not be dyed. In `dye` the area is not really filled. It just calls function `fill` with an additional parameter `z`.

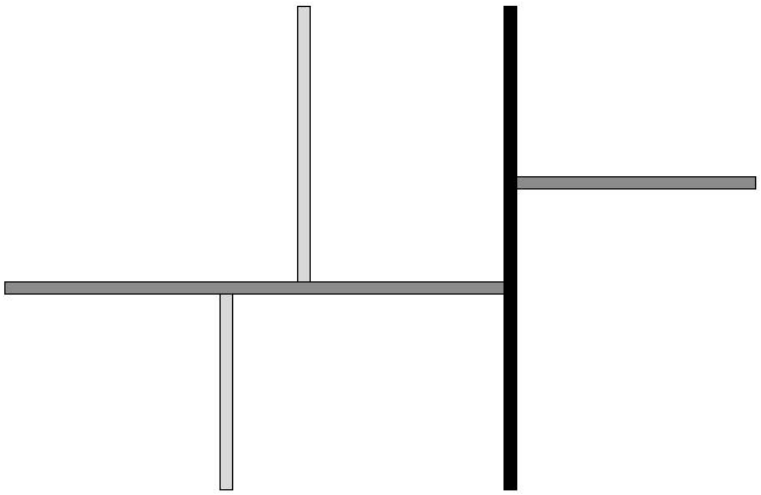


Figure 4.1: First the black vertical bar is drawn by calling `vertical`. Function `horizontal` is called twice. Two horizontal bars are drawn colored dark gray, one left and one right of the vertical bar. Each call of `horizontal` calls `vertical` twice. But only on the left side of the black vertical bar light gray vertical bars are drawn, one above and one below the horizontal bar. There is not enough space for them on the right.

Let's return to the main program. The speed of the turtle is set to `8`, so it will be very fast. But the turtle will be invisible throughout the program caused by `tu.hideturtle()`. The invisible turtle is moved to the correct start position, and function `vertical` is called. The final part of the program is well-known, see section 1.1.2. This is the whole main program:

```
minspan = 120; half = 5; colorset = {1,2,3}
tu.speed(8)
#move the turtle to the start position
tu.up(); tu.hideturtle(); tu.left(90)
vertical(-300,300,-200,200)
# final part
tu.up(); tu.setpos(-300,-250); tu.pencolor((0,0,0))
tu.write('finished!',font = ("Arial",12,"normal"))
tu.mainloop(); tu.done()
```

It still remains to explain how the invisible turtle draws the bars and areas. This is done in function `fill`. The needed parameters are already known. The last one is important for the determination of the color. From the `if`-clause it follows that `c = 0` means black, `c = 1` stands for red, `c = 2` for blue, and `c` ≥ 2 (`else`-branch) stands for yellow. The filling itself is done by drawing a rectangle:

```
tu.setpos(left,bottom)
for i in range(2):
  tu.forward(top-bottom); tu.right(90)
  tu.forward(right-left); tu.right(90)
```

framed by `tu.begin_fill()` and `tu.end_fill()`. That's all.

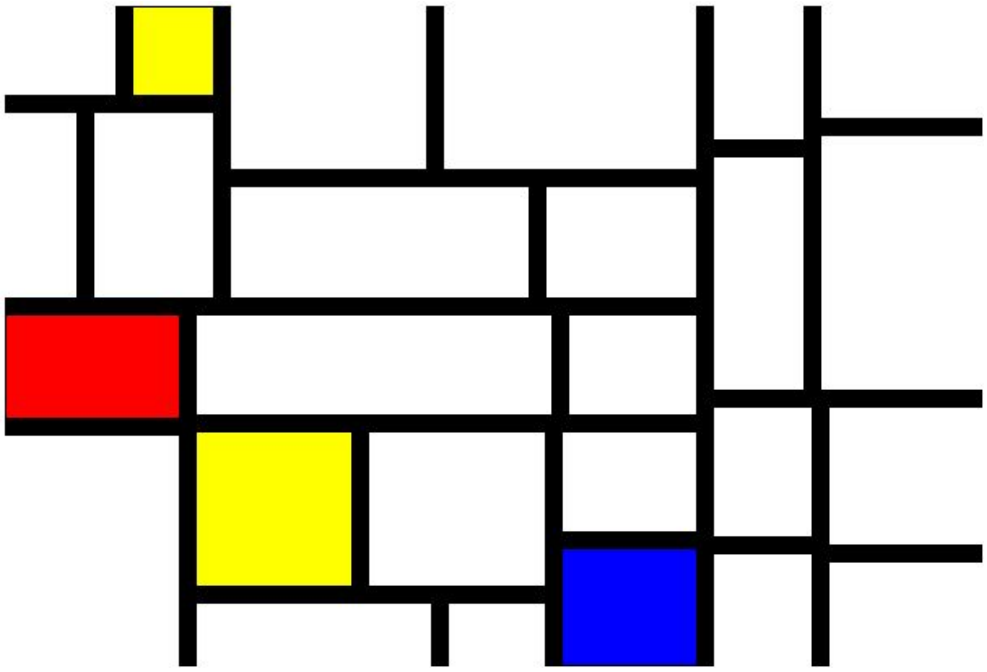


Figure 4.2: An example for a result of program Mondrian

4.2 A Mutually Recursive Term

Let us consider this term:

$$(...((1+2)*3+4)*5+6)*7... \tag{4.1}$$

We are going to compute it by means of mutually recursive functions. Analyzing eq. 4.1 yields that we have to multiply the last number to the rest if it is odd. If the last number is even, we have to add it to the rest of the term. Thus we can denote two short mutually recursive functions:

```
def multiply(i):
    if i > 1:  return i*add(i-1)
    else:  return 1

def add(i):
    return i + multiply(i-1)
```

Function `multiply` takes into account that 1 is also an odd number. In this case `multiply` simply returns 1. Both functions call each other recursively.

Now we consider the main program. After the input of `n` we have to decide whether it is even or odd. For that we use the "modulo" operator of Python. Generally `x % y` gives the rest of `x` by division by `y`. For example $23 \% 5 = 3$ and $35 \% 7 = 0$. A number `n` is even if `n % 2 = 0`. Otherwise it is odd. So we denote the essential part of the main program:

```
if n % 2 == 0:value = add(n)
else:value = multiply(n)
print('value:',value)
```

This is a nice little mutually recursive program for a mathematical function. But do we really need two functions? We can redefine the expression 4.1 as follows:

$$f(n) = \begin{cases} 1 \text{ if } n = 1 \\ n + f(n-1) \text{ if } n \,\%\, 2 = 0 \\ n * f(n-1) \text{ otherwise} \end{cases} \tag{4.2}$$

Using eq. 4.2 we need only one recursive function, and the code gets even simpler. We do not denote the main program because it is trivial. But here is the recursive function:

```
def f(n):
    if n == 1:return 1
    elif n % 2 == 0:return n+f(n-1)
    else:return n*f(n-1)
```

4.3 Shakersort

Bubblesort is a sorting method which is easily to understand, but it works very slowly. A similar method is *Shakersort* [65], and we may suppose that it works faster. The idea is the following: As in Bubblesort [66] the algorithm steps through the elements starting with the last, compares the current element with its predecessor and swaps them if they are in the wrong order, see left column of the scheme below. However, when the first run is finished and the element with the smallest key is at position 1, the direction of sorting is reversed, and it starts at position 2, see the middle column of the scheme. So it is ensured that an element with the largest key stands at the last position. Then the direction of sorting is reversed again, and it starts at the last position but one, see right column of the scheme. Changing the directions and swapping continue until the whole sorting is done. In our example just 8 characters will be sorted alphabetically. In this example sorting is completed after two backward runs and one forward run. But in general more runs are necessary.

sorting backward	sorting forward	sorting backward
E C B F A H G **D**		
E C B F A H **D** G	A **E** C B F D H G	A C B E D F **G** H
E C B F A **D** H G	A C **E** B F D H G	A C B E D **F** G H
E C B F **A** D H G	A C B **E** F D H G	A C B E **D** F G H
E C B **A** F D H G	A C B E **F** D H G	A C B **D** E F G H
E C **A** B F D H G	A C B E D **F** H G	A C **B** D E F G H
E **A** C B F D H G	A C B E D F **H** G	A **B** C D E F G H
A E C B F D H G	A C B E D F G **H**	

We demonstrate this sorting method similarly to mergesort, see section 3.4. To do that we choose 60 capital letters randomly. This is done in the main program. The list `a` is initialized by `[]`.

```
for i in range(n):  # set n = 60
    a.append(rd.randint(65, 90))
    update(a,i)
```

The letters are sorted alphabetically. Whenever a letter is moved, more exactly: swapped with one of its neighbors, function `update` is called. We will not discuss this function in detail. `update(a,kk)` moves the turtle to the right position. Any character that may be at that position is deleted. Then the turtle writes `a[kk]`.

The essential functions are `back` and `forward`. They need three parameters: the list `a`, and `l` and `r` which mark the left and right border of the region where sorting has still to be done. At the beginning, `l = 0` and `r = n-1`; remember that Python starts counting at 0. If there are 60 letters to sort their positions are numbered 0, ..., 59. Function `shakersort` itself just consists of the call `back(0,n-1)`.
Functions `back` and `forward` look very similar, but there are some important differences. We will discuss `back` first. The introduced variable `k` will mark the next *left* border. Initially it is set to `r`, and it remains `r`, if the list is already sorted. The following `for`-loop starts at `r` and ends at `l`. The third parameter of `range` is `-1`, and that means that `j` counts backwards, for example 8,7,6,...,0. If an element with a greater key stands just left of the current element, they are swapped and updated. In this case, `k` is set to `j` because the element swapped to the right must possibly be swapped again. So we are not sure that sorting is already done from position `j` on. Finally the recursive call follows: `if k < r:forward(a,k,r)`. Forward sorting makes sense only if `k < r`. Otherwise the list is already sorted. Here is the listing of `back`:

```
def back(a,l,r):
    k = r
    for j in range(r,l,-1):
      if a[j-1] > a[j]:
        x = a[j-1]; a[j-1] = a[j]; a[j] = x
        update(a,j-1); update(a,j)
        k = j
    if k < r:  forward(a,k,r)
```

In `forward` `k` is initially set to `0`. If the list is already sorted, `k` does not change. The following `for`-loop counts in forward direction. Notice that the right bound is `r+1`. Why is this necessary? Possibly `a[r]` and `a[r-1]` must be swapped. So index `r` must be reached by the loop. But in `range(l,r)` the last index is only `r-1`, a special feature of Python. The last line contains the recursive call: `if l < k:back(a,l,k)`. The case `l >= k` would not make sense and show that sorting is already done. This is the whole listing:

```
def forward(a,l,r):
    k = 0
    for j in range(l,r+1):
      if a[j-1] > a[j]:
        x = a[j-1]; a[j-1] = a[j]; a[j] = x
        update(a,j-1); update(a,j)
        k = j
    if l < k:  back(a,l,k)
```

Is Shakersort significantly better than Bubblesort? We made a small experiment to explore this. With `n` = 60 and `tu.speed(6)` we ran Shakersort and Bubblesort ten times each. The mean value of the time consumed was 113.5 ± 14.5 s for Shakersort and 119.5 ± 10.7 s for Bubblesort. From that we can see that Shakersort was slightly better than Bubblesort but not significantly. It can be shown that the time complexity of both methods is $O(n^2)$, see [65]. If the number of elements is multiplied by 2, 3 etc. the time consumption must be multiplied by 4, 9 etc. Therefore Shakersort is not recommended for sorting large numbers of elements. But the method is funny, isn't it?

4.4 Sierpinski Curves

In 1912 the Polish mathematician Wacław Sierpiński developed the sequence of fractal curves [67] we want to code. The curves of the sequence are closed, and they all lie inside a finite area, e.g. the unit square. But their lengths grow exponentially with the number of step n. The limit for $n \to \infty$ is a curve which runs through every point of an area; it is called "space-filling".

A Sierpiński curve [68] [69] consists of bulges and passages. If `n=1`, a passage is just a line of length `h2`; one of four bulges is colored red in figure 4.3.

In function `bulge` we have to treat the basic case `step = 1` and the recursive case `step > 1` differently. The basic case explains itself. Evidently passing a bulge turns the turtle to the right by 90°.

```
if step == 1:
  tu.left(45);tu.forward(h2)
  tu.right(90);tu.forward(h2)
  tu.right(90);tu.forward(h2)
  tu.left(45)
```

From figure 4.4a it can be seen in which way the code for a bulge has to be replaced for `step > 1`. It is not sufficient just to substitute a simple bulge by three smaller ones. Instead the turtle must turn by 45° to the left and then move along a passage. Then it turns once more by 45° to the left and moves forward. When this is finished three bulges are drawn each of them followed by a forward move. At last `passage` is called again, framed by two more turns by 45° to the left. This is the `else`-branch:

```
else:
  tu.left(45);passage(h,h2,step-1)
  tu.left(45);tu.forward(h2)
```

```
for i in range(3):
  bulge(h,h2,step-1);tu.forward(h2)
tu.left(45); passage(h,h2,step-1)
tu.left(45)
```

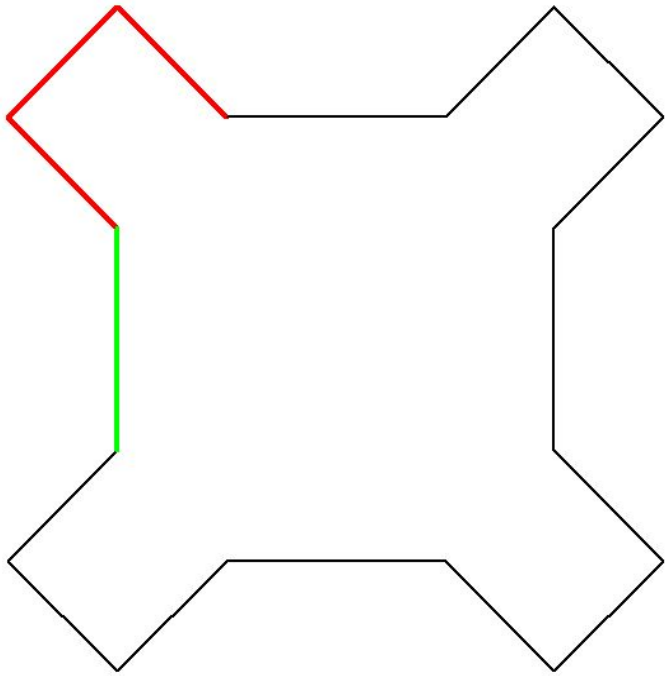


Figure 4.3: Sierpiński curve, step = 1; the green line is a passage, a bulge is colored red.

If `step > 1`, in `passage` the move `tu.forward(h2)` must be replaced by the following (see figure 4.4b):

```
if step > 1:
  passage(h,h2,step-1)
  tu.left(45); tu.forward(2*h)
  bulge(h,h2,step-1)
  tu.forward(2*h); tu.left(45)
  passage(h,h2,step-1)
```

`passage` is called with parameter `step-1` (colored green). Then the turtle turns to the left by 45° and moves forward, black in figure 4.4b. Next `bulge(h,h2,step-1)` is called, colored red in figure 4.4b. Afterwards the turtle turns to the left by 45° and moves forward. Finally `passage` is called again, colored green, and the turtle turns by 45° again. Functions `bulge` and `passage` call each other, thus the functions are mutually recursive.

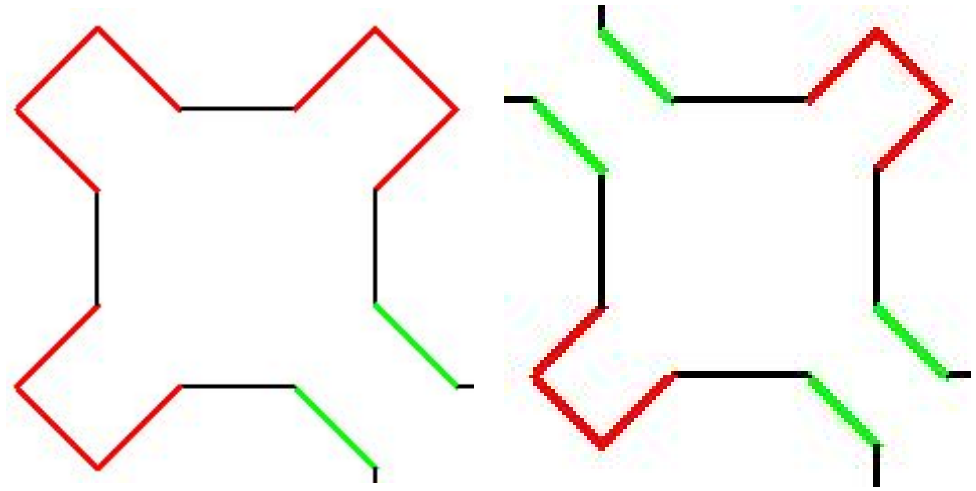


Figure 4.4: **(a)** shows a bulge where n=2. The passages (n=1) are colored green and the bulges of n=1 are colored red. **(b)** shows a passage with `n`=2. The necessary bulges of n=1 are colored red, and the passages of n=1 are green.

The main program is not so short as usual. Between the input of `n` and the function call some preparations are necessary. The initial length `h0 = 200` may depend on the resolution of the reader's screen. Possibly the value must be changed. For drawing a Sierpiński curve of step `n` length `h0` must be adjusted.
Two lengths are used for drawing: `h=round(h0)` and `h2`, the square root of 2 multiplied by `h0`. From figure 4.3 it can be seen that the difference between the right and left sides of the curve is $(4+\sqrt{2})\cdot h$. From figure 4.4a we observe that for `step=2` the above difference is $(8+3\cdot\sqrt{2})\cdot h$. So for each step the original length has to be multiplied by

$$\frac{4+\sqrt{2}}{8+3\cdot\sqrt{2}}$$

So the corresponding lines of code have to be inserted. The final part of the program could be copied from section 1.1.2. This is nearly the whole main program:

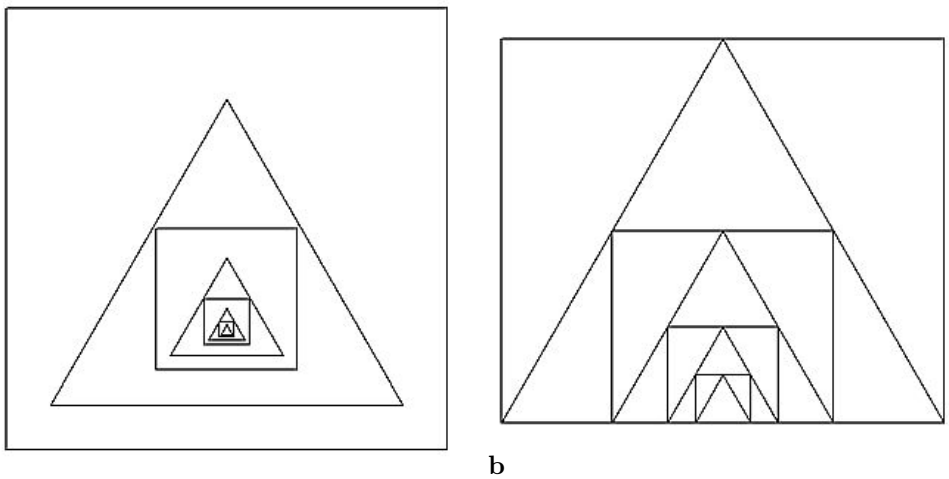
```
n=int(input('n = (1,...,6) '))
h0=200
for i in range(n):
```

```
    h0=h0*(4+math.sqrt(2))/(8+3*math.sqrt(2))
h=round(h0);h2=round(math.sqrt(2)*h0)
tu.width(2);tu.speed(5)
# move to the start position
tu.up();tu.setpos(-100,60);tu.down();tu.left(90)
for i in range(4):
    bulge(h,h2,n);tu.forward(2*h)
# final part, see section 1.1.2
```

This was only one example for a sequence of curves with a space-filling limit. Of course there are many more of them. Some are closed curves and some are not. For more information about space-filling curves see [70]. It should be possible to write a code using turtlegraphics to model Hilbert curves, see [71]. More than two mutually recursive functions are necessary for modeling these curves.

4.5 Exercises

1. **Mutually Recursive Figures**
   (i) Write a program with two mutually recursive functions "square" and "triangle". The result could look like figure 4.5a.
   (ii) Write a program with two mutually recursive functions "triangle" and "rectangle". The result could look like figure 4.5b.



a b

Figure 4.5: (a) Functions `square` and `triangle` call each other. **(b)** Functions `triangle` and `rectagle` call each other.

2. **Simple Figures**
   Write a program which creates hexagons, rectangles and triangles filled with randomly determined colors. The figures should be bounded to the panel. But they may cover each other. Write the program such that function "hexagon" calls "rectangle", "rectangle" calls "triangle", and "triangle" calls "rectangle". Drawing stops after a fixed time, e.g. 25 seconds. Import the module `time`. The time of the computer system is obtained by `time.time()`. A possible solution is shown in figure 4.6, but your solution will look quite differently.

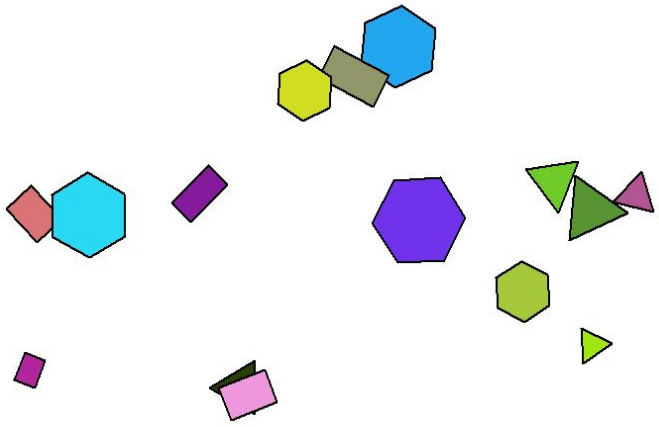


Figure 4.6: Possible solution of exercise 2

3. **Sierpiński curves**
   Color the bulges and passages, e.g. bulges could be red and passages green. Try `n=6`. You will obtain a nice pattern.

4. **Roots and squares**
   Write a program with mutually recursive functions "root" and "square" which computes a term consisting of roots and squares after having entered a number `n`. For example `n=5` yields the following:

$$\sqrt{5 + (4 + \sqrt{(3 + (2 + 1)^2)}\,^2} = 7.7918427$$

5. **Collatz Conjecture [72]**
   A sequence $\{a_n\}$ of positive integer numbers is defined as follows: If a_n is even, $a_{n+1} = n/2$. If a_n is odd, $a_{n+1} = 3 \cdot n + 1$.

The conjecture (L. Collatz, 1937) states that for every $a_0 \epsilon \mathbb{N}$ to start with there exists an index m such that $a_m = 1$. Example: $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

Write a program with two mutually recursive functions to confirm or refute the Collatz conjecture. Try $a_0 = 27$ or 97. Does your program always terminate? What would you conclude if not?

References

[1] Python 3.9 can be downloaded from https://www.python.org/downloads

[2] The concept of the programming language Logo is described in https://el.media.mit.edu/logo-foundation/what_is_logo/logo_primer.html

[3] https://docs.python.org/3/library/turtle.html

[4] https://plato.stanford.edu/entries/nicole-oresme/

[5] https://www.cantorsparadise.com/the-euler-mascheroni-constant-4bd34203aa01

[6] https://en.wikipedia.org/wiki/Euclidean_algorithm

[7] https://socratic.org/questions/what-does-the-alternating-harmonic-series-conv

[8] https://slideplayer.org/slide/2837657/

[9] https://realpython.com/fibonacci-sequence-python/

[10] https://en.wikipedia.org/wiki/Hero_of_Alexandria

[11] https://web2.0calc.com/questions/using-heron-s-method-what-is-the-square-roo

[12] https://en.wikipedia.org/wiki/Catalan_number

[13] https://www.whitman.edu/mathematics/cgt_online/book/section03.05.html

[14] https://en.wikipedia.org/wiki/Noncrossing_partition

[15] https://en.wikipedia.org/wiki/Binomial_coefficient

[16] https://www.geo.fu-berlin.de/en/v/soga/Basics-of-statistics/Discrete-Random-Variables/The-Binomial-Distribution/The-Binomial-Distribution/index.html

[17] https://brilliant.org/wiki/binomial-coefficient/

[18] https://math.libretexts.org/Bookshelves/Linear_Algebra/A_First_Course_in_Linear_Algebra_(Kuttler)/03%3A_Determinants/3.02%3A_Properties_of_Determinants

[19] https://en.wikipedia.org/wiki/Determinant

[20] https://en.wikipedia.org/wiki/Parallelepiped

[21] https://www.purplemath.com/modules/cramers.htm

[22] https://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations

[23] https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem

[24] https://en.wikipedia.org/wiki/Cubic_equation

[25] https://en.wikipedia.org/wiki/Quartic_function

[26] https://www.tandfonline.com/doi/full/10.1080/23311835.2016.1277463

[27] https://en.wikipedia.org/wiki/Horner%27s_method

[28] https://en.wikipedia.org/wiki/Pascal%27s_triangle

[29] https://en.wikipedia.org/wiki/Regula_falsi

[30] https://en.wikipedia.org/wiki/Gaussian_elimination

[31] https://mathimages.swarthmore.edu/index.php/Harter-Heighway_Dragon
The dragon curves developed by Heighway, Harter and Banks are very similar but not quite the same as the example in sec. 3.1.

[32] http://aleph0.clarku.edu/~djoyce/ma114/comb.pdf

[33] https://en.wikipedia.org/wiki/Ackermann_function

[34] https://en.wikipedia.org/wiki/Primitive_recursive_function

[35] https://training.incf.org/sites/default/files/2019-02/03_lester_.pdf

[36] https://www.tcs.ifi.lmu.de/lehre/ss-2019/fsk/material/folien/09

[37] https://www.tutorialspoint.com/data_structures_algorithms/merge_sort_algorithm.htm

[38] https://www.vanishingincmagic.com/puzzles-and-brainteasers/articles/history-of-tower-of-hanoi-problem/

[39] https://www.scientificamerican.com/article/the-tower-of-hanoi/#:~:text=The%20tower%20of%20Hanoi%20(also,mental%20discipline%20of%20young%20priests.

[40] https://www.geeksforgeeks.org/iterative-tower-of-hanoi/

[41] https://www.baeldung.com/cs/towers-of-hanoi-complexity

[42] https://en.wikipedia.org/wiki/Knapsack_problem

[43] https://www.educative.io/blog/0-1-knapsack-problem-dynamic-solution

[44] https://mathworld.wolfram.com/NP-Problem.html

[45] https://deepai.org/machine-learning-glossary-and-terms/nondeterministic-polynomial-time

[46] https://en.wikipedia.org/wiki/Backtracking

[47] https://www.programiz.com/dsa/backtracking-algorithm

[48] http://www.phpsimplex.com/en/simplex_method_example.htm

[49] https://users.mai.liu.se/torla64/MAI0130/Beatrice%202pre.pdf

[50] http://public.tepper.cmu.edu/jnh/karmarkar.pdf

[51] from OPTI2.dvi(uni-siegen.de), unfortunately no longer available

[52] https://en.wikipedia.org/wiki/Z-order_curve The curve of hooks is similar to the Lebesgue curve or Z curve.

[53] https://larryriddle.agnesscott.org/ifs/kcurve/kcurve.htm

[54] http://www.modernscientificpress.com/Journals/ViewArticle.aspx?XBq7Uu+HD/8eRjFUGMqlRSziRNLeU2RixB4ndiJghEnTfTElvDRYUuwGofG1d+5T

[55] https://en.wikipedia.org/wiki/Hofstadter_sequence

[56] https://cs.uwaterloo.ca/journals/JIS/VOL9/Emerson/emerson6.pdf

[57] https://en.wikipedia.org/wiki/Heap%27s_algorithm

[58] https://www.geeksforgeeks.org/heaps-algorithm-for-generating-permutations/

[59] https://www.uni-muenster.de/AMM/num/Vorlesungen/maurer/Uebungen09.pdf

[60] https://en.wikipedia.org/wiki/Piet_Mondrian

[61] https://blog.artsper.com/en/a-closer-look/10-things-to-know-about-piet-mondrian/

[62] https://www.holland.com/global/tourism/search.htm?keyword=Mondrian

[63] https://en.wikipedia.org/wiki/Piet_Mondrian#/media/File:Tableau_I,_by_Piet_Mondriaan.jpg

[64] https://en.wikipedia.org/wiki/Piet_Mondrian#/media/File:Piet_Mondriaan,_1939-1942_-_Composition_10.jpg

[65] https://en.wikipedia.org/wiki/Cocktail_shaker_sort

[66] https://en.wikipedia.org/wiki/Bubble_sort

[67] https://medical-dictionary.thefreedictionary.com/Fractal+curve

[68] https://en.wikipedia.org/wiki/Wac%C5%82aw_Sierpi%C5%84ski

[69] https://en.wikipedia.org/wiki/Sierpi%C5%84ski_curve

[70] https://en.wikipedia.org/wiki/Space-filling_curve

[71] https://www5.in.tum.de/lehre/seminare/oktal/SS03/ausarbeitungen/oberhofer.pdf

[72] https://en.wikipedia.org/wiki/Collatz_conjecture