Washboards in Unpaved Highways as a Complex Dynamic System

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*We present a new model and results for simulation of washboards, or corrugations, in unpaved highways. Our review of published literature shows that the washboard phenomenon on unpaved highways is the result of dynamic interaction between vehicle wheels and road surface, affected by a number of nonlinear variables in the physical system. Rather than attempt to solve the system analytically, we define a set of simple, locally defined rules to describe (1) the wheel jump upon striking an irregularity on the roadway surface and (2) the resulting digging. We use a computer simulation to iterate a mapping algorithm to simulate the effect of multiple vehicles. Finally, we analyze the resulting simulated road surfaces for evidence of complexity using information entropy and chaotic analysis. This approach is able to explain several outstanding questions in the literature, including the irregularity of washboard geometry, the direction of washboard migration, and the determination of washboard pitch, or wavelength. The study also resulted in several observations that are commonly associated with complex dynamic systems, including pattern emergence, sensitive dependence on initial conditions, and for some simulations, evidence of spatial chaos. Our conclusion is that washboards in unpaved highways may be modeled as the manifestation of a complex dynamical system. © 2000 John Wiley & Sons, Inc.**

Key Words: unpaved highways; washboards; corrugations; information entropy; chaos

Since the invention of pneumatic tires for motor vehicles, highway engineers have faced the problem of washboards, or corrugations, developing on unpaved roads. This phenomenon is illustrated by the photograph in Figure 1. The result of repeated digging events, washboards stretch across the road width, perpendicular to traffic, with a wavelength, or pitch, of 300-900 mm and peak-to-trough amplitude up to 50 mm [1]. Because the vehicle's suspension acts as a low-pass filter, a motorist can reduce vibrations by driving faster than the velocity at which the washboards match the suspension's resonant frequency [2], but this has the undesirable effect of reducing the traction avail-

FIGURE 1

Photograph of washboards on an unpaved highway. Photo by first author.

able for braking or steering [3]. Further problems include discomfort to passengers, damage to cargo, and the expense of periodic road regrading.

An early experimental study was conducted by Mather [4,5] at the University of Melbourne, which demonstrated that washboards form with both driven and idling wheels. Riley and Furry [6], of Cornell University, considered a lumped mechanical model of the wheel-surface system that was the basis for a computer simulation in FORTRAN. This work also included a dimensional analysis on 19 input variables and outlined the parameter space in which washboards can form. Heath and Robinson [1] provided an excellent review of work up through the 1970s. In the next decade, two articles came from the University of Nairobi: Stoddart et al. [7] investigated the effect of surface materials, and Misoi and coworkers [2,8] produced a thorough study on vehicle dynamics and road reactions. With a detailed lumped mechanical model, full-scale vehicle testing, and soil particle tracking with time-lapse photography, this 1989 work is the state of the art in understanding washboard dynamics.

Although a mechanical understanding of the washboard phenomenon is well developed, there are several motivations for our research. First, several authors have noted that washboard patterns are not simple sinusoids, but contain irregularities [2,4]. Second, we believe the conventional studies [1,7] fail to account for the spatial occurrence of real washboards, which do not appear uniformly along unpaved highways. Rather, washboards come in groups separated by regions of relatively smooth surface, as shown by the photograph in Figure 1. Third, because granular materials have properties of both discrete solids and continua, they are difficult to model explicitly [2,4,6,7]. Rather than use an empirical load-deflection relationship to account for this effect, we intentionally limit our focus to the processes that we believe are the essential mechanisms of washboard formation. Because there are too many variables for an analytical approach, we construct a synthetic reality that will simulate the essential dynamics.

Accordingly, the goals of this study are to (1) develop a simple, iterated algorithm in discrete, two-dimensional space to simulate washboard patterns in unpaved highways, (2) show qualitative compatibility with published descriptions and field observations of washboard patterns, and (3) investigate the simulated patterns for evidence of complexity. Although this study does not attempt to find a geotechnical solution, it offers a new understanding of the dynamics of washboard formation.

MECHANISTIC MODEL APPROACH

In describing the dynamics of vehicles on unpaved highways, most reports consider vehicle effects and surface effects separately. It is important to note that these effects are generally nonlinear, which means that chaotic behavior may be possible. In terms of vehicle effects, because the rubber material in tires exhibits internal friction, hysteresis, and velocity-dependent damping, the tire stiffness is nonlinear [9]. In terms of surface effects, the reaction force in the soil can be considered as the sum of static forces, viscous forces, and inertial forces. Soil is a visco-elastic medium, so horizontal and vertical displacement of the soil particles, which comprise the road surface, result in motion of the contact patch on the ground [9]. This creates a horizontal shear of the soil, and horizontal shearing of the soil creates vertical sinkage, which is also described by a nonlinear model,

$$w = akz^n \tag{1}$$

where w is total static and dynamic weight pressing on the soil, a is the wheel contact area, k is a coefficient, z is the depth of sinkage, and n is an exponent [9].

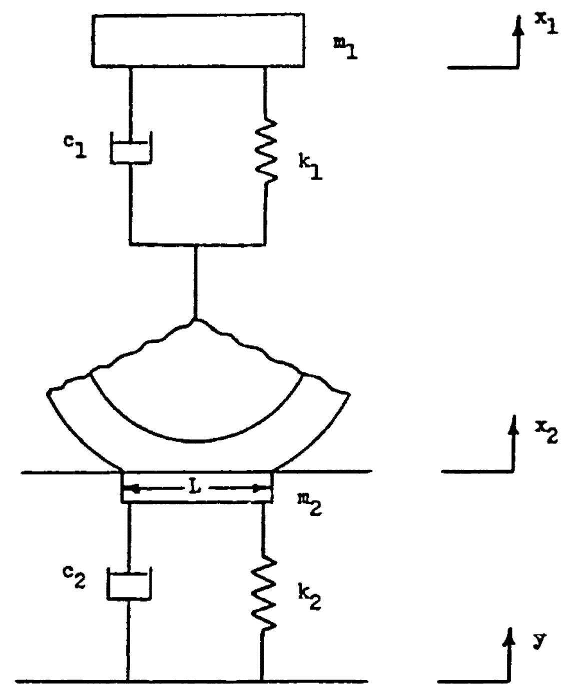
Combining the models for the vehicle movement and the road surface, Riley [9] developed a composite model represented by the lumped spring-mass-damper system in Figure 2. This system, having two degrees of freedom, is described by a system of equations:

$$m_1(d^2x_1/dt^2) = k_1(x_2 - x_1) + c_1(dx_2/dt - dx_1/dt) \tag{2}$$

$$m_2(d^2x_2/dt^2) = k_1(x_1 - x_2) + c_1(dx_1/dt - dx_2/dt) + k_2(y - x_2) + c_2(dy/dt - dx_2/dt) \tag{3}$$

where m_1 is the mass supported by one wheel, m_2 is the effective mass of surface material, k_1 is the stiffness of the suspension, k_2 is the effective stiffness of the surface material, c_1 is the tire damping coefficient, c_2 is the surface material damping coefficient, x_1 is the wheel displacement, x_2 is the surface displacement, and y is the roadway surface. Under certain conditions, a system with two degrees of freedom may exhibit quasi-periodic or chaotic fluctuations [10]. Our discrete, iterated model will attempt to capture this behavior.

FIGURE 2

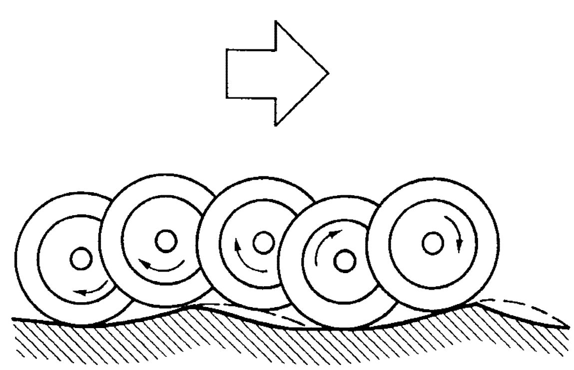


Lumped mechanical model of wheel-surface system [9].

DISCRETE MODEL APPROACH

Our conceptual model for the washboard formation process is shown in Figure 3. A wheel rolls horizontally until it encounters a bump and then jumps for some distance. Even if the wheel does not lose contact with the surface, the bump will cause a reduction in the normal force on the roadway, which could initiate a digging event by wheel slipping. The soil removed by digging piles up behind the landing spot,

FIGURE 3



Conceptual model of washboard formation [1].

in a process known as *kick-back* [8]. Subsequent wheels will then respond to a slightly different road surface.

This conceptual model agrees with the observation that both driven and idling wheels generate washboards [5]. It also agrees with the theoretical and experimental results of Misoi et al., which assert that wheels exert a maximum normal force just before the washboard peak and a minimum normal force in the trough [8]. Accordingly, in the subsequent analysis, the jump length may be considered as the distance from a peak to a trough.

In our simulation, we model the road soil as a set of discrete blocks, as shown in Figure 4. We denote the road elevation as $f(x)$, a discrete function whose value at each x is the surface elevation. We assume that the initial roadway is flat, with $f(x) = 0$, except for a seed point where $f(x_S) = 1$, which triggers the first digging event. In the physical system, this initial disturbance could be the transition from a paved to an unpaved surface, an erosional rivulet, a railroad crossing—anything that would cause the wheel to jump. Subsequent iterations, simulating the jump-dig effect of subsequent wheels, react to the roadway surface $f(x)$ established by the previous iterations. The roadway is modeled as a circular loop of length r, so the last index x_r immediately precedes the first index x_1. This allows immediate feedback from the ith iteration to the $(i + 1)$th iteration.

To simulate the formation of washboards, our basic approach is to identify the local rules that describe the wheel-surface interaction, then use a computer to simulate the cumulative effects of numerous jump-dig events. We define bump height h, jump length L, and a digging function $g(x)$ to describe the surface geometry, wheel-suspension response, and surface reaction, respectively. In general, the jump length will depend on many factors, such as bump height h, bump shape s, tire pressure p, suspension stiffness k, horizontal and vertical velocity (v_x, v_z), and vehicle mass m. Symbolically,

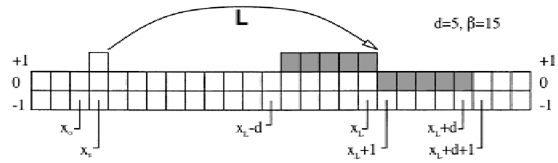
$$L = f(h, s, p, k, v_z, v_x, m, \ldots) \tag{4}$$

Similarly, the digging function $g(x)$ will also generally depend on numerous factors, such as tire pressure, vertical and horizontal velocity, vehicle mass, soil damping coefficient c, and soil water content θ. Therefore, we can write

$$g(x) = f(p, k, v_z, v_x, m, c, \theta, \ldots) \tag{5}$$

Previous authors have addressed this complexity by building physical models [2,4,6,7] or by defining dimensionless groups [9]. Our alternative is to substitute simple expressions for L and $g(x)$.

FIGURE 4



Discrete model of washboard formation.

Let us begin with the determination of L. Considering first the vehicle factors, we know that washboard pitch is proportional to horizontal velocity [4,6,7] and that heavier vehicles produce shorter jumps [7,11]. Stiffer suspensions will respond to externally applied forces more rapidly, resulting in greater vibration frequency [4], which corresponds to shorter wavelength. The jump length L, in general, will depend on these variables as stated in Eq. (4). Rather than attempt to analytically or empirically derive this relationship, we define the dimensionless number:

$$\beta = \alpha v_x / km \tag{6}$$

where v_x is horizontal velocity with units [L/T], k is suspension stiffness with units [F/L] or [M/T^2], m is mass supported by the wheel [M], and α is an unknown coefficient with units [M^2/LT]. In terms of surface factors, we have the experimental observation that "the greater the amplitude of a heap, the longer the following pitch" [4]. We postulate this as a linear scaling to h, which can be derived in the following manner. Because the vertical wheel movement is decoupled from the horizontal movement [2], we can consider the vertical dynamics independently. For a wheel that leaves a bump at $t = 0$ with some initial vertical velocity $v_{z,o}$, the dynamic equation is

$$dz/dt = v_{z,0} - gt \tag{7}$$

where z is directed upward, t is time, and g is the acceleration of gravity [6]. Solving for the trajectory with $z = 0$ at $t = 0$, we have

$$z = t(v_{z,0} - gt/2) \tag{8}$$

which has zeros at $t = \{0, 2v_{z,0}/g\}$, so the wheel returns to the surface at $2v_{z,0}/g$. Assuming a constant horizontal velocity v_x, the jump length L is proportional to $v_{z,0}$. If we make the further assumption that $v_{z,0}$ is directly proportional to h, then L is proportional to h. Therefore, combining the surface and vehicle factors, our model of the jump length is

$$L = \beta h \tag{9}$$

In simulations presented in this article, we use a fixed value of β. Physically, this corresponds to the observation that on a given section of road, vehicle speed is nearly constant [4] and that the suspension stiffness k and mass m of a given vehicle tend to scale together, so their ratio is nearly constant [4]. Because we have defined a discrete, two-dimensional roadway model, β takes only integer values. The bump height h takes on a variety of values, responding to the cumulative effect of multiple wheel-surface interactions over time.

Having defined L, we now turn to the digging function $g(x)$. Washboard patterns, as a whole, can migrate in a direction opposite to the traffic flow [1, 7], and tracer particles migrate *only* in this direction [8]. This means we know the direction of soil displacement. As stated in Eq. (5), the shape of this displacement depends on numerous factors. In this simulation, we are assuming a simple, discrete mapping as follows:

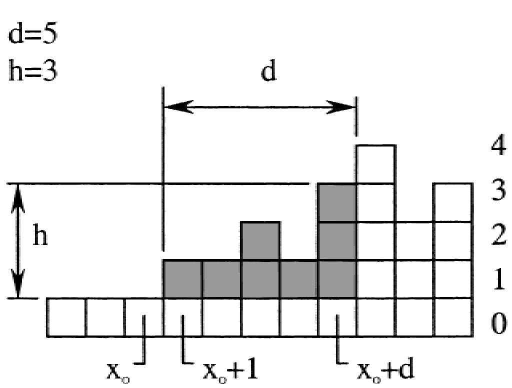
$$g(x_L) = \begin{matrix} f(x) + 1 & (x_L - d) < x \le x_L \\ f(x) - 1 & x_L < x \le (x_L + d) \end{matrix} \qquad (10)$$

where $x_L = x_0 + L$ is the point between the digging and piling zones and d is the characteristic digging width. For illustration, see Figure 4. If we define $x_0 = 10$ units, then $x_L = 25$ units for $\beta = 15$. Like β, d is held constant throughout each simulation.

Now that we have functions for L and $g(x)$, and we have assumed that β and d are constant in each simulation, the only outstanding variable is the bump height h. The bump height h is determined as shown in Figure 5. Stated mathematically,

$$h = \{\max\,[x_0 + 1,\, x_0 + d]\, f(x)\} - f(x_0) \qquad (11)$$

FIGURE 5



Determination of discrete bump height.

where x_0 is the coordinate of the last point before the current bump. In words, the wheel jumps a distance proportional to the bump height, measured as the maximum increase within the characteristic digging width. Once h is known, the jump of length L begins from x_0 as in Figure 4. When the wheel completes the digging cycle, the default road elevation is set to $f(x_L + d + 1)$, which is then compared to subsequent elevations $f(x)$ to find the next bump.

The output of this simulation is an $(n + 1)$ by r matrix $F(i,x)$

$$F = \begin{vmatrix} f_{0,1} & f_{0,2} & f_{0,3} & \cdots & f_{0,r-1} & f_{0,r} \\ f_{1,1} & f_{1,2} & f_{1,3} & \cdots & f_{1,r-1} & f_{1,r} \\ \cdots & & & & & \cdots \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & \cdots & f_{n-1,r-1} & f_{n-1,r} \\ f_{n,1} & f_{n,2} & f_{n,3} & \cdots & f_{n,r-1} & f_{n,r} \end{vmatrix} \qquad (12)$$

where n is the number of vehicle passes, and r is the length of the circular roadway. Thus, the first row of $F(i,x)$ is the initial spatial pattern, and the ith row represents the spatial pattern after the $(i - 1)$th vehicle pass. Any given column of the matrix $F(i,x)$ represents the discrete time history of the elevation at a given point in the road.

This model does not account for several observations in the published literature. First, although low-pressure tires minimize washboard formation [5,6,11], in our model, the digging process $g(x)$ is the same each time, so it does not account for tire pressure. Second, we do not account for the observation that washboard effects are only observed for vehicles traveling above a certain threshold velocity [4]. This is a particular case of the more general observation that washboards form only within upper and lower bounds for system variables such as vehicle speed, tire pressure and vehicle mass, which is neglected in this model [6]. Although the factor β accounts for velocity, stiffness, and mass, it does so indirectly, not explicitly. Finally, there are many variables that have been omitted altogether. In fact, this is the intent of our research. Rather than attempt to account for all the physics involved, we are attempting to build a model from the essential jump-dig mechanism.

MATLAB IMPLEMENTATION

We used MATLAB to implement our model. The basic approach is to roll down the simulated highway, one cell at a time, until the wheel completes one loop, where $x = r$. At this point, the vehicle counter is incremented, and the process repeats until the counter reaches n wheel passes. In each pass, the wheel starts at some known elevation. If the next cell is at an equal or lower elevation, there is no jump, and the wheel rolls forward one unit. If the next cell, indicated on Figure 5 as $x_0 + 1$, has an elevation greater than the present cell, then the wheel will jump. First, we calculate the bump height with Eq. (11). Then, using the notation shown on Figure 4, we add one unit of elevation to the segment $[x_L - d + 1, x_L]$, subtract one unit of elevation from the segment $[x_L + 1, x_L + d]$, corresponding to a single application of the digging function $g(x)$ in discrete space. We then set the current cell to $x_L + d + 1$. Because the road is modeled as a closed loop, it is possible for a wheel to begin a jump at $x < r$, which ends at $x + L - r$, somewhere among the first few roadway cells. As such, the simulation is nothing more than a series of if-then statements, whose output is stored in the matrix $F(i,x)$. The essential mechanism simulated, which leads to pattern formation, is a combination of the local rules and the effect of iteration over time.

SIMULATION RESULTS

Washboard Patterns

To provide a qualitative feeling for the simulation results, Figure 6 shows four patterns generated for various values of β. In each, we have roadway length $r = 20{,}000$; seed point $x_S = 1{,}000$; digging width $d = 100$; and the image is taken at $n = 250$. For clarity, results are shown only for $1{,}000 < x < 5{,}000$ and are plotted at a 1:1 scale. Because a wheel cannot jump instantly upon striking a bump, which would correspond to a jump length of zero, $\beta = 0$ is not allowed.

We see that the patterns, although displaying wave-like structure, are generally irregular. In fact, the spatial pattern varies along the roadway and from iteration to iteration. We believe this explains the observation by Mather [4] that the patterns are not periodic. Although previous studies used the function $\sin(x)$ as a starting point [2,3], we begin with a discrete map of the jump-dig process, which leads to spatial patterns when iterated over space and time. In this way, the washboard structure is an emergent property of the iterative map, one of the hallmarks of self-organized complex systems [12].

In particular, note how Figure 6(d) shows a central zone where the waves are smaller than at the ends. This is qualitatively similar to the photograph in Figure 1, which shows how washboard patterns appear and then disappear, along the travel direction. It is notable that, although the same iterative process is at work throughout the simulated roadway, the model has the ability to qualitatively replicate this observation of spatial intermittency.

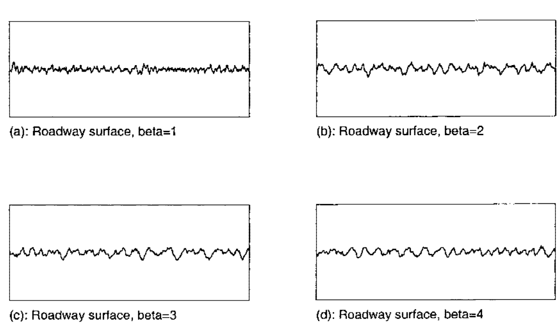
Effect of the Scaling Parameter β

Because of the discrete nature of our washboard model, the simulation eventually enters a regime in which subsequent digging and piling events occur at the same locations. When this occurs, the location of peaks and troughs on the roadway surface will remain constant, but the amplitude of each increases by one unit per iteration. We call this effect *lock-in,* as illustrated in Figure 7.

In Figure 7(a), because $d = 5$ and $\beta = 10$, the jump length will exceed the digging width. This means that $g(x)$ occurs far enough downstream that the initial bump perceived by the next wheel, indicated by an asterisk (*), will be the same. This means the next wheel will jump the same distance as its predecessor, causing the digging and piling to occur in exactly the same locations. Then the pattern will repeat indefinitely, which is the lock-in effect. Calculations show that when $\beta \geq d/2$, it will take only a few iterations before the initial bump becomes large enough to produce long jumps, which leave the initial bump unchanged. As such, there is no feedback from one iteration to the next, and lock-in occurs.

In Figure 7(b), with $d = 5$ and $\beta = 2$, the piling effect of the first wheel, shown in gray fill, will change the bump encountered by the subsequent wheel. This means the next jump will not be identical to the previous, as shown by the dotted hatching, and the spatial pattern will evolve over time. Physically, this represents feedback in the system, which is essential for complex dynamics. In other words,

FIGURE 6



Simulated washboard patterns for various values of β.

FIGURE 7

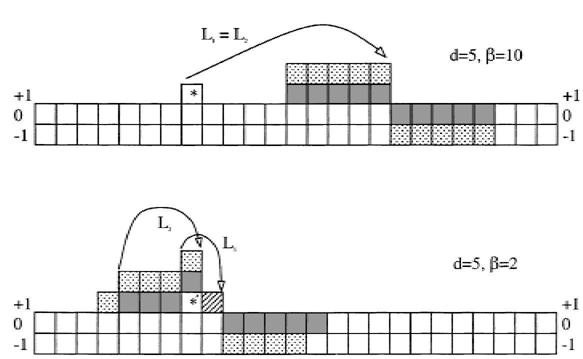


Illustration of numerical dig-in effect.

information about the ith wheel is transmitted, through the roadway, to the $(i+1)$th wheel. This behavior is possible in the range $1 \leq \beta < d/2$.

These observations about the effect of β on lock-in bring up an important limitation. Although lock-in will occur on the second or third iteration for $\beta \geq d/2$, our numerical experiments have shown that it also occurs, after enough iterations, for any value β. This means that the limiting pattern for all simulations is a roadway surface in which arbitrarily high washboard peaks sit adjacent to arbitrarily deep troughs, which is obviously not physical. Therefore, the model is physically plausible only up to a certain number of simulated wheel passes. Because discretization will play a role, at some scale, in any digital model, we have decided to include it as an explicit component of our conceptual model. However, we speculate that lock-in is a side effect of discretization. The details of this process are being left as an open question.

To explore the relationship between β and the emergent pattern, we investigated two macroscopic qualities for various simulations. First, did the washboard pattern fill the road before lock-in? This allows us to separate those patterns that reach lock-in quickly from those that reach lock-in gradually. Second, what was the direction of wave propagation on the simulated surface? We looked into these questions for each of the integer values of β under four distinct discretization scales, corresponding to model road lengths of r = {500; 2,500; 5,000; 10,000}. For each discretization scale, we set the digging width $d = r/50$ and seed point $x_S = r/5$. This exploratory process revealed sensitive dependence on β, summarized in Table 1. As the discretization scale becomes finer, it becomes possible to identify the impact of β on the detailed structure. We see ranges of macroscopic behavior affected by small changes in the input parameter, which is symptomatic of chaotic behavior [12]. After using this process to identify the general region of β, which produces space-filling patterns, subsequent simulations were run with $1 \leq \beta \leq d/4$.

TABLE 1.

Effect of scaling parameter on extent and direction of pattern propagation

Range	Fills Road	Wave Direction
$0 < \beta/d < 0.25$	Yes	Upstream, downstream
$0.25 \leq \beta/d < 0.29$	No	
$0.29 \leq \beta/d \leq 0.30$	Yes	Downstream
$0.30 < \beta/d < 0.40$	No	
$0.40 \leq \beta/d < 0.50$	Yes	Downstream
$\beta/d \geq 0.50$	No	

INFORMATION ENTROPY TO MEASURE WASHBOARD COMPLEXITY

We use two technical metrics to analyze the patterns. The first, discussed in this section, is *information entropy*. The second, discussed in the subsequent section, is *chaotic series* analysis.

Shannon's well-known work on information theory provides a basis for assessing the complexity of patterns using information entropy H as the degree of unpredictability. For a constant signal, H is minimum; for a uniformly distributed random signal, H is maximum. Entropy is calculated by breaking the series into a collection of bins and then calculating the probability distribution for the bins, $P(x)$, constrained by $\Sigma_i P_i = 1$, so P_i is the probability that a measurement falls into bin i. The metric is then calculated with Shannon's formula:

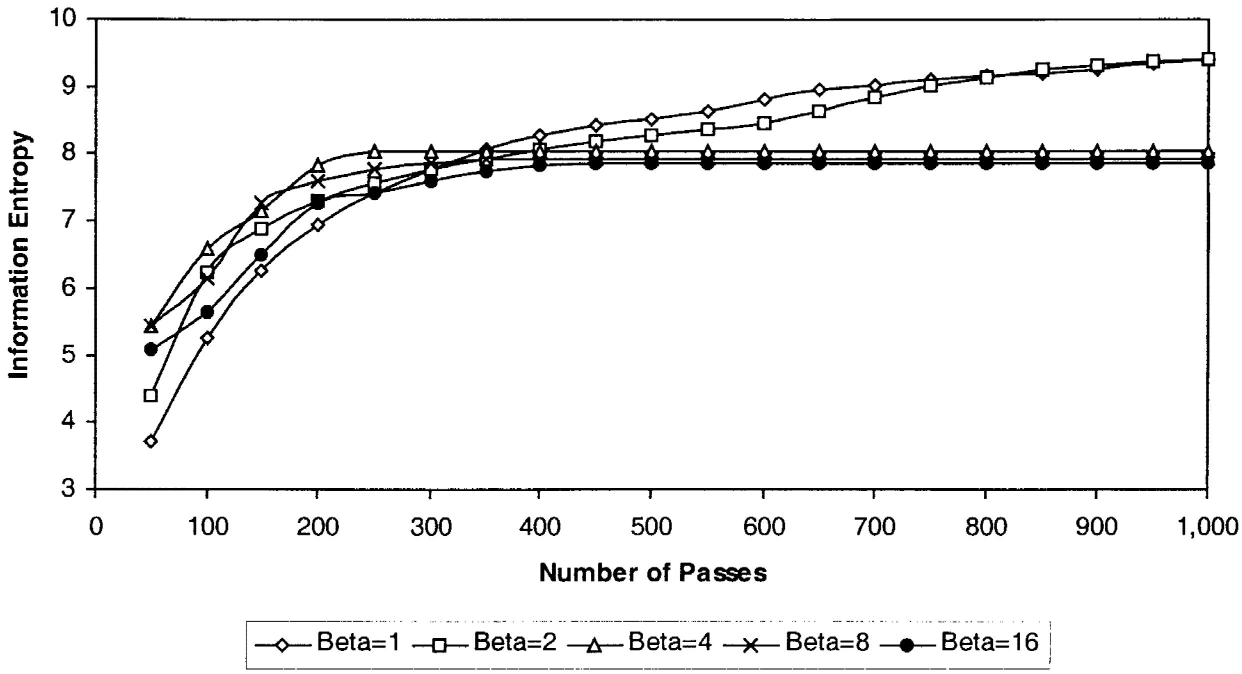
$$H = -\Sigma_i P_i \log_2 P_i \qquad (13)$$

Information entropy depends on the bin spacing. In general, smaller bins will allow greater signal resolution, which means a more complex structure can be evaluated. For continuous signals, increasingly smaller bins generate correspondingly larger calculated entropy. This is not the case for discrete systems. Because the function is defined only at discrete points, we eventually reach the point at which further bin divisions result in empty bins, where $P_i = 0$ for some values of i. According to Eq. (13), these empty bins will have no effect on the entropy. This means, for a discrete series, there will be some limiting value of H, above which further bin divisions will have no effect.

Because the washboard simulation is conducted using a discrete model, we calculated the limiting information entropy of the simulated surface $f(x)$ as a function of wheel passes for various values of β, as shown in Figure 8. In this figure, r = 5,000; x_S = 1,000; d = 100. Our first observation is that the entropy increases in the first 250 wheel passes and then reaches a stable value. Physically, this means that an initially smooth roadway will undergo a period of washboard formation leading to a certain level of complexity. After that time, the physical pattern may continue to evolve over time, but the level of complexity will grow slowly, if at all. For β = {1, 2}, the limiting entropy increases rapidly for $n < 250$ and then increases slowly as a function of subsequent vehicle passes. For β = {4, 8, 16}, the entropy measurement reaches a fixed value, indicating that the pattern has reached lock-in.

A second observation in Figure 8 is that the limiting information entropy, as a function of wheel passes, is monotonically increasing. After achieving a certain amount of complexity, the pattern will continue to display at least as much complexity into the future. Physically, this corresponds to the observation that, unless affected by external factors such as rainfall, washboards do not spontaneously

FIGURE 8



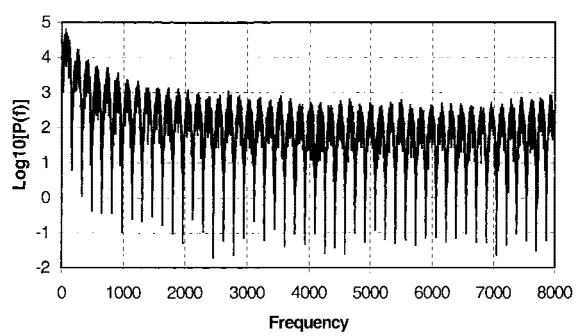
Time evolution of spatial complexity for various values of β.

level themselves. Thus, for certain ranges of the scaling parameter β, a smooth surface is unstable, which leads irreversibly to a corrugated surface.

CHAOTIC ANALYSIS OF SIMULATED PATTERNS

Because the iterated map of washboard formation appeared to display some of the qualities of a complex system, we decided to analyze the simulated spatial distribution of the roadway surface for evidence of deterministic chaos. This analysis was conducted with the software application CSPW [13]. We should note, at the outset, that different simulated roadways generated different dynamical systems characteristics. Like the sensitivity to β, these system metrics depend on the model input parameters. For illustration, we present the results of this analysis applied to the surface shown in Figure 6(b), with road length $r = 20{,}000$ units, seed $x_S = 1{,}000$ units, digging width $d = 100$, and scaling parameter $\beta = 2$.

FIGURE 9

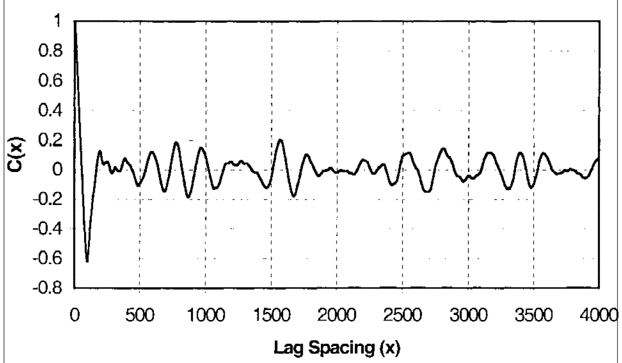


Power spectrum of simulated surface, r = 20,000; x_S = 1,000; β = 2; d = 100; n = 250. The same parameters are used in Figures 10–13.

Like other aperiodic series, the spectral representation does not reveal any clear wave structure, as shown in Figure 9. The autocorrelation function, shown in Figure 10, displays a minimum for separation distance 100 units, which results from the fixed digging width. Thereafter, there is evidence of quasi-periodic behavior, but the autocorrelation itself does not repeat a set pattern for increasing lag spacing. Average mutual information, shown in Figure 11, shows a peak for even values of separation distance. Because of the discrete nature of the model, the possible jump lengths are the set of positive, even integers. This explains the structure of Figure 11. These techniques offer a glimpse of the underlying dynamics.

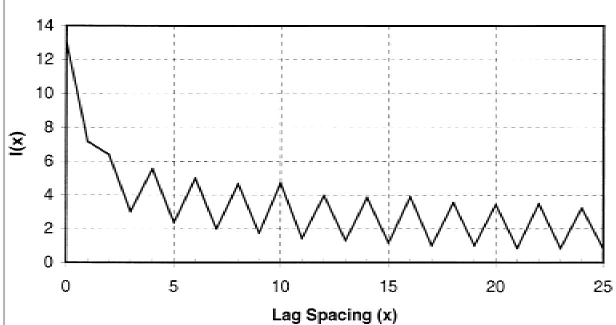
Using the false nearest neighbors (FNN) technique [13], we determined that the global dimension D_E of the phase space required to unfold the attractor is $D_E = 6$, because the curve of

FIGURE 10



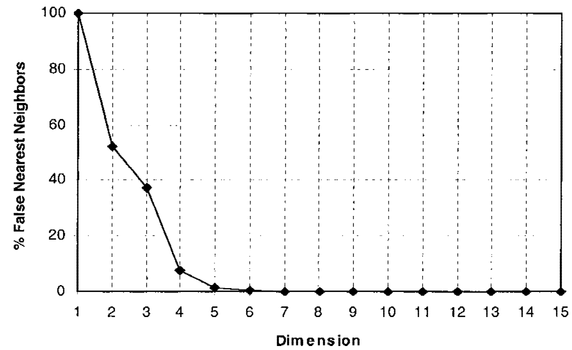
Autocorrelation of simulated surface.

FIGURE 11



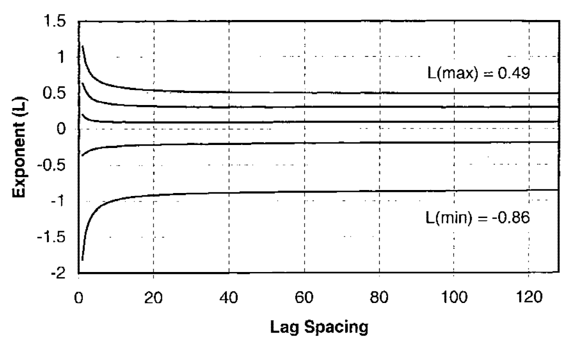
Average mutual information.

FIGURE 12



False nearest neighbors.

FIGURE 13



Local Lyapunov exponents.

the percentage of FNN on Figure 12 drops to zero at the value of the embedding dimension, $D_E = 6$. For this data set, the local dimension D_L of the attractor is $D_L = 5$. Therefore, we calculated five Lyapunov exponents to determine the sensitive dependence on initial conditions, or in other words, the rate at which nearby orbits diverge from each other after small perturbations. Figure 13 demonstrates that the largest Lyapunov exponent is 0.49, the minimum exponent is -0.86, and the sum of the five exponents is negative at -0.17. The combination of these values indicates that the washboard pattern exhibits a nonperiodic, deterministic chaotic evolution that is between regular and stochastic, noisy behavior.

DISCUSSION AND CONCLUSION

This approach offers insight into several outstanding questions in the published literature. First, Mather [4] notes that washboards are not simple sinusoids, but display "appreciable irregularity." We suggest this irregularity results from the iterative work of multiple vehicle passes, each of which is slightly different from the previous. Second, Heath and Robinson [1] note that washboards in real highways tend to migrate upstream, unlike the downstream migration observed in Mather's [4,5] testing apparatus. As shown in Table 1, our simulations account for both pattern migration schemes, depending on the tuning parameter β. Third, Misoi and Carson [2] determined that, unless a vehicle had inoperative suspension dampers, its wheels would not resonate at a frequency corresponding to the washboard wavelength. They argue that because washboard pitch is not determined, in fact, by the suspension's resonant frequency, there is no clear mechanism to determine wavelengths. We suggest that the wavelengths are the emergent property of a dynamical system, determined not by the dynamics of a single jump-dig effect but by the effect of multiple iterations. This is why we observe washboards in our simulation, although our model contains no resonance terms.

Based on information entropy, we learned that the simulated washboards reach a fixed level of complexity, much as washboards on real roadways achieve stable patterns. Analysis of the spatial data structure indicates that, for certain values of the input parameters, the patterns result from a low-dimensional, deterministic system with little stochastic effect. Furthermore, the Lyapunov exponents indicate a chaotic dynamical system. We believe this is evidence that washboards in unpaved highways are the manifestation of a complex dynamical system.

This study offers a new perspective on simulation of washboards in unpaved highways. Rather than account for the detailed physics explicitly, we broke the washboard formation process into two components, jump length L and digging function $g(x)$ and then derived simple expressions for both based on the bump height h. To simulate the effect of numerous wheel-surface interactions, we wrote a simulation program to iterate the $L - g(x)$ mapping and then analyzed various realizations of the model, based on different values of the parameter β and the number of vehicle passes. The essential component of the dynamics, which gives rise to pattern emergence, pattern migration, pattern growth and stability, and sensitive dependence on initial conditions, is not the local rules for jump length and digging response, but the effect of iteration over time. In that sense, iteration transforms a simple model into a simulation which captures many of the essential features of washboards in unpaved highways.

Future Work

An extension of this article would compare simulated patterns with detailed field measurements. Such a data set is being collected by the U.S. Army Corps of Engineers in their Hanover, NH, research laboratory, and it would be straightforward to rescale our parameters r and d to allow comparison with field measurements. For the field data, one could use the same metrics as the present study: information entropy and chaotic pattern analysis. Thus, although the precise shapes of the simulations and field data will differ, one may find that the metrics agree. A further extension would be to refine the model to allow a continuous distribution of block sizes and digging depths, which would account for the effect of discretization on the present simulation.

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