A matrix approach to multilayer optics

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I. SOME DEFINITIONS

Write the electric and magnetic field of an electromagnetic wave in a section of homogeneous (linear, isotropic) medium (with label j), propagating in the $+\hat{z}$ direction, polarized along $\hat{x}$ as

$$\begin{aligned} \tilde{\vec{E}}_j(z,t) &= \hat{x}\tilde{E}_j e^{i(\tilde{k}z-\omega_0 t)}, \\ \tilde{\vec{B}}_j(z,t) &= \hat{y}\frac{\tilde{k}}{\omega_0}\tilde{E}_j e^{i(\tilde{k}z-\omega_0 t)}, \end{aligned} \tag{1}$$

i.e. use complex notation, while it is understood that the actual fields are the real part of each equation:

$$\begin{aligned} \vec{E}_j(z,t) &= \Re(\tilde{\vec{E}}_j(z,t)) = \frac{1}{2}\left(\tilde{\vec{E}}_j(z,t)+cc\right), \\ \vec{B}_j(z,t) &= \Re(\tilde{\vec{B}}_j(z,t)) = \frac{1}{2}\left(\tilde{\vec{B}}_j(z,t)+cc\right), \end{aligned} \tag{2}$$

with cc denoting the complex conjugate of the preceding expression. This allows for compact notation. For instance the complex wave vector $\tilde{k}$ can be written in terms of the complex refractive index $\tilde{n}$,

$$\tilde{k} = \tilde{n}k_0, \tag{3}$$

with

$$\tilde{n} = n + ik, \tag{4}$$

with real part n (the usual refractive index) and imaginary part k (the latter indicating losses in propagation), consistent with the notation used at, e.g., the website refractiveindex.info. Here

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega_0}{c}, \tag{5}$$

is the vacuum wavevector (with λ_0 the wavelength in vacuum, and c the speed of light in vacuum). The vacuum wavevector k_0 is mainly used as a shorthand here for the expression with angular frequency ω_0 (which doesn't change as the wave passes through various media).

Sidenote: the complex dielectric function can be written as

$$\tilde{\epsilon} = \epsilon_0\tilde{n}^2 = \epsilon_0(n^2-k^2+2ink) = \epsilon_0(\epsilon_r + i\epsilon_i). \tag{6}$$

II. THE USUAL IN-OUT MATRIX

At an interface between two linear media (and assuming for simplicity that the magnetic constants of the two media are identical, $\mu_1 = \mu_2$) the usual EM boundary conditions yield amplitude reflection coeffient r, transmission coefficient t at the interface between media 1 and 2, that can be written in terms of

$$\beta = \frac{\tilde{k}_2}{\tilde{k}_1} = \frac{\tilde{n}_2}{\tilde{n}_1}, \tag{7}$$

as

$$t = \frac{2}{1+\beta} = \frac{2\tilde{n}_1}{\tilde{n}_1+\tilde{n}_2} \tag{8}$$

$$r = \frac{1-\beta}{1+\beta} = \frac{\tilde{n}_1-\tilde{n}_2}{\tilde{n}_1+\tilde{n}_2}. \tag{9}$$

The relation between the electric fields of the incoming and outcoming waves at the interface between media 1 and 2 can then be written in matrix form as

$$\begin{pmatrix}\tilde{E}_{2\rightarrow}\\ \tilde{E}_{1\leftarrow}\end{pmatrix} = \begin{pmatrix} t & -r \\ r & \beta t\end{pmatrix}\begin{pmatrix}\tilde{E}_{1\rightarrow}\\ \tilde{E}_{2\leftarrow}\end{pmatrix}. \tag{10}$$

Here $\tilde{E}_{1\rightarrow}$ denotes the amplitude at the side of medium 1 travelling in the $+\hat{z}$ direction, while $\tilde{E}_{1\leftarrow}$ is the amplitude at the side of medium 1 travelling in the $-\hat{z}$ direction (obtained by replacing $\tilde{k}$ by $-\tilde{k}$ in Eqs. (1)), etcetera. Also, to simplify the notation above we have written β instead of β_{12}, and similarly $r = r_{12}$, and $t = t_{12}$. When dealing with multiple interfaces, we will need to include the indices of the layers involved.

Note to self: including a drawing would make things clearer here.

III. INTERFACE MATRIX BETWEEN MEDIA

For the purpose of multilayer systems, it is more convenient to rewrite Eq. (10) into a matrix transforming from the fields in medium 2 to those in medium 1. This is just a transformation of the above linear equations, and yields

$$\begin{pmatrix}\tilde{E}_{1\rightarrow}\\ \tilde{E}_{1\leftarrow}\end{pmatrix} = \begin{pmatrix} \frac{1}{t} & \frac{r}{t} \\ \frac{r}{t} & \frac{r^2}{t}+\beta t\end{pmatrix}\begin{pmatrix}\tilde{E}_{2\rightarrow}\\ \tilde{E}_{2\leftarrow}\end{pmatrix}, \tag{11}$$

with $\tilde{E}_{1\rightarrow}$ (and similar) defined as before.

IV. MATRIX FOR HOMOGENEOUS MEDIUM

For a (linear, isotropic, homogeneous) medium with thickness d, and wavevector $\tilde{k}$, the matrix relating the left-hand side (at low z) to the right-hand side (at higher z, $z_h = z_l + d$), is simply

$$\begin{pmatrix}\tilde{E}_{l\rightarrow}\\ \tilde{E}_{l\leftarrow}\end{pmatrix} = \begin{pmatrix} e^{-i\tilde{k}d} & 0 \\ 0 & e^{+i\tilde{k}d}\end{pmatrix}\begin{pmatrix}\tilde{E}_{h\rightarrow}\\ \tilde{E}_{h\leftarrow}\end{pmatrix}, \tag{12}$$

V. MULTIPLE LAYERS

For a stack of multiple layers, the total reflection and transmission can now be readily calculated (most easily done by the computer) as follows. First multiply the corresponding matrices of the various layers and interfaces, to obtain the (2 by 2) matrix of the total stack. Next, right-multiply this matrix with the vector $(1, 0)$, to represent the situation that light is incident from the left (i.e.: no incident light from the right). This yields a vector with incident and reflected (complex) amplitude, scaled to unit transmitted amplitude. Finally, rescale to unit incident amplitude (and/or intensity) to obtain the reflection and transmission amplitudes (and/or intensities). For the intensities, don't forget to take into account the refractive index of the in- and outgoing media.

In formulas, for a stack of layers, labeled from 0 to m, we have

$$M_{tot} = M_{01}M_1M_{12}M_2 \cdots M_{m-1\,m}, \tag{13}$$

with matrices for the interfaces

$$M_{ij} = \begin{pmatrix} 1/t_{ij} & r_{ij}/t_{ij} \\ r_{ij}/t_{ij} & r_{ij}^2/t_{ij} + \beta_{ij}t_{ij}, \end{pmatrix} \tag{14}$$

and for the propagation within each layer

$$M_j = \begin{pmatrix} e^{-i\tilde{k}_j d_j} & 0 \\ 0 & e^{+i\tilde{k}_j d_j}. \end{pmatrix} \tag{15}$$

To obtain transmission and reflection with light incident from layer 0 (i.e., no incident light from layer m, $E_{m\leftarrow} = 0$, we write

$$\begin{pmatrix} \alpha_{0\rightarrow} \\ \alpha_{0\leftarrow} \end{pmatrix} = M_{tot} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{16}$$

The amplitude transmission coefficient is then simply $1/\alpha_{0\rightarrow}$, while the amplitude reflection coefficient is $\alpha_{0\leftarrow}/\alpha_{0\rightarrow}$.

For intensities of transmission and reflection, we need to account for the local velocities, i.e. for layers 0 and m (and for each of the right- and left-travelling waves separately):

$$I_j = \frac{1}{2}\epsilon_0 n_j c E_j^2. \tag{17}$$

Note also here that we need to take the initial layer (0) and final layer (m) to have no absorption, otherwise total transmission and reflection are ill-defined (since in the presence of absorption, total transmission and reflection would depend on where in layer 0 and m you would you measure them). Thus, with the above we obtain for the intensity transmission coefficient

$$T = \left|\frac{1}{\alpha_{0\rightarrow}}\right|^2 \frac{n_m}{n_0}, \tag{18}$$

intensity reflection coefficient

$$R = \left|\frac{\alpha_{0\leftarrow}}{\alpha_{0\rightarrow}}\right|^2, \tag{19}$$

and total absorbed intensity fraction

$$A = 1 - T - R. \tag{20}$$

VI. LOCALLY ABSORBED POWER

Now, we would also like to have expressions for the locally absorbed power density. Again taking isotropic, linear, homogeneous media, and, for simplicity, nonmagnetic materials, $\mu = \mu_0$, the electromagnetic energy density is

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu_0}B^2\right) \tag{21}$$

and the Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}. \tag{22}$$

The cycle-averaged locally absorbed power density is then

$$p = -\langle \vec{\nabla} \cdot \vec{S} \rangle. \tag{23}$$

Inserting a superposition of left- and right-travelling waves, writing the local amplitudes in Eqs. (1) as $\tilde{E}_{j\rightarrow} = \alpha_{\rightarrow}E_0$ and $\tilde{E}_{j\leftarrow} = \alpha_{\leftarrow}E_0$, (and after more algebra than I would have liked) leads to a relatively simple expression in terms of the (real-valued) shorthand

$$f = \alpha_{\rightarrow}\alpha_{\rightarrow}^* + \alpha_{\leftarrow}\alpha_{\leftarrow}^* + \alpha_{\rightarrow}\alpha_{\leftarrow}^* + \alpha_{\leftarrow}\alpha_{\rightarrow}^*. \tag{24}$$

Namely,

$$\begin{aligned} p &= \frac{E_0^2}{\mu_0\omega_0} nkk_0^2 f \\ &= \epsilon_0 nc E_0^2 k_0 k f. \end{aligned} \tag{25}$$

This result may seem somewhat obvious, in hindsight, and it would be good to have a simple(r) derivation.

The z-dependent $\alpha_{\rightarrow}$ and $\alpha_{\leftarrow}$ can be obtained using essentially the same procedure as in the previous section, namely by taking the product of the relevant matrices from the outgoing side (layer m) up to the position z, right-multiplying with $(1, 0)$ to get the local amplitudes, and scaling to the incident amplitude $\alpha_{0\rightarrow}$ obtained from Eq. (16).

In other words, first calculating

$$\begin{pmatrix} \alpha_{\rightarrow}(z) \\ \alpha_{\leftarrow}(z) \end{pmatrix} = M(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{26}$$

the relative absorbed power density (scaled to the incident intensity obtained from Eq. (16)) is

$$p_{rel}(z) = 2k_0 n_j k_j f(z) \frac{1}{n_0 |\alpha_{0\rightarrow}|^2}. \tag{27}$$

As a consistency check, I have checked for a few examples that the numerically integrated relative local absorption indeed equals the total relative absorption, i.e.

$$\int p_{rel}(z)\, dz = A. \tag{28}$$

So it should all be okay as written here (YMMV :).