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# On MPC based trajectory tracking

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*Abstract*— **This work proposes and investigates a tracking scheme for linear, constrained system based on a combination of model predictive control, virtual references and unknown input observers. In contrast to existing results the proposed approach allows to track a larger class of trajectories exactly: it does not require a reference model, does not need to assumes constant/periodic references or that the reference converges to a steady state. The scheme guarantees under mild conditions recursive feasibility independent of the reference and asymptotically exact tracking. It is computationally tractable, since only a convex quadratically constrained quadratic program or a convex quadratic program needs to be solved at each time step. We outline the applicability and the efficacy of the proposed approach using two examples.**

## I. Introduction

There exist many approaches to design stabilizing model predictive controller for the case of the regulation, i.e. stabilization of a fixed, given set-point, see e.g. [1]–[3]. However even slowly varying references can destroy the stability of the designed predictive controller.

By now a series of MPC approaches for tracking have been proposed and analyzed. In off-set free MPC, see [4]–[7] and the references therein, the mismatch between the current and desired output is considered as a disturbance generated by a known disturbance model. This disturbance can, under certain conditions, be estimated and rejected, which removes the offset between the actual and reference output. For example, if the disturbance is generated by an integrator model (as in [4], [6], [7]), then steady states can be tracked exactly. In comparison the MPC controllers outlined in [5] allow to track a larger class of references with vanishing tracking error, since the disturbance model can be an arbitrary, linear model. However, in off-set free MPC it is challenging to guarantee recursive feasibility, see [4], [5].

Tracking scheme based on so-called virtual steady states [8]–[10] allow to consider piecewise constant references. A further extended approach [11] allows the consideration of periodic references. These schemes maintain under any change of the reference recursive feasibility and guarantee convergence to the reference under certain conditions.

A tracking scheme utilizing the principle of virtual steady states, which is based on the solution of two optimization problems per time instance is proposed in [12]. This scheme guarantees recursive feasibility. However for time-varying references the tracking error might be nonzero, even so the system allows to track the reference exactly.

This work proposes a general tracking scheme for linear, constrained systems. The approach combines the concept of virtual references [8]–[11] with unknown input observer [13]–[15]. It consists of two parts. First from a given output reference corresponding state trajectory and input sequence are estimated. Second the estimated state/input evolution is tracked using a convex optimal control problem featuring two virtual reference trajectories. This allows to guarantee recursive feasibility and tracking of the reference under mild conditions. The work [16] considers the tracking of a priori known, asymptotically constant trajectories.

The main advantages of this approach compared to existing approaches are that (a) it can track a rather general class of trajectories since neither a reference models is employed nor the reference is restricted to be constant or periodic, (b) it is recursive feasible independent of the provided reference for suitably designed terminal regions, (c) it is computationally tractable, since only convex quadratically constrained quadratic programs needs to be solved online.

The remainder of the paper is structured as follow. The next section outlines the considered problem. Sections III and IV contain the main results. They outline and investigate the tracking scheme for the case that the reference is given directly as a state/input reference (Section III) or output reference (Section IV). Section V illustrates the approach.

The notation is standard. For a convex optimization problem $\min_x \mathcal{O}(x)$ s.t. $x \in \Omega$ we denote by $x^\star$ a minimizer. $M^\dagger$ is the Moore-Penrose inverse of the matrix $M$. $\mathbb{N}_{a,b}$ denotes the numbers $a, a+1, \ldots, b$. For a convex set $\Omega$ and a parameter $\alpha \geq 0$ the set $\alpha\Omega$ is given for $\alpha \neq 0$ by $\{x \text{ s.t. } \alpha^{-1}x \in \Omega\}$ and for $\alpha = 0$ by $\alpha\Omega = 0\Omega = \{0\}$.

## II. Problem setup

We consider linear, time-invariant systems given by

$$\tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k, \tag{1}$$

where $\tilde{x}_k \in \mathbb{R}^n$ is the state and $\tilde{u}_k \in \mathbb{R}^p$ the input. The matrices have the appropriate dimensions and $(A, B)$ is assumed to be controllable. The state and the input are constrained to a convex, closed polytope $\Psi$

$$(\tilde{x}_k, \tilde{u}_k) \in \Psi, \tag{2}$$

containing the origin in its interior.

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First we consider tracking of references given as states and inputs: at time instance $k$ the reference given by

$$T_k^{ref} = \left\{ \begin{pmatrix} x_{k|k}^{ref} \\ u_{k|k}^{ref} \end{pmatrix}, \ldots, \begin{pmatrix} x_{k+N-1|k}^{ref} \\ u_{k+N-1|k}^{ref} \end{pmatrix}, x_{k+N|k}^{ref} \right\}, \quad (3)$$

should be tracked, if possible. To achieve this we propose a novel tracking scheme, which can guarantee for arbitrary $\{T_k^{ref}\}$ recursive feasibility and under some mild assumptions that the system (1) tracks the references (3), i.e. that $\tilde{x}_k \to x_{k|k}^{ref}$ and $\tilde{u}_k \to u_{k|k}^{ref}$.

In Section IV we extend the scheme to output tracking. This extension is based on unknown input observers and also guarantees recursive feasibility and under additional assumptions tracking of the reference: $\tilde{y}_k \to y_k^{ref}$.

## III. State input reference tracking MPC

This section proposes a tracking scheme for state-input references as given in (3) and presents conditions guaranteeing recursive feasibility as well as tracking of the reference.

### A. Proposed tracking scheme

The proposed tracking scheme relies on the repeated solution of an optimization problem $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ given below, which depends on the measured state $\tilde{x}_k$ and the provided reference $T_k^{ref}$. Its solution delivers the feedback

$$\tilde{u}_k = u_k^\star, \quad (4)$$

which is applied to the plant.

In the following we first present the optimization problem for the proposed tracking scheme and then discuss it in detail.

*Optimization problem underlying the tracking scheme:* The optimal control problem $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ to be solved in each time instance is given by

$$\min_{\alpha,\beta,\tau,\mathbf{u},\mathbf{x},\mathbf{v}^a,\mathbf{v}^b,\mathbf{z}^a,\mathbf{z}^b} \mathcal{J}(\beta, \tau, \mathbf{u}, \mathbf{x}, \mathbf{v}^a, \mathbf{v}^b, \mathbf{z}^a, \mathbf{z}^b) \quad (5a)$$

subject to

$$x_{i+1} = Ax_i + Bu_i, \in \mathbb{N}_{k,k+N-1} \quad (5b)$$
$$x_k = \tilde{x}_k \quad (5c)$$
$$(x_i, u_i) \in \Psi,\ i \in \mathbb{N}_{k,k+N-1} \quad (5d)$$
$$z_{i+1}^c = Az_i^c + Bv_i^c, i \in \mathbb{N}_{k,k+N+M-1}, c = a, b \quad (5e)$$
$$z_{k+M+N}^c = Az_{k+M+N}^c + Bv_{k+M+N}^c, c = a, b \quad (5f)$$
$$z_i^a = \tau x_{i|k}^{ref},\ i \in \mathbb{N}_{k,k+N} \quad (5g)$$
$$v_i^a = \tau u_{i|k}^{ref},\ i \in \mathbb{N}_{k,k+N-1} \quad (5h)$$
$$0 \le \tau \le 1, \tau \le \frac{\alpha}{1-\epsilon} \quad (5i)$$
$$\alpha + \beta \le 1 - \epsilon,\ \alpha \ge 0,\ \beta \ge 0 \quad (5j)$$
$$x_{k+N} - z_{k+N}^a - z_{k+N}^b \in (1 - \alpha - \beta)\Phi \quad (5k)$$
$$(v_i^a, z_i^a) \in \alpha\Psi,\ i \in \mathbb{N}_{k,k+N+M} \quad (5l)$$
$$(v_i^b, z_i^b) \in \beta\Psi,\ i \in \mathbb{N}_{k,k+N+M} \quad (5m)$$

where the terminal set $\Phi$ is a convex polytope or an ellipsoid, which need to satisfy certain conditions discussed below.

The cost function $\mathcal{J}$ is given by the so-called tracking cost $\mathcal{T}$ and the offset cost $\mathcal{O}$

$$\mathcal{J} = \mathcal{T} + \mathcal{O} \quad (6a)$$
$$\begin{aligned}\mathcal{T} =& (x_{k+N} - z_{k+N}^a - z_{k+N}^b)^T P (x_{k+N} - z_{k+N}^a - z_{k+N}^b) \\ &+ \sum_{i=k}^{k+N-1} (x_i - z_i^a - z_i^b)^T Q (x_i - z_i^a - z_i^b) \\ &+ \sum_{i=k}^{k+N-1} (u_i - v_i^a - v_i^b)^T R (u_i - v_i^a - v_i^b)\end{aligned} \quad (6b)$$
$$\mathcal{O} = w(1 - \tau) \quad (6c)$$

with the weighting matrices $Q = Q^T \ge 0$, $R = R^T > 0$ $P = P^T \ge 0$, weighting $w > 0$ and the design parameters $N > 1$, $M > 1$, $1 > \epsilon > 0$, where $(A, Q^{\frac{1}{2}})$ is observable. The choice/influence of the design parameters $M$ and $\epsilon$ will be discussed in the next subsection.

*Interpretation of the optimization problem:* The proposed tracking scheme features a state trajectory $\mathbf{x}$ and an input sequence $\mathbf{u}$ defined over a horizon $N$

$$\mathbf{u} = \begin{pmatrix} u_k \\ \vdots \\ u_{k+N-1} \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_k \\ \vdots \\ x_{k+N} \end{pmatrix}, \quad (7)$$

which are consistent with the dynamics (1) and measured state $\tilde{x}_k$ due to (5b), (5c). In addition, (5d) guarantees that $\mathbf{x}$, $\mathbf{u}$ satisfies the constraint (2) for $i = k, \ldots, k+N-1$.

Moreover, our proposed scheme features two virtual state and input sequences $(\mathbf{v}^a, \mathbf{z}^a)$ and $(\mathbf{v}^b, \mathbf{z}^b)$ given by

$$\mathbf{v}^c = \begin{pmatrix} v_k^c \\ \vdots \\ v_{k+M+N}^c \end{pmatrix} \quad \mathbf{z}^c = \begin{pmatrix} z_k^c \\ \vdots \\ z_{k+M+N}^c \end{pmatrix} \quad c = a, b. \quad (8)$$

These sequences are trajectories of the system (1) (compare (5e)) ending at $k+N+M$ in steady states, see (5f). However these virtual trajectories do not need to be consistent with the measured state $\tilde{x}_k$: $z_k^a$ and $z_k^b$ can be different from $\tilde{x}_k$.

Note that the first part of $(\mathbf{v}^a, \mathbf{z}^a)$ are a scaled versions of the reference $T_k^{ref}$, c.f. (5g), (5h), (5i). In particular for $\tau = 1$ we have $z_i^a = x_{i|k}^{ref}$ and $v_i^a = u_{i|k}^{ref}$. The constraints (5j), (5k), (5l), (5m) balance in some sense how much of the constraints (2) are used by the terminal constraint (5k) and each of the two virtual trajectories.

The motivation behind using two virtual trajectories is that it will guarantee recursive feasibility and allow under additional assumptions also tracking of the reference. Basically, if the conditions guaranteeing tracking are satisfied, then $\tau$ converges to 1 and $\beta$ to 0, i.e. the first trajectory $\mathbf{v}^a, \mathbf{z}^a$ corresponds to the reference. The second trajectory $\mathbf{v}^b, \mathbf{z}^b$ serves as a back up to guarantee recursive feasibility.

In (6) the tracking cost $\mathcal{T}$ takes the difference between the virtual sequence $\mathbf{v}^a + \mathbf{v}^b$, $\mathbf{z}^a + \mathbf{z}^b$ and the state trajectory $\mathbf{x}$ and input sequence $\mathbf{u}$ into account. The off-set cost $\mathcal{O}$ penalizes derivation of $\tau$ from 1 and $\beta$ from 0. Notice that, if $\mathcal{O} = 0$, then $z_i^a = z_i^a + z_i^b = x_{i|k}^{ref}$ and $v_i^a = v_i^a + v_i^b = u_{i|k}^{ref}$.

So in this case only the difference between $\mathbf{x}$, $\mathbf{u}$ and the reference (3) will be penalized. Moreover, if $\mathbf{x}$, $\mathbf{u}$ tracks the reference perfectly, then $\mathcal{J} = 0$.

While this formulation looks on a first view surprising, we can proof under mild conditions recursive feasibility and a vanishing tracking error.

*Structure of the optimization problem* (5)*:*
The optimization problem (5) is convex. If the terminal set $\Phi$ is a convex polytope, then it is a quadratic program (QP), whereas for an ellipsoidal terminal set it is a quadratically constrained quadratic program (QCQP). Note that for convex QPs/QCQPs there exist many tailored algorithms, we refer for example to [17], [18] for more details. This allows an efficient solution of problem (5), in particular, if the problem structure of (5) is exploited.

### B. Properties of the proposed tracking scheme

As shown in the following the feasibility is independent of the reference. Furthermore for suitably chosen terminal sets and penalties the scheme guarantees recursive feasibility. Finally, conditions to guarantee convergence of the state / input to the reference are outlined.

*1) Influence of the reference on feasibility of* (5)*:*
First let us show that the reference $T_k^{ref}$ does not influence feasibility of the optimization problem $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ (5).

*Proposition 1: (Feasibility independent of reference)*
*The feasibility of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$* (5) *is independent of $T_k^{ref}$, i.e. if $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ is feasible for some $T_k^{ref}$, then for any $\overline{T}_k^{ref}$ $\mathcal{M}(\tilde{x}_k, \overline{T}_k^{ref})$ is feasible.*
The basic idea of the proof is sketched in the Appendix A.1.

*2) Strong feasibility (recursive feasibility):*
Next we focus on recursive feasibility, in particular so-called strong feasibility, see [19]. Strong feasibility means that any suboptimal, but feasible solution of (5) will lead to a feasible problem at the next time instance $k+1$ for any $T_k^{ref}$ and $T_{k+1}^{ref}$, i.e. the references do not influence the recursive feasibility. Moreover, the optimal solution of (5) is not required to guarantee recursive feasibility, cf. [20].

*Definition 1: (Strong feasibility)*
*The tracking scheme* (4), (5) *is called strongly feasible, if for every feasible solution $(\overline{\alpha}, \overline{\beta}, \overline{\tau}, \overline{\mathbf{u}}, \overline{\mathbf{x}}, \overline{\mathbf{v}}^a, \overline{\mathbf{v}}^b, \overline{\mathbf{z}}^a, \overline{\mathbf{z}}^b)$ of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$, the problem $\mathcal{M}(\tilde{x}_{k+1}, T_{k+1}^{ref})$ has at least one feasible solution, where $\tilde{x}_{k+1} = A\tilde{x}_k + B\overline{u}_k$.*
In order to guarantee strong feasibility we assume that the terminal sets and penalties are chosen suitably:

*Assumption 1: (Suitable terminal set & terminal penalty)*
*The terminal penalty $P$ and the terminal set $\Phi$ are chosen such that for some terminal control gain $K$: $\Phi$ contains a neighborhood of the origin and for every $x \in \Phi$ the following conditions are satisfied*

$$(x, Kx) \in \Psi, \quad (A+BK)x \in \Phi$$
$$x^T P x \geq x^T((A+BK)^T P(A+BK) + K^T RK + Q)x.$$

Note that this implies that $\Phi$ is a positive invariant set for the closed loop dynamics $A+BK$ satisfying the constraints and that the cost $x^T P x$ is an upper bound on the infinite horizon cost near the origin.

Different choices of the terminal gain, penalty, set guarantee that these standard assumptions hold, see e.g. [2], [3]. For example one possible choice is to use for $K$ the gain of the unconstrained, infinite horizon LQR with state/input weighting matrices $Q$, $R$ and $P$ as the solution of the LQR Riccati equation and $\Phi$ as a suitable scaled sublevel set of the infinite horizon cost. This choice leads to an ellipsoidal $\Phi$. Note that this choice is possible since we assume $(A, B)$ controllable and $(A, Q^{\frac{1}{2}})$ observable. One choice to obtain a polytopic $\Phi$ is to use the same $K$, $P$ as above and to compute a suitable terminal set $\Phi$ with set theoretic methods [21], e.g. the maximum admissible set [22].

Assumption 1 allows to guarantee recursive feasibility for a feasible, but potentially suboptimal solution of (5):

*Theorem 1: (Strong feasibility of tracking scheme* (5)*)*
*If Assumption 1 holds, then the tracking scheme* (5) *is strongly feasible.*

The full proof is lengthy and avoided for space limitations here, the basic idea is outlined in the Appendix A.2.

*3) Convergence to reference:*
We now outline conditions, which guarantee that the tracking scheme tracks the references $\{T_i^{ref}\}$, i.e. that $\tilde{x}_k \to x_{k|k}^{ref}$ and $\tilde{u}_k \to u_{k|k}^{ref}$. The following is assumed on $\{T_i^{ref}\}$:

*Assumption 2: (Conditions on references $\{T_i^{ref}\}$)*
*For some $l \geq 0$, the references $\{T_i^{ref}\}$ satisfy for each $k \geq l$ the conditions*

$$x_{k+1|k}^{ref} = x_{k+1|k+1}^{ref}, \tag{9a}$$
$$u_{k+1|k}^{ref} = u_{k+1|k+1}^{ref}, \tag{9b}$$
$$x_{k+1|k+1}^{ref} = Ax_{k|k}^{ref} + Bu_{k|k}^{ref} \tag{9c}$$
$$(x_{k|k}^{ref}, u_{k|k}^{ref}) \in (1-\epsilon)\Psi. \tag{9d}$$

*Moreover, for each $T_k^{ref}$ with $k \geq l$ there exists $\overline{x}_{k+N|k}, \dots, \overline{x}_{k+N+M|k}$ and $\overline{u}_{k+N|k}, \dots, \overline{u}_{k+N+M|k}$ with*

$$\overline{x}_{k+N|k} = x_{k+N|k}^{ref} \tag{10a}$$
$$\overline{x}_{j+1|k} = A\overline{x}_{j|k} + B\overline{u}_{j|k},\ j \in \mathbb{N}_{k+N,k+N+M-1} \tag{10b}$$
$$\overline{x}_{k+N+M|k} = A\overline{x}_{k+N+M|k} + B\overline{u}_{k+M|k} \tag{10c}$$
$$(\overline{x}_{i|k}, \overline{u}_{i|k}) \in (1-\epsilon)\Psi,\ i \in \mathbb{N}_{k+N,k+N+M}. \tag{10d}$$

The first condition (9) requires consistency of the supplied references and that $\{x_{k|k}^{ref}, u_{k|k}^{ref}\}$ is a trajectory of the system (1) subject to the tightened constraints (9d). The second part of the assumption implies that one need to reach from each $x_{k|k}^{ref}$ in $M$ steps a steady state, while satisfying the tightened constraints (10d). Clearly, for small $M$ this assumption is rather conservative, e.g. for $M = 0$ each $x_{k|k}^{ref}$ needs to be a steady state of the system and to satisfy (10d). Increasing $M$ expands the set of references satisfying Assumption 2, but also increases the number of optimization variables and thus the computational demand. Note that using a small $\epsilon$ allows to track a larger set of references.

Notice that Assumption 2 does neither imply that there is a (hidden) reference model nor requires that the reference is periodic or converges to a constant; in contrast to existing works.

*Theorem 2: (Convergence to reference)*
*If Assumption 1 and Assumption 2 hold, $N \geq n$ and $\mathcal{M}(\tilde{x}_0, T_0^{ref})$ is feasible, then the tracking scheme* (5) *guarantees that the state and input converge to the reference, i.e. $\tilde{x}_k \to x_{k|k}^{ref}$ and $\tilde{u}_k \to u_{k|k}^{ref}$ for $k \to \infty$.*

Appendix A.3 illustrates the idea of the proof.

Note that one can show that the optimal solution of (5) is not required to guarantee convergence. Instead it is possible to use a suboptimal solution as long as feasibility and a cost decrease (if possible) is guaranteed.

In summary, the tracking scheme (4), (5) guarantees feasibility independent of the reference, strong/recursive feasibility for suitably chosen terminal sets and penalties as well as tracking of suitable references with vanishing tracking error.

## IV. Extension to output tracking

The previous section considered tracking of references given as state and input sequences (3). However often the objective is that the output $y_k$

$$y_k = Cx_k + Du_k, \tag{11}$$

tracks a reference. Thus we extend the previously proposed approach to the output reference case based on unknown input observers [13]–[15][1]. Moreover, we outline conditions on the reference $Y_k^{ref}$ and the system (1), (11), which guarantee tracking, i.e. $y_k \to \hat{y}_{k|k}$.

### A. Proposed output tracking scheme

The main idea is to reconstruct or estimate from the output reference given at each time instance $Y_k^{ref}$ by

$$Y_k^{ref} = \left\{ y_{k|k}^{ref}, y_{k+1|k}^{ref}, \ldots, y_{k+N+L|k}^{ref} \right\} \tag{12}$$

a corresponding state and input sequence of the form (3) and then apply the scheme from the previous section. In (12) $L \geq 0$ is a design parameter to guarantee that (12) is long enough for the estimation of (3).

For the estimation of $\hat{x}_{i|k}^{ref}$ we use an unknown input observer with a delay $L$, $L > 0$ or without delay ($L = 0$). We focus on full order observers of the form (compare [14])

$$\hat{x}_{k+j+1|k}^{ref} = E\hat{x}_{k+j|k}^{ref} + F \begin{pmatrix} y_{k+j+1|k}^{ref} \\ \vdots \\ y_{k+j+L+1|k}^{ref} \end{pmatrix} \tag{13}$$

where $E$ and $F$ are design matrices to guarantee (if possible) stability of the estimation error. This observer is initialized in each time instance $k$ with $\hat{x}_{k|k}^{ref} = \hat{x}_{k|k-1}^{ref}$ and then $\hat{x}_{k+j|k}^{ref}$, $j = 1, \ldots, N$ is calculated.

Afterwards we compute $\hat{u}_{k+j|k}^{ref}$ from

$$\hat{u}_{k+j|k}^{ref} = \begin{pmatrix} B \\ D \end{pmatrix}^{\dagger} \begin{pmatrix} \hat{x}_{k+j+1|k}^{ref} - A\hat{x}_{k+j|k}^{ref} \\ y_{k+j|k}^{ref} - C\hat{x}_{k+j|k}^{ref} \end{pmatrix} \tag{14}$$

c.f. [14]. Finally, we solve $\mathcal{M}(\tilde{x}_k, \hat{T}_k^{ref})$, where

$$\hat{T}_k^{ref} = \left\{ \begin{pmatrix} \hat{x}_{k|k}^{ref} \\ \hat{u}_{k|k}^{ref} \end{pmatrix}, \ldots, \begin{pmatrix} \hat{x}_{k+N-1|k}^{ref} \\ \hat{u}_{k+N-1|k}^{ref} \end{pmatrix}, \hat{x}_{k+N|k}^{ref} \right\}. \tag{15}$$

Algorithm 1 illustrates the overall output tracking scheme.

**Algorithm 1** Proposed output tracking scheme

**Require:** Measured state $\tilde{x}_k$, output reference $Y_k^{ref}$, previous estimate $\hat{x}_{k|k-1}^{ref}$

1: Set $\hat{x}_{k|k}^{ref} = \hat{x}_{k|k-1}^{ref}$
2: Compute $\hat{x}_{k+j|k}^{ref}$ for $j = 1, \ldots, N$ using (13)
3: Compute $\hat{u}_{k+j|k}^{ref}$ for $j = 0, \ldots, N-1$ using (14)
4: Solve $\mathcal{M}(\tilde{x}_k, \hat{T}_k^{ref})$ where $\hat{T}_k^{ref}$ is given by (15)
5: **return** Input $\tilde{u}_k = u_k^\star$, estimate $\hat{x}_{k|k}^{ref}$

### B. Properties of the output tracking scheme

In the following we outline conditions for the output tracking scheme (Algorithm 1) guaranteeing tracking of the reference. First notice that an extension of Proposition 1 and Theorem 1 to the output case is not necessary: the only difference to the previous section is that now we use an estimate $\hat{T}_k^{ref}$ for $T_k^{ref}$. In summary, also for the output tracking scheme we can guarantee feasibility independent of the given output reference and strong feasibility.

To derive conditions to guarantee tracking we make two assumptions. The first assumption (Assumption 3) is an extension of Assumption 2 to the output case. It guarantees that the references $\{Y_k^{ref}\}$ are produced by the system (1), (11) and satisfy a tightened version of the constraints (2). The second assumption requires that the unknown input observer (13) estimates $\hat{x}_{k+j|k}^{ref}$ and $\hat{u}_{k+j|k}^{ref}$ correctly.

*Assumption 3: (Conditions on output reference)*
*For some $l \geq 0$, the references $\{Y_k^{ref}\}$*

$$y_{i|k}^{ref} = y_{i|k+1}^{ref},\ i \in \mathbb{N}_{k+1,k+N+L} \tag{16}$$

*satisfy for every $k \geq l$. For some $\gamma > 0$ and every $k \geq l$ there exist $y_{k|k}^{ref}, y_{k+1|k+1}^{ref} \ldots$ such that*

$$\check{x}_{k+1} = A\check{x}_k + B\overline{u}_k \tag{17a}$$
$$(\check{x}_k, \check{u}_k) \in (1 - \gamma - \epsilon)\Psi \tag{17b}$$
$$y_{k|k}^{ref} = C\check{x}_k + D\check{u}_k \tag{17c}$$

*and for each $k \geq l$ there exists $\overline{x}_{k+N|k}, \ldots, \overline{x}_{k+N+M|k}$ and $\overline{u}_{k+N|k}, \ldots, \overline{u}_{k+N+M|k}$ such that*

$$\overline{x}_{k+N|k} = \check{x}_{k+N} \tag{18a}$$
$$\overline{x}_{j+1|k} = A\overline{x}_{j|k} + B\overline{u}_{j|k},\ j \in \mathbb{N}_{k+N,k+N+M-1} \tag{18b}$$
$$\overline{x}_{k+N+M|k} = A\overline{x}_{k+N+M|k} + B\overline{u}_{k+M|k} \tag{18c}$$
$$(\overline{x}_{i|k}, \overline{u}_{i|k}) \in (1 - \gamma - \epsilon),\ i \in \mathbb{N}_{k+N,k+N+M} \tag{18d}$$

*Assumption 4: (Asymptotic unknown input observer)*
*The dynamic of the estimation error $e_k = \hat{x}_k - x_k$ is asymptotically stable.*

[1]Note that we still assume that the real system state $\tilde{x}_k$ is fully known.

These assumptions provide sufficient conditions to guarantee that the output tracks the reference.

*Corollary 1: (Convergence guarantee to output reference) If $\mathcal{M}(\tilde{x}_0, \hat{T}_0^{ref})$ is feasible, $N \geq n$, Assumptions 1, 3 and 4 hold, then the tracking scheme (Algorithm 1) guarantees tracking of the reference: $y_k \to y_{k|k}^{ref}$ as $k \to \infty$.*

This proof is sketched in the Appendix A.4.

*Remark 1:* (Asymptotic stable unknown input observer) Let us review conditions to guarantee that an asymptotic stable unknown input observer (possibly with delay) exists (i.e. Assumption 4 hold), we refer for more details to [13]–[15] and the references therein. Basically, if an asymptotic stable $E$ and $F$ can be chosen such that

$$e_{k+1} = Ax_k + Bu_k - E\tilde{x}_k - F\begin{pmatrix} y_{k+1} \\ \vdots \\ y_{k+L+1} \end{pmatrix} = Ee_k \quad (19)$$

then the dynamic of the estimation error is $e_{k+1} = Ee_k$ and thus asymptotic stable. These so-called matching conditions can be satisfied for $L = 0$ if the system (1), (11) is *strong* detectable* [13] i.e.

$$\operatorname{rank}\begin{pmatrix} A-zI & B \\ C & D \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} B \\ D \end{pmatrix}, \forall z \in \mathbb{C}, |z| \geq 1 \quad (20a)$$

$$\operatorname{rank}\begin{pmatrix} D & CB \\ 0 & D \end{pmatrix} = \operatorname{rank}(D) + \operatorname{rank}\begin{pmatrix} B \\ D \end{pmatrix}, \quad (20b)$$

hold, see e.g. [13]–[15].

If only (20a) is satisfied (the system is called *strongly detectable* [14]), then an unknown input observer with delay can be designed. If (20a) does not hold, then there exists no (delayed) asymptotic unknown input observer, c.f. [14]. If we consider $p = q$ independent inputs/outputs, i.e. $\operatorname{rank}\begin{pmatrix} C & D \end{pmatrix} = \operatorname{rank}\begin{pmatrix} B^T & D^T \end{pmatrix} = p$, then (20a) requires that the system (1), (11) is minimum phase, i.e. all transmission zeros are within the unit circle.

## V. Examples

Two examples illustrate the proposed tracking scheme.

### A. Double integrator

We consider a double integrator given by

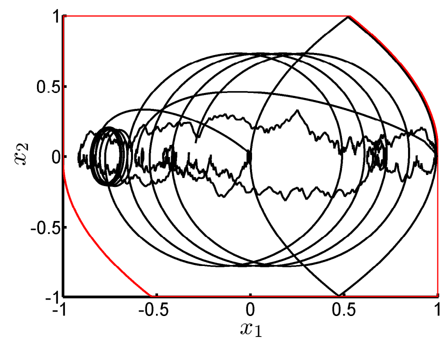
$$A = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} T^2 \\ T \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 0, \quad (21)$$

where $T = 0.05$ and the state and input are constrained by $\begin{pmatrix} -1 \\ -1 \end{pmatrix} \leq x \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $-1 \leq u \leq 1$. Notice that the double integrator is strong* detectable, since $\operatorname{rank}(CB) = 1$ and the system has only one transmission zero at the origin. Therefore we can design a suitable unknown input observer, e.g. based on the procedure outlined in [13]. In detail, we choose $F = \begin{pmatrix} 1 & 20 \end{pmatrix}^T$ to satisfy the matching condition $B = FCB$ and obtain $E$ as $E = A - FCA = \begin{pmatrix} 0 & 0 \\ -20 & 0 \end{pmatrix}$, which is nilpotent (all eigenvalues at zero) and, in particular, asymptotic stable, i.e. Assumption 4 is satisfied.

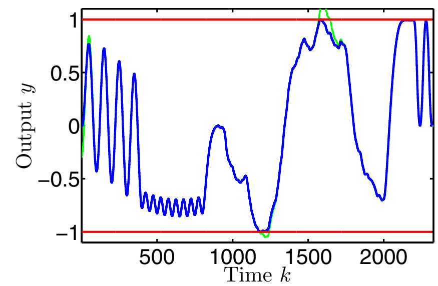
For the tracking setup we choose $N = 20$, $M = 25$ and $Q = I$, $R = I$, $w = 1$ and $\epsilon = 10^{-4}$. For $K$, $P$ we use the LQR controller gain/solution of the corresponding LQR Riccati equation as discussed in Section III-B. As terminal set $\Phi$ we choose the maximum admissible set [22].

For illustration of the tracking behavior we assume that the system starts near the origin and an output reference satisfying (16) and (17) is chosen. Figure 1a shows the reconstructed evolution of the states in the $x(1) - x(2)$ plane. Since the states are most of the times within the area satisfying (18) we expect that the reference is tracked if possible (compare Corollary 1). Indeed this is the case as illustrated by Figure 1b: the output does not violate the constraints and converges (if possible) to the reference. Finally, Figures 1c presents the time evolution of the second state $x(2)$, which satisfies as well as the input $u$ (not shown) the constraints.

(a) State evolutions of $y^{ref}$ reconstructed by unknown input observer (black) and boundary of set defined by (18) (red).




(b) Output $y$ (blue), reference $y^{ref}$ (green), constraints (red).




(c) State $x(2)$ (black), constraints (red).

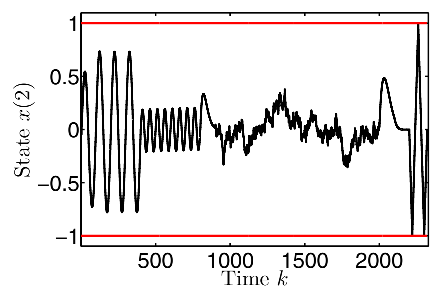



Fig. 1: Plots for double integrator system (Example 1).

### B. Four tank

The second example considers a four tank system. As objective we consider the tracking of the level of the two bottom tanks. We use a linearized model given by

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t) \quad (22)$$

$$\mathcal{A} = \begin{pmatrix} -T_1^{-1} & T_2^{-1} & 0 & 0 \\ 0 & -T_2^{-1} & 0 & 0 \\ 0 & 0 & -T_3^{-1} & T_4^{-1} \\ 0 & 0 & 0 & -T_4^{-1} \end{pmatrix}, \mathcal{B} = 0.1\begin{pmatrix} \xi & 0 \\ 0 & (1-\xi) \\ 0 & \xi \\ (1-\xi) & 0 \end{pmatrix}$$

where $T_i = 0.4\sqrt{h_i}$ depends on the steady state and $\xi$. Here we consider $\xi = 0.6$ and have chosen the steady state such that $T1 = T3 = 2.5$ and $T2 = T4 = 1$. The resulting system is discretized with a sampling time of $0.2$. The input is constrained to the box $\|u\|_\infty \leq 1$ and the states need to satisfy

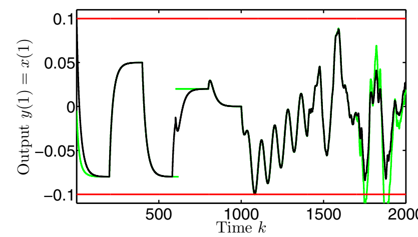
$$\begin{pmatrix} -0.1 \\ -0.05 \\ -0.1 \\ -0.05 \end{pmatrix} \leq x \leq \begin{pmatrix} 0.1 \\ 0.05 \\ 0.1 \\ 0.05 \end{pmatrix}.$$

Since the discretized system is minimum phase (zeros at $0.8464$ and $0.9672$) and $\operatorname{rank}(CB) = 2$, we designed an asymptotic unknown input observer with $L = 0$ in order to satisfy Assumption 4. To complete the tracking setup we choose $N = M = 30$, $Q = I$, $R = I$, $w = 1$ and $\epsilon = 10^{-4}$ and choose $K$, $P$ and $\Phi$ similarly as in the above example.

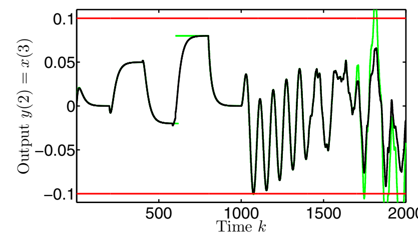
Figure 2 illustrates the results of the output tracking scheme applied to the four tank system. The reference satisfies the continuity condition, except at $k = 600$, where we consider an abrupt change of the reference. Moreover, the first part until $k = 1700$ is chosen such that (17) and (18) are satisfied. From Figure 2a, 2b we observe that the reference is tracked rather well. The exceptions are (a) the begin - we assume the initial state as $\tilde{x}_0 = \begin{pmatrix}0.1 & 0 & 0 & 0\end{pmatrix}^T$, in contrast the reference is initially 0, (b) at and after the abrupt change of the reference at $k = 600$ and (c) due to saturation of the input constraints after $k = 1700$, see Figure 2c.

In summary, our approach allows to track the reference where possible, avoids violation of the constraints and all optimization problems stay feasible due to the guaranteed recursive feasibility.

(a) Output $y(1) = x(1)$ (black), reference (green), constraints (red).




(b) Output $y(2) = x(3)$ (black), reference (green), constraints (red).




(c) Inputs $u(1)$ (black), $u(2)$ (blue).

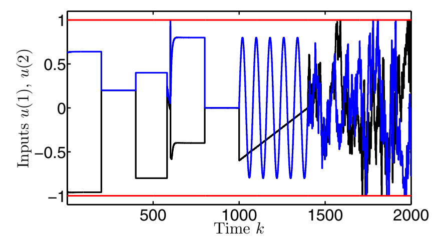



Fig. 2: Plots for four tank system (Example 2).

## VI. Summary and Future working direction

In this work we presented a predictive control approach for reference tracking of linear, constrained system consisting of two parts. First it uses an unknown input observer to estimate from the supplied reference output corresponding input references and state references (if necessary). Second an optimal control problem is solved in order to track these references. We provided conditions to guarantee recursive feasibility and a vanishing tracking error for a large class of trajectories compared to existing results. Examples outlined the applicability of this approach.

Future work considers further analysis, improvements of the proposed approach and a detailed evaluation. On the long run an adaption of the proposed scheme to distributed model predictive control of interconnected systems based on cyclic varying horizons c.f. [23] and [24] or other approaches such as [25], [26] is of interest.

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## APPENDIX

### A. Sketch of Proofs

Due to space limitations we only outline the basic ideas.

*1) Sketch of the proof of Proposition 1:* Let $\overline{\alpha}, \overline{\beta}, \overline{\tau}, \overline{\mathbf{u}}, \overline{\mathbf{x}}, \overline{\mathbf{v}}^a, \overline{\mathbf{v}}^b, \overline{\mathbf{z}}^a, \overline{\mathbf{z}}^b$ be a feasible solution of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$. It is rather straight forward to show that a feasible solution of $\mathcal{M}(\tilde{x}_k, \overline{T}_k^{ref})$ is given by

$$\alpha = \overline{\alpha} + \overline{\beta} \qquad \beta = 0 \qquad \tau = 0 \tag{23a}$$
$$\mathbf{u} = \overline{\mathbf{u}} \qquad \mathbf{x} = \overline{\mathbf{x}} \qquad \mathbf{v}^a = 0 \tag{23b}$$
$$\mathbf{z}^a = 0 \qquad \mathbf{v}^b = \overline{\mathbf{v}}^a + \overline{\mathbf{v}}^b \qquad \mathbf{z}^b = \overline{\mathbf{z}}^a + \overline{\mathbf{z}}^b, \tag{23c}$$

which proofs the proposition.

*2) Sketch of the proof of Theorem 1:* We need to show that for any feasible point $\overline{\alpha}, \overline{\beta}, \overline{\tau}, \overline{\mathbf{u}}, \overline{\mathbf{x}}, \overline{\mathbf{v}}^a, \overline{\mathbf{v}}^b, \overline{\mathbf{z}}^a, \overline{\mathbf{z}}^b$ of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ there exists a feasible point $\alpha^+, \beta^+, \tau^+, \mathbf{u}^+, \mathbf{x}^+, (\mathbf{v}^a)^+, (\mathbf{v}^b)^+, (\mathbf{z}^a)^+, (\mathbf{z}^b)^+$ of $\mathcal{M}(A\tilde{x}_k + B\overline{u}_k, T_{k+1}^{ref})$. A feasible point of $\mathcal{M}(\tilde{x}_{k+1}, T_{k+1}^{ref})$ is given by

$$\alpha^+ = \tau^+ = 0,\ \beta^+ = \overline{\alpha} + \overline{\beta} \tag{24a}$$
$$u_i^+ = \overline{u}_i, i \in \mathbb{N}_{k+1,\dots,k+N-1} \tag{24b}$$
$$u_{k+N}^+ = K(\overline{x}_{k+N} - \overline{z}_{k+N}^a - \overline{z}_{k+N}^b) + \overline{v}_{k+N}^a + \overline{v}_{k+N}^b \tag{24c}$$
$$x_i^+ = \overline{x}_i, i \in \mathbb{N}_{k+1,\dots,k+N} \tag{24d}$$
$$x_{k+N+1}^+ = (A+BK)(\overline{x}_{k+N} - \overline{z}_{k+N}^a - \overline{z}_{k+N}^b) + \overline{z}_{k+N}^a + \overline{z}_{k+N}^b \tag{24e}$$
$$(\mathbf{v}^a)^+ = 0,\ (\mathbf{z}^a)^+ = 0 \tag{24f}$$
$$(v_i^b)^+ = \overline{v}_i^a + \overline{v}_i^b, i \in \mathbb{N}_{k+1,\dots,k+N+M} \tag{24g}$$
$$(v_{k+1+N+M}^b)^+ = \overline{v}_{k+N+M}^a + \overline{v}_{k+N+M}^b \tag{24h}$$
$$(z_i^b)^+ = \overline{z}_i^a + \overline{z}_i^b, i \in \mathbb{N}_{k+1,\dots,k+N+M} \tag{24i}$$
$$(z_{k+1+N+M}^b)^+ = \overline{z}_{k+N+M}^a + \overline{z}_{k+N+M}^b, \tag{24j}$$

which can be verified straightforwardly.

*3) Sketch of the proof of Theorem 2:* The proof considers only time instances $k \geq l$ and consists of three parts. First it is shown without (5c), the optimal solution of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ is given by $\mathcal{J} = \mathcal{T} + \mathcal{O} = 0$ and $\alpha^\star = 1 - \epsilon$, $\beta^\star = 0$, $\tau^\star = 1$.

Second, we need to show that the optimal solution decreases as $k \to \infty$ the cost $\mathcal{J}$ unless $\mathcal{J} = 0$. Basically from the optimal solution of $\mathcal{M}(\tilde{x}_k, T_k^{ref})$ the feasible initial guess

$$\alpha^+ = \alpha^\star,\ \beta^+ = \beta^\star,\ \tau^+ = \tau^\star \tag{25a}$$
$$u_i^+ = u_i^\star, i \in \mathbb{N}_{k+1,\dots,k+N-1} \tag{25b}$$
$$u_{k+N}^+ = K(x_{k+N}^\star - z_{k+N}^\star) + v_{k+N}^\star \tag{25c}$$
$$x_i^+ = x_i^\star, i \in \mathbb{N}_{k+1,\dots,k+N} \tag{25d}$$
$$x_{k+N+1}^+ = (A+BK)(x_{k+N}^\star - z_{k+N}^\star) + z_{k+N}^\star \tag{25e}$$
$$(v_i^a)^+ = \tau^+ u_{i|i}^{ref}, i \in \mathbb{N}_{k+1,\dots,k+N} \tag{25f}$$
$$(z_i^a)^+ = \tau^+ x_{i|i}^{ref}, i \in \mathbb{N}_{k+1,\dots,k+N+1} \tag{25g}$$
$$(v_i^a)^+ = \tau^+ \check{u}_{i|k+N+1}, i \in \mathbb{N}_{k+N+2,\dots,k+1+N+M} \tag{25h}$$
$$(z_i^a)^+ = \tau^+ \check{x}_{i|k+N+1}, i \in \mathbb{N}_{k+N+1,\dots,k+1+N+M} \tag{25i}$$
$$(v_i^b)^+ = (v_i^b)^\star,\ i \in \mathbb{N}_{k+1,\dots,k+N+M} \tag{25j}$$
$$(v_{k+1+N+M}^b)^+ = (v_{k+N+M}^b)^\star \tag{25k}$$
$$(z_i^b)^+ = (z_i^b,)^\star,\ i \in \mathbb{N}_{k+1,\dots,k+N+M} \tag{25l}$$
$$(z_{k+1+N+M}^b)^+ = (z_{k+N+M}^b)^\star \tag{25m}$$

is feasible and does not increase the cost due to Assumption 2, where $z = z^a + z^b$, $v = v^a + v^b$. Moreover, increasing $\tau$ at specific time instances decreases the cost. This is also feasible, because with the above choice $(x_{k+N+W}^\star - (z_{k+N+W}^a)^\star - (z_{k+N+W}^b)^\star) \in \delta\epsilon\Phi$, for any $\delta < 1$ and some $W > 0$. Thus using the optimal solutions of (5) guarantees $\mathcal{J} \to 0$ as $k \to \infty$.

Since $(A, Q^{\frac{1}{2}})$ observable and $R > 0$ $\mathcal{J} \to 0$ as $k \to \infty$ implies that $\tilde{x}_k \to x_{k|k}^{ref}$ and $\tilde{u}_k \to u_{k|k}^{ref}$ for $k \to \infty$.

*4) Sketch of the proof of Corollary 1:* Let $k \geq l$ and define

$$\check{T}_k^{ref} = \left\{ \begin{pmatrix} \check{x}_k \\ \check{u}_k \end{pmatrix}, \dots, \begin{pmatrix} \check{x}_{k+N-1} \\ \check{u}_{k+N-1} \end{pmatrix}, \check{x}_{k+N} \right\}. \tag{26}$$

with $\check{x}$, $\check{u}$ as in Assumption 3. Assumption 3 and Assumption 4 imply that the error $\check{T}_k^{ref}$ and $\hat{T}_k^{ref}$ decreases as $k \to \infty$. Note that there is a $m \geq l$ such that $\hat{T}_k^{ref}$ satisfies Assumption 2 for all $k \geq m$ due to $\gamma > 0$ in Assumption 3.

In combination with Theorem 2 this allows to show $\tilde{x}_k \to \check{x}_{k|k}$ and $\tilde{u}_k \to \check{u}_{k|k}$ and consequently $\tilde{y}_k \to y_{k|k}^{ref}$.