Rewrites as Terms through Justification Logic

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ABSTRACT

Justification Logic is a refinement of modal logic where the modality $\Box A$ is annotated with a reason s for "knowing" A and written $[\![s]\!]A$. The expression s is a proof of A that may be encoded as a lambda calculus term of type A, according to the propositions-as-types interpretation. Our starting point is the observation that terms of type $[\![s]\!]A$ are *reductions* between lambda calculus terms. Reductions are usually encoded as *rewrites* essential tools in analyzing the reduction behavior of lambda calculus and term rewriting systems, such as when studying standardization, needed strategies, Lévy permutation equivalence, etc. We explore a new propositions-as-types interpretation for Justification Logic, based on the principle that terms of type $[\![s]\!]A$ are proof terms encoding reductions (with source s). Note that this provides a logical language to reason about rewrites.

CCS CONCEPTS

• **Theory of computation → Type theory**; *Proof theory*; **Modal and temporal logics**.

KEYWORDS

Lambda calculus, modal logic, Curry-Howard, term rewriting, type systems

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Justification Logic [3, 4, 19] is a modal logic where necessity is indexed by justification expressions. The modal proposition $\Box A$ becomes $[\![s]\!]A$ where the justification expression s is a reason for "knowing" A. Typically, s denotes a proof that attests to the truth of A. An important property of Justification Logic is the *reflection principle*: given a proof of A, one can encode this proof using a justification expression s and prove $[\![s]\!]A$. Most formulations of Justification Logic are in Hilbert style. In that case s above is a combinator, called a *proof polynomial*, encoding a Hilbert style proof of A. This paper proposes to explore the computational significance of Justification Logic via the propositions-as-types methodology. In fact, we focus here on an early precursor of Justification Logic, namely the *Logic of Proofs* (LP) [1, 2]. The Logic of Proofs may be understood as the justification counterpart of S4. All theorems of S4 are also theorems of LP where occurrences of the necessity modality have been suitably annotated with justification expressions. Similarly, dropping the justification expressions of modalities in theorems of Justification Logic yields S4 theorems.

Natural Deduction for the Logic of Proofs. A Natural Deduction presentation for the Logic of Proofs suggests itself through the reflection principle. Consider the following introduction rule for the modality: if s is a Natural Deduction proof of A, then $[\![s]\!]A$ is provable. Here s is a justification expression denoting a Natural Deduction proof. The sequents of our deductive system take the form $\Gamma \vdash A \mid s$, where Γ is a set of hypotheses and the justification expression s encodes the current Natural Deduction proof of the sequent, so that we can express the above reflection principle as an introduction rule: if one proves $\Gamma \vdash A \mid s$, then one may prove $\Gamma \vdash [\![s]\!]A \mid\, !s$. The exclamation mark in "$!s$" records the fact that a modality introduction rule was applied, thus updating our current justification expression. Of course, Γ cannot be any set of hypotheses at all since otherwise A and $[\![s]\!]A$ would be logically equivalent (*i.e.* $A \supset [\![s]\!]A$ and $[\![s]\!]A \supset A$ would both be provable). One could restrict the hypothesis in Γ to be modal expressions, however the resulting system would not be closed under substitution [8]. An alternative is to split them in two disjoint sets, following Bierman and de Paiva [8] and Davies and Pfenning [14]: we use Δ for *modal* hypotheses (those assumed true in all accessible worlds) and Γ for *truth* hypotheses (those assumed true in the current world). Sequents now take the form $\Delta; \Gamma \vdash A \mid s$ and we can recast the above mentioned introduction rule for the modality as follows, where $\emptyset$ denotes an empty set of truth hypotheses:

$$\frac{\Delta; \emptyset \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid\, !s} \tag{1}$$

Although correct from a provability angle, one immediately realizes that, in the presence of this proposed rule, proofs are not closed under normalisation. This is an important requirement towards our goal in uncovering a computational interpretation of $[\![s]\!]A$ since reduction on terms mimics normalisation on proofs. Indeed, normalisation of the proof of $\Delta; \emptyset \vdash A \mid s$ will produce a proof of $\Delta; \emptyset \vdash A \mid t$, for some t different from s. We need some means of relating t back to s.

Towards a Typed Calculus of Rewrites. A *rewrite* is an expression that denotes a sequence of reduction steps from a source term to a

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target term. Consider for example the lambda calculus term $\lambda a.a$ denoting the identity function. Let us abbreviate this term I. The term $(Ib)(Ib)$ reduces in one β-step to $b\,(Ib)$ by contracting the leftmost redex. An expression denoting this reduction step would be the rewrite:

$$\mathbf{ba}(a.a, b)\,(Ib) : (Ib)(Ib) \rhd b(Ib)$$

The expression $\mathbf{ba}(a.a, b)\,(Ib)$ models the above mentioned reduction step. The symbol '**ba**' is called a *rule symbol* since it witnesses that a rewrite rule was applied (β in this case). The occurrence of $\mathbf{ba}(a.a, b)$ tells us that a β-reduction step reduced the leftmost of the two redexes in $(Ib)(Ib)$. The term $(Ib)(Ib)$ to the left of the triangle is the source of the rewrite and $b(Ib)$ on the right the target. We could continue reduction from the target $b(Ib)$ to obtain bb. The rewrite encoding both steps would then be written:

$$\mathbf{ba}(a.a, b)\,Ib; b\,\mathbf{ba}(a.a, b) : (Ib)(Ib) \rhd bb$$

the semi-colon denoting composition of rewrites. Rewrites are useful in studying properties of lambda calculus reduction, where the rewrite steps themselves are of interest. It is particularly useful due to the presence of *syntactic accidents*: two possible steps may explain $I(Ib) \to_\beta Ib$ both of which may be modeled with distinct rewrites.

Returning to our discussion on (1) on obtaining some means of relating t back to s, and given that proofs are reflected as justification expressions in the logic, it seems natural to reify normalisation steps as rewrites. This suggests that:

$$[\![s]\!]A \text{ is the type of } \textit{rewrites with source } s.$$

Or in terms of a deduction rule:

$$\frac{\Delta; \emptyset \vdash \rho : s \rhd t : A}{\Delta; \Gamma \vdash !(\rho, s, t) : [\![\overline{s}]\!]A}$$

A new sequent $\Delta; \emptyset \vdash \rho : s \rhd t : A$ types rewrites rather than terms. It states that if ρ is a rewrite from source term s to target term t, then $!(\rho, s, t)$ is a term. Since s may have occurrences of rewrites and we are only interested in their source, $!(\rho, s, t)$ is given type $[\![\overline{s}]\!]A$ where $\overline{s}$ projects the source of all rewrites in s (for example, in the case of $!(\sigma, p, q)$ it returns $!\overline{p}$). The rest of this article is devoted to developing this propositions-as-types interpretation for the Logic of Proofs, called the *Typed Rewrite Calculus* (TRC), based on the ideas discussed above.

Rewrites for Higher-Order Rewriting. Although this paper is not about higher-order term rewriting (HOR), HOR is its starting point. First-order term rewrite systems are sets of rewrite rules over first-order terms. The latter are constructed from a given signature of function symbols with arity. An example is the term rewrite system consisting of two rewrite rules: $\{f(x) \to x, a \to b\}$, where f has arity 1 and a and b have arity 0. Rewrites for first-order term rewrite systems are introduced in [30] and [29, Ch.8]. They are defined as first-order terms where the signature is extended with additional function symbols for the rules (*rule symbols*) and represent (equational) proofs that a term reduces to another term. For example, the reduction sequence $f(a) \to f(b) \to b$ is represented as $\varrho(a); \vartheta$, where the rule symbol ϱ of arity 1, is a function symbol denoting application of the first rule and ϑ, of arity 0, is a symbol denoting application of the second rule. Rewrites without rule symbols model a trivial reduction step over itself. For example, a in the above mentioned rewrite $\varrho(a); \vartheta$ models the trivial step from a to itself.[1] Rewrites are equipped with simple structural properties stating, for example, that composition is associative and that, if ρ is a rewrite from source s to target t, then $\rho; t$ is equal to ρ.

HOR systems [29, Ch.11] are rewrite systems with binders, the paradigmatic example being the lambda calculus and its binder 'λ'. Rewrite rules are pairs of simply typed lambda calculus with constants, following the HOAS approach [24]. For example, the β rule as a HOR system is represented as $\mathsf{app}(\mathsf{lam}(\lambda x.yx), z) \to yz$. Here we have function symbols $\mathsf{app} : \iota \supset \iota \supset \iota$ and $\mathsf{lam} : (\iota \supset \iota) \supset \iota$, for ι some base type. Terms are assumed to be in β-normal form.[2] Rewrites were extended to HOR by S. Bruggink [12, 13] following the same idea as in the first-order case, namely adding an additional rule symbol constant for each rewrite rule. Also, they are associated a source and target term. Structural equality of rewrites, as in the first order case, is also adopted so that, for example, $\rho; t$ equates to t, where this time the trivial step t is a simply typed term. Note that, in particular, a variable x is equated to $x; x$. This has the unfortunate consequence of admitting the unsound equation [12, 13]:

$$\vartheta =_\beta (\lambda x.x)\vartheta = (\lambda x.x; x)\vartheta =_\beta \vartheta; \vartheta$$

The rewrite $\vartheta; \vartheta$ is incorrect since the source a and target b of ϑ do not coincide. Lambda-calculus substitution is incompatible with rewrite composition. To avoid this obstacle, the author [12, 13] opted to drop composition from the set of simply typed constants, and reintroduce it as an additional notational layer outside the typed framework. Left pending was the problem of providing a type-system for rewrites with binders that included rewrite composition.[3] We devised our *Typed Rewrite Calculus* with this issue in mind and believe it can remedy the problem. We embed rewrites into the typed lambda calculus by assigning them a modal type and then introduce a notion of substitution on modal variables compatible with rewrite composition. In particular, we have (we use $\underline{b}$ for the trivial step from b to b, also $\underline{u}$ is a rewrite variable):

$$\begin{array}{rcl} !(\vartheta, a, b) & = & !(\vartheta; \underline{b}, a, b) \\ & = & let\ u \triangleq !(\vartheta, a, b)\ in\ !(\underline{u}, u, u) \\ & = & let\ u \triangleq !(\vartheta, a, b)\ in\ !(\underline{u}; \underline{u}, u, u) \\ & = & !(\vartheta; \underline{b}; \underline{b}, a, b) \\ & = & !(\vartheta, a, b) \end{array}$$

Although this was the initial motivation for our work, our paper focusses on the *Typed Rewrite Calculus* which we believe to be of interest in itself. Exploring the use of the TRC as a framework for rewrites in HOR is left to future work.

Summary of Contributions and Structure.

- A novel propositions-as-types presentation for the Logic of Proofs based on rewrites as terms.
- A notion of reduction on rewrites we dub extension.

[1]This is slightly more general than our example above where, in $\mathbf{ba}(a.s, t)$, s and t are terms, not rewrites. Although not required for reasoning about sequences of rewrite steps, allowing rewrites that contain rule symbols to be nested provides a concise notation reduction steps that reduce multiple, simultaneous redexes.

[2]In fact, rewrites are restricted to higher-order patterns in $\beta\overline{\eta}$ normal form, where $\overline{\eta}$ denotes the standard notion restricted η-expansion. However, these details are not important right now.

[3]"*Because we introduced proof terms [=rewrites] in particular for easily defining meta-operations on reductions, we have two options: do not allow nested compositions or do not consider proof terms modulo $\beta\eta$-equivalence. I have opted for the first solution. It is left to further research to devise an elegant form of proof term which includes nested compositions but does not have the problem observed above.*" [13, Pg.30].

- Meta-theoretic properties of substitution (*cf.* Lem. 2.3, Lem. 2.4), subject reduction for rewrite extension (*cf.* Prop. 3.7) and strong normalization for rewrite extension (*cf.* Cor. 3.12).

Sec. 1 introduces the terms and rewrites. The type system for TRC is presented in Sec. 2. Extension of rewrites is discussed in Sec. 3. We present related work in Sec. 4 and conclude in Sec. 5. Proofs are relegated to an extended report ebonelli.github.io/files/rc.pdf.

1 TERMS AND REWRITES (AS TERMS)

This section presents (untyped) terms and rewrites. Types will be considered in Sec. 2.

Terms and Rewrites. **Terms** ($\mathbb{T}$) and **Rewrites** ($\mathbb{R}$) are defined by the following mutually recursive grammar:

$$\begin{array}{rcl} s,t & ::= & a \mid u \mid \lambda a.s \mid s\,t \mid !(\rho,s,t) \mid let\ u \doteq s\ in\,t \\ \rho,\sigma & ::= & \underline{a} \mid \underline{u} \mid \mathbf{ba}(a.s,t) \mid \mathbf{bb}(s,u.t) \mid \rho;\sigma \mid \lambda a.\rho \mid \rho\,\sigma \\ & & \mid \langle \rho|_s \sigma\rangle \mid let\ u \doteq \rho\ in\,\sigma \end{array}$$

Terms include the usual lambda calculus expressions consisting of term variables a, abstraction $\lambda a.s$ and application $s\,t$. There are also three new ones. A term of the form $!(\rho,s,t)$ denotes a rewrite from source term s to target term t. Variable u is a rewrite variable of sort term. When a rewrite ρ is substituted into a term, this variable will potentially be replaced with either the source or the target of ρ, as will be made clear in the upcoming definition of substitution of rewrites. The term $let\ u \doteq s\ in\,t$ denotes rewrite composition. For example, the term $let\ v \doteq b\ in\ let\ u \doteq a\ in\,!(\underline{u}\,\underline{v}, u\,v, u\,v)$ will evaluate b to obtain a term $!(\rho,s,t)$ and a to obtain $!(\sigma,p,q)$ and then compose the rewrites ρ and σ to build a rewrite $\rho\,\sigma$ from the application $s\,p$ to $t\,q$. After appropriate substitutions the resulting term will be $!(\rho\,\sigma, s\,p, t\,q)$.

Rewrites denote reduction between a source and target term. The rewrite $\underline{a}$ denotes the identity reduction over term a. Rewrite $\underline{u}$ is the same only that it, moreover, is subject to be replaced by rewrite substitution. Rewrite $\mathbf{ba}(a.s,t)$ models a β-reduction step from term $(\lambda a.s)\,t$ to term $s\{a/t\}$, the latter denoting the capture-avoiding substitution of all free occurrences of a in s by t (defined below). The rewrite $\mathbf{bb}(!(\rho,s,t),u.r)$ similarly will stand for a reduction step involving a redex of the form $let\ u \doteq !(\rho,s,t)\ in\,r$, where u in r is to be substituted by ρ, s and t; further details will be supplied later. As mentioned in the introduction, the rewrite $\rho;\sigma$ denotes composition of reductions. Not all such rewrites are reasonable since the target of ρ may not coincide with the source of σ. Making this precise requires a definition of source and target of a rewrite, a topic we address below. The remaining rewrites denote reduction under a term constructor: $\lambda a.\rho$ is for reduction under an abstraction, $\rho\,\sigma$ for reduction under an application, $let\ u \doteq \rho\ in\,\sigma$ for reduction under a let and $\langle \rho|_s \sigma\rangle$ for reduction under a bang term constructor, where s is assumed to be the source of ρ.[4] Reduction under a term of the form $!(\rho,s,t)$ is interpreted as extending ρ with additional "work" as captured by the rewrite σ. In fact, $\langle \rho|_s \sigma\rangle$ will be considered valid only if the target of ρ coincides with the source of σ.

Free term variables and **free rewrite variables** are defined as expected. Worthy of mention are the clauses: $\mathsf{ftv}(!(\rho,s,t)) := \emptyset$ and $\mathsf{ftv}(let\ u \doteq s\ in\,t) := \mathsf{ftv}(s) \cup \mathsf{ftv}(t)$. The former owes to the fact that term variables represent truth hypothesis in the current world and hence, as is standard, cannot occur free in the term introduced by the modal type Also, $\mathsf{frv}(!(\rho,s,t)) := \mathsf{frv}(\rho) \cup \mathsf{frv}(s) \cup \mathsf{frv}(t)$ and $\mathsf{frv}(let\ u \doteq s\ in\,t) := \mathsf{frv}(s) \cup \mathsf{frv}(t) \setminus \{u\}$.

The subset of rewrites called **trivial rewrites** ($\mathbb{R}_1^-$) is inductively characterized as follows:

$$\mathbb{s} \quad ::= \quad \underline{a} \mid \underline{u} \mid \lambda a.\mathbb{s} \mid \mathbb{s}\,\mathbb{s} \mid \langle \rho|_t \mathbb{s}\rangle \mid let\ u \doteq \mathbb{s}\ in\,\mathbb{s}$$

Any term s can be cast as a trivial rewrite $\underline{s}$ (written $\mathbb{s}$), the latter denoting the identity reduction over itself (*cf.* Lem. 1.3) as follows:

$$\begin{array}{rcl} \underline{a} & := & \underline{a} \\ \underline{u} & := & \underline{u} \\ \underline{\lambda a.s} & := & \lambda a.\underline{s} \\ \underline{s\,t} & := & \underline{s}\,\underline{t} \\ \underline{!(\rho,s,t)} & := & \langle \rho|_s \underline{t}\rangle \\ \underline{let\ u \doteq s\ in\,t} & := & let\ u \doteq \underline{s}\ in\,\underline{t} \end{array}$$

In the clause for $\underline{!(\rho,s,t)}$ above, a trivial rewrite from $!(\rho,s,t)$ to itself consists in the rewrite $\langle \rho|_s \underline{t}\rangle$ that extends ρ with the trivial for the target of ρ.

Substitution. We next introduce three notions of substitution, where o below denotes an **object** ($\mathbb{O}$) defined simply as the union of terms and rewrites:

Substitution of term variables	$s\{a/t\}$
Substitution of rewrite variables over trivial rewrites	$\mathbb{r}\{u/\rho_s^t\}$
Moded substitution of rewrite variables	$\mathrm{o}\{u/^{\mathrm{m}}\rho_s^t\}$

Substitution of term variables is defined as expected. It is worth mentioning that it does not propagate to rewrites since rewrites do not have occurrences of free term variables, as may be seen from looking at the clause defining $!(\rho,s,t)\{a/r\}$.

$$\begin{array}{rcl} b\{a/r\} & := & \begin{cases} r, & a = b \\ a, & a \neq b \end{cases} \\ u\{a/r\} & := & u \\ (\lambda b.s)\{a/r\} & := & \lambda b.s\{a/r\} \\ (s\,t)\{a/r\} & := & s\{a/r\}\,t\{a/r\} \\ !(\rho,s,t)\{a/r\} & := & !(\rho,s,t) \\ (let\ u \doteq s\ in\,t)\{a/r\} & := & let\ u \doteq s\{a/r\}\ in\,t\{a/r\} \end{array}$$

Substitution of rewrite variables into rewrites must be done with some care. Consider the term $\underline{u};\underline{u}$, which is well-formed since $\underline{u}$ is a rewrite from u to itself. Let ρ be a rewrite from a source s to target t. As discussed in the introduction, naive definition of $(\underline{u};\underline{u})\{u/\rho\}$ could end up producing $\rho;\rho$ which is not well-formed in the sense that the source and target of ρ may not coincide. Our notion of substitution will produce $\rho;\mathbb{t}$. Alternatively, one could produce $\mathbb{s};\rho$. However, substituting ρ at the beginning or end makes no difference since both $\rho;\mathbb{t}$ and $\mathbb{s};\rho$ should be equated to ρ anyhow. This will indeed be the case once we have introduced structural equivalence on rewrites (Fig. 1). What is clear is that only one copy of ρ should be substituted and that either prefixing or postfixing it makes no difference.

Another observation we make is that when substituting in $\underline{u};\underline{u}$ we replace each copy of $\underline{u}$ by *different* objects. The first occurrence gets replaced by ρ but the second one gets replaced by a trivial rewrite, namely $\mathbb{t}$. Accordingly, we split substitution of rewrite variables in two: one that substitutes ρ itself and another one that substitutes the

[4]There is an abuse of notation here since "λa" is used both as a term constructor, to build an abstraction, and as a rewrite constructor, to build a rewrite that denotes reduction under an abstraction. The context should prove sufficient to avoid confusion.

source or target of ρ cast as a trivial rewrite. The former is written $\mathfrak{r}\{u/\rho_s^t\}$ and the latter $\mathfrak{o}\{u/^{\mathsf{m}}\rho_s^t\}$ where m stands for either src or tgt. In particular, $\underline{u}\{u/\rho_s^t\} = \rho$, $\underline{u}\{u/^{\mathsf{src}}\rho_s^t\} = \mathfrak{s}$ and $\underline{u}\{u/^{\mathsf{tgt}}\rho_s^t\} = \mathfrak{t}$. Note also that defining $(\sigma_1;\sigma_2)\{u/^{\mathsf{m}}\rho_s^t\} = \sigma_1\{u/^{\mathsf{m}}\rho_s^t\};\sigma_2\{u/^{\mathsf{m}}\rho_s^t\}$ is correct but not so for $\mathfrak{r}\{u/\rho_s^t\}$. As a final observation, both of these notions of substitution are mutually recursive. Substitution of Rewrite Variables over trivial rewrites is defined as:

$$
\begin{array}{rcl}
\underline{a}\{u/\rho_s^t\} & := & \underline{a} \\
\underline{v}\{u/\rho_s^t\} & := & \begin{cases} \rho, & u = v \\ \underline{v}, & u \neq v \end{cases} \\
(\lambda a.\mathfrak{s})\{u/\rho_s^t\} & := & \lambda a.\mathfrak{s}\{u/\rho_s^t\} \\
(\mathfrak{p}\,\mathfrak{q})\{u/\rho_s^t\} & := & \mathfrak{p}\{u/\rho_s^t\}\,\mathfrak{q}\{u/\rho_s^t\} \\
\langle\sigma|_p\mathfrak{q}\rangle\{u/\rho_s^t\} & := & \langle\sigma'|_{p\{u/^{\mathsf{src}}\rho_s^t\}}\mathfrak{q}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle \\
 & & \text{where } \sigma' = \mathfrak{p}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\} \\
(let\ v \mathrel{\hat{=}} \mathfrak{p}\ in\ \mathfrak{q})\{u/\rho_s^t\} & := & let\ v \mathrel{\hat{=}} \mathfrak{p}\{u/\rho_s^t\}\ in\ \mathfrak{q}\{u/\rho_s^t\}
\end{array}
$$

Notice the clause for $\langle\sigma|_p\mathfrak{q}\rangle$. Substitution prepends a copy of the source of σ in which ρ has been substituted (*cf.* rewrite $\mathfrak{p}\{u/\rho_s^t\}$ above) and updates σ so that all occurrences of u in σ are replaced with the target of ρ (*cf.* rewrite $\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}$ above). For the latter it relies on moded substitution defined below. Similar updates are applied to the source term p and trivial rewrite $\mathfrak{q}$. Perhaps worth mentioning is that the resulting rewrite is also a trivial rewrite: $\rho \in \mathbb{R}_1^-$ implies $\rho\{u/^{\mathsf{m}}\sigma_p^q\} \in \mathbb{R}_1^-$.

Moded Substitution of Rewrite Variables over rewrites is defined as follows:[5]

$$
\begin{array}{rcl}
\underline{a}\{u/^{\mathsf{m}}\rho_s^t\} & := & \underline{a} \\
\underline{v}\{u/^{\mathsf{m}}\rho_s^t\} & := & \begin{cases} \mathfrak{s}, & u = v \wedge \mathsf{m} = \mathsf{src} \\ \mathfrak{t}, & u = v \wedge \mathsf{m} = \mathsf{tgt} \\ \underline{v}, & u \neq v \end{cases} \\
\mathbf{ba}(a.p, q)\{u/^{\mathsf{m}}\rho_s^t\} & := & \mathbf{ba}(a.p\{u/^{\mathsf{m}}\rho_s^t\}, q\{u/^{\mathsf{m}}\rho_s^t\}) \\
\mathbf{bb}(p, v.q)\{u/^{\mathsf{m}}\rho_s^t\} & := & \mathbf{bb}(p\{u/^{\mathsf{m}}\rho_s^t\}, v.q\{u/^{\mathsf{m}}\rho_s^t\}) \\
(\lambda a.\rho)\{u/^{\mathsf{m}}\rho_s^t\} & := & \lambda a.\rho\{u/^{\mathsf{m}}\rho_s^t\} \\
(\sigma\,\tau)\{u/^{\mathsf{m}}\rho_s^t\} & := & \sigma\{u/^{\mathsf{m}}\rho_s^t\}\,\tau\{u/^{\mathsf{m}}\rho_s^t\} \\
\langle\sigma|_p\tau\rangle\{u/^{\mathsf{m}}\rho_s^t\} & := & \langle\sigma'|_{p\{u/^{\mathsf{src}}\rho_s^t\}}\tau\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle \\
 & & \text{where } \sigma' = \mathfrak{p}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\} \\
(\sigma;\tau)\{u/^{\mathsf{m}}\rho_s^t\} & := & \sigma\{u/^{\mathsf{m}}\rho_s^t\};\tau\{u/^{\mathsf{m}}\rho_s^t\} \\
(let\ v \mathrel{\hat{=}} \sigma\ in\ \tau)\{u/^{\mathsf{m}}\rho_s^t\} & := & let\ v \mathrel{\hat{=}} \sigma\{u/^{\mathsf{m}}\rho_s^t\}\ in\ \tau\{u/^{\mathsf{m}}\rho_s^t\}
\end{array}
$$

Notice how, in the clause for $\underline{v}$, it is the source s and target t that are substituted, prior to having being cast as trivial rewrites. Also, moded substitution still needs access to ρ itself (not just its source and target); it is used in the clause for $\langle\sigma|_p\tau\rangle$. One final comment on the above definition is that in the clause for $\sigma;\tau$ it is safe to distribute moded substitution over σ and τ.

Finally, moded Substitution of Rewrite Variables over terms is defined as:

[5] $\sigma\{u/^{\mathsf{m}}\rho_s^t\}$ is defined for both $\mathsf{m} = \mathsf{src}$ and $\mathsf{m} = \mathsf{tgt}$, however we only use this notion of substitution for $\mathsf{m} = \mathsf{tgt}$.

$$
\begin{array}{rcl}
a\{u/^{\mathsf{m}}\rho_s^t\} & := & a \\
v\{u/^{\mathsf{m}}\rho_s^t\} & := & \begin{cases} s, & v = u \wedge \mathsf{m} = \mathsf{src} \\ t, & v = u \wedge \mathsf{m} = \mathsf{tgt} \\ v, & v \neq u \end{cases} \\
(\lambda a.r)\{u/^{\mathsf{m}}\rho_s^t\} & := & \lambda a.r\{u/^{\mathsf{m}}\rho_s^t\} \\
(p\,q)\{u/^{\mathsf{m}}\rho_s^t\} & := & p\{u/^{\mathsf{m}}\rho_s^t\}\,q\{u/^{\mathsf{m}}\rho_s^t\} \\
!(\sigma, p, q)\{u/^{\mathsf{m}}\rho_s^t\} & := & !(\sigma', p\{u/^{\mathsf{src}}\rho_s^t\}, q\{u/^{\mathsf{tgt}}\rho_s^t\}) \\
 & & \text{where } \sigma' = \mathfrak{p}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\} \\
(let\ v \mathrel{\hat{=}} p\ in\ q)\{u/^{\mathsf{m}}\rho_s^t\} & := & let\ v \mathrel{\hat{=}} p\{u/^{\mathsf{m}}\rho_s^t\}\ in\ q\{u/^{\mathsf{m}}\rho_s^t\}
\end{array}
$$

For example, consider $!(\tau\,\underline{u}, s_1\,u, s_2\,u)$ where τ has source s_1 and target s_2. Then $!(\tau\,\underline{u}, s_1\,u, s_2\,u)\{u/^{\mathsf{m}}\rho_{t_1}^{t_2}\} = !(\mathfrak{s}_1\,\rho;\tau\,\mathfrak{t}_2, s_1\,t_1, s_2\,t_2)$, for $\mathsf{m} = \mathsf{src}$ or $\mathsf{m} = \mathsf{tgt}$.

Some basic, but subtle to prove, properties for substitution are presented below, after introducing structural equivalence.

Structural Equivalence and Well-Formedness. As mentioned, a rewrite may not have a source and target. If it does we say it is *well-formed*. For example, if a and b are distinct variables, then $\underline{a};\underline{b}$ is not well-formed. More generally, for $\rho;\sigma$ to be well-formed the target of ρ must coincide with the source of σ (similar requirements apply to $!(\rho, s, t)$ and $\langle\rho|_s\sigma\rangle$). This leads us to consider how terms are to be compared. Since terms may include rewrites, we need to consider rewrite comparison too.

One reasonable property is that composition be associative: rewrites $(\rho;\sigma);\tau$ and $\rho;(\sigma;\tau)$ should be considered equivalent. Similarly, $\rho;\mathfrak{t}$ should be considered equivalent to ρ, assuming that ρ and $\mathfrak{t}$ are composable (in which case t should be equivalent to the target of ρ, though it may not be identical to it). Another example of rewrite equivalence is as follows. Let I be the term $\lambda b.b$ and consider the lambda calculus reduction $\lambda a.I(\underline{Ia}) \to_\beta \lambda a.\underline{Ia} \to_\beta \lambda a.a$, where the redex being reduced in each step is underlined. It can be represented via the rewrite $\lambda a.I\,\mathbf{ba}(b.b, a);\lambda a.\mathbf{ba}(b.b, a)$. However, the same reduction sequence could also have been represented as $\lambda a.(I\,\mathbf{ba}(b.b, a);\mathbf{ba}(b.b, a))$. Such minor, structural variations are absorbed through *structural equivalence.*

Definition 1.1 (Source/Target Predicate and Structural Equivalence). The source/target (ST) predicate $\bullet : \bullet \rhd \bullet \subseteq \mathbb{R} \times \mathbb{T} \times \mathbb{T}$ is defined mutually recursively with structural equivalence $\simeq \subseteq \mathbb{O} \times \mathbb{O}$ via the rules in Fig. 1.[6] If $\rho : s \rhd t$ holds then we say that ρ has source s and target t. There are two structural equivalence judgements:

$s \simeq t$	**Structurally equivalent terms**
$\rho \simeq \sigma : s \rhd t$	**Structurally equivalent rewrites**

If $s \simeq t$, then we say s and t are structurally equivalent terms. If $\rho \simeq \sigma : s \rhd t$, then we say ρ and σ are structurally equivalent rewrites with source s and target t. In that case both $\rho : s \rhd t$ and $\sigma : s \rhd t$ (Lem. 1.4).

The rules defining the ST-predicate $\bullet : \bullet \rhd \bullet$ (whose names are prefixed with ST in Fig. 1) are quite expected. We comment on ST-Bang. As already mentioned, $\langle\rho|_s\sigma\rangle$ is a rewrite denoting reduction under a term of the form $!(\rho, s, r)$ and consists of the additional "work" with which ρ is extended. The additional work is represented by the rewrite σ whose source must coincide with the target of ρ (modulo structural equivalence). The source and target of $\langle\rho|_s\sigma\rangle$ are $!(\rho, s, r)$ and $!(\rho;\sigma, s, t)$.

[6] The congruence rules for $\simeq$ have been omitted.

$$\frac{}{\underline{a} : a \rhd a}\ \text{ST-TVar} \qquad \frac{}{\underline{u} : u \rhd u}\ \text{ST-RVar}$$

$$\frac{}{\mathbf{ba}(a.s,t) : (\lambda a.s)\, t \rhd s\{a/t\}}\ \text{ST-}\beta \qquad \frac{}{\mathbf{bb}(!(\rho,s,t),u.r) : \mathit{let}\ u \doteq\, !(\rho,s,t)\ \mathit{in}\, r \rhd r\{u/{}^{\mathrm{tgt}}\rho^{t}_{s}\}}\ \text{ST-}\beta_{\Box}$$

$$\frac{\rho : s \rhd t}{\lambda a.\rho : \lambda a.s \rhd \lambda a.t}\ \text{ST-Abs} \qquad \frac{\rho : s_1 \rhd t_1 \quad \sigma : s_2 \rhd t_2}{\rho\, \sigma : s_1\, s_2 \rhd t_1\, t_2}\ \text{ST-App} \qquad \frac{\rho : s_1 \rhd t_1 \quad \sigma : s_2 \rhd t_2}{\mathit{let}\ u \doteq \rho\ \mathit{in}\, \sigma : \mathit{let}\ u \doteq s_1\ \mathit{in}\, s_2 \rhd \mathit{let}\ u \doteq t_1\ \mathit{in}\, t_2}\ \text{ST-Let}$$

$$\frac{\rho : r \rhd s \quad \sigma : s \rhd t}{\rho;\sigma : r \rhd t}\ \text{ST-Comp} \qquad \frac{\rho : s \rhd r \quad \sigma : r \rhd t}{\langle \rho|_{s}\sigma\rangle : !(\rho,s,r) \rhd !(\rho;\sigma,s,t)}\ \text{ST-Bang} \qquad \frac{r \simeq r' \quad \rho : r' \rhd s' \quad s' \simeq s}{\rho : r \rhd s}\ \text{ST-SEq}$$

$$\frac{}{\underline{a} \simeq \underline{a} : a \rhd a}\ \text{EqR-Refl-TVar} \qquad \frac{}{\underline{u} \simeq \underline{u} : u \rhd u}\ \text{EqR-Refl-RVar}$$

$$\frac{}{\mathbf{ba}(a.s,t) \simeq \mathbf{ba}(a.s,t) : (\lambda a.s)\, t \rhd s\{a/t\}}\ \text{EqR-Refl-}\beta \qquad \frac{}{\mathbf{bb}(!(\rho,s,t),u.r) \simeq \mathbf{bb}(!(\rho,s,t),u.r) : \mathit{let}\ u \doteq\, !(\rho,s,t)\ \mathit{in}\, r \rhd r\{u/{}^{\mathrm{tgt}}\rho^{t}_{s}\}}\ \text{EqR-Refl-}\beta_{\Box}$$

$$\frac{\rho : p \rhd q \quad \mathsf{t} : q \rhd r}{\rho;\mathsf{t} \simeq \rho : p \rhd r}\ \text{EqR-IdR} \qquad \frac{\mathsf{s} : p \rhd q \quad \rho : q \rhd r}{\mathsf{s};\rho \simeq \rho : p \rhd r}\ \text{EqR-IdL}$$

$$\frac{\sigma : p \rhd q \quad \rho : q \rhd r \quad \tau : r \rhd s}{(\sigma;\rho);\tau \simeq \sigma;(\rho;\tau) : p \rhd s}\ \text{EqR-Ass} \qquad \frac{\rho : p \rhd q \quad \sigma : q \rhd r}{\lambda a.\rho;\lambda a.\sigma \simeq \lambda a.(\rho;\sigma) : \lambda a.p \rhd \lambda a.r}\ \text{EqR-Abs}$$

$$\frac{\rho_1 : p_1 \rhd q_1 \quad \rho_2 : q_1 \rhd r_1 \quad \sigma_1 : p_2 \rhd q_2 \quad \sigma_2 : q_2 \rhd r_2}{(\rho_1\, \sigma_1);(\rho_2\, \sigma_2) \simeq (\rho_1;\rho_2)\,(\sigma_1;\sigma_2) : p_1\, p_2 \rhd r_1\, r_2}\ \text{EqR-App}$$

$$\frac{\rho_1 : p_1 \rhd q_1 \quad \rho_2 : q_1 \rhd r_1 \quad \sigma_1 : p_2 \rhd q_2 \quad \sigma_2 : q_2 \rhd r_2}{\mathit{let}\ u \doteq \rho_1\ \mathit{in}\, \sigma_1;\mathit{let}\ u \doteq \rho_2\ \mathit{in}\, \sigma_2 \simeq \mathit{let}\ u \doteq \rho_1;\rho_2\ \mathit{in}\, \sigma_1;\sigma_2 : \mathit{let}\ u \doteq p_1\ \mathit{in}\, p_2 \rhd \mathit{let}\ u \doteq r_1\ \mathit{in}\, r_2}\ \text{EqR-Let}$$

$$\frac{\rho : p \rhd q \quad \sigma : q \rhd r \quad \tau : r \rhd s}{\langle \rho|_{p}\sigma\rangle;\langle \rho;\sigma|_{p}\tau\rangle \simeq \langle \rho|_{p}\sigma;\tau\rangle : !(\rho,p,q) \rhd !(\rho;\sigma;\tau,p,s)}\ \text{EqR-BangR} \qquad \frac{s \simeq s' \quad \rho \simeq \sigma : s' \rhd t' \quad t' \simeq t}{\rho \simeq \sigma : s \rhd t}\ \text{EqR-SEq}$$

$$\frac{}{a \simeq a}\ \text{EqT-TVar} \qquad \frac{}{u \simeq u}\ \text{EqT-RVar} \qquad \frac{s \simeq t}{\lambda a.s \simeq \lambda a.t}\ \text{EqT-Abs} \qquad \frac{s \simeq p \quad t \simeq q}{s\, t \simeq p\, q}\ \text{EqT-App}$$

$$\frac{s \simeq p \quad t \simeq q \quad \rho \simeq \sigma : s \rhd t}{!(\rho,s,t) \simeq !(\sigma,p,q)}\ \text{EqT-Bang} \qquad \frac{s \simeq p \quad t \simeq q}{\mathit{let}\ u \doteq s\ \mathit{in}\, t \simeq \mathit{let}\ u \doteq p\ \mathit{in}\, q}\ \text{EqT-Let}$$

Figure 1: Source/Target Predicate and Structural Equivalence of Rewrites and Terms

The rules defining structural equivalence of terms (those whose names are prefixed with EqT in Fig. 1) are as expected. The rules defining structural equivalence of rewrites (those whose names are prefixed with EqR in Fig. 1) are similar to the ones one has in first-order term rewriting (*cf.* Def. 8.3.1. in [29]) except for two important differences are as follows. The first is the need to rely on structural equivalence on terms to define structural equivalence on rewrites, given that terms and rewrites are mutually dependent. The other is the presence of terms as rewrites $!(\rho,s,t)$, rewrites on such terms $\langle \rho|_{p}\sigma\rangle$ and the equation $\langle \rho|_{p}\sigma\rangle;\langle \rho;\sigma|_{p}\tau\rangle \simeq \langle \rho|_{p}\sigma;\tau\rangle$. The latter states how two rewrites under a bang may be composed. Given $!(\rho,p,q)$, a rewrite σ extending ρ must be composable with ρ and produces $!(\rho;\sigma,p,r)$ as target. A further rewrite extending $\rho;\sigma$, say τ, will produce term $!((\rho;\sigma);\tau,p,s)$ as target.

We next mention some lemmata on structural equivalence. The first one is that the source and target are unique modulo structural equivalence. It is straightforward to prove.

Lemma 1.2 (Uniqueness of Source and Target). *If $\rho : s \rhd t$ and $\rho : p \rhd q$, then $s \simeq p$ and $t \simeq q$.*

The lemma below states that a trivial rewrite is a step over itself:

Lemma 1.3. *$\mathsf{s} : p \rhd q$ implies $p \simeq q \simeq s$.*

The next result states that the rewrites related by structural equivalence have the same source and target. Its proof relies on Lem. 1.3:

Lemma 1.4. *$\rho \simeq \sigma : s \rhd t$ implies $\rho : s \rhd t$ and $\sigma : s \rhd t$.*

The term-as-a-trivial-rewrite operation $\underline{\bullet}$ is compatible with structural equivalence:

Lemma 1.5. *$s \simeq t$ implies $\underline{s} \simeq \underline{t} : s \rhd s$.*

Finally, substitution is compatible with structural equivalence too. For substitution of term variables this is proved by induction on $s \simeq t$:

Lemma 1.6 (Structural Equivalence is closed under substitution of term variables). *Suppose $s \simeq t$ and $p \simeq q$. Then $s\{a/p\} \simeq t\{a/q\}$.*

For substitution of rewrite variables, the result is broken down into four items all of which are proved by simultaneous induction:

Lemma 1.7 (Structural Equivalence is closed under substitution of rewrite variables). *Suppose $\tau \simeq \upsilon : p \rhd q$. Then*

(a) $\rho \simeq \sigma : s \rhd t$ implies $\rho\{u/^{\mathsf{m}}\tau^q_p\} \simeq \sigma\{u/^{\mathsf{m}}\upsilon^q_p\} : s\{u/^{\mathsf{m}}\tau^q_p\} \rhd t\{u/^{\mathsf{m}}\tau^q_p\}$.

(b) $s \simeq t$ implies $s\{u/^{\mathsf{m}}\tau^q_p\} \simeq t\{u/^{\mathsf{m}}\upsilon^q_p\}$.

(c) $s \simeq t$ implies $\mathfrak{s}\{u/\tau^q_p\} \simeq \mathfrak{t}\{u/\upsilon^q_p\} : s\{u/^{\mathsf{src}}\tau^q_p\} \rhd s\{u/^{\mathsf{tgt}}\tau^q_p\}$

(d) $s \simeq t$ implies $\mathfrak{s}\{u/^{\mathsf{m}}\tau^q_p\} \simeq \mathfrak{t}\{u/^{\mathsf{m}}\upsilon^q_p\} : s\{u/^{\mathsf{m}}\tau^q_p\} \rhd s\{u/^{\mathsf{m}}\tau^q_p\}$

Having introduced the ST-predicate and structural equivalence we can now precisely state when terms and rewrites are well-formed.

Definition 1.8 (Well-formed Terms and Rewrites).

(a) $s \in \mathbb{T}$ is **well-formed** if for all subexpressions of s of the form $!(\rho, p, q)$, (ρ, p, q) is well-formed.

(b) $(\rho, s, t) \in \mathbb{R} \times \mathbb{T} \times \mathbb{T}$ is **well-formed** iff $\rho : s \rhd t$ and s and t are well-formed.

$\rho \in \mathbb{R}$ is **well-formed** if there exist s and t such that (ρ, s, t) is well-formed.

For example, $\underline{a}; \underline{b}$ is not well-formed, however $\mathbf{ba}(a.a, b)$ and $\underline{a}; \underline{a}$ are. The triple $(\mathbf{ba}(a.!(\underline{b}; \underline{c}, a, a), b), (\lambda a.!(\underline{b}; \underline{c}, a, a))b, !(\underline{b}; \underline{c}, a, a))$ is not well-formed. Indeed, even though we do have $\mathbf{ba}(a.!(\underline{b}; \underline{c}, a, a), b) : (\lambda a.!(\underline{b}; \underline{c}, a, a))b \rhd !(\underline{b}; \underline{c}, a, a)$ the source term $(\lambda a.!(\underline{b}; \underline{c}, a, a))b$ is not well-formed (since $\underline{b}; \underline{c} : a \rhd a$ does not hold).
Well-formedness is preserved by structural equivalence, a fact that relies on Lem. 1.4

Lemma 1.9 (Structural Equivalence Preserves Well-Formedness). *If s is well-formed and $s \simeq t$, then t is well-formed. Similarly, if (ρ, s, t) is well-formed and $\rho \simeq \sigma : s \rhd t$, then (σ, s, t) is well-formed.*

We conclude the section with two important results on commutation of substitutions. We assume for these results that our objects are well-formed[7]. The first one concerns commutation of term and rewrite substitutions.

Lemma 1.10 (Commutation of Rewrite Substitution with Term Substitution). *Suppose $a \notin \mathsf{ftv}(\rho, s, t)$.*

$$\underline{p}\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} = \underline{p}\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\}$$

where both occurrences of m are either both src or both tgt.

The second is about commutation of rewrite substitutions and requires some care. First note that when $\bullet\{u/^{\mathsf{m}}\rho^t_s\}$ commutes "over" $\bullet\{v/^{\mathsf{m}}\mu^q_p\}$ in the expression $\mathrm{o}\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\}$, a copy of ρ has to be prefixed in front of μ. This is witnessed in item (a) of Lem. 1.11 below. We comment on item (b) of Lem. 1.11, below, after having analyzed a sample proof case for item (a) which motivates the need for it.

Lemma 1.11 (Commutation of Rewrite Substitution). *Let o be any object (i.e. term or rewrite) and suppose $v \notin \mathsf{frv}(\rho, s, t)$.*

(a) Suppose all occurrences of m below are either all src or all tgt. Then,

$$\begin{array}{rl} & \mathrm{o}\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} \\ \simeq & \mathrm{o}\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

(b) If $\mathrm{o} \in \mathbb{R}_1$, then

$$\begin{array}{rl} & \mathrm{o}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}; \\ & \mathrm{o}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} \\ \simeq & \mathrm{o}\{u/^{\mathsf{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}; \\ & \mathrm{o}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

Item (b) is motivated by analyzing the following proof case for item (a). Suppose $\mathrm{o} = !(\sigma, r_1, r_2)$ and let us introduce the following abbreviations:

$$\begin{array}{rcl} \alpha^{\mathsf{m}} & := & \bullet\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\ \alpha & := & \bullet\{v/\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

We seek to prove:

$$!(\sigma, r_1, r_2)\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} \simeq !(\sigma, r_1, r_2)\{u/^{\mathsf{m}}\rho^t_s\}\alpha^{\mathsf{m}}$$

We reason as in Fig. 2 where ($\star$) is the property that the function that casts a term as a rewrite commutes with rewrite substitution ($\underline{p\{u/^{\mathsf{m}}\rho^t_s\}} = \underline{p}\{u/^{\mathsf{m}}\rho^t_s\}$). The topmost box signals exactly where we apply item (b) above. Consider the case where $r_1 = v$. If one just considers the left argument of the composition inside the box, namely $\underline{r_1}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}$, then the resulting term would be $\mathfrak{p}\{u/\rho^t_s\}$. If we now take the left argument of the composition in the second box, namely $\underline{r_1}\{u/^{\mathsf{src}}\rho^t_s\}\alpha$, then we have $\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}$. Clearly these rewrites are not equivalent. However, when the entire composed rewrites inside the boxes are considered, then we do obtain structurally equivalent rewrites.

2 TYPES

This section presents the type system for TRC. As mentioned in the introduction, types will include the modal type $[\![\boldsymbol{s}]\!]A$ where $\boldsymbol{s}$ is a so called source-expression.

Types and Typing Judgements. **Source-terms (s-terms)** are terms where all references to rewrites are omitted (note the use of boldface to denote s-terms):

$$\boldsymbol{s}, \boldsymbol{t} \quad ::= \quad \boldsymbol{a} \mid \boldsymbol{u} \mid \boldsymbol{\lambda a.s} \mid \boldsymbol{s}\,\boldsymbol{t} \mid \boldsymbol{!s} \mid \boldsymbol{let}\ \boldsymbol{u} \mathrel{\hat{=}} \boldsymbol{s}\ \boldsymbol{in}\ \boldsymbol{t}$$

The s-term underlying a term can be obtained from the following translation:

$$\begin{array}{rcl} \overline{a} & := & \boldsymbol{a} \\ \overline{u} & := & \boldsymbol{u} \\ \overline{\lambda a.s} & := & \boldsymbol{\lambda a.}\overline{s} \\ \overline{s\,t} & := & \overline{s}\,\overline{t} \\ \overline{!(\rho, s, t)} & := & \boldsymbol{!}\overline{s} \\ \overline{let\ u \mathrel{\hat{=}} s\ in\ t} & := & \boldsymbol{let}\ \boldsymbol{u} = \overline{s}\ \boldsymbol{in}\ \overline{t} \end{array}$$

[7] Lem. 1.10 in fact holds without this assumption, but Lem. 1.11 relies on it.

$$
\begin{array}{lll}
= & !(\sigma, r_1, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} & \\
= & !(\underline{r}_1\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{m}}\rho_s^t\} & \\
= & !(r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; (\underline{r}_1\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{tgt}}\rho_s^t\}, q\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\
= & !(\overline{\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}}\{u/\rho_s^t\}; (\underline{r}_1\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}), r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & (\star) \\
\simeq & !(\boxed{(\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \underline{r}_1\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\})}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\
\simeq & !(\boxed{(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}})}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \text{(item (b)} \\
\simeq & !((\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}); \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(item (a))} \\
= & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}), r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\
= & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r_1}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\
= & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r_1}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & (\star) \\
= & !(\overline{\underline{r_1}\{u/\rho_s^t\}}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{m}} & \\
= & !(\sigma, r_1, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} &
\end{array}
$$

Figure 2: Commutation of substitution of rewrite variables - Sample proof case

Substitution over s-terms is defined as follows:

$$
\begin{array}{rcl}
\boldsymbol{a}\{u/\boldsymbol{p}\} & := & \boldsymbol{a} \\
\boldsymbol{v}\{u/\boldsymbol{p}\} & := & \begin{cases} \boldsymbol{p}, & u = v \\ \boldsymbol{v}, & u \neq v \end{cases} \\
(\boldsymbol{\lambda a.s})\{u/\boldsymbol{p}\} & := & \boldsymbol{\lambda a.s}\{u/\boldsymbol{p}\} \\
(\boldsymbol{s}\,\boldsymbol{t})\{u/\boldsymbol{p}\} & := & \boldsymbol{s}\{u/\boldsymbol{p}\}\,\boldsymbol{t}\{u/\boldsymbol{p}\} \\
(\boldsymbol{!s})\{u/\boldsymbol{p}\} & := & \boldsymbol{!s}\{u/\boldsymbol{p}\} \\
(\boldsymbol{let}\ \boldsymbol{u} \mathrel{\mathring{=}} \boldsymbol{s}\ \boldsymbol{in}\ \boldsymbol{t})\{u/\boldsymbol{p}\} & := & \boldsymbol{let}\ \boldsymbol{u} = \boldsymbol{s}\{u/\boldsymbol{p}\}\ \boldsymbol{in}\ \boldsymbol{t}\{u/\boldsymbol{p}\}
\end{array}
$$

The set of **propositions** ($\mathbb{P}$) is defined as follows:

$$A, B \quad ::= \quad P \mid A \supset B \mid [\![\boldsymbol{s}]\!]A$$

where P ranges over some set of propositional variables and $\boldsymbol{s}$ is an s-term. Substitution of s-terms in types is defined as follows:

$$
\begin{array}{rcl}
P\{u/\boldsymbol{s}\} & := & P \\
(A \supset B)\{u/\boldsymbol{s}\} & := & A\{u/\boldsymbol{s}\} \supset B\{u/\boldsymbol{s}\} \\
([\![\boldsymbol{p}]\!]A)\{u/\boldsymbol{s}\} & := & [\![\boldsymbol{p}\{u/\boldsymbol{s}\}]\!]A\{u/\boldsymbol{s}\}
\end{array}
$$

We write Δ for a set of rewrite hypotheses and Γ for a set of term hypotheses. If $\Delta = \{u_1 : A_1, \ldots, u_n : A_n\}$, then we write $\mathsf{dom}(\Delta)$ for the set $\{u_1, \ldots, u_n\}$, $\mathsf{frv}(\Delta)$ for $\bigcup_{i \in 1..n} \mathsf{frv}(A_i)$, and similarly for $\mathsf{ftv}(\Delta)$. There are two typing judgements:

$\Delta; \Gamma \vdash s : A$	**Term typing judgement**
$\Delta; \Gamma \vdash \rho : s \rhd t : A$	**Rewrite typing judgement**

A term typing judgement $\Delta; \Gamma \vdash s : A$ is **well-formed** if, (1) $\mathsf{dom}(\Delta) \cap \mathsf{frv}(\Delta, \Gamma) = \emptyset$ and (2) $\mathsf{dom}(\Gamma) \cap \mathsf{ftv}(\Delta, \Gamma) = \emptyset$. Similarly for the rewrite typing judgement. These conditions state that the labels of the hypothesis are fresh.

Type System. The type system for TRC is given by the rules of Fig. 3. A judgement is **derivable**, indicated with $\Vdash \Delta; \Gamma \vdash s : A$, if it is provable using these rules. Moreover, we write $\Vdash_\pi \Delta; \Gamma \vdash s : A$ if it is derivable with derivation π. This notation applies to rewrite typing judgements too.

We next comment on the salient typing rules. The Bang rule was motivated in the introduction. Note that the type of $!(\rho, s, t)$ is $[\![\overline{s}]\!]A$. The rule R-Bang types the rewrite that denotes reduction inside a term of the form $!(\rho, s, r)$. Reduction under such a term corresponds to extending ρ with some additional work σ. The source of $\langle \rho|_s \sigma \rangle$ is $!(\rho, s, r)$ and the target is $!(\rho; \sigma, s, t)$.

Example 2.1. For any source term $\boldsymbol{p}$, we can give a derivation of the proposition:

$$[\![\boldsymbol{p}]\!]A \supset [\![\boldsymbol{!p}]\!][\![\boldsymbol{p}]\!]A$$

It is presented in Fig. 4 where we omit some of the rule names to save space. Also, $\Delta := u : A$ and $\Gamma := a : [\![\boldsymbol{p}]\!]A$.

Other theorems of TRC are:

- $[\![\boldsymbol{s}]\!](A \supset B) \supset [\![\boldsymbol{p}]\!]A \supset [\![\boldsymbol{s}\,\boldsymbol{p}]\!]B$
- $[\![\boldsymbol{s}]\!]A \supset A$

These may be seen as annotated versions of the S4 theorems:

- $\Box(A \supset B) \supset \Box A \supset \Box B$
- $\Box A \supset A$

Remark 1. *If we drop all annotations in the modality in theorems of TRC, then we obtain theorems of (minimal) S4. This follows from observing that applying this forgetful function on the typing rules, yields the system for S4 presented in [14]. Similarly, if we drop all references to rewrites we can prove all theorems of LP. This stems from observing that by performing this transformation on the typing rules, yields the Hypothetical Logic of Proofs [10].*

Basic Metatheory of TRC. This section presents some basic metatheoretic results on TRC. First note that $\Vdash \Delta; \Gamma \vdash \rho : s \rhd t : A$ implies $\rho : s \rhd t$ (*i.e.* the triple (ρ, s, t) satisfies the ST-predicate). In fact, $\Vdash \Delta; \Gamma \vdash \rho : s \rhd t : A$ implies (ρ, s, t), s and t are well-formed (*cf.* Def. 1.8) and, similarly, $\Vdash \Delta; \Gamma \vdash s : A$ implies s is well-formed. Typable terms can be recast as typable trivial rewrites.

Lemma 2.2 (Term as Trivial Rewrite). *If* $\Vdash_\pi \Delta; \Gamma \vdash s : A$, *then also* $\Vdash \Delta; \Gamma \vdash \mathfrak{s} : s \rhd s : A$.

The proof is by induction on π. We consider here two of the interesting cases. The first one is when $\Delta; \Gamma \vdash s : A$ is $\Delta; \Gamma \vdash !(\rho, s, t) : [\![\overline{s}]\!]B$ and π ends in

$$\frac{\Delta; \emptyset \vdash s, t : B \quad \Delta; \emptyset \vdash \rho : s \rhd t : B}{\Delta; \Gamma \vdash !(\rho, s, t) : [\![\overline{s}]\!]B}\ \mathsf{Bang}$$

$$\frac{a : A \in \Gamma}{\Delta;\Gamma \vdash a : A}\,\text{TVar} \qquad \frac{\Delta;\Gamma, a : A \vdash s : B}{\Delta;\Gamma \vdash \lambda a.s : A \supset B}\,\text{Abs} \qquad \frac{\Delta;\Gamma \vdash s : A \supset B \quad \Delta;\Gamma \vdash t : A}{\Delta;\Gamma \vdash s\,t : B}\,\text{App}$$

$$\frac{u : A \in \Delta}{\Delta;\Gamma \vdash u : A}\,\text{RVar} \qquad \frac{\Delta;\emptyset \vdash r, s : A \quad \Delta;\emptyset \vdash \rho : r \rhd s : A}{\Delta;\Gamma \vdash !(\rho, r, s) : \llbracket \overline{r} \rrbracket A}\,\text{Bang} \qquad \frac{\Delta;\Gamma \vdash s : \llbracket \boldsymbol{p} \rrbracket A \quad \Delta, u : A;\Gamma \vdash t : C}{\Delta;\Gamma \vdash \mathit{let}\ u \doteq s\ \mathit{in}\ t : C\{u/\boldsymbol{p}\}}\,\text{Let}$$

$$\frac{a : A \in \Gamma}{\Delta;\Gamma \vdash \underline{a} : a \rhd a : A}\,\text{R-Refl-TVar} \qquad \frac{u : A \in \Delta}{\Delta;\Gamma \vdash \underline{u} : u \rhd u : A}\,\text{R-Refl-RVar}$$

$$\frac{\Delta;\emptyset \vdash s, r, t : A \quad \Delta;\emptyset \vdash \rho : s \rhd r : A \quad \Delta;\emptyset \vdash \sigma : r \rhd t : A}{\Delta;\Gamma \vdash \langle \rho|_s \sigma \rangle : !(\rho, s, r) \rhd !(\rho;\sigma, s, t) : \llbracket \overline{s} \rrbracket A}\,\text{R-Bang} \qquad \frac{\Delta;\Gamma \vdash \rho : r \rhd s : A \quad \Delta;\Gamma \vdash \sigma : s \rhd t : A}{\Delta;\Gamma \vdash \rho;\sigma : r \rhd t : A}\,\text{R-Trans}$$

$$\frac{\Delta;\Gamma, a : A \vdash s : B \quad \Delta;\Gamma \vdash t : A}{\Delta;\Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\,t \rhd s\{a/t\} : B}\,\text{R-}\beta$$

$$\frac{\Delta;\emptyset \vdash \rho : s \rhd t : A \quad \Delta, u : A;\Gamma \vdash r : C}{\Delta;\Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \doteq !(\rho, s, t)\ \mathit{in}\ r \rhd r\{u/{}^{\mathrm{tgt}}\rho^{t}_{s}\} : C\{u/\overline{s}\}}\,\text{R-}\beta_{\Box}$$

$$\frac{\Delta;\Gamma, a : A \vdash \rho : s \rhd t : B}{\Delta;\Gamma \vdash \lambda a.\rho : \lambda a.s \rhd \lambda a.t : A \supset B}\,\text{R-Abs} \qquad \frac{\Delta;\Gamma \vdash \rho : s_1 \rhd t_1 : A \supset B \quad \Delta;\Gamma \vdash \sigma : s_2 \rhd t_2 : A}{\Delta;\Gamma \vdash \rho\,\sigma : s_1\,s_2 \rhd t_1\,t_2 : B}\,\text{R-App}$$

$$\frac{\Delta;\Gamma \vdash \rho : s_1 \rhd t_1 : \llbracket \boldsymbol{p} \rrbracket A \quad \Delta, u : A;\Gamma \vdash \sigma : s_2 \rhd t_2 : C}{\Delta;\Gamma \vdash \mathit{let}\ u \doteq \rho\ \mathit{in}\ \sigma : \mathit{let}\ u \doteq s_1\ \mathit{in}\ s_2 \rhd \mathit{let}\ u \doteq t_1\ \mathit{in}\ t_2 : C\{u/\boldsymbol{p}\}}\,\text{R-Let}$$

$$\frac{\Delta;\Gamma \vdash s : A \quad s \simeq t}{\Delta;\Gamma \vdash t : A}\,\text{SEq-T} \qquad \frac{\Delta;\Gamma \vdash \rho : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q}{\Delta;\Gamma \vdash \sigma : p \rhd q : A}\,\text{SEq-R}$$

Figure 3: Typing Rules

$$\dfrac{\emptyset;\Gamma \vdash a : \llbracket \boldsymbol{p} \rrbracket A \qquad \dfrac{\dfrac{\Delta;\emptyset \vdash \mathtt{u} : u \rhd u : A}{\Delta;\emptyset \vdash !(\mathtt{u}, u, u) : \llbracket \boldsymbol{u} \rrbracket A} \quad \dfrac{\Delta;\emptyset \vdash \mathtt{u};\mathtt{u} : u \rhd u : A}{\Delta;\emptyset \vdash !(\mathtt{u};\mathtt{u}, u, u) : \llbracket \boldsymbol{u} \rrbracket A} \quad \dfrac{\dfrac{}{u : A;\emptyset \vdash u : A} \quad \dfrac{}{\Delta;\emptyset \vdash \mathtt{u} : u \rhd u : A}}{\Delta;\emptyset \vdash \langle \mathtt{u}|_u \mathtt{u} \rangle : !(\mathtt{u}, u, u) \rhd !(\mathtt{u};\mathtt{u}, u, u) : \llbracket \boldsymbol{u} \rrbracket A}\,\text{R-Bang}}{\Delta;\Gamma \vdash !(\langle \mathtt{u}|_u \mathtt{u} \rangle, !(\mathtt{u}, u, u), !(\mathtt{u};\mathtt{u}, u, u)) : \llbracket !\boldsymbol{u} \rrbracket \llbracket \boldsymbol{u} \rrbracket A}\,\text{Bang}}{\dfrac{\emptyset;\Gamma \vdash \mathit{let}\ u \doteq a\ \mathit{in}\ !(\langle \mathtt{u}|_u \mathtt{u} \rangle, !(\mathtt{u}, u, u), !(\mathtt{u};\mathtt{u}, u, u)) : (\llbracket !\boldsymbol{u} \rrbracket \llbracket \boldsymbol{u} \rrbracket A)\{u/\boldsymbol{p}\}}{\emptyset;\emptyset \vdash \lambda a.\mathit{let}\ u \doteq a\ \mathit{in}\ !(\langle \mathtt{u}|_u \mathtt{u} \rangle, !(\mathtt{u}, u, u), !(\mathtt{u};\mathtt{u}, u, u)) : \llbracket \boldsymbol{p} \rrbracket A \supset \llbracket !\boldsymbol{p} \rrbracket \llbracket \boldsymbol{p} \rrbracket A}\,\text{Abs}}\,\text{Let}$$

Figure 4: Sample Type Derivation

Given $\Vdash \Delta;\emptyset \vdash t : B$, we may apply the IH[8] to obtain a derivation of $\Delta;\emptyset \vdash \mathtt{t} : t \rhd t : B$. Then we deduce

$$\frac{\Delta;\emptyset \vdash s, t : B \quad \Delta;\emptyset \vdash \rho : s \rhd t : B \quad \Delta;\emptyset \vdash \mathtt{t} : t \rhd t : B}{\Delta;\Gamma \vdash \langle \rho|_s \mathtt{t} \rangle : !(\rho, s, t) \rhd !(\rho;\mathtt{t}, s, t) : \llbracket \overline{s} \rrbracket B}\,\text{R-Bang}$$

We conclude that the judgement

$$\Delta;\Gamma \vdash \langle \rho|_s \mathtt{t} \rangle : !(\rho, s, t) \rhd !(\rho, s, t) : \llbracket \overline{s} \rrbracket B$$

is derivable from SEq-R. The other interesting case is when the derivation of $\Delta;\Gamma \vdash s : A$ ends in

$$\frac{\Delta;\Gamma \vdash t : A \quad t \simeq s}{\Delta;\Gamma \vdash s : A}\,\text{SEq-T}$$

In this case we resort to the IH, Lem. 1.5 and rule SEq-R.

Next we present two substitution lemmas. The first is straightforward to prove (it uses Lem. 1.6). The second (Lem. 2.4) however, is subtle and has guided the notion of substitution on rewrites that we presented in Sec. 1.

Lemma 2.3 (Term Substitution). *Suppose $\Delta;\Gamma, a : A \vdash s : B$ and $\Delta;\Gamma \vdash t : A$. Then $\Delta;\Gamma \vdash s\{a/t\} : B$.*

The second substitution lemma (Lem. 2.4) starts by assuming that $\Vdash \Delta;\emptyset \vdash \rho : s \rhd t : A$, $\Vdash \Delta;\emptyset \vdash s : A$ and $\Vdash \Delta;\emptyset \vdash t : A$. Note that typability of s and t from typability of ρ (upcoming Lem. 2.5) is proved with the help of Lem. 2.4 itself, so we have to assume typability of all three objects at this point. We use S below to denote either a term or a rewrite subject.

[8] This shows why we have included the judgement $\Delta;\emptyset \vdash s, t : B$ in the hypothesis of Bang: it allows for structural induction on the derivation of a term.

Lemma 2.4 (Rewrite Substitution). *Suppose* $\Vdash \Delta;\emptyset \vdash \rho : s \rhd t : A$, $\Vdash \Delta;\emptyset \vdash s : A$ *and* $\Vdash \Delta;\emptyset \vdash t : A$. *Suppose, moreover, also that* $\Vdash \Delta, u : A;\Gamma \vdash S : B$.

(a) $S = \sigma : p \rhd q$ *implies*
$$\Vdash \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho^t_s\} : p\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$
(b) $S = \sigma : p \rhd q$ *implies*
$$\Vdash \Delta, u : A;\Gamma \vdash p : B \text{ and } \Vdash \Delta, u : A;\Gamma \vdash q : B.$$
(c) $S = p$ *implies*
$$\Vdash \Delta;\Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}.$$
(d) $S = p$ *implies*
$$\Vdash \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$
(e) $S = p$ *implies*
$$\Vdash \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}.$$

The proof is by simultaneous induction on $\Vdash \Delta, u : A;\Gamma \vdash S : B$.
We conclude this section with a result that states that the source and target of a typable rewrite are typable. The proof is by induction on the derivation of $\Delta;\Gamma \vdash \rho : s \rhd t : A$ and relies on the Term Substitution Lemma (Lem. 2.3(c)), Term as a Trivial Rewrite Lemma (Lem. 2.2), and the Rewrite Substitution Lemma (Lem. 2.4).

Lemma 2.5. $\Vdash \Delta;\Gamma \vdash \rho : s \rhd t : A$ *implies* $\Vdash \Delta;\Gamma \vdash s : A$ *and also* $\Vdash \Delta;\Gamma \vdash t : A$.

One of the key cases in the proof is when the derivation of the typing judgement $\Delta;\Gamma \vdash \rho : s \rhd t : A$ ends in an instance of the rule R-$\beta_\Box$:
$$\frac{\Delta;\emptyset \vdash \rho_1 : p \rhd q : A \quad \Delta, u : A;\Gamma \vdash r : C}{\Delta;\Gamma \vdash \mathbf{bb}(!(\rho_1, p, q), u.r) : \mathit{let}\ u \doteq !(\rho_1, p, q)\ \mathit{in}\ r \rhd r\{u/^{\mathsf{tgt}}\rho_1{}^q_p\} : C\{u/\overline{p}\}}$$
By the IH on $\Delta;\emptyset \vdash \rho_1 : p \rhd q : A$, we deduce $\Delta;\emptyset \vdash p : A$ and $\Delta;\emptyset \vdash q : A$. This allows us to use the Rewrite Substitution Lemma (Lem. 2.4) for typing $r\{u/^{\mathsf{tgt}}\rho_1{}^q_p\}$. For typing $\mathit{let}\ u \doteq !(\rho_1, p, q)\ \mathit{in}\ r$ we use Bang, then Let.

3 REWRITE EXTENSION

In the *Typed Rewrite Calculus* rather than reduction on terms we have *extension of rewrites*. Extension is similar to reduction in the lambda calculus but it applies to rewrites, it is defined *modulo* structural equivalence, and it leaves a trail. A rewrite σ extends a rewrite ρ if σ results from appending a rewrite step to ρ, modulo structural equivalence. For example, given the rewrite $I(I\underline{a}) : I(Ia) \rhd I(Ia)$ one has the following extension sequence of rewrites to normal form:
$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)) : I(Ia) \rhd Ia \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array} \quad (2)$$
The rewrite $I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a)$ is in normal form since it cannot be extended further. This section proves two results on rewrite extension, namely that is preserves typability (*cf.* Prop. 3.7) and that it is strongly normalizing (*cf.* Cor. 3.12). Confluence fails to hold for trivial reasons (as explained below).

Rewrite and Term Extension. We now define rewrite extension formally. Since terms may contain rewrites, we introduce two extension judgements:
$$\begin{array}{rl} r \rightarrowtail s & \textbf{Term extension} \\ \rho : r \rhd s \rightarrowtail \sigma : p \rhd q & \textbf{Rewrite extension} \end{array}$$
Term r extends to s, written $r \rightarrowtail s$, iff:
$$\exists r', s' \text{ s.t. } r \simeq r' \mapsto s' \simeq s$$
Rewrite ρ extends to σ, written $\rho : r \rhd s \rightarrowtail \sigma : p \rhd q$, iff:
$$\exists \rho', \sigma' \text{ s.t. } \rho \simeq \rho' : r \rhd s \text{ and } \rho' : r \rhd s \mapsto \sigma' : r \rhd q \text{ and } \sigma' \simeq \sigma : r \rhd q$$
The judgements for one-step extension $r \mapsto s$ and $\rho : r \rhd s \mapsto \sigma : r \rhd q$ are defined by the rules of Fig. 5. The rules above the horizontal line apply to terms and the rules below it to rewrites. These rules are mostly self-explanatory. For example, E-β, states that if the "current" rewrite is of the form $\rho : s \rhd (\lambda a.t_1)\,t_2$, then it can be extended by adding a witness to a β-rewrite step that is sourced at its target, namely $\rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}$. Perhaps worth mentioning is that in the congruence rule for $\langle \rho|_r\sigma\rangle$, namely E-BangR, it is σ that may be extended, but not ρ.
We will only be interested in extension on well-formed terms and well-formed rewrites. Term and rewrite extension preserves well-formedness (*cf.* Def. 1.8):

Lemma 3.1 (Extension preserves well-formedness). $s \rightarrowtail t$ *and* s *well-formed implies* t *well-formed. Similarly,* $\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ *and* (ρ, s, t) *well-formed implies* (ρ', s, t') *well-formed.*

We next set up some auxiliary notions and results required for proving that extension of rewrites does preserve types. We begin with the definition of a **step rewrite**, a rewrite that corresponds to one reduction step. In other words, a rewrite that models the contraction of exactly one redex.

Definition 3.2 (Step Rewrite). Step rewrites are defined by the following grammar:
$$\begin{array}{rcl} \xi & ::= & \mathbf{ba}(a.s, r) \mid \mathbf{bb}(s, u.r) \mid \lambda a.\xi \mid \xi\,\mathfrak{s} \mid \mathfrak{s}\,\xi \\ & & \mid \mathit{let}\ u \doteq \xi\ \mathit{in}\,\mathfrak{s} \mid \mathit{let}\ u \doteq \mathfrak{s}\ \mathit{in}\,\xi \mid \langle \rho|_s\xi\rangle \end{array}$$

The next result formalizes what is intuitively clear from the definition of extension, namely that extending a rewrite consists in suffixing a step:

Lemma 3.3 (Extension adds a step). $\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ *implies there exists* ξ *s.t.* $\rho' \simeq \rho; \xi$. *Moreover,* (ρ, s, t) *well-formed implies* (ξ, t, t') *well-formed.*

In our upcoming proof of Extension Reduction (Prop. 3.7) we need to extract the suffixed step from the extension of a rewrite, and analyze its form. These steps will be broken down into a step context and redex.

Definition 3.4 (Step Contexts). Step contexts are defined by the following grammar:
$$\mathsf{C} \quad ::= \quad \Box \mid \mathsf{C}\,\mathfrak{s} \mid \mathfrak{s}\,\mathsf{C} \mid \lambda a.\mathsf{C} \mid \mathit{let}\ u \doteq \mathsf{C}\ \mathit{in}\,\mathfrak{s} \mid \mathit{let}\ u \doteq \mathfrak{s}\ \mathit{in}\,\mathsf{C} \mid \langle \rho|_s\mathsf{C}\rangle$$
There are three notions of filling the hole of a step context. Simple replacement of a rewrite ρ for the hole is written $\mathsf{C}\langle\rho\rangle$. Such a replacement produces a rewrite. Then we have *source filling* and *target filling*. The former is denoted $\mathsf{C}[p]^{\mathsf{src}}$ and the latter $\mathsf{C}[\rho, p, q]^{\mathsf{tgt}}$. These notions of filling produce terms. They are used in conjunction to denote the source and target of the rewrite $\mathsf{C}\langle\rho\rangle$ (*cf.* Lem. 3.5). Both are defined below:

$$\frac{s \mapsto s'}{\lambda a.s \mapsto \lambda a.s'}\ \text{E-AbsT} \qquad \frac{s \mapsto s'}{s\,t \mapsto s'\,t}\ \text{E-AppTL} \qquad \frac{t \mapsto t'}{s\,t \mapsto s\,t'}\ \text{E-AppTR} \qquad \frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{!(\rho, s, t) \mapsto !(\rho', s, t')}\ \text{E-BangT}$$

$$\frac{s \mapsto s'}{let\ u \doteq s\ in\,t \mapsto let\ u \doteq s'\ in\,t}\ \text{E-LetTL} \qquad \frac{t \mapsto t'}{let\ u \doteq s\ in\,t \mapsto let\ u \doteq s\ in\,t'}\ \text{E-LetTR}$$

$$\frac{}{\rho : s \rhd (\lambda a.t_1)\,t_2 \mapsto \rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}}\ \text{E-}\beta$$

$$\frac{}{\rho : s \rhd let\ u \doteq !(\sigma, p, q)\ in\,t \mapsto \rho; \mathbf{bb}(!(\sigma, p, q), u.t) : s \rhd t\{u/^{\mathrm{tgt}}\sigma_p^q\}}\ \text{E-}\beta_\Box$$

$$\frac{\rho : s \rhd t \mapsto \rho : s \rhd t'}{\lambda a.\rho : \lambda a.s \rhd \lambda a.t \mapsto \lambda a.\rho' : \lambda a.s \rhd \lambda a.t'}\ \text{E-AbsR}$$

$$\frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\langle \rho|_r \sigma\rangle : !(\rho, r, s) \rhd !(\rho; \sigma, r, t) \mapsto \langle \rho|_r \sigma'\rangle : !(\rho, r, s) \rhd !(\rho; \sigma', r, t')}\ \text{E-BangR} \qquad \frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\rho; \sigma : r \rhd t \mapsto \rho; \sigma' : r \rhd t'}\ \text{E-Trans}$$

$$\frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{\rho\,\sigma : s\,p \rhd t\,q \mapsto \rho'\,\sigma : s\,p \rhd t'\,q}\ \text{E-AppRL} \qquad \frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\rho\,\sigma : p\,s \rhd q\,t \mapsto \rho\,\sigma' : p\,s \rhd q\,t'}\ \text{E-AppRR}$$

$$\frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{let\ u \doteq \rho\ in\,\sigma : let\ u \doteq s\ in\,p \rhd let\ u \doteq t\ in\,q \mapsto let\ u \doteq \rho'\ in\,\sigma : let\ u \doteq s\ in\,p \rhd let\ u \doteq t'\ in\,q}\ \text{E-LetRL}$$

$$\frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{let\ u \doteq \rho\ in\,\sigma : let\ u \doteq p\ in\,s \rhd let\ u \doteq q\ in\,t \mapsto let\ u \doteq \rho\ in\,\sigma' : let\ u \doteq p\ in\,s \rhd let\ u \doteq q\ in\,t'}\ \text{E-LetRR}$$

Figure 5: Rewrite Extension

$$\begin{array}{rcl}
\Box[t]^{\mathrm{src}} & ::= & t \\
(\mathrm{C}\ \mathfrak{s})[t]^{\mathrm{src}} & ::= & \mathrm{C}[t]^{\mathrm{src}}\,s \\
(\mathfrak{s}\ \mathrm{C})[t]^{\mathrm{src}} & ::= & s\,\mathrm{C}[t]^{\mathrm{src}} \\
(\lambda \mathrm{a}.\mathrm{C})[t]^{\mathrm{src}} & ::= & \lambda a.\mathrm{C}[t]^{\mathrm{src}} \\
(let\ \mathrm{u} \doteq \mathrm{C}\ in\,\mathfrak{s})[t]^{\mathrm{src}} & ::= & let\ u \doteq \mathrm{C}[t]^{\mathrm{src}}\ in\,s \\
(let\ \mathrm{u} \doteq \mathfrak{s}\ in\,\mathrm{C})[t]^{\mathrm{src}} & ::= & let\ u \doteq s\ in\,\mathrm{C}[t]^{\mathrm{src}} \\
\langle \rho|_{\mathrm{s}}\mathrm{C}\rangle[t]^{\mathrm{src}} & ::= & !(\rho, s, \mathrm{C}[t]^{\mathrm{src}})
\end{array}$$

$$\begin{array}{rcl}
\Box[\rho, p, q]^{\mathrm{tgt}} & ::= & q \\
(\mathrm{C}\ \mathfrak{s})[\rho, p, q]^{\mathrm{tgt}} & ::= & \mathrm{C}[\rho, p, q]^{\mathrm{tgt}}\,s \\
(\mathfrak{s}\ \mathrm{C})[\rho, p, q]^{\mathrm{tgt}} & ::= & s\,\mathrm{C}[\rho, p, q]^{\mathrm{tgt}} \\
(\lambda \mathrm{a}.\mathrm{C})[\rho, p, q]^{\mathrm{tgt}} & ::= & \lambda a.\mathrm{C}[\rho, p, q]^{\mathrm{tgt}} \\
(let\ \mathrm{u} \doteq \mathrm{C}\ in\,\mathfrak{s})[\rho, p, q]^{\mathrm{tgt}} & ::= & let\ u \doteq \mathrm{C}[\rho, p, q]^{\mathrm{tgt}}\ in\,s \\
(let\ \mathrm{u} \doteq \mathfrak{s}\ in\,\mathrm{C})[\rho, p, q]^{\mathrm{tgt}} & ::= & let\ u \doteq s\ in\,\mathrm{C}[\rho, p, q]^{\mathrm{tgt}} \\
\langle \sigma|_{\mathrm{s}}\mathrm{C}\rangle[\rho, p, q]^{\mathrm{tgt}} & ::= & !(\sigma; \mathrm{C}\langle\rho\rangle, s, \mathrm{C}[\rho, p, q]^{\mathrm{tgt}})
\end{array}$$

The interesting clause in the filling operations above is when the step context is $\langle \sigma|_s \mathrm{C}\rangle$. In particular, in the case of target filling, note how, in addition to actually inserting the target term q (as may be seen from the case for $\Box$), it suffixes a copy of the argument step itself: $\sigma; \mathrm{C}\langle\rho\rangle$. For example, if $\mathrm{C} = \langle \sigma|_s \Box\rangle$, then the source of $\mathrm{C}\langle \mathbf{ba}(a.p, q)\rangle$ will be $\mathrm{C}[(\lambda a.p)\,q]^{\mathrm{src}} = !(\sigma, s, (\lambda a.p)\,q)$ and its target $\mathrm{C}[\mathbf{ba}(a.p, q), (\lambda a.p)\,q, p\{a/q\}]^{\mathrm{tgt}} = !(\sigma; \mathbf{ba}(a.p, q), s, p\{a/q\})$

Lemma 3.5 (Form of a Step). *Let ξ be a well-formed step rewrite. Then one of the two following hold.*

(a) $\xi = \mathrm{C}\langle \mathbf{ba}(a.s, t)\rangle$ *and*

$$\mathrm{C}\langle \mathbf{ba}(a.s, t)\rangle : \mathrm{C}[(\lambda a.s)\,t]^{\mathrm{src}} \rhd \mathrm{C}[\mathbf{ba}(a.s, t), (\lambda a.s)\,t, s\{a/t\}]^{\mathrm{tgt}}$$

(b) $\xi = \mathrm{C}\langle \mathbf{bb}(!(\rho, p, q), u.r)\rangle$ *and*

$$\mathrm{C}\langle \mathbf{bb}(!(\rho, p, q), u.r)\rangle : \mathrm{C}[let\ u \doteq !(\rho, p, q)\ in\,r]^{\mathrm{src}} \rhd$$
$$\mathrm{C}[\mathbf{bb}(!(\rho, p, q), u.r), let\ u \doteq !(\rho, p, q)\ in\,r, r\{u/^{\mathrm{tgt}}\rho_p^q\}]^{\mathrm{tgt}}$$

The proof is by induction on ξ. The only interesting case is when $\xi = \langle \sigma|_m \xi'\rangle$. Since ξ is well-formed we know $\sigma : m \rhd n$ and $\xi' : n \rhd o$ for some m, n, o. By the IH on ξ' either case (a) or (b) holds. Assume it is (a) (the case for (b) is similar and hence omitted) then $\xi' = \mathrm{C}'\langle \mathbf{ba}(a.s, r)\rangle$, for some C', a, s, r. By Lem. 1.2, $n \simeq \mathrm{C}'[(\lambda a.s)\,t]^{\mathrm{src}}$. Then $\sigma : m \rhd \mathrm{C}'[(\lambda a.s)\,t]^{\mathrm{src}}$. But then

$$\langle \sigma|_m \mathrm{C}'\langle \mathbf{ba}(a.s, r)\rangle\rangle : p \rhd q$$

where

$$p := !(\sigma, m, \mathrm{C}[(\lambda a.s)\,t]^{\mathrm{src}})$$
$$q := !(\sigma; \mathrm{C}\langle \mathbf{ba}(a.s, r)\rangle, m, \mathrm{C}[\mathbf{bb}(!(\rho, p, q), u.r), let\ u \doteq !(\rho, p, q)\ in\,r, r\{u/^{\mathrm{tgt}}\rho_p^q\}]^{\mathrm{tgt}})$$

Which concludes the case.

Finally, for our Subject Extension result, it will not suffice to break down a step rewrite into its components, as described above, but also to ensure typability. Typability of the source of a step suffices to type the step itself.

Lemma 3.6 (Step Typability). *$\xi : s \rhd t$ and $\Delta; \Gamma \vdash s : A$ implies $\Delta; \Gamma \vdash \xi : s \rhd t : A$.*

We are now in condition to prove the main result of this section, namely that extension preserves types for both terms and rewrites.

Proposition 3.7 (Subject Extension).

(a) $\Vdash_\pi \Delta;\Gamma \vdash s : A$ *and* $s \rightarrowtail s'$ *implies* $\Vdash \Delta;\Gamma \vdash s' : A$.
(b) $\Vdash_\pi \Delta;\Gamma \vdash \rho : s \rhd t : A$ *and* $\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ *implies* $\Vdash \Delta;\Gamma \vdash \rho' : s \rhd t' : A$.

We first prove (by induction on π) that

(a) $\Vdash_\pi \Delta;\Gamma \vdash s : A$ and $s \mapsto s'$ implies $\Vdash \Delta;\Gamma \vdash s' : A$.
(b) $\Vdash_\pi \Delta;\Gamma \vdash \rho : s \rhd t : A$ and $\rho : s \rhd t \mapsto \rho' : s \rhd t'$ implies $\Vdash \Delta;\Gamma \vdash \rho' : s \rhd t' : A$.

Then we conclude from the fact that $\Vdash \Delta;\Gamma \vdash s : A$ and $s \simeq s'$ implies $\Vdash \Delta;\Gamma \vdash s' : A$ via SEq-T. Similarly, $\Vdash \Delta;\Gamma \vdash \rho : s \rhd t : A$ and $\rho \simeq \rho' : s \rhd t$ implies $\Vdash \Delta;\Gamma \vdash \rho' : s \rhd t : A$ via SEq-R. We focus on three interesting cases:

- The derivation ends in:

$$\frac{\Delta;\emptyset \vdash r, s : A \quad \Delta;\emptyset \vdash \rho_1 : r \rhd s : A}{\Delta;\Gamma \vdash !(\rho_1, r, s) : [\![\overline{r}]\!]A}\ \mathsf{Bang}$$

  Then $\rho_1 : r \rhd s \mapsto \rho_1' : r \rhd s'$. By the IH we have

$$\Delta;\emptyset \vdash \rho_1' : r \rhd s' : A \tag{3}$$

  By Lem. 2.5 on (3) $\Delta;\emptyset \vdash s' : A$. Thus we can use Bang to deduce $\Delta;\Gamma \vdash !(\rho_1', r, s') : [\![\overline{r}]\!]A$.

- The derivation ends in:

$$\frac{\Delta;\emptyset \vdash s, r, t : A \quad \Delta;\emptyset \vdash \rho_1 : s \rhd r : A \quad \Delta;\emptyset \vdash \rho_2 : r \rhd p : A}{\Delta;\Gamma \vdash \langle \rho_1|_s \rho_2\rangle : !(\rho_1, s, r) \rhd !(\rho_1;\rho_2, s, p) : [\![\overline{s}]\!]A}\ \mathsf{R\text{-}Bang}$$

  Then $\rho' = \langle \rho_1|_s \rho_2'\rangle$ and $\langle \rho_1|_s \rho_2\rangle \mapsto \langle \rho_1|_s \rho_2'\rangle$ follows from $\rho_2 : r \rhd p \mapsto \rho_2' : r \rhd p'$. By the IH $\Delta;\emptyset \vdash \rho_2' : r \rhd p' : A$. By Lem. 2.5 $\Delta;\emptyset \vdash p' : A$. Using R-Bang we deduce

$$\Delta;\Gamma \vdash \langle \rho_1|_s \rho_2'\rangle : !(\rho_1, s, r) \rhd !(\rho_1;\rho_2', s, p') : [\![\overline{s}]\!]A$$

- The derivation ends in:

$$\frac{\Delta;\Gamma, a : B \vdash p : A \quad \Delta;\Gamma \vdash q : B}{\Delta;\Gamma \vdash \mathbf{ba}(a.p, q) : (\lambda a.p)\, q \rhd p\{a/q\} : A}\ \mathsf{R\text{-}\beta}$$

  Suppose $\mathbf{ba}(a.p, q) : (\lambda a.p)\, q \rhd p\{a/q\} \mapsto \rho'$. Then by Lem. 3.3, $\rho' \simeq \mathbf{ba}(a.p, q);\xi$ for some step ξ. By Lem. 3.5, one of the two following hold.

  (a) $\xi = \mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle$ and
$$\mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle : \mathsf{C}[(\lambda a.m)\, n]^{\mathsf{src}} \rhd \mathsf{C}[\mathbf{ba}(b.m, n), (\lambda b.m)\, n, m\{b/n\}]^{\mathsf{tgt}}$$

  (b) or $\xi = \mathsf{C}\langle \mathbf{bb}(!(\sigma, r_1, r_2), u.n)\rangle$ and
$$\mathsf{C}\langle \mathbf{bb}(!(\sigma, r_1, r_2), u.n)\rangle : \mathsf{C}[\mathit{let}\ u \doteq !(\sigma, r_1, r_2)\ \mathit{in}\ n]^{\mathsf{src}} \rhd \mathsf{C}[\mathbf{bb}(!(\sigma, r_1, r_2), u.n), \mathit{let}\ u \doteq !(\sigma, r_1, r_2)\ \mathit{in}\ n, n\{u/^{\mathsf{tgt}}\sigma_{r_1}^{r_2}\}]^{\mathsf{tgt}}$$

  By Lem. 2.5 $p\{a/q\}$ is typable. That is, $\Delta;\Gamma \vdash p\{a/q\} : A$. By Lem. 3.6, in the first case above we obtain

$$\Delta;\Gamma \vdash \mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle : o_1 \rhd o_2 : A$$

  where

$$\begin{array}{lcl} o_1 & := & \mathsf{C}[(\lambda b.m)\, n]^{\mathsf{src}} \\ o_2 & := & \mathsf{C}[\mathbf{ba}(b.m, n), (\lambda b.m)\, n, m\{b/n\}]^{\mathsf{tgt}} \end{array}$$

  Similarly, we obtain

$$\Delta;\Gamma \vdash \mathsf{C}\langle \mathbf{bb}(!(\sigma, r_1, r_2), u.n)\rangle : o_1 \rhd o_2 : A$$

  in the second case above, where

$$\begin{array}{lcl} o_1 & := & \mathsf{C}[\mathit{let}\ u \doteq !(\sigma, r_1, r_2)\ \mathit{in}\ n]^{\mathsf{src}} \\ o_2 & := & \mathsf{C}[\mathbf{bb}(!(\sigma, r_1, r_2), u.n), \mathit{let}\ u \doteq !(\sigma, r_1, r_2)\ \mathit{in}\ n, n\{u/^{\mathsf{tgt}}\sigma_{r_1}^{r_2}\}]^{\mathsf{tgt}} \end{array}$$

  We conclude from Trans.

Confluence and Strong Normalization. Rewrite extension is certainly not confluent. For example, the rewrite $I(I\underline{a}) : I(Ia) \rhd I(Ia)$ from above, in addition to be extended as depicted in (2), can also be extended as follows:

$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & \mathbf{ba}(b.b, Ia) : I(Ia) \rhd Ia \\ \rightarrowtail & \mathbf{ba}(b.b, Ia);\mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array}$$

Clearly $I(\mathbf{ba}(b.b, a));\mathbf{ba}(b.b, a) \not\simeq \mathbf{ba}(b.b, Ia);\mathbf{ba}(b.b, a)$. This is expected since structural equivalence does not include permutation of redexes as in Lévy permutation equivalence. However, it is strongly normalizing. This is proved by a simple mapping, called the *target mapping*, of rewrite extension steps to β-steps in the simply typed lambda calculus.

$$\begin{array}{rcl} U(a) & := & a \\ U(u) & := & u \\ U(\lambda a.s) & := & \lambda a.U(s) \\ U(s\, t) & := & U(s)\, U(t) \\ U(!(\rho, s, t)) & := & U(t) \\ U(\mathit{let}\ u \doteq s\ \mathit{in}\ t) & := & (\lambda u.U(t))\, U(s) \\ \\ U(\rho : s \rhd t) & := & U(t) \\ \\ U(P) & := & P \\ U(A \supset B) & := & U(A) \supset U(B) \\ U([\![s]\!]A) & := & U(A) \end{array}$$

First two simple results, both proved by induction, relate the target mapping, substitution and extension.

Lemma 3.8.

(a) $U(r\{a/s\}) = U(r)\{a := U(s)\}$ *for any term* r.
(b) $U(r\{u/^{\mathsf{tgt}}\rho_s^t\}) = U(r)\{u := U(t)\}$
(c) $U(A) = U(A\{u/\boldsymbol{t}\})$ *for any type* A.

Lemma 3.9. *(a) If* $r \simeq r'$ *then* $U(r) = U(r')$.
(b) If $\rho \simeq \sigma : r \rhd s$ *then* $U(\rho : r \rhd s) = U(s) = U(\sigma : r \rhd s)$.

Rewrite extension steps map to β-reduction steps. The proof is by induction using Lem. 3.9 and Lem. 3.8.

Lemma 3.10. *(a)* $r \rightarrowtail s$ *implies* $U(r) \to_\beta U(s)$
(b) $\rho : r \rhd s \rightarrowtail \sigma : p \rhd q$ *implies* $U(\rho : r \rhd s) \to_\beta U(\sigma : p \rhd q)$

Finally, we address preservation of typability. If $\Delta = u_1 : A_1, \ldots, u_n : A_n$ is a set of rewrite hypotheses and $\Gamma = a_1 : B_1, \ldots, a_m : B_m$ is a set of term hypotheses, let us write $U(\Delta;\Gamma)$ for the typing context $u_1 : U(A_1), \ldots, u_n : U(A_n), a_1 : U(B_1), \ldots, a_n : U(B_n)$ in $\lambda^{\to}$.

Lemma 3.11.

(a) $\Vdash_\pi \Delta;\Gamma \vdash s : A$ *implies* $U(\Delta;\Gamma) \vdash U(s) : U(A)$ *in* $\lambda^{\to}$.
(b) $\Vdash_\pi \Delta;\Gamma \vdash \rho : s \rhd t : A$ *implies* $U(\Delta;\Gamma) \vdash U(\rho : s \rhd t) : U(A)$ *in* $\lambda^{\to}$.

Proof. The proof is by simultaneous induction on the derivation π. Most cases are straightforward by resorting to the IH when appropriate. Some of the interesting cases are:

- Bang: Let $\Delta;\Gamma \vdash !(\rho, r, s) : [\![\overline{r}]\!]A$ be derived from $\Delta;\emptyset \vdash r, s : A$ and $\Delta;\emptyset \vdash \rho : r \rhd s : A$. By IH on the last premise we have that $U(\Delta;\emptyset) \vdash U(\rho : r \rhd s) : U(A)$. Moreover, $U(\rho : r \rhd s) = U(s)$ and $U([\![\overline{r}]\!]A) = U(A)$, so we may conclude by weakening.
- Let: Let $\Delta;\Gamma \vdash \mathit{let}\ u \doteq s\ \mathit{in}\ t : C\{u/\boldsymbol{p}\}$ be derived from $\Delta;\Gamma \vdash s : [\![\boldsymbol{p}]\!]A$ and $\Delta, u : A;\Gamma \vdash t : C$. By IH we have that $U(\Delta;\Gamma) \vdash U(s) : U([\![\boldsymbol{p}]\!]A) = U(A)$ and that $U(\Delta;\Gamma), u : U(A) \vdash U(t) : U(C)$. So $U(\Delta;\Gamma) \vdash U(\mathit{let}\ u \doteq s\ \mathit{in}\ t) = (\lambda u.U(t))\,U(s) : U(C)$. We conclude by Lem. 3.8, given that $U(C) = U(C\{u/\boldsymbol{p}\})$.
- R-Bang: Let $\Delta;\Gamma \vdash \langle\rho|_s\sigma\rangle : !(\rho, s, r) \rhd !(\rho;\sigma, s, t) : [\![\overline{s}]\!]A$ be derived from $\Delta;\emptyset \vdash s, r, t : A$ and $\Delta;\emptyset \vdash \rho : s \rhd r : A$ and $\Delta;\emptyset \vdash \sigma : r \rhd t : A$. By IH we have that $U(\Delta;\emptyset) \vdash U(t) : U(A)$. Moreover, $U(\langle\rho|_s\sigma\rangle : !(\rho, s, r) \rhd !(\rho;\sigma, s, t)) = U(!(\rho;\sigma, s, t)) = U(t)$ and $U([\![\overline{s}]\!]A) = U(A)$, so we may conclude by weakening.
- R-β: Let $\Delta;\Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\,t \rhd s\{a/t\} : B$ be derived from $\Delta;\Gamma, a : A \vdash s : B$ and $\Delta;\Gamma \vdash t : A$. By IH, $U(\Delta;\Gamma), a : U(A) \vdash U(s) : U(B)$ and $U(\Delta;\Gamma) \vdash U(t) : U(A)$. So by the (standard) substitution lemma $U(\Delta;\Gamma) \vdash U(s)\{a := U(t)\} : U(B)$. Moreover $U(\mathbf{ba}(a.s, t) : (\lambda a.s)\,t \rhd s\{a/t\}) = U(s\{a/t\}) = U(s)\{a := U(t)\}$ by Lem. 3.8, which concludes this case.
- R-$\beta_\Box$: Let

$$\Delta;\Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \doteq !(\rho, s, t)\ \mathit{in}\ r \rhd r\{u/^{\mathrm{tgt}}\rho^t_s\} : C\{u/\overline{s}\}$$

be derived from $\Delta;\emptyset \vdash \rho : s \rhd t : A$ and $\Delta, u : A;\Gamma \vdash r : C$. By IH $U(\Delta;\emptyset) \vdash U(\rho : s \rhd t) : U(A)$ and $U(\Delta;\emptyset), u : U(A) \vdash U(r) : U(C)$. From the first condition we have that $U(\Delta;\emptyset) \vdash U(t) : U(A)$ holds by definition. By the (standard) substitution lemma we have that $U(\Delta;\emptyset) \vdash U(r)\{u := U(t)\} : U(C)$. Moreover,

$$\begin{array}{cll} & U(\mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \doteq !(\rho, s, t)\ \mathit{in}\ r \rhd r\{u/^{\mathrm{tgt}}\rho^t_s\}) & \\ = & U(r\{u/^{\mathrm{tgt}}\rho^t_s\}) & \\ = & U(r)\{u := U(t)\} & \text{by Lem. 3.8} \end{array}$$

and $U(C) = U(C\{u/\overline{s}\})$ by Lem. 3.8, so by weakening we conclude this case.

□

The following is an immediate consequence of Lem. 3.11, Lem. 3.10, and strong normalization of the simply typed lambda calculus [6].

Corollary 3.12. *Rewrite extension is strongly normalizing.*

4 RELATED WORK

Propositions-as-types for modal logic has an extensive body of literature which would be impossible to summarize here. We refer the reader to [15, 18] for further references. We focus instead on Justification Logic and the Logic of Proofs. Artemov introduced LP in [1, 2]. It was presented as the missing link between the provability interpretation of classical S4 and provability in PA (Peano Arithmetic). The more general setting of Justification Logic was presented in [4]. A recent survey is [16] and recent texts [3, 19]. For Natural Deduction and Sequent Calculus presentations consult [2, 5, 11]. Computational interpretation of proofs in JL is studied in [5, 7, 9, 25–27]. The first-order logic of proofs is studied in [28]. Regarding rewrites, as mentioned, they are studied in [29, 30] for first-order rewriting and are dubbed *proof terms*. See Rem. 8.3.25 in [29] for additional references. They are closely related to Meseguer's Rewriting Logic [22]. Proof terms are used as a tool to prove various properties of first-order term rewriting systems (such as that various notions of equivalence of reductions coincide). A theory of proof terms for the typed lambda calculus was developed by Hilken [17]; however proof terms themselves are not reified as terms. An extension to arbitrary higher-order term rewriting systems was given by Bruggink in [12] with the drawbacks discussed in the introduction. A notion of proof term was also developed for infinitary (first-order) rewriting [20] and used for studying equivalence of infinitary reductions [21].

Dependent types (DT) [23] includes types that depend on terms and corresponds to the propositions-as-types interpretation of first-order logic. DT lacks the primitive finite reflection principle from LP. However, the Logic of Proofs is similar to DT in that the modality also depends on terms. This leads one to ponder on the suitability of resorting to a rule such as DT's conversion rule to circumvent the issue from the introduction, namely that normalisation of the proof of $\Delta;\emptyset \vdash A \,|\, s$ in (1) will produce a proof of $\Delta;\emptyset \vdash A \,|\, t$, for some t different from s. The conversion rule from DT states that from $\Gamma \vdash A$ and $A \equiv B$, one infers $\Gamma \vdash B$, where $A \equiv B$ is some notion of equivalence of types that typically includes conversion for terms. This seems problematic in at least two aspects. Our sequents have the form $\Gamma;\Delta \vdash A \,|\, s$, where s keeps track of the current derivation under construction. Conversion would replace A with an equivalent type B, resulting in $\Gamma;\Delta \vdash B \,|\, s$. However s, whose role is to encode the current derivation being constructed, would have to be updated to reflect the application of conversion, to say $eq(s, e)$ where e is an encoding of the derivation of $A \equiv B$:

$$\frac{\Gamma;\Delta \vdash A \,|\, s \quad A \equiv B \,|\, e}{\Gamma;\Delta \vdash B \,|\, eq(s, e)}\ \mathsf{Conv}$$

Since $!s$ from (1) and $eq(s, e)$ are distinct, the issue with closure under normalisation remains. Another potential issue with a conversion like rule to address the issue with (1) is that in hypothesis of the form $[\![s]\!]A$ there is no assumption that s is a proof of A (in fact, it may not be typable at all). The reason for this is that the Logic of Proofs is capable of realizing all IS4 theorems. Take, for instance, the IS4 theorem $\Box(\neg\Box\bot)$ where $\neg A$ abbreviates $A \supset \bot$. In our TRC, there are s and t such that $[\![s]\!](\neg[\![t]\!]\bot)$ is derivable[9]. Note, however, that there is no proof of $\bot$ given that the system is consistent. A conversion like rule cannot assume even typability of s in $[\![s]\!]A$.

5 CONCLUSION

We present a novel propositions-as-types interpretation of the Logic of Proofs, dubbed the *Typed Rewrite Calculus* or TRC, in which reductions between terms are reified as terms. Consider simply typed terms s and t and a reduction from s to t. Such reductions may be expressed as terms too, called rewrites or proof terms. An example rewrite ρ is $\mathbf{ba}(a.a, b)\,Ib; b\,\mathbf{ba}(a.a, b)$ (*cf.* the introduction), denoting the reduction sequence from $(Ib)(Ib)$ to bb. What is the type of a rewrite? If A is the type of the source $(Ib)(Ib)$, then we propose the modal type $[\![(Ib)(Ib)]\!]A$ as the type of ρ. More generally, $[\![s]\!]A$ is the type of rewrites with source s. The salient term in our term assignment for the Logic of Proofs is $!(\rho, s, t)$ denoting a reduction from source term s to target t via rewrite ρ. We assign it a modal type. Reduction under a "!" is understood as extending

[9]Take s to be $\lambda a^{[\![t]\!]\bot}.\mathit{let}\ u \doteq a\ \mathit{in}\ u$, for any t, where we have decorated the type of a for clarity.

the rewrite ρ with further work σ, leading to the rewrite $\langle\rho|_s\sigma\rangle$. We devise a notion of structural equivalence for our rewrites that includes composition of rewrites such as $\langle\rho|_\sigma s\rangle$. We then introduce a notion of "reduction" on rewrites that we call extension. Extension is proved to preserve types.

As mentioned in the introduction, it seems worthwhile to revisit rewrites for higher-order rewriting using the TRC as type system for typing rewrites. Such rewrites would allow an analysis of Lévy permutation equivalence and projection equivalence for HOR in a fully typed setting. One would expect to prove equivalence of both these notions, *i.e.* that equivalence of rewrites via permutation equivalence and via projection coincide. Also of interest, once that is in place, is to prove a notion of algebraic confluence: any two reductions to normal form are Lévy permutation equivalent.

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A TERMS AND REWRITES

A.1 Terms and Rewrites

Definition A.1 (Free Variables). The set of free truth and validity variables of a preobject o are defined as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{ftv}(a) & := & \{a\} & \mathsf{frv}(a) & := & \emptyset \\
\mathsf{ftv}(u) & := & \emptyset & \mathsf{frv}(u) & := & \{u\} \\
\mathsf{ftv}(\lambda a.s) & := & \mathsf{ftv}(s) \setminus \{a\} & \mathsf{frv}(\lambda a.s) & := & \mathsf{frv}(s) \\
\mathsf{ftv}(s\,t) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(t) & \mathsf{frv}(s\,t) & := & \mathsf{frv}(s) \cup \mathsf{frv}(t) \\
\mathsf{ftv}(!(\rho, s, t)) & := & \emptyset & \mathsf{frv}(!(\rho, s, t)) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(s) \cup \mathsf{frv}(t) \\
\mathsf{ftv}(let\ u \doteq s\ in\, t) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(t) & \mathsf{frv}(let\ u \doteq s\ in\, t) & := & \mathsf{frv}(s) \cup \mathsf{frv}(t) \setminus \{u\}
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{ftv}(\underline{a}) & := & \{a\} & \mathsf{frv}(\underline{a}) & := & \emptyset \\
\mathsf{ftv}(\underline{u}) & := & \emptyset & \mathsf{frv}(\underline{u}) & := & \{u\} \\
\mathsf{ftv}(\mathbf{ba}(a.s, r)) & := & \mathsf{ftv}(s) \setminus \{a\} \cup \mathsf{ftv}(r) & \mathsf{frv}(\mathbf{ba}(a.s, r)) & := & \mathsf{frv}(s) \cup \mathsf{frv}(r) \\
\mathsf{ftv}(\mathbf{bb}(s, u.r)) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(r) & \mathsf{frv}(\mathbf{bb}(s, u.r)) & := & \mathsf{frv}(s) \cup \mathsf{frv}(r) \setminus \{u\} \\
\mathsf{ftv}(\rho; \sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\rho; \sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(\lambda a.\rho) & := & \mathsf{ftv}(\rho) \setminus \{a\} & \mathsf{frv}(\lambda a.\rho) & := & \mathsf{frv}(\rho) \\
\mathsf{ftv}(\rho\,\sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\rho\,\sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(\langle \rho|_s \sigma \rangle) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(s) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\langle \rho|_s \sigma \rangle) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(s) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(let\ u \doteq \rho\ in\, \sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(let\ u \doteq \rho\ in\, \sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \setminus \{u\}
\end{array}
$$

A.2 Structural Equivalence

We write $\Vdash_\pi \rho : s \rhd t$ if the judgement $\rho : s \rhd t$ is derivable with derivation π.

Lemma A.2 (Generation for Source/Target).

(a) If $\underline{a} : s \rhd t$, then $s \simeq a$ and $t \simeq a$.
(b) If $\underline{u} : s \rhd t$, then $s \simeq u$ and $t \simeq u$.
(c) If $\mathbf{ba}(a.p, q) : s \rhd t$, then $s \simeq (\lambda a.p)\, q$ and $t \simeq p\{a/q\}$.
(d) If $\mathbf{bb}(!(\rho, p, q), u.r) : s \rhd t$, then $s \simeq let\ u \doteq !(\rho, p, q)\ in\, r$ and $t \simeq r\{u/^{\mathsf{tgt}}\rho_p^q\}$.
(e) If $\lambda a.\rho : s \rhd t$, then there exists p, q such that $s \simeq \lambda a.p$, $t \simeq \lambda a.q$ and $\rho : p \rhd q$.
(f) If $\sigma\,\tau : s \rhd t$, then there exists p_1, p_2, q_1, q_2 such that $s \simeq p_1\, p_2$ and $t \simeq q_1\, q_2$ and $\sigma : p_1 \rhd q_1$ and $\tau : p_2 \rhd q_2$.
(g) If $let\ u \doteq \rho\ in\, \sigma : s \rhd t$, then there exists p_1, p_2, q_1, q_2 such that $s \simeq let\ u \doteq p_1\ in\, p_2$ and $t \simeq let\ u \doteq q_1\ in\, q_2$ and $\rho : p_1 \rhd q_1$ and $\sigma : p_2 \rhd q_2$.
(h) If $\sigma; \tau : s \rhd t$ and there exists r s.t. $\sigma : s \rhd r$ and $\tau : r \rhd t$.
(i) If $\langle \sigma|_p \tau \rangle : s \rhd t$, then there exists r, q such that $s \simeq !(\sigma, p, r)$ and $t \simeq !(\sigma; \tau, p, q)$ and $\sigma : p \rhd r$ and $\tau : r \rhd q$.

Proof. By induction on π in the assumption $\Vdash_\pi \rho : s \rhd t$ and using the IH and transitivity of $\simeq$ in the ST-Eq case. □

Lemma. [Uniqueness of Source and Target – Lem. 1.2] If $\rho : s \rhd t$ and $\rho : p \rhd q$, then $s \simeq p$ and $t \simeq q$.

Proof. By induction on ρ using Lem. A.2. □

Lemma. (Lem 1.3) $\mathfrak{s} : p \rhd q$ implies $p \simeq q \simeq s$.

Proof. By induction on s.

- $s = a$ and $\underline{a} : p \rhd q$. We conclude from Lem. A.2.
- $s = u$. Same as in the previous case.
- $s = \lambda a.s_1$ and $\mathfrak{s} : p \rhd q$. Note first that $\mathfrak{s} = \underline{\lambda a.s_1} = \lambda a.\underline{s_1}$. By Lem. A.2, there exist p', q', π' s.t. $p \simeq \lambda a.p'$, $q \simeq \lambda a.q'$ and $\Vdash \underline{s_1} : p' \rhd q'$. By the IH $p' \simeq q' \simeq s_1$. Then $p \simeq \lambda a.p' = \lambda a.q' \simeq \lambda a.s_1$.
- $s = s\,t$ and $s = let\ u \doteq s\ in\, t$. Similar to the previous case.
- $s = !(\sigma, m, n)$. First note that $\underline{!(\sigma, m, n)} = \langle \sigma|_m \mathfrak{n} \rangle$. By Lem. A.2 there exist p', q', π' s.t. $p \simeq !(\sigma, m, p')$, $q \simeq !(\sigma; \mathfrak{n}, m, q')$ and $\sigma : m \rhd p'$ and $\mathfrak{n} : p' \rhd q'$. By the IH on the latter, $p' \simeq q' \simeq n$. Then $p \simeq !(\sigma, m, p') \simeq !(\sigma; \mathfrak{n}, m, q') \simeq q$.

□

Lemma. [Lem. 1.4] $\rho \simeq \sigma : s \rhd t$ implies $\rho : s \rhd t$ and $\sigma : s \rhd t$.

Proof. By induction on the derivation of $\rho \simeq \sigma : s \rhd t$. For the cases EqR-IdR and EqR-IdL we use Lem. 1.3. For the case EqR-SEq we use ST-SEq. □

Lemma. [Lem. 1.5] $s \simeq t$ implies $\underline{s} \simeq \underline{t} : s \rhd s$.

Proof. By induction on the derivation of $s \simeq t$. □

Lemma. [Structural Equivalence is closed under substitution of term variables – Lem. 1.6] Suppose $s \simeq t$ and $p \simeq q$. Then $s\{a/p\} \simeq t\{a/q\}$.

Proof. By induction on the derivation of $s \simeq t$. □

Lemma. [Structural Equivalence is closed under substitution of rewrite variables – Lem. 1.7] Suppose $\tau \simeq \upsilon : p \rhd q$. Then

- $\rho \simeq \sigma : s \rhd t$ implies $\rho\{u/^{\mathsf{m}}\tau_p^q\} \simeq \sigma\{u/^{\mathsf{m}}\upsilon_p^q\} : s\{u/^{\mathsf{m}}\tau_p^q\} \rhd t\{u/^{\mathsf{m}}\tau_p^q\}$.
- $s \simeq t$ implies $s\{u/^{\mathsf{m}}\tau_p^q\} \simeq t\{u/^{\mathsf{m}}\upsilon_p^q\}$.
- $\mathfrak{s}\{u/\tau_p^q\} \simeq \mathfrak{s}\{u/\upsilon_p^q\} : s\{u/^{\mathsf{src}}\tau_p^q\} \rhd s\{u/^{\mathsf{tgt}}\tau_p^q\}$

Proof. By simultaneous induction on $\rho \simeq \sigma : s \rhd t$ and $s \simeq t$ for the first two items. We use induction on s for the third item. □

Lemma. [Structural Equivalence Preserves Well-Formedness – Lem. 1.9] If s is well-formed and $s \simeq t$, then t is well-formed. Similarly, if (ρ, s, t) is well-formed and $\rho \simeq \sigma : s \rhd t$, then (σ, s, t) is well-formed.

Proof. By induction on the derivation of $s \simeq t$ and $\rho \simeq \sigma : s \rhd t$. It uses Lem. 1.4. □

A.3 Substitution Commutation Results

All objects (*i.e.* terms and rewrites) are assumed well-formed.

Lemma A.3. *Suppose $u \notin \mathrm{fv}(\mathsf{o})$ and $a \notin \mathrm{fv}(r)$.*
(a) $\mathsf{o}\{u/^{\mathsf{m}}\rho_s^t\} \simeq \mathsf{o}$.
(b) $\mathsf{o} \in \mathbb{R}_1^-$ *implies* $\mathsf{o}\{u/\rho_s^t\} \simeq \mathsf{o}$.
(c) $r\{a/s\} = r$.

Proof. The third item is by induction on r. We focus on the other two. We prove them simultaneously by induction on the size of o. For the first six cases below, the second item holds trivially since $\mathsf{o} \notin \mathbb{R}_1$.

- $\mathsf{o} = a$. Then $LHS = a = RHS$.
- $\mathsf{o} = w$. If $w \neq v, u$, then $LHS = w = RHS$. Otherwise, if $w = v$, then $LHS = v = RHS$. The case where $w = u$ is not possible by hypothesis.
- $\mathsf{o} = \lambda a.p_1$.

$$\begin{array}{lll} & LHS & \\ = & (\lambda a.p_1)\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} & \\ \simeq & \lambda a.p_1 & \text{(IH/1)} \\ = & RHS & \end{array}$$

- $\mathsf{o} = p_1\, p_2$. By the IH.
- $\mathsf{o} = \mathit{let}\ v \mathrel{\hat{=}} p_1\ \mathit{in}\, p_2$. By the IH.
- $\mathsf{o} = !(\sigma, p, q)$. We reason as follows

$$\begin{array}{lll} & LHS & \\ = & !(\sigma, p, q)\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(\mathfrak{p}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}, p\{u/^{\mathsf{src}}\rho_s^t\}, q\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !(\mathfrak{p}; \sigma, p, q) & \text{(IH/2,IH/1 3-times)} \\ \simeq & !(\sigma, p, q) & \\ = & RHS & \end{array}$$

  Note that the IH/2 is applied to $\mathfrak{p}$ rather than p. Hence the reason why we perform induction over the size of o (both $\mathfrak{p}$ and p have the same size) and not the structure. Note also that in order to determine that $\mathfrak{p}; \sigma \simeq \sigma : p \rhd q$ we rely on well-formedness.
- $\mathsf{o} = \underline{a}$. For both items we have $LHS = \underline{a} = RHS$.
- $\mathsf{o} = \underline{w}$. Then $w \neq v$ by hypothesis and $LHS = \underline{v} = RHS$ for both items.
- $\mathsf{o} = \mathbf{ba}(a.p_1, p_2)$. The second item holds trivially since $\mathsf{o} \notin \mathbb{R}_1$. For the first item

$$\begin{array}{lll} & LHS & \\ = & (\mathbf{ba}(a.p_1, p_2))\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \mathbf{ba}(a.p_1\{u/^{\mathsf{m}}\rho_s^t\}, p_2\{u/^{\mathsf{m}}\rho_s^t\}) & \\ \simeq & \mathbf{ba}(a.p_1, p_2) & \text{(IH/1 twice)} \\ = & RHS & \end{array}$$

- $\mathsf{o} = \mathbf{bb}(p_1, w.p_2)$. Similar to the previous case.
- $\mathsf{o} = \langle \sigma|_p\tau \rangle$. For item 1 we reason as follows.

$$\begin{array}{lll} & LHS & \\ = & \langle\sigma|_p\tau\rangle\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \langle\mathfrak{p}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}|_{q\{u/^{\mathsf{src}}\rho_s^t\}}\tau\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\ \simeq & \langle\mathfrak{p};\sigma|_q\tau\rangle & \text{(IH/2,IH/1 3-times)} \\ = & RHS & \end{array}$$

For item 2 $\langle\sigma|_p\tau\rangle = \langle\sigma|_p\mathfrak{r}\rangle$ for some term r and we reason as for item 1.

- $\mathfrak{o} = \lambda a.\sigma$. For the first item we have:

$$\begin{array}{lll} & LHS & \\ = & (\lambda a.\sigma)\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \lambda a.\sigma\{u/^{\mathsf{m}}\rho_s^t\} & \\ \simeq & \lambda a.\sigma & \text{(IH/1)} \\ = & RHS & \end{array}$$

For the second item $\sigma = \mathfrak{p}$ for some pre-term p. We reason as above but use IH/2.

- $\mathfrak{o} = \sigma;\tau$. The second item holds trivially since $\mathfrak{o} \notin \mathbb{R}_1$. The first item follows from the IH.
- $\mathfrak{o} = \sigma\,\tau$. We use the IH for both items.
- $\mathfrak{o} = let\ v \triangleq \sigma\ in\,\tau$. We use the IH for both items.

□

Lemma A.4. *$\rho \in \mathbb{R}_1^-$ implies $\rho\{u/^{\mathsf{m}}\sigma_p^q\} \in \mathbb{R}_1^-$.*

Proof. Suppose $\rho = \mathfrak{r}$. We proceed by induction on $\mathfrak{r}$:

- $\mathfrak{r} = \underline{a}$. Then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma_p^q\} = \underline{a}$ and we conclude.
- $\mathfrak{r} = \underline{w}$. If $w \neq u$, then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma_p^q\} = \underline{w}$. If $w = u$ and $\mathsf{m} = src$ then $\mathfrak{r}\{u/^{\mathsf{src}}\sigma_p^q\} = \mathfrak{p} \in \mathbb{R}_1^-$. The case where $\mathsf{m} = tgt$ is similar.
- $\mathfrak{r} = \lambda a.\mathfrak{s}$. Then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma_p^q\} = \lambda a.\mathfrak{s}\{u/^{\mathsf{m}}\sigma_p^q\}$. By the IH $\mathfrak{s}\{u/^{\mathsf{m}}\sigma_p^q\} \in \mathbb{R}_1^-$, say $\mathfrak{s}\{u/^{\mathsf{m}}\sigma_p^q\} = \mathfrak{s}'$, and hence $\lambda a.\mathfrak{s}' \in \mathbb{R}_1^-$ too.
- $\mathfrak{r} = \mathfrak{s}\,\mathfrak{t}$. We use the IH.
- $\mathfrak{r} = \langle\sigma|_s\mathfrak{t}\rangle$. We reason as follows:

$$\begin{array}{lll} & \langle\sigma|_s\mathfrak{t}\rangle\{u/^{\mathsf{m}}\sigma_p^q\} & \\ = & \langle\mathfrak{s}\{u/\sigma_p^q\};\sigma\{u/^{\mathsf{tgt}}\sigma_p^q\}|_{s\{u/^{\mathsf{src}}\sigma_p^q\}}\mathfrak{t}\{u/^{\mathsf{tgt}}\sigma_p^q\}\rangle & \\ = & \langle\mathfrak{s}\{u/\sigma_p^q\};\sigma\{u/^{\mathsf{tgt}}\sigma_p^q\}|_{s\{u/^{\mathsf{src}}\sigma_p^q\}}\mathfrak{t}'\rangle & \text{(IH)} \\ \in & \mathbb{R}_1^- & \end{array}$$

- $\mathfrak{r} = let\ u \triangleq \mathfrak{s}\ in\,\mathfrak{t}$. We use the IH.

□

Lemma. (Commutation of Rewrite Substitution with Term Substitution – Lem. 1.10) Suppose $a \notin \mathsf{fv}(\rho, s, t)$.

$$p\{u/^{\mathsf{m}}\rho_s^t\}\{a/q\{u/^{\mathsf{m}}\rho_s^t\}\} = p\{a/q\}\{u/^{\mathsf{m}}\rho_s^t\}$$

Proof. By induction on p.

- $p = b$. If $a \neq b$, then $LHS = b = RHS$. Otherwise,

$$\begin{array}{ll} & LHS \\ = & a\{u/^{\mathsf{m}}\rho_s^t\}\{a/q\{u/^{\mathsf{m}}\rho_s^t\}\} \\ = & a\{a/q\{u/^{\mathsf{m}}\rho_s^t\}\} \\ = & q\{u/^{\mathsf{m}}\rho_s^t\} \\ = & a\{a/q\}\{u/^{\mathsf{m}}\rho_s^t\} \\ = & RHS \end{array}$$

- $p = v$. If $u \neq v$, then $LHS = v = RHS$. Otherwise, if $u = v$ and $\mathsf{m} = src$ (the case where $\mathsf{m} = tgt$ is similar and omitted), we have

$$\begin{array}{lll} & LHS & \\ = & s\{a/q\{u/^{\mathsf{src}}\rho_s^t\}\} & \\ = & s & \text{(Lem. A.3(c))} \\ = & u\{a/q\}\{u/^{\mathsf{src}}\rho_s^t\} & \\ = & RHS & \end{array}$$

- $p = \lambda a.p_1$. By the IH.
- $p = p_1\,p_2$. By the IH.
- $p = !(\sigma, o, r)$. We reason as follows:

$$
\begin{array}{rl}
& LHS \\
= & !(\sigma, o, r)\{u/^{\mathrm{m}}\rho^t_s\}\{a/q\{u/^{\mathrm{m}}\rho^t_s\}\} \\
= & !(\mathfrak{o}\{u/\rho^t_s\}; \sigma\{u/^{\mathrm{tgt}}\rho^t_s\}, o\{u/^{\mathrm{src}}\rho^t_s\}, r\{u/^{\mathrm{tgt}}\rho^t_s\})\{a/q\{u/^{\mathrm{m}}\rho^t_s\}\} \\
= & !(\mathfrak{o}\{u/\rho^t_s\}; \sigma\{u/^{\mathrm{tgt}}\rho^t_s\}, o\{u/^{\mathrm{src}}\rho^t_s\}, r\{u/^{\mathrm{tgt}}\rho^t_s\}) \\
= & !(\sigma, o, r)\{u/^{\mathrm{m}}\rho^t_s\} \\
= & !(\sigma, o, r)\{a/q\}\{u/^{\mathrm{m}}\rho^t_s\} \\
= & RHS
\end{array}
$$

- $p = let\ v \circeq p_1\ in\, p_2$.

$$
\begin{array}{rl}
& LHS \\
= & (let\ v \circeq p_1\ in\, p_2)\{u/^{\mathrm{m}}\rho^t_s\}\{a/q\{u/^{\mathrm{m}}\rho^t_s\}\} \\
= & let\ v \circeq p_1\{u/^{\mathrm{m}}\rho^t_s\}\{a/q\{u/^{\mathrm{m}}\rho^t_s\}\}\ in\, p_2\{u/^{\mathrm{m}}\rho^t_s\}\{a/q\{u/^{\mathrm{m}}\rho^t_s\}\} \\
= & let\ v \circeq p_1\{a/q\}\{u/^{\mathrm{m}}\rho^t_s\}\ in\, p_2\{a/q\}\{u/^{\mathrm{m}}\rho^t_s\} \\
= & (let\ v \circeq p_1\ in\, p_2)\{a/q\}\{u/^{\mathrm{m}}\rho^t_s\} \\
= & RHS
\end{array}
$$

□

Lemma A.5. *Let $p \in \mathbb{T}$. Then $\underline{p\{u/^{\mathrm{m}}\rho^t_s\}} = \underline{p}\{u/^{\mathrm{m}}\rho^t_s\}$.*

Proof. By induction on p.

- $p = a$. Then $LHS = a = RHS$.
- $p = v$. If $u \neq v$, then $LHS = v = RHS$. Otherwise, if $u = v$ and $\mathrm{m} = src$ (the case where $\mathrm{m} = tgt$ is similar and omitted), we have
$$LHS = \underline{s} = \underline{u}\{u/^{\mathrm{src}}\rho^t_s\}$$
- $p = \lambda a.p_1$. We use the IH and $\underline{\lambda a.p_1} := \lambda a.\underline{p_1}$
- $p = p_1\, p_2$. We use the IH and $\underline{p_1\, p_2} := \underline{p_1}\, \underline{p_2}$.
- $p = !(\sigma, o, r)$. We reason as follows.
$$
\begin{array}{rl}
& LHS \\
= & \underline{!(\sigma, o, r)\{u/^{\mathrm{m}}\rho^t_s\}} \\
= & \underline{!(\mathfrak{o}\{u/\rho^t_s\}; \sigma\{u/^{\mathrm{tgt}}\rho^t_s\}, o\{u/^{\mathrm{src}}\rho^t_s\}, r\{u/^{\mathrm{tgt}}\rho^t_s\})} \\
= & \langle \mathfrak{o}\{u/\rho^t_s\}; \sigma\{u/^{\mathrm{tgt}}\rho^t_s\}|_{o\{u/^{\mathrm{src}}\rho^t_s\}} \underline{r\{u/^{\mathrm{tgt}}\rho^t_s\}}\rangle
\end{array}
$$
$$
\begin{array}{rl}
& RHS \\
= & \underline{!(\sigma, o, r)}\{u/^{\mathrm{m}}\rho^t_s\} \\
= & \langle \sigma|_o \mathfrak{r}\rangle\{u/^{\mathrm{m}}\rho^t_s\} \\
= & \langle \mathfrak{o}\{u/\rho^t_s\}; \sigma\{u/^{\mathrm{tgt}}\rho^t_s\}|_{o\{u/^{\mathrm{src}}\rho^t_s\}} \underline{r}\{u/^{\mathrm{tgt}}\rho^t_s\}\rangle
\end{array}
$$
We conclude from the IH that $\underline{r\{u/^{\mathrm{tgt}}\rho^t_s\}} = \underline{r}\{u/^{\mathrm{tgt}}\rho^t_s\}$ and hence LHS=RHS.
- $p = let\ v \circeq p_1\ in\, p_2$.
$$
\begin{array}{rll}
& LHS & \\
= & \underline{(let\ v \circeq p_1\ in\, p_2)\{u/^{\mathrm{m}}\rho^t_s\}} & \\
= & \underline{let\ v \circeq p_1\{u/^{\mathrm{m}}\rho^t_s\}\ in\, p_2\{u/^{\mathrm{m}}\rho^t_s\}} & \\
= & let\ v \circeq \underline{p_1\{u/^{\mathrm{m}}\rho^t_s\}}\ in\, \underline{p_2\{u/^{\mathrm{m}}\rho^t_s\}} & \\
= & let\ v \circeq \underline{p_1}\{u/^{\mathrm{m}}\rho^t_s\}\ in\, \underline{p_2}\{u/^{\mathrm{m}}\rho^t_s\} & (IH) \\
= & (let\ v \circeq \underline{p_1}\ in\, \underline{p_2})\{u/^{\mathrm{m}}\rho^t_s\} & \\
= & \underline{(let\ v \circeq p_1\ in\, p_2)}\{u/^{\mathrm{m}}\rho^t_s\} & \\
= & RHS &
\end{array}
$$

□

Lemma. [Commutation of Rewrite Substitution – Lem. 1.11] Let o be any object (*i.e.* term or rewrite) and suppose $v \notin \mathrm{fv}(\rho, s, t)$.

(a) Suppose all occurrences of m below are either all src or all tgt. Then,
$$\mathrm{o}\{v/^{\mathrm{m}}\mu^q_p\}\{u/^{\mathrm{m}}\rho^t_s\} \simeq \mathrm{o}\{u/^{\mathrm{m}}\rho^t_s\}\{v/^{\mathrm{m}}\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathrm{tgt}}\rho^t_s\}^{q\{u/^{\mathrm{tgt}}\rho^t_s\}}_{p\{u/^{\mathrm{src}}\rho^t_s\}}\}$$

(b) If $\mathrm{o} \in \mathbb{R}_1$, then
$$\mathrm{o}\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\}; \mathrm{o}\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\}$$
$$\simeq$$
$$\mathrm{o}\{u/^{\mathrm{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathrm{tgt}}\rho^t_s\}^{q\{u/^{\mathrm{tgt}}\rho^t_s\}}_{p\{u/^{\mathrm{src}}\rho^t_s\}}\}; \mathrm{o}\{u/\rho^t_s\}\{v/^{\mathrm{tgt}}\mathfrak{p}\{u/\rho^t_s\}; \mu\{u/^{\mathrm{tgt}}\rho^t_s\}^{q\{u/^{\mathrm{tgt}}\rho^t_s\}}_{p\{u/^{\mathrm{src}}\rho^t_s\}}\}$$

Proof. We prove both items simultaneously by induction on the size of $\mathfrak{o}$. For the first six cases below, the second item holds trivially since $\mathfrak{o} \notin \mathbb{R}_1$.

- $\mathfrak{o} = a$. Then $LHS = a = RHS$.
- $\mathfrak{o} = w$. If $w \neq v, u$, then $LHS = w = RHS$. Otherwise, if $w = v$ and $\mathsf{m} = src$ (the case where $\mathsf{m} = tgt$ is similar and omitted) we have

$$\begin{array}{cl} & LHS \\ = & v\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\} \\ = & p\{u/^{\mathsf{src}}\rho_s^t\} \\ = & v\{v/^{\mathsf{src}}\mathfrak{o}\{u/\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} \\ = & v\{u/^{\mathsf{src}}\rho_s^t\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} \\ = & RHS \end{array}$$

  if $w = u$, we have

$$\begin{array}{cll} & LHS & \\ = & u\{u/^{\mathsf{m}}\rho_s^t\} & \\ \simeq & u\{u/^{\mathsf{m}}\rho_s^t\}\{v/^{\mathsf{m}}\mathfrak{o}\{u/\rho_s^t\}_{o\{u/^{\mathsf{src}}\rho_s^t\}}^{o\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \text{(Lem. A.3)} \\ = & RHS & \end{array}$$

- $\mathfrak{o} = \lambda a.r_1$. We use the IH.
- $\mathfrak{o} = r_1\, r_2$. We use the IH.
- $\mathfrak{o} = !(\sigma, r_1, r_2)$.
  Below we write α^{m} to abbreviate the substitution $\bullet\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\}$ and α to abbreviate the substitution $\bullet\{v/\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{s}}}^{q\{u/^{\mathsf{t}}}$

$$\begin{array}{cll} & LHS & \\ = & !(\sigma, r_1, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(\mathfrak{r}_1\{v/\mu_p^q\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};(\underline{r_1}\{v/\mu_p^q\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{tgt}}\rho_s^t\}, q\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ = & !(\overline{\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}}\{u/\rho_s^t\};(\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}), r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !((\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\});\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !((\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}});\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \text{(IH/2)} \\ \simeq & !((\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}});\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(IH/1 three times)} \\ = & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;(\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}), r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\ = & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;(\underline{r_1}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\ = & !(\overline{\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha};(\underline{r_1}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(Lem. A.5)} \\ = & !(\overline{\underline{r_1}\{u/\rho_s^t\}};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{m}} & \\ = & !(\sigma, r_1, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} & \\ = & RHS & \end{array}$$

- $\mathfrak{o} = let\ v \mathrel{\mathring{=}} r_1\ in\, r_2$.

$$\begin{array}{cl} & LHS \\ = & (let\ w \mathrel{\mathring{=}} r_1\ in\, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} \\ = & let\ w \mathrel{\mathring{=}} r_1\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\}\ in\, r_2\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} \\ = & let\ w \mathrel{\mathring{=}} r_1\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}}\ in\, r_2\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} \\ = & (let\ w \mathrel{\mathring{=}} r_1\ in\, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} \\ = & RHS \end{array}$$

- $\mathfrak{o} = \underline{a}$. Then for item 1 we have $LHS = \underline{a} = RHS$. For item 2

$$\begin{array}{cl} & \underline{a}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{a}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} \\ \simeq & \underline{a};\underline{a} \\ \simeq & \underline{a}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{a}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} \end{array}$$

- $\mathfrak{o} = \underline{w}$. Item 1. If $w \neq v, u$, then $LHS = \underline{w} = RHS$. Suppose $w = v$ and $\mathsf{m} = src$ (the case $\mathsf{m} = tgt$ is similar and omitted). Then we have

$$
\begin{array}{rll}
& LHS & \\
= & \underline{v}\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\} & \\
= & \mathfrak{p}\{u/^{\mathsf{src}}\rho_s^t\} & \\
= & \underline{p\{u/^{\mathsf{src}}\rho_s^t\}} & \\
= & \underline{v}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & \underline{v}\{u/^{\mathsf{src}}\rho_s^t\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & RHS &
\end{array}
$$

if $w = u$ and $\mathsf{m} = src$ (the case $\mathsf{m} = tgt$ is similar and omitted), we have

$$
\begin{array}{rll}
& LHS & \\
= & \underline{u}\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\} & \\
= & \underline{u}\{u/^{\mathsf{src}}\rho_s^t\} & \\
= & \mathfrak{s} & \\
\simeq & \mathfrak{s}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \text{(Lem. A.3)} \\
= & \underline{u}\{u/^{\mathsf{src}}\rho_s^t\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & RHS &
\end{array}
$$

For item 2 we consider two cases. Suppose $w = u$:

$$
\begin{array}{rll}
& LHS & \\
= & \underline{u}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{u}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
= & \underline{u}\{u/\rho_s^t\};\underline{u}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
= & \rho;\mathfrak{t} & \\
\simeq & \rho & \\
\simeq & \mathfrak{s};\rho & \\
\simeq & \underline{u}\{u/^{\mathsf{src}}\rho_s^t\};\underline{u}\{u/\rho_s^t\} & \\
\simeq & \underline{u}\{u/^{\mathsf{src}}\rho_s^t\}\{v/\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\};\underline{u}\{u/\rho_s^t\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \text{(Lem. A.3)} \\
= & RHS &
\end{array}
$$

Then $w = v$:

$$
\begin{array}{rll}
& LHS & \\
= & \underline{v}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{v}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
= & \mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
\simeq & (\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\});\underline{q\{u/^{\mathsf{tgt}}\rho_s^t\}} & \\
= & \underline{v}\{v/\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\};\underline{v}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & \underline{v}\{u/^{\mathsf{src}}\rho_s^t\}\{v/\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\};\underline{v}\{u/\rho_s^t\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & RHS &
\end{array}
$$

- $\mathfrak{o} = \mathbf{ba}(a.r_1, r_2)$. Item 2 holds trivially since $\mathfrak{o} \notin \mathbb{R}_1$. For item 1 we use the IH.

$$
\begin{array}{rll}
& LHS & \\
= & \mathbf{ba}(a.r_1, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} & \\
= & \mathbf{ba}(a.r_1\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\}, r_2\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\}) & \\
\simeq & \mathbf{ba}(a.r_1\{u/^{\mathsf{m}}\rho_s^t\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\}, r_2\{u/^{\mathsf{m}}\rho_s^t\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\}) & \text{(IH/1 twice)} \\
\simeq & \mathbf{ba}(a.r_1, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}_{p\{u/^{\mathsf{src}}\rho_s^t\}}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}\} & \\
= & RHS &
\end{array}
$$

- $\mathfrak{o} = \mathbf{bb}(r_1, w.r_2)$. Item 2 holds trivially since $\mathfrak{o} \notin \mathbb{R}_1$. For item 1 we use the IH, as in the previous case.
- $\mathfrak{o} = \langle\sigma|_r\tau\rangle$. For the first item we reason as follows:

$$
\begin{array}{rll}
& \mathit{LHS} & \\
= & \langle \sigma |_r \tau \rangle \{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} & \\
= & \langle \mathfrak{r}\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}} \tau\{v/^{\mathsf{tgt}}\mu_p^q\}\rangle\{u/^{\mathsf{m}}\rho_s^t\} & \\
= & \langle r\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; (\mathfrak{r}\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \tau\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\
= & \langle \overline{\underline{r}\{v/^{\mathsf{src}}\mu_p^q\}}\{u/\rho_s^t\}; (\mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\})|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \tau\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \text{(Lem. A.5)} \\
\simeq & \langle (\underline{r}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}); \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \tau\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\
\simeq & \langle (\underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}); \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{r}\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \text{(IH/2)} \\
\simeq & \langle (\underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}); \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \tau\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \text{(IH/1 three times)} \\
= & \langle \underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}})|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \tau\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle \underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \tau\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle r\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \tau\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \text{(Lem. A.5)} \\
= & \langle \overline{\underline{r}\{u/\rho_s^t\}}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{u/^{\mathsf{src}}\rho_s^t\}} \tau\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle\alpha^{\mathsf{m}} & \\
= & \langle \sigma |_r \tau \rangle \{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} & \\
= & \mathit{RHS} &
\end{array}
$$

For item 2, $\langle \sigma |_r \tau \rangle = \langle \sigma |_r \mathfrak{o} \rangle$ for term o the target of σ. We reason as follows:

$$
\begin{array}{rll}
& \mathit{LHS} & \\
= & \langle \sigma |_r \mathfrak{o} \rangle \{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \langle \sigma |_r \mathfrak{o} \rangle \{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
\simeq & \langle \sigma |_r \mathfrak{o} \rangle \{u/^{\mathsf{src}}\rho_s^t\}\alpha; \langle \sigma |_r \mathfrak{o} \rangle \{u/\rho_s^t\}\alpha^{\mathsf{tgt}} & (\star) \\
= & \mathit{RHS} &
\end{array}
$$

Step $(\star)$ follows from proving

$$
\langle \sigma |_r \mathfrak{o} \rangle \{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\} \simeq \langle \sigma |_r \mathfrak{o} \rangle \{u/^{\mathsf{src}}\rho_s^t\}\alpha \tag{4}
$$

and

$$
\langle \sigma |_r \mathfrak{o} \rangle \{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq \langle \sigma |_r \mathfrak{o} \rangle \{u/\rho_s^t\}\alpha^{\mathsf{tgt}} \tag{5}
$$

separately. For (4) we reason as follows (the case (5) is similar).

$$
\begin{array}{rll}
= & \langle \sigma |_r \mathfrak{o} \rangle \{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\} & \\
= & \langle \mathfrak{r}\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}} \mathfrak{o}\{v/^{\mathsf{tgt}}\mu_p^q\}\rangle\{u/\rho_s^t\} & \\
= & \langle r\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; (\mathfrak{r}\{v/\mu_p^q\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{o}\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\
= & \langle \overline{\underline{r}\{v/^{\mathsf{src}}\mu_p^q\}}\{u/\rho_s^t\}; (\mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}; \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\})|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{o}\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\
= & \langle (\underline{r}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}); \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{o}\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \\
\simeq & \langle (\underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}); \sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{o}\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle & \text{(IH/2)} \\
= & \langle (\underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \underline{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}); \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \mathfrak{o}\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \text{(IH/1 three times)} \\
= & \langle \underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}\{v/^{\mathsf{tgt}}\mathfrak{o}\{u/\rho_s^t\}^{o\{u/^{\mathsf{tgt}}\rho_s^t\}}_{o\{u/^{\mathsf{src}}\rho_s^t\}}\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}})|_{q\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \mathfrak{o}\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle \underline{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}} \mathfrak{o}\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle r\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\underline{r}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{tgt}}} \mathfrak{o}\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}\rangle & \text{(Lem. A.5)} \\
= & \langle \overline{\underline{r}\{u/\rho_s^t\}}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}|_{r\{u/^{\mathsf{src}}\rho_s^t\}} \mathfrak{o}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle\alpha & \\
= & \langle \sigma |_r \mathfrak{o} \rangle \{u/^{\mathsf{src}}\rho_s^t\}\alpha &
\end{array}
$$

- $\mathfrak{o} = \lambda a.\sigma$. Item 1 follows from the IH/1. In the case of item 2, $\lambda a.\sigma = \lambda a.\mathfrak{r}$ for some term r. We reason as follows:

$$
\begin{array}{rll}
& \mathit{LHS} & \\
= & (\lambda a.\mathfrak{r})\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; (\lambda a.\mathfrak{r})\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
= & \lambda a.\mathfrak{r}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \lambda a.\mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} & \\
\simeq & \lambda a.(\mathfrak{r}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}; \mathfrak{r}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\
\simeq & \lambda a.(\mathfrak{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \mathfrak{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(IH/2 4 times)} \\
\simeq & \lambda a.\mathfrak{r}\{u/^{\mathsf{src}}\rho_s^t\}\alpha; \lambda a.\mathfrak{r}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} & \\
= & (\lambda a.\mathfrak{r})\{u/^{\mathsf{src}}\rho_s^t\}\alpha; (\lambda a.\mathfrak{r})\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} & \\
= & \mathit{RHS} &
\end{array}
$$

- $\mathfrak{o} = \sigma;\tau$. Item 1 follows from the IH. Item 2 is immediate since $\mathfrak{o} \notin \mathbb{R}_1$.
- $\mathfrak{o} = \sigma\,\tau$. Item 1 follows from the IH. For item 2 we reason as follows, where $\sigma\,\tau = \underline{r_1}\,\mathfrak{r}_2$ for some term r_1 and r_2.

$$
\begin{array}{lll}
 & \mathit{LHS} & \\
= & (\underline{r_1}\,\mathfrak{r}_y)\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\};(\underline{r_1}\,\mathfrak{r}_y)\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\} & \\
= & \underline{r_1}\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\}\,\mathfrak{r}_y\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\};\underline{r_1}\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\}\,\mathfrak{r}_y\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\} & \\
\simeq & (\underline{r_1}\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\};\underline{r_1}\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\})\,(\mathfrak{r}_y\{v/^{\mathrm{src}}\mu^q_p\}\{u/\rho^t_s\};\mathfrak{r}_y\{v/\mu^q_p\}\{u/^{\mathrm{tgt}}\rho^t_s\}) & \\
\simeq & (\underline{r_1}\{u/^{\mathrm{src}}\rho^t_s\}\alpha;\underline{r_1}\{u/\rho^t_s\}\alpha^{\mathrm{tgt}})\,(\mathfrak{r}_y\{u/^{\mathrm{src}}\rho^t_s\}\alpha;\mathfrak{r}_y\{u/\rho^t_s\}\alpha^{\mathrm{tgt}}) & \text{(IH/2 4 times)} \\
\simeq & \underline{r_1}\{u/^{\mathrm{src}}\rho^t_s\}\alpha\,\mathfrak{r}_y\{u/^{\mathrm{src}}\rho^t_s\}\alpha;\underline{r_1}\{u/\rho^t_s\}\alpha^{\mathrm{tgt}}\,\mathfrak{r}_y\{u/\rho^t_s\}\alpha^{\mathrm{tgt}} & \\
= & (\underline{r_1}\,\mathfrak{r}_y)\{u/^{\mathrm{src}}\rho^t_s\}\alpha;(\underline{r_1}\,\mathfrak{r}_y)\{u/\rho^t_s\}\alpha^{\mathrm{tgt}} & \\
= & \mathit{RHS} &
\end{array}
$$

- $\mathfrak{o} = \mathit{let}\ w \doteq \sigma\ \mathit{in}\,\tau$. Item 1 follows from the IH. For item 2, $\mathit{let}\ w \doteq \sigma\ \mathit{in}\,\tau = \mathit{let}\ w \doteq \underline{r_1}\ \mathit{in}\,\mathfrak{r}_y$ for some r_1 and r_2. We reason as in the previous item, using the IH/2.

□

B TYPES

Sample proof of $[\![\boldsymbol{p}]\!](A \supset B) \supset [\![\boldsymbol{s}]\!]A \supset [\![\boldsymbol{p}\,\boldsymbol{s}]\!]B$, where $\Delta := u : A \supset B, v : A$ and $\Gamma := b : [\![\boldsymbol{s}]\!]A$.

$$
\dfrac{\emptyset;\Gamma \vdash b : [\![\boldsymbol{s}]\!]A \qquad \dfrac{\emptyset;\Gamma \vdash a : [\![\boldsymbol{p}]\!](A \supset B) \qquad \dfrac{\dfrac{\dfrac{}{\Delta;\emptyset \vdash \mathfrak{u} : u \rhd u : A \supset B} \qquad \dfrac{}{\Delta;\emptyset \vdash \mathfrak{v} : v \rhd v : A}}{\Delta;\emptyset \vdash \mathfrak{u}\,\mathfrak{v} : u\,v \rhd u\,v : B}\,\text{R-App}}{\Delta;\emptyset \vdash !(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : [\![\overline{u\,v}]\!]B}\,\text{Bang}}{\emptyset;\Gamma \vdash \mathit{let}\ u \doteq a\ \mathit{in}\,!(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : ([\![\boldsymbol{u}\,\boldsymbol{v}]\!]B)\{u/\boldsymbol{p}\}}\,\text{Let}}{\emptyset;\Gamma \vdash \mathit{let}\ v \doteq b\ \mathit{in}\ \mathit{let}\ u \doteq a\ \mathit{in}\,!(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : [\![\boldsymbol{p}\,\boldsymbol{v}]\!]B\{v/\boldsymbol{s}\}}\,\text{Let}
$$

Lemma B.1 (Weakening). *(a)* $\Delta;\Gamma \vdash s : B$ *and* $u \notin \mathit{dom}(\Delta)$*, implies* $\Delta, u : A;\Gamma \vdash s : B$
(b) $\Delta;\Gamma \vdash \rho : s \rhd t : B$ *and* $u \notin \mathit{dom}(\Delta)$*, implies* $\Delta, u : A;\Gamma \vdash \rho : s \rhd t : B$
(c) $\Delta;\Gamma \vdash s : B$ *and* $a \notin \mathit{dom}(\Gamma)$*, implies* $\Delta;\Gamma, a : A \vdash s : B$
(d) $\Delta;\Gamma \vdash \rho : s \rhd t : B$ *and* $a \notin \mathit{dom}(\Gamma)$*, implies* $\Delta;\Gamma, a : A \vdash \rho : s \rhd t : B$

Proof. By induction on the corresponding derivations of each item. □

Lemma. [Term as Trivial Rewrite – Lem. 2.2] $\Vdash \Delta;\Gamma \vdash s : A$ implies $\Vdash \Delta;\Gamma \vdash \mathfrak{s} : s \rhd s : A$.

Proof. By induction on the derivation of $\Delta;\Gamma \vdash s : A$. □

Lemma. [Term Substitution – Lem. 2.3] Suppose $\Delta;\Gamma, a : A \vdash s : B$ and $\Delta;\Gamma \vdash t : A$. Then $\Delta;\Gamma \vdash s\{a/t\} : B$.

Proof. By induction on the derivation of $\Delta;\Gamma, a : A \vdash s : B$. It relies on Lem. 1.6. □

Lemma B.2. $\overline{p\{u/^{\mathrm{src}}\rho^t_s\}} = \overline{p}\{u/\overline{s}\}$

Proof. By induction on p.

- $p = a$. Then $\overline{p\{u/^{\mathrm{src}}\rho^t_s\}} = a = \overline{p}\{u/\overline{s}\}$.
- $p = v$. If $v \neq u$, then $\overline{p\{u/^{\mathrm{src}}\rho^t_s\}} = v = \overline{p}\{u/\overline{s}\}$. If $p = u$, then $\overline{p\{u/^{\mathrm{src}}\rho^t_s\}} = \overline{s} = \boldsymbol{u}\{u/\overline{s}\} = \overline{p}\{u/\overline{s}\}$
- $p = \lambda a.s$, $p = s\,t$ and $p = \mathit{let}\ u \doteq s\ \mathit{in}\,t$ follow by using the IH.
- $p =!(\sigma, q, r)$. Then

$$
\begin{array}{lll}
 & \overline{p\{u/^{\mathrm{src}}\rho^t_s\}} & \\
= & \overline{!(\sigma, q, r)\{u/^{\mathrm{src}}\rho^t_s\}} & \\
= & \overline{!(\mathfrak{q}\{u/\rho^t_s\};\sigma\{u/^{\mathrm{tgt}}\rho^t_s\}, q\{u/^{\mathrm{src}}\rho^t_s\}, r\{u/^{\mathrm{tgt}}\rho^t_s\})} & \\
= & !\overline{q\{u/^{\mathrm{src}}\rho^t_s\}} & \\
= & !\overline{q}\{u/\overline{s}\} & \text{(IH)} \\
= & \overline{!(\sigma, q, r)}\{u/\overline{s}\} & \\
= & \overline{p}\{u/\overline{s}\} &
\end{array}
$$

□

Lemma B.3. *Suppose* $v \notin \mathrm{frv}(\boldsymbol{s})$*. Then*

$$\boldsymbol{p}\{v/\boldsymbol{t}\}\{u/\boldsymbol{s}\} = \boldsymbol{p}\{u/\boldsymbol{s}\}\{v/\boldsymbol{t}\{u/\boldsymbol{s}\}\}$$

Proof. By induction on $\boldsymbol{p}$.

- $p = \boldsymbol{a}$. $LHS = \boldsymbol{a} = RHS$.
- $p = \boldsymbol{w}$. If $\boldsymbol{w} = \boldsymbol{u}$, then $LHS = \boldsymbol{p}\{v/\boldsymbol{t}\}\{u/\boldsymbol{s}\} = \boldsymbol{u}\{u/\boldsymbol{s}\} = \boldsymbol{s}$ and $RHS = \boldsymbol{p}\{u/\boldsymbol{s}\}\{v/\boldsymbol{t}\{u/\boldsymbol{s}\}\} = \boldsymbol{s}\{v/\boldsymbol{t}\{u/\boldsymbol{s}\}\} = \boldsymbol{s}$.
If $\boldsymbol{w} = \boldsymbol{v}$ then $LHS = \boldsymbol{v}\{v/\boldsymbol{t}\}\{u/\boldsymbol{s}\} = \boldsymbol{t}\{u/\boldsymbol{s}\}$ and $RHS = \boldsymbol{p}\{u/\boldsymbol{s}\}\{v/\boldsymbol{t}\{u/\boldsymbol{s}\}\} = \boldsymbol{v}\{v/\boldsymbol{t}\{u/\boldsymbol{s}\}\} = \boldsymbol{t}\{u/\boldsymbol{s}\}$.
If $\boldsymbol{w} \neq \boldsymbol{u}$ and $\boldsymbol{w} \neq \boldsymbol{v}$, then we conclude immediately.
- $p = \boldsymbol{\lambda a.s}$, $p = \boldsymbol{s\,t}$, $p = \boldsymbol{!s}$, $p = \boldsymbol{let\ u \doteq s\ in\ t}$. We use the IH.

□

Lemma B.4. *$s \simeq t$ implies $s\{u/^{\mathsf{m}}\rho_p^q\} \simeq t\{u/^{\mathsf{m}}\rho_p^q\}$.*

Proof. By induction on the derivation of $s \simeq t$. □

Lemma [Rewrite Substitution Lemma – Lem. 2.4] Suppose $\Vdash \Delta; \emptyset \vdash \rho : s \rhd t : A$, $\Vdash \Delta; \emptyset \vdash s : A$ and $\Vdash \Delta; \emptyset \vdash t : A$. Suppose $\Vdash \Delta, u : A; \Gamma \vdash S : B$.

(a) $S = \sigma : p \rhd q$ implies
$$\Vdash \Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\}.$$
(b) $S = \sigma : p \rhd q$ implies
$$\Vdash \Delta, u : A; \Gamma \vdash p : B \text{ and } \Vdash \Delta, u : A; \Gamma \vdash q : B.$$
(c) $S = p$ implies
$$\Vdash \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\}.$$
(d) $S = p$ implies
$$\Vdash \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\}.$$
(e) $S = p$ implies
$$\Vdash \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\}.$$

Proof. By simultaneous structural induction on the derivations of $\Vdash \Delta, u : A; \Gamma \vdash S : B$.

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:
$$\frac{a : B \in \Gamma}{\Delta, u : A; \Gamma \vdash a : B}\ \mathsf{TVar}$$
Then $u \notin \mathsf{frv}(B)$. The first two items holds trivially. For the other three we reason as follows:
  - $\Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash a : B$.
  - $\Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \underline{a} : a \rhd a : B$.
  - $\Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \underline{a} : a \rhd a : B$.
- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:
$$\frac{\Delta, u : A; \Gamma, a : B_1 \vdash p_1 : B_2}{\Delta, u : A; \Gamma \vdash \lambda a.p_1 : B_1 \supset B_2}\ \mathsf{Abs}$$
Then $u \notin \mathsf{frv}(B_1)$. The first two items holds trivially. For item (c) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1\{u/\overline{s}\} \supset B_2\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1 \supset B_2\{u/\overline{s}\} \end{array}$$
The latter is derivable from
$$\Vdash \Delta; \Gamma, a : B_1 \vdash p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_2\{u/\overline{s}\}$$
which we obtain from the IH w.r.t. (c).
For item (d) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\}b \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_1\{u/\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_1\{u/\overline{s}\} \supset B_2\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_1\{u/\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_1 \supset B_2\{u/\overline{s}\} \end{array}$$
The latter is derivable from
$$\Vdash \Delta; \Gamma, a : B_1 \vdash \mathfrak{p}_1\{u/\rho_s^t\} : p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_2\{u/\overline{s}\}$$
which we obtain from the IH w.r.t. (d).
For (e) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} : \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1\{u/\overline{s}\} \supset B_2\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} : \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd \lambda a.\mathfrak{p}_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1 \supset B_2\{u/\overline{s}\} \end{array}$$
The latter is derivable from

$$\Vdash \Delta; \Gamma, a : B_1 \vdash \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\} : p_1\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{m}}\rho^t_s\} : B_2\{u/\overline{s}\}$$

which we obtain from the IH w.r.t. (e).

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash p_1 : C \supset B \quad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash p_1\, p_2 : B}\,\mathsf{App}$$

The first two items holds trivially. For item (c) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\} : (C \supset B)\{u/\overline{s}\} \\ \Vdash \Delta; \Gamma \vdash p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (c).
For item (d) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathfrak{p}_1\{u/\rho^t_s\}\, \mathfrak{p}_2\{u/\rho^t_s\} : p_1\{u/^{\mathsf{src}}\rho^t_s\}\, p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{tgt}}\rho^t_s\}\, p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta; \Gamma \vdash \mathfrak{p}_1\{u/\rho^t_s\} : p_1\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{tgt}}\rho^t_s\} : (C \supset B)\{u/\overline{s}\} \\ \Vdash \Delta; \Gamma \vdash \mathfrak{p}_2\{u/\rho^t_s\} : p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (d).
For item (e) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\}\, \mathfrak{p}_2\{u/^{\mathsf{m}}\rho^t_s\} : p_1\{u/^{\mathsf{m}}\rho^t_s\}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{m}}\rho^t_s\}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta; \Gamma \vdash \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\} : p_1\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{m}}\rho^t_s\} : (C \supset B)\{u/\overline{s}\} \\ \Vdash \Delta; \Gamma \vdash \mathfrak{p}_2\{u/^{\mathsf{m}}\rho^t_s\} : p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (e).

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:

$$\frac{v : B \in (\Delta, u : A)}{\Delta, u : A; \Gamma \vdash v : B}\,\mathsf{VVar}$$

Note that $u \notin \mathsf{frv}(B)$. The first two items holds trivially. If $u \neq v$ then, for items (c), (d) and (e) we reason as follows:
  – $\Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash v : B$.
  – $\Delta; \Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \underline{v} : v \rhd v : B$.
  – $\Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \underline{v} : v \rhd v : B$

If $u = v$, then
  – For item (c) we consider two cases. If $\mathsf{m} = \mathsf{src}$, then
  $$\Delta; \Gamma \vdash p\{u/^{\mathsf{src}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash s : B$$
  Moreover, the latter is derivable from the hypothesis and Weakening (Lem. B.1). If $\mathsf{m} = \mathsf{tgt}$, then
  $$\Delta; \Gamma \vdash p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash t : B$$
  and the latter is derivable from the hypothesis and Weakening (Lem. B.1) too.
  – For item (d) we have: $\Delta; \Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \rho : s \rhd t : B$. The latter is derivable from the hypothesis and Weakening (Lem. B.1).
  – For (e) and $\mathsf{m} = src$ (the case $\mathsf{m} = tgt$ is similar and omitted) we have: $\Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} = \Delta; \Gamma \vdash \mathfrak{s} : s \rhd s : B$. The latter follows from the hypothesis, Weakening (Lem. B.1) and the Term as Trivial Rewrite (Lem. 2.2).

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:

$$\frac{\Delta, u : A; \emptyset \vdash o, r : D \quad \Delta, u : A; \emptyset \vdash \tau : o \rhd r : D}{\Delta, u : A; \Gamma \vdash !(\tau, o, r) : [\![\overline{o}]\!]D}\,\mathsf{Bang}$$

Items (a) and (b) hold trivially. For item (c) we reason as follows. By the hypothesis we know $\Delta, u : A; \emptyset \vdash o : D$ and $\Delta, u : A; \emptyset \vdash r : D$. This allows us to apply the IH w.r.t. (d), to deduce

$$\Vdash \Delta; \emptyset \vdash \mathfrak{o}\{u/\rho^t_s\} : o\{u/^{\mathsf{src}}\rho^t_s\} \rhd o\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\overline{s}\} \tag{6}$$

By the IH w.r.t (a),

$$\Vdash \Delta; \emptyset \vdash \tau\{u/^{\mathsf{tgt}}\rho^t_s\} : o\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\overline{s}\} \tag{7}$$

By the IH w.r.t. (c) twice we have:

$$\Delta;\emptyset \vdash o\{u/^{\mathsf{src}}\rho^t_s\} : D\{u/\bar{s}\} \text{ and } \Delta;\emptyset \vdash r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\} \tag{8}$$

We can derive the following:

$$\dfrac{\begin{array}{c}\Delta;\emptyset \vdash o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\} \\ \Vdash_\pi \Delta;\emptyset \vdash \mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\} : o\{u/^{\mathsf{src}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\}\end{array}}{\Delta;\Gamma \vdash !(\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : \llbracket\overline{o\{u/^{\mathsf{src}}\rho^t_s\}}\rrbracket D\{u/\bar{s}\}}\ \text{Bang}$$

where π is the derivation:

$$\dfrac{\begin{array}{c}\Delta;\emptyset \vdash \mathfrak{o}\{u/\rho^t_s\} : o\{u/^{\mathsf{src}}\rho^t_s\} \rhd o\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\} \\ \Delta;\emptyset \vdash \tau\{u/^{\mathsf{tgt}}\rho^t_s\} : o\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\}\end{array}}{\Delta;\emptyset \vdash \mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\} : o\{u/\bar{s}\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\}}\ \text{R-Trans}$$

Note that

$$!(\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) = !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\}$$

and

$$\begin{array}{cll} & \llbracket\overline{o\{u/^{\mathsf{src}}\rho^t_s\}}\rrbracket D\{u/\bar{s}\} & \\ = & \llbracket\bar{o}\{u/\bar{s}\}\rrbracket D\{u/\bar{s}\} & (\textit{Lem. B.2}) \\ = & (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} & \end{array}$$

This concludes the proof of item (c).
For item (d) we reason as follows. Note that $\underline{!(\tau, o, r)} = \langle\tau|_o\mathfrak{r}\rangle$.

$$\begin{array}{cl} & \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\tau|_o\mathfrak{r}\rangle\{u/\rho^t_s\} : !(\tau, o, r)\{u/^{\mathsf{src}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle : !(\tau, o, r)\{u/^{\mathsf{src}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle : !(\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !(\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \end{array}$$

Recall from above that $\Delta, u : A;\emptyset \vdash r : D$. This allows us to apply the IH w.r.t. (e), to deduce

$$\Vdash \Delta;\emptyset \vdash \mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} : r\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\} \tag{9}$$

Consider the following abbreviations:

$$\begin{array}{rcl} \mathfrak{o}_u & := & \mathfrak{o}\{u/\rho^t_s\} \\ \tau^{\mathsf{tgt}}_u & := & \tau\{u/^{\mathsf{tgt}}\rho^t_s\} \\ \mathfrak{r}^{\mathsf{tgt}}_u & := & \mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} \end{array}$$

Then, for example, $\mathfrak{o}\{u/\rho^t_s\};\tau\{u/^{\mathsf{tgt}}\rho^t_s\}$ is just $\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u$. We can derive:

$$\dfrac{\dfrac{\begin{array}{c}\Vdash_\pi \Delta;\emptyset \vdash \mathfrak{o}_u;\tau^{\mathsf{tgt}}_u : o\{u/^{\mathsf{src}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\} \\ \Delta;\emptyset \vdash \mathfrak{r}^{\mathsf{tgt}}_u : r\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/\bar{s}\}\end{array}}{\Delta;\emptyset \vdash \langle\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}^{\mathsf{tgt}}_u\rangle : !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !((\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u);\mathfrak{r}^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : \llbracket\overline{o\{u/^{\mathsf{src}}\rho^t_s\}}\rrbracket D\{u/\bar{s}\}}\ \text{R-Bang}}{\Delta;\emptyset \vdash \langle\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}^{\mathsf{tgt}}_u\rangle : !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : \llbracket\overline{o\{u/^{\mathsf{src}}\rho^t_s\}}\rrbracket D\{u/\bar{s}\}}\ \text{SEq-R}$$

Moreover, from Lem. A.4, substitution preserves membership in $\mathbb{R}_1$. Thus $\mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} \in \mathbb{R}_1$ and hence $(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u);\mathfrak{r}^{\mathsf{tgt}}_u \simeq \mathfrak{o}_u;\tau^{\mathsf{tgt}}_u$.
For item (e) we reason as follows.

$$\begin{array}{cl} & \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\tau|_o\mathfrak{r}\rangle\{u/^{\mathsf{m}}\rho^t_s\} : !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}^{\mathsf{tgt}}_u\rangle : !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \\ = & \Delta;\Gamma \vdash \langle\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}^{\mathsf{tgt}}_u\rangle : !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\} \end{array}$$

We have already shown that the judgment

$$\Delta;\Gamma \vdash \langle\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}^{\mathsf{tgt}}_u\rangle : !(\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !((\mathfrak{o}_u;\tau^{\mathsf{tgt}}_u);\mathfrak{r}^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : (\llbracket\bar{o}\rrbracket D)\{u/\bar{s}\}$$

is derivable.

- The derivation of $\Delta, u : A;\Gamma \vdash p : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma \vdash p_1 : \llbracket\boldsymbol{m}\rrbracket D \quad \Delta, u : A, v : D;\Gamma \vdash p_2 : C}{\Delta, u : A;\Gamma \vdash \mathit{let}\ v \doteq p_1\ \mathit{in}\, p_2 : C\{v/\boldsymbol{m}\}}\ \text{Let}$$

where $u \notin \mathsf{frv}(D)$. The first two items holds trivially. For item (c) we reason as follows:

$$\begin{array}{rl} & \Delta;\Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{v/\boldsymbol{m}\}\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta;\Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\} : ([\![\boldsymbol{m}]\!]D)\{u/\overline{s}\} \\ \Vdash \Delta, v : D;\Gamma \vdash p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (c), an application of Let leading to (note $D\{u/\overline{s}\} = D$):

$$\dfrac{\begin{array}{c}\Delta;\Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\} : [\![\boldsymbol{m}\{u/\overline{s}\}]\!]D \\ \Delta, v : D;\Gamma \vdash p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\}\end{array}}{\Delta;\Gamma \vdash \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\}\{v/\boldsymbol{m}\{u/\overline{s}\}\}}\ \mathsf{Let}$$

We conclude from the substitution lemma Lem. B.3 that

$$\begin{array}{rl} & C\{v/\boldsymbol{m}\}\{u/\overline{s}\} \\ = & C\{u/\overline{s}\}\{v/\boldsymbol{m}\{u/\overline{s}\}\} \end{array}$$

For item (d) we reason as follows:

$$\begin{array}{rl} & \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \mathit{let}\ v \doteq \mathfrak{p}_1\{u/\rho^t_s\}\ \mathit{in}\, \mathfrak{p}_2\{u/\rho^t_s\} : \mathit{let}\ v \doteq p_1\{u/^{\mathsf{src}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd \mathit{let}\ v \doteq p_1\{u/^{\mathsf{tgt}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta;\Gamma \vdash \mathfrak{p}_1\{u/\rho^t_s\} : p_1\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{tgt}}\rho^t_s\} : [\![\boldsymbol{m}\{u/\overline{t}\}]\!]D \\ \Vdash \Delta;\Gamma \vdash \mathfrak{p}_2\{u/\rho^t_s\} : p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (d), an application of R-Let and Lem. B.3.
For item (e) we reason as follows:

$$\begin{array}{rl} & \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \mathit{let}\ v \doteq \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, \mathfrak{p}_2\{u/^{\mathsf{m}}\rho^t_s\} : \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \mathit{let}\ v \doteq \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, \mathfrak{p}_2\{u/^{\mathsf{m}}\rho^t_s\} : \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd \mathit{let}\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ \mathit{in}\, p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{v/\boldsymbol{m}\}\{u/\overline{s}\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \Vdash \Delta;\Gamma \vdash \mathfrak{p}_1\{u/^{\mathsf{m}}\rho^t_s\} : p_1\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{m}}\rho^t_s\} : [\![\boldsymbol{m}\{u/\overline{s}\}]\!]D \\ \Vdash \Delta;\Gamma \vdash \mathfrak{p}_2\{u/^{\mathsf{m}}\rho^t_s\} : p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/\overline{s}\} \end{array}$$

which we obtain from the IH w.r.t. (e), an application of R-Let and Lem. B.3.

- The derivation of $\Delta, u : A;\Gamma \vdash p : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma \vdash q : B \quad q \simeq p}{\Delta, u : A;\Gamma \vdash p : B}\ \mathsf{SEq\text{-}T}$$

The first two items hold immediately. For (c)

$$\Vdash \Delta;\Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}.$$

follows from the IH ($\Vdash \Delta;\Gamma \vdash q\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}$), Lem. B.4 and an application of SEq-T.
For (d) we must prove

$$\Vdash \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$

From the IH w.r.t. (d) we have

$$\Vdash \Delta;\Gamma \vdash \mathfrak{q}\{u/\rho^t_s\} : q\{u/^{\mathsf{src}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$

From Lem. 1.7(c) and $q \simeq p$, we have $\mathfrak{q}\{u/\rho^t_s\} \simeq \mathfrak{p}\{u/\rho^t_s\} : q\{u/^{\mathsf{src}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\}$. Also, from Lem. B.4, we have $q\{u/^{\mathsf{src}}\rho^t_s\} \simeq p\{u/^{\mathsf{src}}\rho^t_s\}$. and $q\{u/^{\mathsf{tgt}}\rho^t_s\} \simeq p\{u/^{\mathsf{tgt}}\rho^t_s\}$. Thus

$$\dfrac{\begin{array}{c}\Delta;\Gamma \vdash \mathfrak{q}\{u/\rho^t_s\} : q\{u/^{\mathsf{src}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\} \\ \mathfrak{q}\{u/\rho^t_s\} \simeq \mathfrak{p}\{u/\rho^t_s\} : q\{u/^{\mathsf{src}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} \\ q\{u/^{\mathsf{src}}\rho^t_s\} \simeq p\{u/^{\mathsf{src}}\rho^t_s\} \\ q\{u/^{\mathsf{tgt}}\rho^t_s\} \simeq p\{u/^{\mathsf{tgt}}\rho^t_s\}\end{array}}{\Delta;\Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}}\ \mathsf{SEq\text{-}R}$$

For (e) we must show

$$\Vdash \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}.$$

By the IH

$$\Vdash \Delta;\Gamma \vdash \mathfrak{q}\{u/^{\mathsf{m}}\rho^t_s\} : q\{u/^{\mathsf{m}}\rho^t_s\} \rhd q\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/\overline{s}\}.$$

From Lem. 1.7(b) and $q \simeq p$, we deduce $q\{u/^{\mathsf{m}}\rho_s^t\} \simeq p\{u/^{\mathsf{m}}\rho_s^t\}$. From Lem. 1.7(d) and $q \simeq p$, we deduce $\mathfrak{q}\{u/^{\mathsf{m}}\rho_s^t\} \simeq \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : q\{u/^{\mathsf{m}}\rho_s^t\} \rhd q\{u/^{\mathsf{m}}\rho_s^t\}$. We conclude with an application of SEq-R.

$$\frac{\begin{array}{c}\Delta;\Gamma \vdash \mathfrak{q}\{u/^{\mathsf{m}}\rho_s^t\} : q\{u/^{\mathsf{m}}\rho_s^t\} \rhd q\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\} \\ \mathfrak{q}\{u/^{\mathsf{m}}\rho_s^t\} \simeq \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : q\{u/^{\mathsf{m}}\rho_s^t\} \rhd q\{u/^{\mathsf{m}}\rho_s^t\} \\ q\{u/^{\mathsf{m}}\rho_s^t\} \simeq p\{u/^{\mathsf{m}}\rho_s^t\}\end{array}}{\Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/\overline{s}\}}\ \text{SEq-R}$$

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{a : B \in \Gamma}{\Delta, u : A; \Gamma \vdash \underline{a} : a \rhd a : B}\ \text{R-Refl-TVar}$$

The last three items are immediate. For (a) and (b) we have ($u \notin \mathsf{frv}(B)$):
– $\Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta;\Gamma \vdash \underline{a} : a \rhd a : B$.
– $\Delta, u : A; \Gamma \vdash a : B$.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{v : B \in (\Delta, u : A)}{\Delta, u : A; \Gamma \vdash \underline{v} : v \rhd v : B}\ \text{R-Refl-VVar}$$

The last three items are immediate. For (a) ($u \notin \mathsf{frv}(B)$):
– Suppose $u \neq v$. Then $\Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta;\Gamma \vdash \underline{v} : v \rhd v : B$.
– Suppose $u = v$. Then $\Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} = \Delta;\Gamma \vdash \mathfrak{t} : t \rhd t : B$. The latter is derivable from the Term as Trivial Rewrite Lemma (Lem. 2.2), the hypothesis and Weakening (Lem. B.1).

For item (b) we have $\Delta, u : A; \Gamma \vdash v : B$.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \emptyset \vdash o, r, n : C \quad \Delta, u : A; \emptyset \vdash \tau : o \rhd r : C \quad \Delta, u : A; \emptyset \vdash \mu : r \rhd n : C}{\Delta, u : A; \Gamma \vdash \langle \tau|_{\mathfrak{o}}\mu\rangle : !(\tau, o, r) \rhd !(\tau;\mu, o, n) : [\![\overline{o}]\!]C}\ \text{R-Bang}$$

The last three items are immediate. For (b) we have:

$$\frac{\Delta, u : A; \emptyset \vdash o, r : C \quad \Delta, u : A; \emptyset \vdash \tau : o \rhd r : C}{\Delta, u : A; \Gamma \vdash !(\tau, o, r) : [\![\overline{o}]\!]C}\ \text{Bang}$$

and since

$$\frac{\Delta, u : A; \emptyset \vdash \tau : o \rhd r : C \quad \Delta, u : A; \emptyset \vdash \mu : r \rhd n : C}{\Delta, u : A; \emptyset \vdash \tau;\mu : o \rhd n : C}\ \text{R-Trans}$$

then

$$\frac{\Delta, u : A; \emptyset \vdash o, n : C \quad \Delta, u : A; \emptyset \vdash \tau;\mu : o \rhd n : C}{\Delta, u : A; \Gamma \vdash !(\tau;\mu, o, n) : [\![\overline{o}]\!]C}\ \text{Bang}$$

We now address items (a) and (b). Consider the following abbreviations:

$$\begin{array}{rcl}\mathfrak{o}_u & := & \mathfrak{o}\{u/\rho_s^t\} \\ \tau_u^{\mathsf{tgt}} & := & \tau\{u/^{\mathsf{tgt}}\rho_s^t\} \\ \mu_u^{\mathsf{tgt}} & := & \mu\{u/^{\mathsf{tgt}}\rho_s^t\}\end{array}$$

For item (a) we reason as follows:

$$\begin{array}{cl} & \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \langle \tau|_{\mathfrak{o}}\mu\rangle\{u/^{\mathsf{tgt}}\rho_s^t\} : !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd !(\tau;\mu, o, n)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\overline{o}]\!]C)\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \langle \mathfrak{o}_u;\tau_u^{\mathsf{tgt}}|_{o\{u/^{\mathsf{src}}\rho_s^t\}}\mu_u^{\mathsf{tgt}}\rangle : !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd !(\tau;\mu, o, n)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\overline{o}]\!]C)\{u/\overline{s}\} \\ = & \Delta;\Gamma \vdash \langle \mathfrak{o}_u;\tau_u^{\mathsf{tgt}}|_{o\{u/^{\mathsf{src}}\rho_s^t\}}\mu_u^{\mathsf{tgt}}\rangle : !(\mathfrak{o}_u;\tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) \rhd !(\mathfrak{o}_u;(\tau;\mu)\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) : ([\![\overline{o}]\!]C)\{u/\overline{s}\}\end{array}$$

We conclude just as in the Bang case. For (b), it is similar.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash \sigma_1 : p \rhd r : B \quad \Delta, u : A; \Gamma \vdash \sigma_2 : r \rhd q : B}{\Delta, u : A; \Gamma \vdash \sigma_1;\sigma_2 : p \rhd q : B}\ \text{R-Trans}$$

The last four items are immediate. Item (b) follows from the IH. For item (a) we reason as above.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma, a : C \vdash p_1 : B \quad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathbf{ba}(a.p_1, p_2) : (\lambda a.p_1)\, p_2 \rhd p_1\{a/p_2\} : B}\ \text{R-}\beta$$

The last three items are immediate. Also, $u \notin \mathsf{frv}(C)$. For (b) we have:

$$\frac{\dfrac{\Delta, u : A; \Gamma, a : C \vdash p_1 : B}{\Delta, u : A; \Gamma \vdash \lambda a.s : C \supset B}\ \text{Abs} \qquad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash (\lambda a.p_1)\, p_2 : B}\ \text{App}$$

Also, $\Vdash\ \Delta, u : A; \Gamma \vdash p_1\{a/p_2\} : B$ follows from $\Vdash\ \Delta, u : A; \Gamma, a : C \vdash p_1 : B$, $\Vdash\ \Delta, u : A; \Gamma \vdash p_2 : C$ and the Truth Substitution Lemma (Lem. 2.3).
For item (a) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash \sigma\{u/^{\mathrm{tgt}}\rho_s^t\} : p\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd q\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{ba}(a.p_1, p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} : ((\lambda a.p_1)\, p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\}, p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : ((\lambda a.p_1)\, p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\}, p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : (\lambda a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\})\, p_2\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \end{array}$$

By the IH w.r.t (c):
– $\Vdash \Delta; \Gamma, a : C \vdash p_1\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\}$,
– $\Vdash \Delta; \Gamma \vdash p_2\{u/^{\mathrm{tgt}}\rho_s^t\} : C\{u/\overline{s}\} = \Delta; \Gamma \vdash p_2\{u/^{\mathrm{tgt}}\rho_s^t\} : C$

$$\frac{\dfrac{\Delta; \Gamma, a : C \vdash p_1\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\}}{\Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\} : C \supset B\{u/\overline{s}\}}\ \text{Abs} \qquad \Delta; \Gamma \vdash p_2\{u/^{\mathrm{tgt}}\rho_s^t\} : C}{\Delta; \Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\}, p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : (\lambda a.p_1\{u/^{\mathrm{tgt}}\rho_s^t\})\, p_2\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_1\{u/^{\mathrm{tgt}}\rho_s^t\}\{a/p_2\{u/^{\mathrm{tgt}}\rho_s^t\}\} : B\{u/\overline{s}\}}\ \text{R-}\beta$$

Note that, $p_1\{u/^{\mathrm{tgt}}\rho_s^t\}\{a/p_2\{u/^{\mathrm{tgt}}\rho_s^t\}\} = p_1\{a/p_2\}\{u/^{\mathrm{tgt}}\rho_s^t\}$ follows from the Commutation of Validity Substitution with Truth Substitution Lemma (Lem. 1.10).
- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \emptyset \vdash \tau : o \rhd r : D \quad \Delta, u : A, v : D; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathbf{bb}(!(\tau, o, r), v.p_2) : \mathit{let}\ v \doteq\, !(\tau, o, r)\ \mathit{in}\, p_2 \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\} : C\{v/\overline{o}\}}\ \text{R-}\beta_\Box$$

The last three items are immediate. Also, $u \notin \mathsf{frv}(D)$. Also, by IH w.r.t to (b) applied to $\Delta, u : A; \emptyset \vdash \tau : o \rhd r : D$ we know:

$$\Delta, u : A; \emptyset \vdash o, r : D \tag{10}$$

For (b) we have:

$$\frac{\dfrac{\Delta, u : A; \emptyset \vdash o, r : D \quad \Delta, u : A; \emptyset \vdash \tau : o \rhd r : D}{\Delta, u : A; \Gamma \vdash !(\tau, o, r) : [\![\overline{o}]\!]D}\ \text{Bang} \qquad \Delta, u : A, v : D; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathit{let}\ v \doteq\, !(\tau, o, r)\ \mathit{in}\, p_2 : C\{v/\overline{o}\}}\ \text{Let}$$

To deduce that the following judgement is derivable:

$$\Delta, u : A; \Gamma \vdash p_2\{v/^{\mathrm{tgt}}\tau_o^r\} : C\{v/\overline{o}\}$$

we resort to the IH w.r.t. (c).
Consider the following abbreviations:

$$\begin{array}{l} \mathfrak{o}_u := \mathfrak{o}\{u/\rho_s^t\} \\ \tau_u^{\mathrm{tgt}} := \tau\{u/^{\mathrm{tgt}}\rho_s^t\} \\ \mu := \mathbf{bb}(!(\mathfrak{o}_u; \tau_u^{\mathrm{tgt}}, o\{u/^{\mathrm{src}}\rho_s^t\}, r\{u/^{\mathrm{tgt}}\rho_s^t\}), v.p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) \end{array}$$

For (a) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash \sigma\{u/^{\mathrm{tgt}}\rho_s^t\} : p\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd q\{u/^{\mathrm{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{bb}(!(\tau, o, r), v.p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} : (\mathit{let}\ v \doteq\, !(\tau, o, r)\ \mathit{in}\, p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\}\{u/^{\mathrm{tgt}}\rho_s^t\} : C\{v/\overline{o}\}\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{bb}(!(\tau, o, r)\{u/^{\mathrm{tgt}}\rho_s^t\}, v.p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : (\mathit{let}\ v \doteq\, !(\tau, o, r)\ \mathit{in}\, p_2)\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\}\{u/^{\mathrm{tgt}}\rho_s^t\} : C\{v/\overline{o}\}\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{bb}(!(\tau, o, r)\{u/^{\mathrm{tgt}}\rho_s^t\}, v.p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : \mathit{let}\ v \doteq\, !(\tau, o, r)\{u/^{\mathrm{tgt}}\rho_s^t\}\ \mathit{in}\, p_2\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\}\{u/^{\mathrm{src}}\rho_s^t\} : C\{v/\overline{o}\}\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mathbf{bb}(!(\mathfrak{o}_u; \tau_u^{\mathrm{tgt}}, o\{u/^{\mathrm{src}}\rho_s^t\}, r\{u/^{\mathrm{tgt}}\rho_s^t\}), v.p_2\{u/^{\mathrm{tgt}}\rho_s^t\}) : \mathit{let}\ v \doteq\, !(\tau, o, r)\{u/^{\mathrm{tgt}}\rho_s^t\}\ \mathit{in}\, p_2\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\}\{u/^{\mathrm{src}}\rho_s^t\} : C\{v/\overline{o}\}\{u/\overline{s}\} \\ = & \Delta; \Gamma \vdash \mu : \mathit{let}\ v \doteq\, !(\mathfrak{o}_u; \tau_u^{\mathrm{tgt}}, o\{u/^{\mathrm{src}}\rho_s^t\}, r\{u/^{\mathrm{tgt}}\rho_s^t\})\ \mathit{in}\, p_2\{u/^{\mathrm{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathrm{tgt}}\tau_o^r\}\{u/^{\mathrm{src}}\rho_s^t\} : C\{v/\overline{o}\}\{u/\overline{s}\} \end{array}$$

$$\dfrac{\begin{array}{c}\Delta;\Gamma \vdash \mathfrak{d}_u;\tau_u^{\mathsf{tgt}} : o\{u/^{\mathsf{src}}\rho_s^t\} \rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D \\ \Delta, v : D;\Gamma \vdash p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C\{u/\overline{s}\}\end{array}}{\Delta, u : A;\Gamma \vdash \mu : \mathit{let}\ v \mathrel{\hat{=}} !(\mathfrak{d}_u;\tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\})\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{u/^{\mathsf{tgt}}\rho_s^t\}\{v/^{\mathsf{tgt}}\mathfrak{d}_u;\tau_u^{\mathsf{tgt}}{}^{r\{u/^{\mathsf{tgt}}\rho_s^t\}}_{o\{u/^{\mathsf{src}}\rho_s^t\}}\} : C\{u/\overline{s}\}\{v/\overline{o\{u/^{\mathsf{src}}\rho_s^t\}}\}}\ \text{R-}\beta_{\Box}$$

First note that

$$\begin{array}{cll} & C\{u/\overline{s}\}\{v/\overline{o\{u/^{\mathsf{src}}\rho_s^t\}}\} & \\ = & C\{u/\overline{s}\}\{v/\overline{o}\{u/\overline{s}\}\} & (\mathit{Lem.}\ B.2) \\ = & C\{v/\overline{o}\}\{u/\overline{s}\} & (\mathit{Lem.}\ B.3) \end{array}$$

Also, note that

$$\begin{array}{cll} & p_2\{u/^{\mathsf{tgt}}\rho_s^t\}\{v/^{\mathsf{tgt}}\mathfrak{d}_u;\tau_u^{\mathsf{tgt}}{}^{r\{u/^{\mathsf{tgt}}\rho_s^t\}}_{o\{u/^{\mathsf{src}}\rho_s^t\}}\} & \\ = & p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{tgt}}\rho_s^t\} & (\mathit{Lem.}\ 1.11) \end{array}$$

- The derivation of $\Delta, u : A;\Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma, a : B_1 \vdash \sigma_1 : p_1 \rhd q_1 : B_2}{\Delta, u : A;\Gamma \vdash \lambda a.B_1 : \lambda a.p_1 \rhd \lambda a.q_1 : B_1 \supset B_2}\ \text{R-Abs}$$

  The last three items are immediate. For item (b) we conclude from the IH and an application of Abs that $\Vdash \Delta, u : A;\Gamma \vdash \lambda a.B_1 : B_1 \supset B_2$. Likewise for $\Vdash \Delta, u : A;\Gamma \vdash \lambda a.B_1 : B_1 \supset B_2$. For item (a), we proceed as in the previous cases.
- The derivation of $\Delta, u : A;\Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma \vdash \sigma_1 : p_1 \rhd q_1 : C \supset B \quad \Delta, u : A;\Gamma \vdash \sigma_2 : p_2 \rhd q_2 : C}{\Delta, u : A;\Gamma \vdash \sigma_1\,\sigma_2 : p_1\,p_2 \rhd q_1\,q_2 : B}\ \text{R-App}$$

  The last three items are immediate. For item (b) we conclude from the IH (twice) and an application of App that $\Vdash \Delta, u : A;\Gamma \vdash p_1\,p_2 : B$. Likewise for $\Vdash \Delta, u : A;\Gamma \vdash q_1\,q_2 : B$. For item (a), we proceed as in the previous cases.
- The derivation of $\Delta, u : A;\Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma \vdash \sigma_1 : p_1 \rhd q_1 : [\![\boldsymbol{m}]\!]D \quad \Delta, u : A, v : D;\Gamma \vdash \sigma_2 : p_2 \rhd q_2 : C}{\Delta, u : A;\Gamma \vdash \mathit{let}\ v \mathrel{\hat{=}} \sigma_1\ \mathit{in}\, \sigma_2 : \mathit{let}\ v \mathrel{\hat{=}} p_1\ \mathit{in}\, p_2 \rhd \mathit{let}\ v \mathrel{\hat{=}} q_1\ \mathit{in}\, q_2 : C\{v/\boldsymbol{m}\}}\ \text{R-Let}$$

  The last three items are immediate. For item (b) we conclude from the IH (twice) and an application of Let that $\Vdash \Delta, u : A;\Gamma \vdash \mathit{let}\ v \mathrel{\hat{=}} p_1\ \mathit{in}\, p_2 : C\{v/\boldsymbol{m}\}$. Likewise for $\Vdash \Delta, u : A;\Gamma \vdash \mathit{let}\ v \mathrel{\hat{=}} q_1\ \mathit{in}\, q_2 : C\{v/\boldsymbol{m}\}$. For item (a), we proceed as in the previous cases.
- The derivation of $\Delta, u : A;\Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\dfrac{\Delta, u : A;\Gamma \vdash \tau : m \rhd n : B \quad \tau \simeq \sigma : m \rhd n \quad s \simeq p \quad t \simeq q}{\Delta, u : A;\Gamma \vdash \sigma : p \rhd q : B}\ \text{SEq-R}$$

  The last three items are immediate. For (a) we must prove:

$$\Vdash \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\}.$$

  From the IH we have

$$\Vdash \Delta;\Gamma \vdash \tau\{u/^{\mathsf{tgt}}\rho_s^t\} : m\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd n\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\}.$$

  From Lem. 1.7(a) and $\tau \simeq \sigma : m \rhd n$ we deduce $\tau\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : m\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd n\{u/^{\mathsf{tgt}}\rho_s^t\}$. From Lem. 1.7(b) and $m \simeq p$ (resp. $n \simeq q$) we deduce $m\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq p\{u/^{\mathsf{tgt}}\rho_s^t\}$ (resp. $n\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq q\{u/^{\mathsf{tgt}}\rho_s^t\}$). We conclude with:

$$\dfrac{\begin{array}{c}\Delta;\Gamma \vdash \tau\{u/^{\mathsf{tgt}}\rho_s^t\} : m\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd n\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\} \\ \tau\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : m\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd n\{u/^{\mathsf{tgt}}\rho_s^t\} \\ m\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq p\{u/^{\mathsf{tgt}}\rho_s^t\} \\ n\{u/^{\mathsf{tgt}}\rho_s^t\} \simeq q\{u/^{\mathsf{tgt}}\rho_s^t\}\end{array}}{\Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/\overline{s}\}}\ \text{SEq-R}$$

  For (b) we must prove:

$$\Vdash \Delta, u : A;\Gamma \vdash p : B \text{ and } \Vdash \Delta, u : A;\Gamma \vdash q : B.$$

  This follows from the IH and SEq-T.

$\square$

Lemma. [Lem. 2.5] $\Vdash \Delta;\Gamma \vdash \rho : s \rhd t : A$ implies $\Vdash \Delta;\Gamma \vdash s : A$ and $\Vdash \Delta;\Gamma \vdash t : A$.

Proof. By induction on the derivation of $\Delta;\Gamma \vdash \rho : s \rhd t : A$.

- R-Refl-TVar and R-Refl-VVar. We conclude immediately from TVar and VVar.

- The derivation ends in:

$$\frac{\Delta;\emptyset \vdash r,s,t : A \quad \Delta;\emptyset \vdash \rho_1 : r \rhd s : A \quad \Delta;\emptyset \vdash \rho_2 : s \rhd t : A}{\Delta;\Gamma \vdash \langle \rho_1|_r \rho_2\rangle : !(\rho_1,r,s) \rhd !(\rho_1;\rho_1,r,t) : [\![\overline{r}]\!]A} \text{ R-Bang}$$

  Then for $!(\rho_1,r,s)$ we derive:

$$\frac{\Delta;\emptyset \vdash r,s : A \quad \Delta;\emptyset \vdash \rho_1 : r \rhd s : A}{\Delta;\Gamma \vdash !(\rho_1,r,s) : [\![\overline{r}]\!]A} \text{ Bang}$$

  For $!(\rho_1;\rho_2,r,t)$ we obtain the following derivation π:

$$\frac{\Delta;\emptyset \vdash r,t : A \quad \dfrac{\Delta;\emptyset \vdash \rho_1 : r \rhd s : A \quad \Delta;\emptyset \vdash \rho_2 : s \rhd t : A}{\Delta;\emptyset \vdash \rho_1;\rho_2 : r \rhd t : A} \text{ R-Trans}}{\Delta;\Gamma \vdash !(\rho_1;\rho_2,r,t) : [\![\overline{r}]\!]A} \text{ Bang}$$

- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s \rhd r : A \quad \Delta;\Gamma \vdash \rho_2 : r \rhd t : A}{\Delta;\Gamma \vdash \rho_1;\rho_2 : s \rhd t : A} \text{ R-Trans}$$

  We conclude from the IH.
- The derivation ends in:

$$\frac{\Delta;\Gamma, a : A \vdash p : B \quad \Delta;\Gamma \vdash q : A}{\Delta;\Gamma \vdash \mathbf{ba}(a.p,q) : (\lambda a.p)\, q \rhd p\{a/q\} : B} \text{ R-}\beta$$

  For $(\lambda a.p)\, q$ we use Abs and then App. For $p\{a/q\}$ we use the Truth Substitution Lemma (Lem. 2.3).
- The derivation ends in:

$$\frac{\Delta;\emptyset \vdash \rho_1 : p \rhd q : A \quad \Delta, u : A;\Gamma \vdash r : C}{\Delta;\Gamma \vdash \mathbf{bb}(!(\rho_1,p,q),u.r) : let\ u \doteq !(\rho_1,p,q)\ in\, r \rhd r\{u/^{\mathrm{tgt}}\rho_1{}_p^q\} : C\{u/\overline{p}\}} \text{ R-}\beta_\Box$$

  By the IH on $\Delta;\emptyset \vdash \rho_1 : p \rhd q : A$, we deduce $\Delta;\emptyset \vdash p : A$ and $\Delta;\emptyset \vdash q : A$. For $let\ u \doteq !(\rho_1,p,q)\ in\, r$ we use Bang, then Let. For $r\{u/^{\mathrm{tgt}}\rho_1{}_p^q\}$ we use the Validity Substitution Lemma (Lem. 2.4).
- The derivation ends in:

$$\frac{\Delta;\Gamma, a : A \vdash \rho_1 : p \rhd q : B}{\Delta;\Gamma \vdash \lambda a.\rho_1 : \lambda a.p \rhd \lambda a.q : A \supset B} \text{ R-Abs}$$

  We conclude from the IH and an application of Abs.
- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s_1 \rhd t_1 : A \supset B \quad \Delta;\Gamma \vdash \rho_2 : s_2 \rhd t_2 : A}{\Delta;\Gamma \vdash \rho_1\, \rho_2 : s_1\, s_2 \rhd t_1\, t_2 : B} \text{ R-App}$$

  We conclude from the IH (twice) and an application of App.
- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s_1 \rhd t_1 : [\![\boldsymbol{p}]\!]A \quad \Delta, u : A;\Gamma \vdash \rho_2 : s_2 \rhd t_2 : C}{\Delta;\Gamma \vdash let\ u \doteq \rho_1\ in\, \rho_2 : let\ u \doteq s_1\ in\, s_2 \rhd let\ u \doteq t_1\ in\, t_2 : C\{u/\boldsymbol{p}\}} \text{ R-Let}$$

  We conclude from the IH (twice) and an application of Let.
- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \sigma : p \rhd q : A \quad \sigma \simeq \rho : p \rhd q \quad p \simeq s \quad q \simeq t}{\Delta;\Gamma \vdash \rho : s \rhd t : A} \text{ SEq-R}$$

  We resort to the IH and SEq-T.

□

Lemma B.5 (Generation for Structural Equivalence on Terms). *(a)* $s \simeq \lambda a.t$ *implies* $s = \lambda a.r$ *and* $r \simeq t$.
(b) $\lambda a.t \simeq s$ *implies* $s = \lambda a.r$ *and* $t \simeq r$.
(c) $s \simeq t_1\, t_2$ *implies* $s = s_1\, s_2$ *and* $s_1 \simeq t_1$ *and* $s_2 \simeq t_2$.
(d) $t_1\, t_2 \simeq s$ *implies* $s = s_1\, s_2$ *and* $t_1 \simeq s_1$ *and* $t_2 \simeq s_2$.
(e) $s \simeq let\ u \doteq t_1\ in\, t_2$ *implies* $s = let\ u \doteq s_1\ in\, s_2$ *and* $s_1 \simeq t_1$ *and* $s_2 \simeq t_2$.
(f) $let\ u \doteq t_1\ in\, t_2 \simeq s$ *implies* $s = let\ u \doteq s_1\ in\, s_2$ *and* $t_1 \simeq s_1$ *and* $t_2 \simeq s_2$.
(g) $s \simeq !(\rho,t_1,t_2)$ *implies* $s = !(\sigma,s_1,s_2)$ *and* $s_1 \simeq t_1$ *and* $s_2 \simeq t_2$ *and* $\rho \simeq \sigma : s_1 \rhd s_2$.
(h) $!(\rho,t_1,t_2) \simeq s$ *implies* $s = !(\sigma,s_1,s_2)$ *and* $t_1 \simeq s_1$ *and* $t_2 \simeq s_2$ *and* $\rho \simeq \sigma : t_1 \rhd t_2$.
(i) $a \simeq s$ *implies* $s = a$.
(j) $s \simeq a$ *implies* $a = s$.
(k) $u \simeq s$ *implies* $s = u$.
(l) $u \simeq s$ *implies* $u = s$.

Proof. By induction on the derivation of $p \simeq q$. Also, the cases for symmetry and transitivity are immediate. The remaining cases are:

- The derivation ends in:
$$\frac{s \simeq t}{\lambda a.s \simeq \lambda a.t}\ \text{EqT-Abs}$$
Then only case (a) or (b) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q}{s\,t \simeq p\,q}\ \text{EqT-App}$$
Then only case (c) or (d) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q}{\mathit{let}\ u \doteq s\ \mathit{in}\ t \simeq \mathit{let}\ u \doteq p\ \mathit{in}\ q}\ \text{EqT-Let}$$
Then only case (e) or (f) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q \quad \rho \simeq \sigma : s \rhd t}{!(\rho, s, t) \simeq !(\sigma, p, q)}\ \text{EqT-Bang}$$
Then only case (g) or (h) applies and we conclude immediately.
- The derivation ends in:
$$\frac{}{a \simeq a}\ \text{EqT-TVar}$$
Then only case (i) or (j) applies and we can conclude immediately.
- The derivation ends in:
$$\frac{}{u \simeq u}\ \text{EqT-RVar}$$
Then only case (k) or (l) applies and we can conclude immediately.

□

Lemma B.6. *$s \simeq s'$ implies $\overline{s} = \overline{s'}$.*

Proof. By induction on the derivation of $s \simeq s'$. □

Lemma B.7 (Generation for Typability on Terms).

(a) $\Delta; \Gamma \vdash \lambda a.s : A$ implies there exists s', A', B s.t. $\Delta; \Gamma, a : B \vdash s' : A$ and $s \simeq s'$ and $A = B \supset A'$.
(b) $\Delta; \Gamma \vdash s\,t : A$ implies there exists s', t', B s.t. $\Delta; \Gamma \vdash s' : B \supset A$ and $\Delta; \Gamma \vdash t' : B$ and $s \simeq s'$ and $t \simeq t'$.
(c) $\Delta; \Gamma \vdash \mathit{let}\ u \doteq s\ \mathit{in}\ t : A$ implies there exists $s', t', B, A', \boldsymbol{p}$ s.t. $\Delta; \Gamma \vdash s' : [\![\boldsymbol{p}]\!]B$ and $\Delta, u : B; \Gamma \vdash t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $A = A'\{u/\boldsymbol{p}\}$.
(d) $\Delta; \Gamma \vdash !(\rho, s, t) : A$ implies there exists s', t', A', ρ' s.t. $\Delta; \emptyset \vdash s', t' : A'$ and $\Delta; \emptyset \vdash \rho' : s' \rhd t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $\rho \simeq \rho' : s \rhd t$ and $A = [\![\overline{s'}]\!]A'$.
(e) $\Delta; \Gamma \vdash a : A$ implies $a : A \in \Gamma$.
(f) $\Delta; \Gamma \vdash u : A$ implies $u : A \in \Delta$.

Proof. Consider the first item. The derivation can end in one of two cases:

- Abs and $A = A_1 \supset A_2$:
$$\frac{\Delta; \Gamma, a : A_1 \vdash s : A_2}{\Delta; \Gamma \vdash \lambda a.s : A_1 \supset A_2}\ \text{Abs}$$
We take $s' := s$, $B := A_1$ and conclude.
- SEq-T'
$$\frac{\Delta; \Gamma \vdash t : A \quad t \simeq \lambda a.s}{\Delta; \Gamma \vdash \lambda a.s : A}\ \text{SEq-T'}$$
By Lem. B.5 on $t \simeq \lambda a.s$, there exists t' s.t. $t = \lambda a.t'$ and, moreover, $t' \simeq s$. By the IH on $\Delta; \Gamma \vdash t : A$ there exists t'', A', B' s.t. $\Delta; \Gamma, a : B' \vdash t'' : A'$ and $t'' \simeq t'$ and $A = B' \supset A'$. We set $s' := t''$, $B := B'$ and notice that $t'' \simeq s$, and $A = B' \supset A'$.

Consider the second item. The derivation can end in one of two cases:

- App:
$$\frac{\Delta; \Gamma \vdash s : B \supset A \quad \Delta; \Gamma \vdash t : B}{\Delta; \Gamma \vdash s\,t : A}\ \text{App}$$
We take $s' := s$, $t' := t$, $A' := A$ and conclude.

- SEq-T'

$$\frac{\Delta;\Gamma \vdash r : A \quad r \simeq s\,t}{\Delta;\Gamma \vdash s\,t : A}\ \text{SEq-T'}$$

By Lem. B.5 on $r \simeq s\,t$, there exists r_1, r_2 s.t. $r = r_1\,r_2$ and, moreover, $r_1 \simeq s$ and $r_2 \simeq t$. By the IH on $\Delta;\Gamma \vdash r : A$ there exists r_1', r_2', B' s.t. $\Delta;\Gamma \vdash r_1' : B' \supset A$ and $\Delta;\Gamma \vdash r_2' : B'$ and $r_1' \simeq r_1$ and $r_2' \simeq r_2$. We set $s' := r_1'$, $t' := r_2'$, $B := B'$, and notice that $s \simeq r_1'$, $t \simeq r_2'$.

Consider the third item. The derivation can end in one of two cases:

- Let and $A = C\{u/\boldsymbol{p}\}$:

$$\frac{\Delta;\Gamma \vdash s : [\![\boldsymbol{p}]\!]D \quad \Delta, u : D;\Gamma \vdash t : C}{\Delta;\Gamma \vdash \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t : C\{u/\boldsymbol{p}\}}\ \text{Let}$$

We take $s' := s$, $t' := t$, $B := D$, $A' := C$.

- SEq-T'

$$\frac{\Delta;\Gamma \vdash r : A \quad r \simeq \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t}{\Delta;\Gamma \vdash \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t : A}\ \text{SEq-T'}$$

By Lem. B.5 on $r \simeq \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t$, there exists r_1, r_2 s.t. $r = \mathit{let}\ u \mathrel{\hat{=}} r_1\ \mathit{in}\ r_2$ and, moreover, $r_1 \simeq s$ and $r_2 \simeq t$. By the IH on $\Delta;\Gamma \vdash r : A$ there exists $r_1', r_2', B, A'\rho, p, q$ s.t. $\Delta;\Gamma \vdash s' : [\![\boldsymbol{p}]\!]B$ and $\Delta, u : B;\Gamma \vdash t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $A = A'\{u/\boldsymbol{p}\}$. We set $s' := r_1'$, $t' := r_2'$, $B := B'$ and notice that $s \simeq r_1'$, $t \simeq r_2'$.

Consider the fourth item. The derivation can end in one of two cases:

- Bang and $A = [\![\rho, s, t]\!]B$:

$$\frac{\Delta;\emptyset \vdash s, t : B \quad \Delta;\emptyset \vdash \rho : s \rhd t : B}{\Delta;\Gamma \vdash !(\rho, s, t) : [\![\overline{s}]\!]B}\ \text{Bang}$$

We take $s' := s$, $t' := t$, $A' := B$, $\rho' := \rho$.

- SEq-T'

$$\frac{\Delta;\Gamma \vdash r : A \quad r \simeq !(\rho, s, t)}{\Delta;\Gamma \vdash !(\rho, s, t) : A}\ \text{SEq-T'}$$

By Lem. B.5 on $r \simeq !(\rho, s, t)$, there exists s_1, t_1, ρ_1 s.t. $r = !(\rho_1, s_1, t_1)$ and $s_1 \simeq s$, $t_1 \simeq t$ and $\rho \simeq \rho_1 : s_1 \rhd t_1$. By the IH on $\Delta;\Gamma \vdash r : A$ there exists s_2, t_2, A'', ρ_2 s.t. $\Delta;\emptyset \vdash s_2, t_2 : A''$ and $\Delta;\emptyset \vdash \rho_2 : s_2 \rhd t_2 : A''$ and $s_2 \simeq s_1$, $\rho_2 \simeq \rho_1 : s_2 \rhd t_2$, and $A = [\![\overline{s_2}]\!]A''$. We set $s' := s_2$, $t' := t_2$, $A' := A''$ and notice that $s \simeq s_2$ and $t \simeq t_2$ and $\rho \simeq \rho_2 : s_2 \rhd t_2$.

Consider the fifth item. The derivation can end in one of two cases:

- TVar and

$$\frac{a : A \in \Gamma}{\Delta;\Gamma \vdash a : A}\ \text{TVar}$$

We conclude immediately.

- SEq-T'

$$\frac{\Delta;\Gamma \vdash s : A \quad s \simeq a}{\Delta;\Gamma \vdash a : A}\ \text{SEq-T'}$$

By Lem. B.5 on $s \simeq a$, we have that $s = a$ so we conclude by the IH.

The sixth item is similar to the fifth item, as the derivation can only end in RVar or SEq-T'. □

Lemma. [Step Typability – Lem. 3.6] $\xi : s \rhd t$ and $\Delta;\Gamma \vdash s : A$ implies $\Delta;\Gamma \vdash \xi : s \rhd t : A$.

Proof. By induction on the derivation of $\xi : s \rhd t$.

- ST-TVar: Then $\underline{a} : a \rhd a$ and $\Delta;\Gamma \vdash a : A$. By Lem. B.7 $a : A \in \Gamma$. By R-Refl-TVar $\Delta;\Gamma \vdash \underline{a} : a \rhd a : A$.
- ST-RVar: Similar to the ST-TVar case.
- ST-β: Then $\mathbf{ba}(a.s, t) : (\lambda a.s)t \rhd s\{a/t\}$ and $\Delta;\Gamma \vdash (\lambda a.s)\,t : A$. By Lem. B.7 there exist s', t', B such that $\Delta;\Gamma, a : B \vdash s' : A$ and $\Delta;\Gamma \vdash t' : B$ and $s \simeq s'$ and $t \simeq t'$. We may apply (SEq-T) to obtain $\Delta, a : B;\Gamma \vdash s : A$ and $\Delta;\Gamma \vdash t : B$. This allows us to apply R-β to conclude $\Delta;\Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\,t \rhd s\{a/t\} : A$ as required.
- ST-$\beta_\Box$: Then $\mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \mathrel{\hat{=}} !(\rho, s, t)\ \mathit{in}\ r \rhd r\{u/^{\mathrm{tgt}}\rho_s^t\}$ and $\Delta;\Gamma \vdash \mathit{let}\ u \mathrel{\hat{=}} !(\rho, s, t)\ \mathit{in}\ r : C$. By Lem. B.7 there exist $\rho', s', t', r', A, A', C', \sigma, p, q$ s.t. $\Delta;\emptyset \vdash s', t' : A'$ and $\Delta;\emptyset \vdash \rho' : s' \rhd t' : A'$ and $\Delta, u : A;\Gamma \vdash r' : C'$ and $s \simeq s'$ and $t \simeq t'$ and $r \simeq r'$ and $\rho \simeq \rho' : s \rhd t$ and $C = C'\{u/\overline{s'}\}$. From $\Delta;\emptyset \vdash \rho' : s' \rhd t' : A'$ by (SEq-R) we have that $\Delta;\emptyset \vdash \rho : s \rhd t : A$. Also by (SEq-T), from $\Delta, u : A;\Gamma \vdash r' : C'$ we have that $\Delta, u : A;\Gamma \vdash r : C'$. Hence we may apply R-$\beta_\Box$ to obtain that $\Delta;\Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \mathrel{\hat{=}} !(\rho, s, t)\ \mathit{in}\ r \rhd r\{u/^{\mathrm{tgt}}\rho_s^t\} : C\{u/\overline{s}\}$.

□

B.1 Strong Normalization

Lemma. [Lem. 3.8]

(a) $U(r\{a/s\}) = U(r)\{a := U(s)\}$ for any term r.
(b) $U(r\{u/^{\mathsf{tgt}}\rho^t_s\}) = U(r)\{u := U(t)\}$
(c) $U(A) = U(A\{u/\boldsymbol{t}\})$ for any type A.

Proof. The first and third items are straightforward by induction on r and A respectively. The second item is by induction on r. The interesting cases are:
• Rewrite variable. If $r = u$, then $U(u\{u/^{\mathsf{tgt}}\rho^t_s\}) = U(t) = U(u)\{u := U(t)\}$. If $r = v \neq u$, then $U(v\{u/^{\mathsf{tgt}}\rho^t_s\}) = U(v) = U(v)\{u := U(t)\}$.
• Bang, $r = !(\sigma, p, q)$. Then, if we write σ' for $\mathfrak{p}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}$:

$$\begin{array}{lll} & U(!(\sigma,p,q)\{u/^{\mathsf{tgt}}\rho^t_s\}) & \\ = & U(!(\sigma', p\{u/^{\mathsf{src}}\rho^t_s\}, q\{u/^{\mathsf{tgt}}\rho^t_s\})) & \\ = & U(q\{u/^{\mathsf{tgt}}\rho^t_s\}) & \\ = & U(q)\{u := U(t)\} & \text{by IH} \\ = & U(!(\sigma,p,q))\{u := U(t)\} & \end{array}$$

□

Lemma. [Lem. 3.9]

(a) If $r \simeq r'$ then $U(r) = U(r')$.
(b) If $\rho \simeq \sigma : r \rhd s$ then $U(\rho : r \rhd s) = U(s) = U(\sigma : r \rhd s)$.

Proof. The first item can be checked easily by induction on the derivation of $r \simeq r'$. The second item is immediate by definition. □

Lemma. [Lem. 3.10]

(a) $r \rightarrowtail s$ implies $U(r) \to_\beta U(s)$
(b) $\rho : r \rhd s \rightarrowtail \sigma : p \rhd q$ implies $U(\rho : r \rhd s) \to_\beta U(\sigma : p \rhd q)$

Proof. We prove the second item first. By Lem. 3.9, it suffices to show that if $\rho : r \rhd s \mapsto \sigma : p \rhd q$ then $U(\rho : r \rhd s) \to_\beta U(\sigma : p \rhd q)$. Proceed by induction on the derivation of $\rho : r \rhd s \mapsto \sigma : p \rhd q$. Most cases are straightforward by resorting to the IH when appropriate, there are two interesting cases:
• E-β: Let $\rho : s \rhd (\lambda a.t_1)\, t_2 \mapsto \rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}$. Then:

$$\begin{array}{lll} & U(\rho : s \rhd (\lambda a.t_1)\, t_2) & \\ = & (\lambda a.U(t_1))\, U(t_2) & \\ \to_\beta & U(t_1)\{a := U(t_2)\} & \\ = & U(t_1\{a/t_2\}) & \text{by Lem. 3.8} \\ = & U(\rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}) & \end{array}$$

• E-$\beta_\Box$: Let

$$\rho : s \rhd \mathit{let}\, u \doteq !(\rho, p, q)\, \mathit{in}\, t \mapsto \rho; \mathbf{bb}(!(\rho, p, q), u.t) : s \rhd t\{u/^{\mathsf{tgt}}\sigma^q_p\}$$

Then:

$$\begin{array}{lll} & U(\rho : s \rhd \mathit{let}\, u \doteq !(\rho, p, q)\, \mathit{in}\, t) & \\ = & (\lambda u.U(t))\, U(q) & \\ \to_\beta & U(t)\{u := U(q)\} & \\ = & U(t\{u/^{\mathsf{tgt}}\sigma^q_p\}) & \text{by Lem. 3.8} \\ = & U(\rho; \mathbf{bb}(!(\rho, p, q), u.t) : s \rhd t\{u/^{\mathsf{tgt}}\sigma^q_p\}) & \end{array}$$

To prove the first item, by Lem. 3.9 it suffices to show that if $r \mapsto s$ then $U(r) \to_\beta U(s)$. The proof then follows by induction on the derivation of $r \mapsto s$, resorting to the second item of this lemma in the E-BangT case. □

Lemma. [Lem. 3.11]

(a) $\Vdash_\pi \Delta; \Gamma \vdash s : A$ implies $U(\Delta;\Gamma) \vdash U(s) : U(A)$ in $\lambda^{\to}$.
(b) $\Vdash_\pi \Delta; \Gamma \vdash \rho : s \rhd t : A$ implies $U(\Delta;\Gamma) \vdash U(\rho : s \rhd t) : U(A)$ in $\lambda^{\to}$.

Proof. The proof is by simultaneous induction on the derivation π. Most cases are straightforward by resorting to the IH when appropriate. The interesting cases are:
• Bang: Let $\Delta; \Gamma \vdash !(\rho, r, s) : [\![\overline{r}]\!]A$ be derived from $\Delta; \emptyset \vdash r, s : A$ and $\Delta; \emptyset \vdash \rho : r \rhd s : A$. By IH on the last premise we have that $U(\Delta;\emptyset) \vdash U(\rho : r \rhd s) : U(A)$. Moreover, $U(\rho : r \rhd s) = U(s)$ and $U([\![\overline{r}]\!]A) = U(A)$, so we may conclude by weakening.
• Let: Let $\Delta; \Gamma \vdash \mathit{let}\, u \doteq s\, \mathit{in}\, t : C\{u/\boldsymbol{p}\}$ be derived from $\Delta; \Gamma \vdash s : [\![\boldsymbol{p}]\!]A$ and $\Delta, u : A; \Gamma \vdash t : C$. By IH we have that $U(\Delta;\Gamma) \vdash U(s) : U([\![\boldsymbol{p}]\!]A) = U(A)$ and that $U(\Delta;\Gamma), u : U(A) \vdash U(t) : U(C)$. So $U(\Delta;\Gamma) \vdash U(\mathit{let}\, u \doteq s\, \mathit{in}\, t) = (\lambda u.U(t))\, U(s) : U(C)$. We conclude by Lem. 3.8, given that

$U(C) = U(C\{u/\boldsymbol{p}\})$.

• R-Bang: Let $\Delta; \Gamma \vdash \langle \rho|_s \sigma \rangle : !(\rho, s, r) \rhd !(\rho; \sigma, s, t) : [\![\bar{s}]\!]A$ be derived from $\Delta; \emptyset \vdash s, r, t : A$ and $\Delta; \emptyset \vdash \rho : s \rhd r : A$ and $\Delta; \emptyset \vdash \sigma : r \rhd t : A$. By IH we have that $U(\Delta; \emptyset) \vdash U(t) : U(A)$. Moreover, $U(\langle \rho|_s \sigma \rangle : !(\rho, s, r) \rhd !(\rho; \sigma, s, t)) = U(!(\rho; \sigma, s, t)) = U(t)$ and $U([\![\bar{s}]\!]A) = U(A)$, so we may conclude by weakening.

• R-β: Let $\Delta; \Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\, t \rhd s\{a/t\} : B$ be derived from $\Delta; \Gamma, a : A \vdash s : B$ and $\Delta; \Gamma \vdash t : A$. By IH, $U(\Delta; \Gamma), a : U(A) \vdash U(s) : U(B)$ and $U(\Delta; \Gamma) \vdash U(t) : U(A)$. So by the (standard) substitution lemma $U(\Delta; \Gamma) \vdash U(s)\{a := U(t)\} : U(B)$. Moreover $U(\mathbf{ba}(a.s, t) : (\lambda a.s)\, t \rhd s\{a/t\}) = U(s\{a/t\}) = U(s)\{a := U(t)\}$ by Lem. 3.8, which concludes this case.

• R-$\beta_\Box$: Let

$$\Delta; \Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \triangleq !(\rho, s, t)\ \mathit{in}\, r \rhd r\{u/^{\mathrm{tgt}}\rho^t_s\} : C\{u/\bar{s}\}$$

be derived from $\Delta; \emptyset \vdash \rho : s \rhd t : A$ and $\Delta, u : A; \Gamma \vdash r : C$. By IH $U(\Delta; \emptyset) \vdash U(\rho : s \rhd t) : U(A)$ and $U(\Delta; \emptyset), u : U(A) \vdash U(r) : U(C)$. From the first condition we have that $U(\Delta; \emptyset) \vdash U(t) : U(A)$ holds by definition. By the (standard) substitution lemma we have that $U(\Delta; \emptyset) \vdash U(r)\{u := U(t)\} : U(C)$. Moreover,

$$\begin{array}{rll} & U(\mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \triangleq !(\rho, s, t)\ \mathit{in}\, r \rhd r\{u/^{\mathrm{tgt}}\rho^t_s\}) & \\ = & U(r\{u/^{\mathrm{tgt}}\rho^t_s\}) & \\ = & U(r)\{u := U(t)\} & \text{by Lem. 3.8} \end{array}$$

and $U(C) = U(C\{u/\bar{s}\})$ by Lem. 3.8, so by weakening we conclude this case.

• R-Let: Similar to the Let case.

• SEq-T: Let $\Delta; \Gamma \vdash t : A$ be derived from $\Delta; \Gamma \vdash s : A$ and $s \simeq t$. By IH we have that $U(\Delta; \Gamma) \vdash U(s) : U(A)$ and moreover by Lem. 3.9 $U(s) = U(t)$, which concludes this case.

• SEq-R: Let $\Delta; \Gamma \vdash \sigma : p \rhd q : A$ be derived from $\Delta; \Gamma \vdash \rho : s \rhd t : A$ and $\rho \simeq \sigma : s \rhd t$ and $s \simeq p$ and $t \simeq q$. By IH we have that $U(\Delta; \Gamma) \vdash U(\sigma : s \rhd t) : U(A)$, that is, $U(\Delta; \Gamma) \vdash U(t) : U(A)$. Moreover by Lem. 3.9 $U(t) = U(q)$, so $U(\Delta; \Gamma) \vdash U(q) : U(A)$, and therefore $U(\Delta; \Gamma) \vdash U(\sigma : p \rhd q) : U(A)$, which concludes this case. $\Box$