Rewrites as Terms via Justification Logic[1]

Eduardo Bonelli

Stevens Institute of Technology

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[1] Joint work with Pablo Barenbaum (UBA, Argentina)

Birds Eye View

Term Rewriting $\longrightarrow$ Rewrites $\longrightarrow$ Typed Rewrites

a.k.a. Proof Terms

Term Rewrite System

Signature $\longrightarrow$ Terms $\longrightarrow$ Rewrite Rules $\longrightarrow$ Reduction Sequence

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	First-order rewriting	Higher-order rewriting
Signature	Symbols with arity	Typed constants
Terms	First-order terms	Simply typed terms with constants (*i.e.* terms with binders)

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This case next

Rewrites – First-Order Rewriting

Signature

$$a^0,\ b^0,\ f^1,\ g^1$$

Rewrite rules

$$f(x) \mapsto_1 x \qquad\qquad a \mapsto_2 b$$

Reduction sequence

$$f(a) \to f(b) \to b$$

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Reduction sequence as a term?
Convenient: strategies
(*e.g.*needness, standardization),
equivalence of reductions, etc.

Rewrites – First-Order Rewriting

Signature

Rule symbols

$$a^0,\ b^0,\ f^1,\ g^1,\ \varrho^1,\ \vartheta^0,\ ;^2$$

Rewrite rules

$$\varrho(x) : f(x) \mapsto_1 x \qquad \vartheta : a \mapsto_2 b$$

Reduction sequence

$$f(a) \to f(b) \to b$$

Rewrites – First-Order Rewriting

Signature

$$a^0,\ b^0,\ f^1,\ g^1,\ \varrho^1,\ \vartheta^0,\ ;^2$$

Rule symbols

Sequential composition

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Rewrite

$$f(a) \to f(b) \to b \qquad f(\vartheta);\varrho(b)$$

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Reduction sequence

Syntactic accidents: $\varrho(f(a))$

	Rewrite
$f(a) \to f(b) \to b$	$f(\vartheta);\varrho(b)$
$f(f(a)) \to f(a)$	$f(\varrho(a))$

Rewrites – First-Order Rewriting

Rule symbols

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$$a^0,\ b^0,\ f^1,\ g^1,\ \varrho^1,\ \vartheta^0,\ ;^2$$

Sequential composition

Rewrite rules

$$\varrho(x) : f(x) \mapsto_1 x \qquad \vartheta : a \mapsto_2 b$$

Syntactic accidents: $\varrho(f(a))$

Reduction sequence

Multistep

Rewrite

$$f(a) \to f(b) \to b \qquad f(\vartheta);\varrho(b)$$

$$f(f(a)) \to f(a) \qquad f(\varrho(a))$$

$$f(f(a)) \to f(a) \to a \qquad \varrho(\varrho(a))$$

Rewrites – First-Order Rewriting

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Rule symbols

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$f(f(a)) \to f(a)$	$f(\varrho(a))$
$f(f(a)) \to f(a) \to a$	$\varrho(\varrho(a))$
$f(a) \to f(a)$	$f(a)$

Empty Rewrite

Rewrites – First-Order Rewriting

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$f(f(a)) \to f(a)$	$f(\varrho(a))$
$f(f(a)) \to f(a) \to a$	$\varrho(\varrho(a))$
$f(a) \to f(a)$	$f(a)$
$g(f(f(a))) \to g(f(a)) \to g(a)$	$g(\varrho(f(a)));g(\varrho(a))$

Empty Rewrite

Structural Equality: $g\,(\varrho(f(a))); g\,(\varrho(a)) \simeq g\,(\varrho(f(a)); \varrho(a))$

Term Rewrite Systems

Signature $\longrightarrow$ Terms $\longrightarrow$ Rewrite Rules $\longrightarrow$ Reduction Sequence

	First-order rewriting	Higher-order rewriting
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Terms	First-order terms	Simply typed terms with constants (*i.e.* terms with binders)

This case next

Higher Order Rewriting (HOR) [Nipkow's HRS '91]

Constants

base type

$$\mathsf{app} : \iota \supset \iota \supset \iota \qquad \mathsf{lam} : (\iota \supset \iota) \supset \iota$$

Rewrite rule

$$\mathsf{app}(\mathsf{lam}(\lambda x. Y\, x), Z) \mapsto Y\, Z$$

Reduction sequence

$$\mathsf{lam}(\lambda v.\mathsf{app}(\mathsf{lam}(\lambda x.x), v)) \qquad \text{“}\lambda v. I\, v\text{”}$$

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\mapsto & \mathsf{lam}(\lambda v.(\lambda w.w)\, v) \quad \text{replace} & \\
=_\beta & \mathsf{lam}(\lambda v.v) \quad \text{reduce} & \text{“}\lambda v.v\text{”}
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\end{array}$$

In short

$$\begin{array}{rl}
 & \mathsf{lam}(\lambda v.\mathsf{app}(\mathsf{lam}(\lambda x.x), v)) \\
\to & \mathsf{lam}(\lambda v.v)
\end{array}$$

Rewrites for HOR [Hilken96,Bruggink03,08]

Rewrite System

$$\mathsf{app}(\mathsf{lam}(\lambda x.Y\,x), Z) \mapsto Y\,Z$$

Rewrites for HOR [Hilken96,Bruggink03,08]

Rule symbol $\beta : (\iota \supset \iota) \supset \iota \supset \iota$ (type of LHS)

Rewrite System

$$\beta : \mathsf{app}(\mathsf{lam}(\lambda x.Y\,x), Z) \mapsto Y\,Z$$

Rewrites for HOR [Hilken96,Bruggink03,08]

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Reduction sequence

$$\mathsf{lam}(\lambda v.\, \mathsf{app}(\mathsf{lam}(\lambda x.x), \mathsf{app}(\mathsf{lam}(\lambda w.w), v))\,) \quad \text{“}\lambda v. I\,(I\,v)\text{”}$$

Rewrite

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Rewrite

$$\mathsf{lam}\,(\,\lambda v\,.\beta(\lambda x.x, \mathsf{app}(\mathsf{lam}(\lambda w.w), v)))$$

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Rewrite

$$\begin{array}{ll} & \mathsf{lam}\ (\ \lambda v\ .\beta(\lambda x.x, \mathsf{app}(\mathsf{lam}(\lambda w.w), v)));\ \mathsf{lam}\ (\ \lambda v\ .\beta(\lambda w.w, v)) \\ \simeq & \mathsf{lam}\ (\ \lambda v\ .\beta(\lambda w.w, \mathsf{app}(\mathsf{lam}(\lambda w.w, v)));\beta(\lambda w.w, v)) \end{array}$$

Structural Equality

Lambda Calculus Substitution and Rewrite Composition

Structural equality includes

$$\begin{array}{rcll} f(\rho); f(\sigma) & \simeq & f(\rho;\sigma) & \\ (\rho;\sigma);\tau & \simeq & \rho;(\sigma;\tau) & \\ \rho; t & \simeq & \rho & (\star) \end{array}$$

Lambda Calculus Substitution and Rewrite Composition

Structural equality includes

$$
\begin{array}{rcll}
f(\rho); f(\sigma) & \simeq & f(\rho; \sigma) & \\
(\rho; \sigma); \tau & \simeq & \rho; (\sigma; \tau) & \\
\rho; t & \simeq & \rho & (\star)
\end{array}
$$

Term is an empty rewrite over itself

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An instance of $(\star)$ is

$$x; x \simeq x$$

Unfortunate consequence

$$\vartheta =_\beta (\lambda x.x)\vartheta \simeq (\lambda x.x; x)\vartheta =_\beta \vartheta; \vartheta$$

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Unfortunate consequence

$$\vartheta =_\beta (\lambda x.x)\vartheta \simeq (\lambda x.x;x)\vartheta =_\beta \vartheta;\vartheta$$

Lambda Calculus substitution is incompatible with rewrite composition (*e.g.* $\vartheta : a \mapsto b$)

Towards a Typed Theory of Rewrites

What is the type of a rewrite?

- Should rewrites have the same type as terms (over which they verse)?

$$\beta(\lambda w.w, v) : \iota$$

How should rewrites be substituted?

- Should substitution of terms and rewrites coincide?

$$(\lambda x.x; x)\vartheta =_\beta \vartheta; \vartheta$$

This talk:

- Attempts to answer these questions
- Is not about higher-order rewriting and we will not be devising rewrites for higher-order rewriting
- We will, hopefully, lay some foundations for doing so

Logic of Proofs

"*s is a proof of A*"

$$\Box A \quad \longrightarrow \quad [\![s]\!]A$$

Introduced by S. Artemov as solution to observation by Gödel

$$\mathrm{IPL} \hookrightarrow \mathrm{S4} \hookrightarrow \mathrm{LP} \hookrightarrow \mathrm{PA}$$

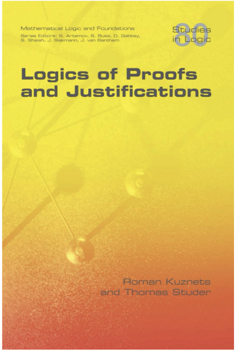
Reading $\Box A$ as $\exists x.Proof(x, \ulcorner A \urcorner)$ problematic

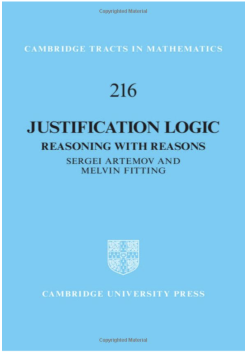
$$\vdash_{\mathrm{S4}} \Box(\neg\Box\perp)$$

- Observed by Gödel [Gödel:1933] who posed two problems:
  1. modal logic of formal provability predicate $\exists x.Proof(x, \ulcorner A \urcorner)$
  2. exact intended provability semantics for S4
- Both have been addressed
  1. Solovay [Solovay:1976] (completeness of Löb's logic)
  2. Artemov [Artemov:1994]

Logic of Proofs and Justification Logic

LP is a precursor of the more general (and recent) Justification Logic





S4 $\longrightarrow$ Logic of Proofs

Propositions

$$A, B \quad ::= \quad P \mid A \supset B \mid [\![\ ]\!]A$$

Axioms

$$\begin{array}{ll} A0 & \text{Axioms of CPL} \\ A1 & [\![\ ]\!](A \supset B) \supset [\![\ ]\!]A \supset [\![\qquad]\!]B \\ A2 & [\![\ ]\!]A \supset [\![\quad]\!][\![\ ]\!]A \\ A3 & [\![\ ]\!]A \supset A \end{array}$$

Rules

$$\begin{array}{ll} MP & \vdash A \supset B \text{ and } \vdash A, \text{ implies } \vdash B \\ Nec & \vdash A \text{ implies } \vdash [\![\ ]\!]A \end{array}$$

S4 $\longrightarrow$ Logic of Proofs

Proof polynomials

$$s, t \quad ::= \quad x \mid c \mid s \cdot t \mid !s \mid s + t$$

Propositions

$$A, B \quad ::= \quad P \mid A \supset B \mid [\![s]\!]A$$

Axioms

$$\begin{array}{ll}
A0 & \text{Axioms of CPL} \\
A1 & [\![s]\!](A \supset B) \supset [\![t]\!]A \supset [\![s \cdot t]\!]B \\
A2 & [\![s]\!]A \supset [\![!s]\!][\![s]\!]A \\
A3 & [\![s]\!]A \supset A \\
A4 & [\![s]\!]A \supset [\![s + t]\!]A, [\![t]\!]A \supset [\![s + t]\!]A
\end{array}$$

Rules

$$\begin{array}{ll}
MP & \vdash A \supset B \text{ and } \vdash A, \text{ implies } \vdash B \\
Nec & \vdash [\![c]\!]A, A \text{ an instance of an axiom } A0 - A4
\end{array}$$

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

Sample Derivation

$$\Box A \lor \Box B \supset \Box(A \lor B)$$

$\vdash A \supset A \lor B$ CPL

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

$\vdash A \supset A \vee B$	CPL
$\vdash [\![a]\!](A \supset A \vee B)$	Nec

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

$\vdash A \supset A \vee B$	CPL
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$[\![x]\!]A \vdash [\![\, a \cdot x \,]\!](A \vee B)$	Hyp,A1,MP*2

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

$\vdash A \supset A \vee B$	CPL
$\vdash [\![a]\!](A \supset A \vee B)$	Nec
$[\![x]\!]A \vdash [\![\, a \cdot x \,]\!](A \vee B)$	Hyp,A1,MP*2
$\vdash B \supset A \vee B$	CPL

Sample Derivation

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$\vdash B \supset A \vee B$	CPL
$\vdash [\![b]\!](B \supset A \vee B)$	Nec

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

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$\vdash [\![a]\!](A \supset A \vee B)$	Nec
$[\![x]\!]A \vdash [\![a \cdot x]\!](A \vee B)$	Hyp,A1,MP*2
$\vdash B \supset A \vee B$	CPL
$\vdash [\![b]\!](B \supset A \vee B)$	Nec
$[\![y]\!]B \vdash [\![b \cdot y]\!](A \vee B)$	Hyp,A1,MP*2

Sample Derivation

$$\Box A \vee \Box B \supset \Box(A \vee B)$$

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$\vdash [\![a]\!](A \supset A \vee B)$	Nec
$[\![x]\!]A \vdash [\![\, a \cdot x \,]\!](A \vee B)$	Hyp,A1,MP*2
$\vdash B \supset A \vee B$	CPL
$\vdash [\![b]\!](B \supset A \vee B)$	Nec
$[\![y]\!]B \vdash [\![\, b \cdot y \,]\!](A \vee B)$	Hyp,A1,MP*2
$[\![x]\!]A \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	A4

Sample Derivation

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$[\![y]\!]B \vdash [\![\, b \cdot y \,]\!](A \vee B)$	Hyp,A1,MP*2
$[\![x]\!]A \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	A4
$[\![y]\!]B \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	A4

Sample Derivation

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$[\![x]\!]A \vdash [\![\, a \cdot x \,]\!](A \vee B)$	Hyp,A1,MP*2
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$[\![y]\!]B \vdash [\![\, b \cdot y \,]\!](A \vee B)$	Hyp,A1,MP*2
$[\![x]\!]A \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	A4
$[\![y]\!]B \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	A4
$[\![x]\!]A \vee [\![y]\!]B \vdash [\![a \cdot x + b \cdot y]\!](A \vee B)$	CPL Reasoning

Sample Properties

- Internalization

$$\overline{B_m} \vdash C \text{ implies } \overline{[\![x_m]\!]B_m} \vdash [\![t(\overline{x_m})]\!]C.$$

$$\begin{array}{c} B_1 \ldots B_m \\ \vdots\ \pi \\ C \end{array} \quad \longrightarrow \quad \begin{array}{c} [\![x_1]\!]B_1 \ldots [\![x_m]\!]B_m \\ \vdots\ \mathit{lift}(\pi) \\ [\![t_\pi]\!]C \end{array}$$

- Multi-conclusion

$$\vdash [\![s]\!]A \wedge [\![t]\!]B \supset [\![s+t]\!]A \wedge [\![s+t]\!]B$$

- Disjunctive Property [Krupski 2006]

$$\vdash [\![s]\!]A \vee [\![t]\!]B \text{ iff } \vdash [\![s]\!]A \text{ or } \vdash [\![t]\!]B$$

- Realization

$$\vdash_{S4} A \text{ implies } \vdash A^r, \text{ for } \bullet^r \text{ a normal realization}$$

Sample Properties

Design principle towards Natural Deduction for LP

- Internalization

$$\overline{B_m} \vdash C \text{ implies } \overline{[\![x_m]\!]B_m} \vdash [\![t(\overline{x_m})]\!]C.$$

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$$\vdash_{S4} A \text{ implies } \vdash A^r, \text{ for } \bullet^r \text{ a normal realization}$$

Natural Deduction for (Minimal) LP

$$\Delta ; \Gamma \vdash A$$

"A is true under validity hypothesis Δ *and truth hypothesis* Γ "

$$\frac{\Delta ; \emptyset \vdash A}{\Delta ; \Gamma \vdash \Box A}$$

Natural Deduction for (Minimal) LP

$$\Delta; \Gamma \vdash A \mid s$$

"*A is true under validity hypothesis* Δ *and truth hypothesis* Γ *with proof* s"

$$\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A}$$

Natural Deduction for (Minimal) LP

$$\Delta; \Gamma \vdash A \mid s$$

"*A is true under validity hypothesis* Δ *and truth hypothesis* Γ *with proof s*"

Internalization as design principle for Introduction Rule

$$\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A} \qquad \frac{\Delta; \emptyset \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s} \, \Box\mathrm{I}'$$

Natural Deduction for (Minimal) LP

$$\Delta; \Gamma \vdash A \mid s$$

"*A is true under validity hypothesis* Δ *and truth hypothesis* Γ *with proof s*"

Internalization as design principle for Introduction Rule

$$\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A} \qquad \frac{\Delta; \emptyset \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s}\ \Box\mathrm{I}'$$

Natural Deduction for (Minimal) LP

$$\frac{a : A \in \Gamma}{\Delta; \Gamma \vdash A \mid a}\ \mathsf{TVar} \qquad \frac{\Delta; \Gamma, a : A \vdash B \mid s}{\Delta; \Gamma \vdash A \supset B \mid \lambda a.s}\ \mathsf{Abs}$$

$$\frac{\Delta; \Gamma \vdash A \supset B \mid s \quad \Delta; \Gamma \vdash A \mid t}{\Delta; \Gamma \vdash B \mid s\, t}\ \mathsf{App}$$

$$\frac{u : A \in \Delta}{\Delta; \Gamma \vdash A \mid u}\ \mathsf{RVar} \qquad \frac{\Delta; \emptyset \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s}\ \Box I'$$

$$\frac{\Delta; \Gamma \vdash [\![r]\!]A \mid s \quad \Delta, u : A; \Gamma \vdash C \mid t}{\Delta; \Gamma \vdash C\{u/r\} \mid let\ u \mathrel{\circeq} s\ in\ t}\ \mathsf{Let}$$

Correct from a provability angle, not closed under normalisation

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \Delta;\emptyset\vdash A\mid s\end{array}}{\Delta;\Gamma\vdash[\![s]\!]A\mid\,!s}\;\Box\mathrm{I}'$$

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

$$\dfrac{\begin{array}{c}\vdots\ \pi\\ \Delta;\emptyset \vdash A \mid s\end{array}}{\Delta;\Gamma \vdash [\![s]\!]A \mid\, !s}\ \Box\mathsf{I'} \quad\longrightarrow\quad \dfrac{\begin{array}{c}\vdots\ \pi'\\ \Delta;\emptyset \vdash A \mid \boldsymbol{t}\end{array}}{\Delta;\Gamma \vdash [\![\boldsymbol{S}]\!]A \mid\, !s}\ \Box\mathsf{I'}$$

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

$$\dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset \vdash A \mid s\end{array}}{\Delta;\Gamma \vdash [\![s]\!]A \mid\ !s}\ \Box\mathrm{I'} \quad\longrightarrow\quad \dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset \vdash A \mid t\end{array}}{\Delta;\Gamma \vdash [\![s]\!]A \mid\ !s}\ \Box\mathrm{I'}$$

Need some means of relating t back to s

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

Need some means of relating t back to s

$$\dfrac{\begin{matrix}\vdots\ \pi \\ \Delta; \emptyset \vdash A \mid s\end{matrix}}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s}\ \Box\text{I'} \longrightarrow \dfrac{\begin{matrix}\vdots\ \pi \\ \Delta; \emptyset \vdash A \mid \boldsymbol{t}\end{matrix}}{\Delta; \Gamma \vdash [\![\boldsymbol{S}]\!]A \mid !s}\ \Box\text{I'}$$

Reify the normalisation step as a rewrite ρ

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

$$\dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset\vdash A \mid s\end{array}}{\Delta;\Gamma\vdash [\![s]\!]A \mid\, !s}\ \Box\mathsf{I}' \quad\longrightarrow\quad \dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset\vdash A \mid t\end{array}}{\Delta;\Gamma\vdash [\![s]\!]A \mid\, !s}\ \Box\mathsf{I}'$$

Need some means of relating t back to s

Reify the normalisation step as a rewrite ρ

Rewrite

$$\dfrac{\Delta;\emptyset\vdash \rho : s \rhd t : A}{\Delta;\Gamma\vdash\, !(\rho, s, t) : [\![\bar{s}]\!]A}\ \Box\mathsf{I}$$

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

Need some means of relating t back to s

$$\dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset \vdash A \mid s\end{array}}{\Delta;\Gamma \vdash [\![s]\!]A \mid\ !s}\ \Box\mathrm{I}' \quad \longrightarrow \quad \dfrac{\begin{array}{c}\vdots\ \pi \\ \Delta;\emptyset \vdash A \mid t\end{array}}{\Delta;\Gamma \vdash [\![\boldsymbol{S}]\!]A \mid\ !s}\ \Box\mathrm{I}'$$

Reify the normalisation step as a rewrite ρ

Rewrite

$$\dfrac{\Delta;\emptyset \vdash \rho : s \rhd t : A}{\Delta;\Gamma \vdash\ !(\rho, s, t) : [\![\overline{s}]\!]A}\ \Box\mathrm{I}$$

Term

Thesis: $[\![s]\!]A$ is the type of rewrites with source s

$$\frac{\begin{array}{c}\vdots\ \pi\\ \Delta;\emptyset \vdash A \mid s\end{array}}{\Delta;\Gamma \vdash [\![s]\!]A \mid\ !s}\ \Box\mathsf{I}' \quad \longrightarrow \quad \frac{\begin{array}{c}\vdots\ \pi\\ \Delta;\emptyset \vdash A \mid t\end{array}}{\Delta;\Gamma \vdash [\![S]\!]A \mid\ !s}\ \Box\mathsf{I}'$$

Need some means of relating t back to s

Reify the normalisation step as a rewrite ρ

Rewrite

$$\frac{\Delta;\emptyset \vdash \rho : s \rhd t : A}{\Delta;\Gamma \vdash\ !(\rho, s, t) : [\![\bar{s}]\!]A}\ \Box\mathsf{I}$$

Type of all rewrites rooted at s

Term

Typing Terms

$$\frac{a : A \in \Gamma}{\Delta; \Gamma \vdash a : A}\ \text{TVar} \qquad \frac{\Delta; \Gamma, a : A \vdash s : B}{\Delta; \Gamma \vdash \lambda a.s : A \supset B}\ \text{Abs}$$

$$\frac{\Delta; \Gamma \vdash s : A \supset B \quad \Delta; \Gamma \vdash t : A}{\Delta; \Gamma \vdash s\, t : B}\ \text{App}$$

$$\frac{u : A \in \Delta}{\Delta; \Gamma \vdash u : A}\ \text{RVar} \qquad \frac{\Delta; \emptyset \vdash s, t : A \quad \Delta; \emptyset \vdash \rho : s \rhd t : A}{\Delta; \Gamma \vdash !(\rho, s, t) : [\![\bar{s}]\!]A}\ \text{Bang}$$

$$\frac{\Delta; \Gamma \vdash s : [\![\boldsymbol{p}]\!]A \quad \Delta, u : A; \Gamma \vdash t : C}{\Delta; \Gamma \vdash \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t : C\{u/\boldsymbol{p}\}}\ \text{Let}$$

Typing Rewrites (Axioms)

Never substituted

May be substituted by a rewrite

$$\frac{a : A \in \Gamma}{\Delta; \Gamma \vdash \underline{a} : a \rhd a : A}\ \text{R-Refl-TVar} \qquad \frac{u : A \in \Delta}{\Delta; \Gamma \vdash \underline{u} : u \rhd u : A}\ \text{R-Refl-RVar}$$

$$\frac{\Delta; \Gamma, a : A \vdash s : B \quad \Delta; \Gamma \vdash t : A}{\Delta; \Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\, t \rhd s\{a/t\} : B}\ \text{R-}\beta$$

$$\frac{\Delta; \emptyset \vdash \rho : s \rhd t : A \quad \Delta, u : A; \Gamma \vdash r : C}{\Delta; \Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : let\ u \mathrel{\hat{=}} !(\rho, s, t)\ in\ r \rhd r\{u/^{\mathrm{tgt}} \rho_s^t\} : C\{u/\overline{s}\}}\ \text{R-}\beta_\Box$$

What is substitution of rewrites?

Typing Rewrites (Sample)

Reduction			*Rewrite*
$\lambda a.s$	$\longrightarrow$	$\lambda a.s'$	$\lambda a.\rho$
$s\ t$	$\longrightarrow$	$s'\ t'$	$\rho\ \sigma$
$let\ \ u \mathrel{\mathring{=}} s\ in\ t$	$\longrightarrow$	$let\ \ u \mathrel{\mathring{=}} s'\ in\ t'$	$let\ \ u \mathrel{\mathring{=}} \rho\ in\ \sigma$
$!(\rho, s, t)$	$\longrightarrow$	???	???

Typing Rewrites (Sample)

$$
\begin{array}{rcll}
\multicolumn{3}{c}{\textit{Reduction}} & \textit{Rewrite} \\
\lambda a.s & \longrightarrow & \lambda a.s' & \lambda a.\rho \\
s\,t & \longrightarrow & s'\,t' & \rho\,\sigma \\
\mathit{let}\ u \mathrel{\mathring{=}} s\ \mathit{in}\ t & \longrightarrow & \mathit{let}\ u \mathrel{\mathring{=}} s'\ \mathit{in}\ t' & \mathit{let}\ u \mathrel{\mathring{=}} \rho\ \mathit{in}\ \sigma \\
!(\rho, s, t) & \longrightarrow & !(\rho;\sigma, s, t) & \langle \rho|_s \sigma \rangle
\end{array}
$$

$$
\frac{\Delta;\emptyset \vdash s, r, t : A \quad \Delta;\emptyset \vdash \rho : s \rhd r : A \quad \Delta;\emptyset \vdash \sigma : r \rhd t : A}{\Delta;\Gamma \vdash \langle \rho|_s \sigma \rangle : !(\rho, s, r) \rhd !(\rho;\sigma, s, t) : [\![\bar{s}]\!]A}\ \text{R-Bang}
$$

Typing Rewrites (Sample)

$$\frac{\Delta; \Gamma \vdash s : A \qquad \boxed{s \simeq t}}{\Delta; \Gamma \vdash t : A}\ \mathsf{SEq\text{-}T}$$

$$\frac{\Delta; \Gamma \vdash \rho : s \rhd t : A \qquad \boxed{\rho \simeq \sigma : s \rhd t} \qquad \boxed{s \simeq p} \qquad \boxed{t \simeq q}}{\Delta; \Gamma \vdash \sigma : p \rhd q : A}\ \mathsf{SEq\text{-}R}$$

Typing Rewrites (Sample)

$$\frac{\Delta; \Gamma \vdash s : A \quad \boxed{s \simeq t}}{\Delta; \Gamma \vdash t : A}\ \text{SEq-T}$$

$$\frac{\Delta; \Gamma \vdash \rho : s \rhd t : A \quad \boxed{\rho \simeq \sigma : s \rhd t} \quad \boxed{s \simeq p} \quad \boxed{t \simeq q}}{\Delta; \Gamma \vdash \sigma : p \rhd q : A}\ \text{SEq-R}$$

Desired property:

$$\Delta; \Gamma \vdash s : A \text{ implies } \Delta; \Gamma \vdash \mathfrak{s} : s \rhd s : A$$

Example:

$$\frac{\Delta; \emptyset \vdash s, r : A \quad \Delta; \emptyset \vdash \rho : s \rhd r : A \quad \Delta; \emptyset \vdash \mathfrak{r} : r \rhd r : A}{\Delta; \Gamma \vdash \langle \rho|_s \mathfrak{r} \rangle : !(\rho, s, r) \rhd !(\rho; \mathfrak{r}, s, r) : [\![\overline{s}]\!] A}\ \text{R-Bang}$$

$s \simeq t$ (Sample)

$$\frac{s \simeq p \quad t \simeq q \quad \rho \simeq \sigma : s \rhd t}{!(\rho, s, t) \simeq !(\sigma, p, q)}\ \text{EqT-Bang}$$

$\rho \simeq \sigma : s \rhd t$ (Sample)

$$\frac{\rho : p \rhd q \quad \mathfrak{t} : q \rhd r}{\rho ; \mathfrak{t} \simeq \rho : p \rhd r} \text{ EqR-IdR} \qquad \frac{\sigma : p \rhd q \quad \rho : q \rhd r \quad \tau : r \rhd s}{(\sigma ; \rho) ; \tau \simeq \sigma ; (\rho ; \tau) : p \rhd s} \text{ EqR-Ass}$$

$$\frac{\rho : p \rhd q \quad \sigma : q \rhd r}{\lambda a.\rho ; \lambda a.\sigma \simeq \lambda a.(\rho ; \sigma) : \lambda a.p \rhd \lambda a.r} \text{ EqR-Abs}$$

$$\frac{\rho : p \rhd q \quad \sigma : q \rhd r \quad \tau : r \rhd s}{\langle \rho |_p \sigma \rangle ; \langle \rho ; \sigma |_p \tau \rangle \simeq \langle \rho |_p \sigma ; \tau \rangle : !(\rho, p, q) \rhd !(\rho ; \sigma ; \tau, p, s)} \text{ EqR-BangR}$$

$$\frac{s \simeq s' \quad \rho \simeq \sigma : s' \rhd t' \quad t' \simeq t}{\rho \simeq \sigma : s \rhd t} \text{ EqR-SEq}$$

$\rho \simeq \sigma : s \rhd t$ (Sample)

Empty rewrite

$$\dfrac{\rho : p \rhd q \quad \mathfrak{t} : q \rhd r}{\rho ; \mathfrak{t} \simeq \rho : p \rhd r}\ \text{EqR-IdR} \qquad \dfrac{: q \rhd r \quad \tau : r \rhd s}{(\sigma ; \rho) ; \tau \simeq \sigma ; (\rho ; \tau) : p \rhd s}\ \text{EqR-Ass}$$

$$\dfrac{\rho : p \rhd q \quad \sigma : q \rhd r}{\lambda a.\rho ; \lambda a.\sigma \simeq \lambda a.(\rho ; \sigma) : \lambda a.p \rhd \lambda a.r}\ \text{EqR-Abs}$$

$$\dfrac{\rho : p \rhd q \quad \sigma : q \rhd r \quad \tau : r \rhd s}{\langle \rho |_p \sigma \rangle ; \langle \rho ; \sigma |_p \tau \rangle \simeq \langle \rho |_p \sigma ; \tau \rangle : !(\rho, p, q) \rhd !(\rho ; \sigma ; \tau, p, s)}\ \text{EqR-BangR}$$

$$\dfrac{s \simeq s' \quad \rho \simeq \sigma : s' \rhd t' \quad t' \simeq t}{\rho \simeq \sigma : s \rhd t}\ \text{EqR-SEq}$$

Substitution of Rewrites

$\rho : s \rhd t$

$$(\underline{u}; \underline{u})\{u/\rho\}$$

Careful!

$$(\underline{u}; \underline{u})\{u/\rho\} \not\simeq \rho; \rho$$

Substitution of rewrites

t reified as an empty rewrite

$$(\underline{u}; \underline{u})\{u/\rho\} = \rho; \mathfrak{t}; \mathfrak{t} \simeq \rho$$

Substitution of Rewrites

$\rho : s \triangleright t$

$$(\underline{u}; \underline{u})\{u/\rho\}$$

Careful!

$$(\underline{u}; \underline{u})\{u/\rho\} \not\simeq \rho; \rho$$

t reified as an empty rewrite

Substitution of rewrites

$$(\underline{u}; \underline{u})\{u/\rho\} = \rho; \mathfrak{t}; \mathfrak{t} \simeq \rho$$

More generally

m is src or tgt

$$\langle \sigma |_p \tau \rangle \{u/^{\mathsf{m}} \rho_s^t\} := \langle \qquad ; \qquad | \qquad \rangle$$

$\tau : q \triangleright r$

$\sigma : p \triangleright q$

Substitution of Rewrites

$\rho : s \rhd t$

$$(\underline{u}; \underline{u})\{u/\rho\}$$

Careful!

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$$(\underline{u}; \underline{u})\{u/\rho\} = \rho; \mathfrak{t}; \mathfrak{t} \simeq \rho$$

More generally

m is src or tgt

$$\langle \sigma|_p \tau \rangle \{u/^{\mathsf{m}} \rho_s^t\} := \langle \mathfrak{p}\{u/\rho_s^t\}; \qquad |_{p\{u/^{\mathsf{src}} \rho_s^t\}} \qquad \rangle$$

$\tau : q \rhd r$

$\sigma : p \rhd q$

Substitution of Rewrites

$\rho : s \rhd t$

$$(\underline{u}; \underline{u})\{u/\rho\}$$

Careful!

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Substitution of rewrites

t reified as an empty rewrite

$$(\underline{u}; \underline{u})\{u/\rho\} = \rho; \mathfrak{t}; \mathfrak{t} \simeq \rho$$

More generally

m is src or tgt

$$\langle \sigma |_p \tau \rangle \{u/^{\mathsf{m}} \rho_s^t\} := \langle \mathfrak{p}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}} \rho_s^t\}|_{p\{u/^{\mathsf{src}} \rho_s^t\}} \qquad \rangle$$

$\tau : q \rhd r$

$\sigma : p \rhd q$

Substitution of Rewrites

$\rho : s \rhd t$

$$(\underline{u}; \underline{u})\{u/\rho\}$$

Careful!

$$(\underline{u}; \underline{u})\{u/\rho\} \not\simeq \rho; \rho$$

Substitution of rewrites

t reified as an empty rewrite

$$(\underline{u}; \underline{u})\{u/\rho\} = \rho; \mathfrak{t}; \mathfrak{t} \simeq \rho$$

More generally

m is src or tgt

$$\langle \sigma|_p \tau\rangle\{u/^{\mathsf{m}} \rho_s^t\} := \langle \mathfrak{p}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}} \rho_s^t\}|_{p\{u/^{\mathsf{src}} \rho_s^t\}} \tau\{u/^{\mathsf{tgt}} \rho_s^t\}\rangle$$

$\tau : q \rhd r$

$\sigma : p \rhd q$

Substitution of Rewrites

Rewrite Substitution on Empty Rewrites

$$\underline{a}\{u/\rho^t_s\} := \underline{a}$$

$$\underline{v}\{u/\rho^t_s\} := \begin{cases} \rho, & u = v \\ \underline{v}, & u \neq v \end{cases}$$

$$\langle\sigma|_p\mathfrak{q}\rangle\{u/\rho^t_s\} := \langle\mathfrak{p}\{u/\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{p\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{q}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle$$

No clause for $\sigma;\tau$

Moded rewrite substitution ($\mathsf{m} = \mathsf{src}/\mathsf{tgt}$)

$$\underline{a}\{u/^{\mathsf{m}}\rho^t_s\} := \underline{a}$$

$$\underline{v}\{u/^{\mathsf{m}}\rho^t_s\} := \begin{cases} \mathfrak{s}, & u = v \wedge \mathsf{m} = \mathsf{src} \\ \mathfrak{t}, & u = v \wedge \mathsf{m} = \mathsf{tgt} \\ \underline{v}, & u \neq v \end{cases}$$

$$\langle\sigma|_p\tau\rangle\{u/^{\mathsf{m}}\rho^t_s\} := \langle\mathfrak{p}\{u/\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{p\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle$$

$$(\sigma;\tau)\{u/^{\mathsf{m}}\rho^t_s\} := \sigma\{u/^{\mathsf{m}}\rho^t_s\}; \tau\{u/^{\mathsf{m}}\rho^t_s\}$$

Example ($\tau : p \rhd q$)

$$!(\tau\,\underline{u}, p\,u, q\,u)\{u/^{\mathsf{src}}\rho^t_s\} = !(\mathfrak{p}\,\rho; \tau\,\mathfrak{t}, p\,s, q\,t)$$

Typing is Closed Under Substitution of Rewrites

Suppose

$$\Delta; \emptyset \vdash \rho : s \rhd t : A \quad \Delta; \emptyset \vdash s : A \quad \Delta; \emptyset \vdash t : A$$

1. $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ implies
$$\Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho^t_s\} : p\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$
2. $\Delta, u : A; \Gamma \vdash p : B$ implies
$$\Delta; \Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/\overline{s}\}.$$

Rewrite Extension

Typed Lambda Calculus

- Subject Reduction

$$\Gamma \vdash s : A \text{ and } s \to_\beta t \text{ implies } \Gamma \vdash t : A$$

- Strong Normalization

β terminates on typed terms

Typed Rewrite Calculus

Rewrite Extension

- Subject Extension

$$\Delta;\Gamma \vdash \rho : s \rhd t : A \text{ and } \rho : s \rhd t \rightarrowtail \rho' : s \rhd t' \text{ implies}$$
$$\Delta;\Gamma \vdash \rho' : s \rhd t' : A.$$

- Strong Normalization

Rewrite extension terminates on typed rewrites

Rewrite Extension

$$\rho : r \rhd s \rightarrowtail \sigma : r \rhd q$$

Example 1

$$\begin{array}{rl} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)) : I(Ia) \rhd Ia \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array}$$

Example 2

$$\begin{array}{rl} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & \mathbf{ba}(b.b, Ia) : I(Ia) \rhd Ia \\ \rightarrowtail & \mathbf{ba}(b.b, Ia); \mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array}$$

Rewrite Extension

$$\rho : r \triangleright s \rightarrowtail \sigma : r \triangleright q$$

Confluence could be regained by including permutation of redexes into $\simeq$

Example 1

$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \triangleright I(Ia) \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)) : I(Ia) \triangleright Ia \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a) : I(Ia) \triangleright a \end{array}$$

Example 2

$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \triangleright I(Ia) \\ \rightarrowtail & \mathbf{ba}(b.b, Ia) : I(Ia) \triangleright Ia \\ \rightarrowtail & \mathbf{ba}(b.b, Ia); \mathbf{ba}(b.b, a) : I(Ia) \triangleright a \end{array}$$

Rewrite Extension

$\exists r', s'$ s.t. $r \simeq r' \mapsto s' \simeq s$

$r \rightarrowtail s$ Term extension

$\rho : r \rhd s \rightarrowtail \sigma : r \rhd q$ Rewrite extension

$\exists \rho', \sigma'$ s.t. $\rho \simeq \rho' : r \rhd s$ and
$\rho' : r \rhd s \mapsto \sigma' : r \rhd q$ and
$\sigma' \simeq \sigma : r \rhd q$

Sample Rules ($s \mapsto t$ and $\rho : s \triangleright t \mapsto \sigma : p \triangleright q$)

$$\frac{\rho : s \triangleright t \mapsto \rho' : s \triangleright t'}{!(\rho, s, t) \mapsto !(\rho', s, t')}\ \text{E-BangT}$$

$$\frac{}{\rho : s \triangleright (\lambda a.t_1)\, t_2 \mapsto \rho; \mathbf{ba}(a.t_1, t_2) : s \triangleright t_1\{a/t_2\}}\ \text{E-}\beta$$

$$\frac{\sigma : s \triangleright t \mapsto \sigma' : s \triangleright t'}{\langle \rho|_r \sigma \rangle : !(\rho, r, s) \triangleright !(\rho; \sigma, r, t) \mapsto \langle \rho|_r \sigma' \rangle : !(\rho, r, s) \triangleright !(\rho; \sigma', r, t')}\ \text{E-BangR}$$

$$\frac{\sigma : s \triangleright t \mapsto \sigma' : s \triangleright t'}{\rho; \sigma : r \triangleright t \mapsto \rho; \sigma' : r \triangleright t'}\ \text{E-Trans}$$

Summary

What is the type of a rewrite?

- Rewrites have a modal type
- Thesis:

$$[\![s]\!]A \text{ is the type of rewrites with source } s$$

- Manifested through the Logic of Proofs
- Further details: PPDP'20 paper (`https://ebonelli.github.io`)

TRC and Dependent Types

$$\frac{\Delta; \emptyset \vdash A \mid \cancel{s}t}{\Delta; \Gamma \vdash [\![s]\!] A \mid !s} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \simeq B}{\Gamma \vdash B}\,\text{Conv}$$

2

TRC and Dependent Types

$[\![s]\!]A' \simeq [\![t]\!]B'$ if
$s \simeq t$ and $A' \simeq B'$

$$\frac{\Delta; \emptyset \vdash A \mid \cancel{s}t}{\Delta; \Gamma \vdash [\![s]\!]A \mid\ !s} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \simeq B}{\Gamma \vdash B}\,\mathrm{Conv}$$

2

TRC and Dependent Types

$[\![s]\!]A' \simeq [\![t]\!]B'$ if
$s \simeq t$ and $A' \simeq B'$

$$\frac{\Delta; \emptyset \vdash A \mid \cancel{s}\,t}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \simeq B}{\Gamma \vdash B}\ \text{Conv}$$

What can we assume about s in a proposition $[\![s]\!]C$?

[2] Take s to be $\lambda a^{[\![t]\!]\perp}.let\ u \mathrel{\hat{=}} a\ in\ u$

TRC and Dependent Types

$[\![s]\!]A' \simeq [\![t]\!]B'$ if $s \simeq t$ and $A' \simeq B'$

$$\frac{\Delta;\emptyset \vdash A \mid \cancel{s}\,t}{\Delta;\Gamma \vdash [\![s]\!]A \mid !s} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \simeq B}{\Gamma \vdash B}\,\mathrm{Conv}$$

What can we assume about s in a proposition $[\![s]\!]C$?

- s is a proof of C? No.
    - S4 theorem $\Box(\neg\Box\bot)$
    - TRC theorem[2] $[\![s]\!](\neg[\![t]\!]\bot)$
- s is "typable"? No.
  Eg. (introspection) $[\![s]\!]A \supset [\![!s]\!][\![s]\!]A$ reads as "if we assume that s is a proof of A, then $!s$ is a proof that s is a proof of A".

$$[\![\lambda a.a\,a]\!]A \supset [\![!(\lambda a.a\,a)]\!][\![\lambda a.a\,a]\!]A$$

[2] Take s to be $\lambda a^{[\![t]\!]\bot}.let\ u \mathrel{\hat{=}} a\ in\ u$

Rewrites for HOR Revisited

$$\vartheta =_\beta (\lambda x.x)\vartheta \simeq (\lambda x.x;x)\vartheta =_\beta \vartheta;\vartheta$$

$$\begin{array}{lll} !(\vartheta, a, b) & \simeq & !(\vartheta; \underline{b}, a, b) \\ & =_{\beta_\Box} & let\ \ u \mathrel{\mathring{=}} !(\vartheta, a, b)\ in\, !(\underline{u}, u, u) \\ & \simeq & let\ \ u \mathrel{\mathring{=}} !(\vartheta, a, b)\ in\, !(\underline{u}; \underline{u}, u, u) \\ & \simeq & !(\vartheta; \underline{b}; \underline{b}, a, b) \\ & \simeq & !(\vartheta, a, b) \end{array}$$

Future Work

- Rewrites
  - Revisit rewrites for HOR: Still work to do
  - Equivalence of Rewrites
    - Permutation Equivalence
    - Projection Equivalence: The HOAS approach also presents issues when defining projection equivalence of rewrites
- Natural Deduction for LP
  - Self-referring Propositions and ND
  - S4 proof decoration

Thank you!