Rewrites as Terms through Justification Logic

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ABSTRACT

Justification Logic is a refinement of modal logic where the modality $\Box A$ is annotated with a reason s for "knowing" A and written $[\![s]\!]A$. The expression s is a proof of A that may be encoded as a lambda calculus term of type A, according to the propositions-as-types interpretation. Our starting point is the observation that terms of type $[\![s]\!]A$ are *reductions* between lambda calculus terms. Reductions are usually encoded as *rewrites*, also called *proof terms*, essential tools in analyzing the reduction behavior of lambda calculus and term rewriting systems, such as when studying standardization, needed strategies, Lévy permutation equivalence, etc. We explore a new propositions-as-types interpretation for Justification Logic, based on the principle that terms of type $[\![s]\!]A$ are proof terms encoding reductions (with source s). Note that this provides a logical language to reason about proof terms.

KEYWORDS

Lambda calculus, modal logic, Curry-Howard, term rewriting, type systems

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Justification Logic [3, 4, 14] is a modal logic where necessity is indexed by justification expressions. The modal proposition $\Box A$ becomes $[\![s]\!]A$ where the justification expression s is a reason for "knowing" A. Typically, s denotes a proof that attests to the truth of A. An important property of Justification Logic is the *reflection principle*: given a proof of A, one can encode this proof using a justification expression s and prove $[\![s]\!]A$. Most formulations of Justification Logic are in Hilbert style. In that case s above is a combinator, called a *proof polynomial*, encoding a Hilbert style proof of A. This paper proposes to explore the computational significance of Justification Logic via the propositions-as-types methodology. In fact, we focus here on an early precursor of Justification Logic, namely the *Logic of Proofs* (LP) [1, 2]. The Logic of Proofs may be understood as the justification counterpart of S4. All theorems of S4 are also theorems of LP where occurrences of the necessity modality have been suitably annotated with justification expressions. Similarly, dropping the justification expressions of modalities in theorems of Justification Logic yields S4 theorems.

Natural Deduction for the Logic of Proofs. A Natural Deduction presentation for the Logic of Proofs suggests itself through the reflection principle. Consider the following introduction rule for the modality: if s is a Natural Deduction proof of A, then $[\![s]\!]A$ is provable. Here s is a justification expression denoting a Natural Deduction proof. The sequents of our deductive system take the form $\Gamma \vdash A \mid s$, where Γ is a set of hypotheses and the justification expression s encodes the current Natural Deduction proof of the sequent, so that we can express the above reflection principle as an introduction rule as: if one proves $\Gamma \vdash A \mid s$, then one may prove $\Gamma \vdash [\![s]\!]A \mid !s$. The exclamation mark in "$!s$" records the fact that a modality introduction rule was applied, thus updating our current justification expression. Of course, Γ cannot be any set of hypotheses at all since otherwise A and $[\![s]\!]A$ would be logically equivalent (*i.e.* $A \supset [\![s]\!]A$ and $[\![s]\!]A \supset A$ would both be provable). Rather than restrict the hypotheses in Γ to be modal expressions we split them in two disjoint sets, following Pfenning et al [11]: we use Δ for *modal* hypotheses (those assumed true in all accessible worlds) and Γ for *truth* hypotheses (those assumed true in the current world). Sequents now take the form $\Delta; \Gamma \vdash A \mid s$ and we can recast the above mentioned introduction rule for the modality as follows, where the "$\cdot$" denotes an empty set of truth hypotheses:

$$\frac{\Delta; \cdot \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s} \qquad (1)$$

Although correct from a provability angle, one immediately realizes that, in the presence of this proposed rule, proofs are no longer closed under normalisation. This is an important requirement towards our goal in uncovering a computational interpretation of $[\![s]\!]A$ since reduction on terms mimics normalisation on proofs. Indeed, normalisation of the proof of $\Delta; \cdot \vdash A \mid s$ will produce a proof of $\Delta; \cdot \vdash A \mid t$, for some t different from s. We need some means of relating t back to s.

Towards a Calculus of Rewrites. A rewrite is an expression that denotes a sequence of reduction steps from a source term to a target term. Consider for example the lambda calculus term $\lambda a.a$ denoting the identity function. Let us abbreviate this term I. The term $(Ib)(Ib)$ reduces in one β-step to $b\,(Ib)$ by contracting the leftmost redex. An expression denoting this reduction step would be the rewrite:

$$\mathbf{ba}(a.a, b)\,(Ib) : (Ib)(Ib) \rhd b(Ib)$$

The expression $\mathbf{ba}(a.a, b)\,(Ib)$ models the above mentioned reduction step. The occurrence of $\mathbf{ba}(a.a, b)$ tells us that a β-reduction

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step reduced the leftmost of the two redexes in $(Ib)(Ib)$. The term $(Ib)(Ib)$ to the left of the triangle is the source of the rewrite and $b(Ib)$ on the right the target. We could continue reduction from the target $b(Ib)$ to obtain bb. The reduction sequence encoding both steps would then be written:

$$\mathbf{ba}(a.a, b)\, Ib; b\, \mathbf{ba}(a.a, b) : (Ib)(Ib) \rhd bb$$

the semi-colon denoting composition of rewrites. Returning to our discussion on (1) on obtaining some means of relating t back to s, and given that proofs are reflected as justification expressions in the logic, it seems natural to reify normalisation steps as rewrites. This suggests that:

$$[\![s]\!]A \text{ is the type of } \textit{rewrites with source } s.$$

Or in terms of a deduction rule:

$$\frac{\Delta; \cdot \vdash \rho : s \rhd t : A}{\Delta; \Gamma \vdash !(\rho, s, t) : [\![s]\!]A}$$

A new sequent $\Delta; \cdot \vdash \rho : s \rhd t : A$ types rewrites rather than terms. It states that if ρ is a rewrite from source term s to target term t, then $!(\rho, s, t)$ is a term. This rule is not quite right in fact but hopefully suffices for the reader to get the gist of the technical development that follows: a novel propositions-as-types interpretation for the Logic of Proofs, called the Rewrite Calculus (RC), based on the ideas discussed above.

The main contributions of this work are:

- Terms and Rewrites as mutually dependent objects.
- A propositions-as-types presentation for the Logic of Proofs based on rewrites as terms.
- A notion of reduction on rewrites we dub extension.
- Fundamental meta-theoretic properties of substitution and subject reduction for extension of rewrites.

Structure of the paper. Sec. 1 introduces the terms and rewrites. The type system for RC is presented in Sec. 2. Extension of rewrites is discussed in Sec. 3. We conclude and present related work in Sec. 4. Proofs are relegated to an appendix. Some figures and expressions are color coded for better legibility. An extended version of this report is available at ebonelli.github.io/files/rc.pdf

1 TERMS AND REWRITES (AS TERMS)

This section presents (untyped) terms and rewrites. Types will be considered in Sec. 2.

Terms and Rewrites. **Terms** ($\mathbb{T}^-$) and **rewrites** ($\mathbb{R}^-$) are defined by the following mutually recursive grammar:

$$\begin{array}{rcl} s, t & ::= & a \mid u \mid \lambda a.s \mid s\, t \mid !(\rho, s, t) \mid \textit{let } u \doteq s \textit{ in } t \\ \rho, \sigma & ::= & \underline{a} \mid \underline{u} \mid \mathbf{ba}(a.s, r) \mid \mathbf{bb}(s, u.r) \mid \rho; \sigma \mid \lambda a.\rho \mid \rho\, \sigma \\ & & \mid \langle \rho |_s \sigma \rangle \mid \textit{let } u \doteq \rho \textit{ in } \sigma \end{array}$$

Terms include the usual lambda calculus expressions consisting of term variables a, abstraction $\lambda a.s$ and application $s\,t$. There are also three new ones. A term of the form $!(\rho, s, t)$ denotes a rewrite from source term s to target term t. Variable u is a rewrite variable of sort term. When a rewrite ρ is substituted into a term, this variable will potentially be replaced with either the source or the target of ρ, as will be made clear in the upcoming definition of substitution of rewrites. The term $\textit{let } u \doteq s \textit{ in } t$ denotes rewrite composition. For example, the term $\textit{let } v \doteq b \textit{ in let } u \doteq a \textit{ in } !(\underline{u}\,\underline{v}, u\,v, u\,v)$ will evaluate b to obtain a term $!(\rho, s, t)$ and a to obtain $!(\sigma, p, q)$ and then compose the rewrites ρ and σ to build a rewrite $\rho\,\sigma$ from the application $s\,p$ to $t\,q$. After appropriate substitutions the resulting term will be $!(\rho\,\sigma, s\,p, t\,q)$.

Rewrites denote reduction between a source and target term. The rewrite $\underline{a}$ denotes the identity reduction over term a. Rewrite $\underline{u}$ is the same only that it, moreover, is subject to be replaced by rewrite substitution. Rewrite $\mathbf{ba}(a.s, r)$ denotes a β-reduction step from term $(\lambda a.s)\,t$ to term $s\{a/t\}$, the latter denoting the capture-avoiding substitution of all free occurrences of a in s by t (defined below). The rewrite $\mathbf{bb}(!(\rho, s, t), u.r)$ similarly will stand for a reduction step involving a redex of the form $\textit{let } u \doteq !(\rho, s, t) \textit{ in } r$, where u in r is to be substituted by ρ, s and t; further details will be supplied later. As mentioned in the introduction, the rewrite $\rho; \sigma$ denotes composition of reductions. Not all such rewrites are reasonable since the target of ρ may not coincide with the source of σ. Making this precise requires a definition of source and target of a rewrite, a topic we address below. The remaining rewrites denote reduction under a term constructor: $\lambda a.\rho$ is for reduction under an abstraction, $\rho\,\sigma$ for reduction under an application, $\textit{let } u \doteq \rho \textit{ in } \sigma$ for reduction under a let and $\langle \rho |_s \sigma \rangle$ for reduction under a bang term constructor, where s is assumed to be the source of ρ. The latter merits additional comments. Reduction under a term of the form $!(\rho, s, t)$ is interpreted as extending ρ with additional "work" as captured by the rewrite σ. In fact, $\langle \rho |_s \sigma \rangle$ will be considered valid only if the target of ρ coincides with the source of σ.

Free term variables and **free rewrite variables** are defined as expected. Worthy of mention are the clauses: $\mathsf{ftv}(!(\rho, s, t)) := \emptyset$ and $\mathsf{ftv}(\textit{let } u \doteq s \textit{ in } t) := \mathsf{ftv}(s) \cup \mathsf{ftv}(t)$. The former owes to the fact that term variables represent truth hypothesis in the current world and hence, as is standard, cannot occur free in the term introduced by the modal type Also, $\mathsf{frv}(!(\rho, s, t)) := \mathsf{frv}(\rho) \cup \mathsf{frv}(s) \cup \mathsf{frv}(t)$ and $\mathsf{frv}(\textit{let } u \doteq s \textit{ in } t) := \mathsf{frv}(t) \setminus \{u\}$.

The subset of rewrites called **unit rewrites** ($\mathbb{R}_1^-$) is inductively characterized as follows:

$$\mathfrak{s} \quad ::= \quad \underline{a} \mid \underline{u} \mid \lambda a.\mathfrak{s} \mid \mathfrak{s}\,\mathfrak{s} \mid \langle \rho |_t \mathfrak{s} \rangle \mid \textit{let } u \doteq \mathfrak{s} \textit{ in } \mathfrak{s}$$

Any term s can be cast as a unit rewrite $\underline{s}$ (written $\mathfrak{s}$), the latter denoting the identity reduction over itself (*cf.* Lem. 1.3) as follows:

$$\begin{array}{rcl} \underline{a} & := & \underline{a} \\ \underline{u} & := & \underline{u} \\ \underline{\lambda a.s} & := & \lambda a.\underline{s} \\ \underline{s\,t} & := & \underline{s}\,\underline{t} \\ \underline{!(\rho, s, t)} & := & \langle \rho |_s \underline{t} \rangle \\ \underline{\textit{let } u \doteq s \textit{ in } t} & := & \textit{let } u \doteq \underline{s} \textit{ in } \underline{t} \end{array}$$

Substitution. We next introduce three notions of substitution, where o below denotes an **object** ($\mathbb{O}^-$) defined simply as the union of terms and rewrites:

Substitution of term variables	$s\{a/t\}$
Substitution of rewrite variables over unit rewrites	$\mathfrak{r}\{u/\rho_s^t\}$
Moded substitution of rewrite variables	$\mathrm{o}\{u/^{\mathsf{m}}\rho_s^t\}$

Substitution of term variables is defined as expected. It is worth mentioning that it does not propagate to rewrites since rewrites do not have occurrences of free term variables, as may be seen from looking at the clause defining $!(\rho, s, t)\{a/r\}$.

$$
\begin{array}{rcl}
b\{a/r\} & := & \begin{cases} r, & a = b \\ a, & a \neq b \end{cases} \\
u\{a/r\} & := & u \\
(\lambda b.s)\{a/r\} & := & \lambda b.s\{a/r\} \\
(s\,t)\{a/r\} & := & s\{a/r\}\,t\{a/r\} \\
!(\rho, s, t)\{a/r\} & := & !(\rho, s, t) \\
(let\ u \doteq s\ in\ t)\{a/r\} & := & let\ u \doteq s\{a/r\}\ in\ t\{a/r\}
\end{array}
$$

Substitution of rewrite variables into rewrites must be done with some care. Consider the term $\underline{u};\underline{u}$, which is well-formed since $\underline{u}$ is a rewrite from u to itself. Let ρ be a rewrite from a source s to target t. A naive definition of $(\underline{u};\underline{u})\{u/\rho\}$ could end up producing $\rho;\rho$ which is not well-formed in the sense that the source and target of ρ may not coincide. Our notion of substitution will produce $\rho;\mathfrak{t}$. Alternatively, one could produce $\mathfrak{s};\rho$. However, substituting ρ at the beginning or end makes no difference since both $\rho;\mathfrak{t}$ and $\mathfrak{s};\rho$ should be equated to ρ anyhow. This will indeed be the case once we have introduced structural equivalence on rewrites (Fig. 1). What is clear is that only one copy of ρ should be substituted and that either prefixing or postfixing it makes no difference.
Another observation we make is that when substituting in $\underline{u};\underline{u}$ we replace each copy of $\underline{u}$ by *different* objects. The first occurrence gets replaced by ρ but the second one gets replaced by a unit rewrite, namely $\mathfrak{t}$. Accordingly, we split substitution of rewrite variables in two: one that substitutes ρ itself and another one that substitutes the source or target of ρ cast as a unit rewrite. The former is written $\mathfrak{r}\{u/\rho^t_s\}$ and the latter $\mathfrak{o}\{u/^{\mathsf{m}}\rho^t_s\}$ where m stands for either src or tgt. In particular, $\underline{u}\{u/\rho^t_s\} = \rho$, $\underline{u}\{u/^{\mathsf{src}}\rho^t_s\} = \mathfrak{s}$ and $\underline{u}\{u/^{\mathsf{tgt}}\rho^t_s\} = \mathfrak{t}$. Note also that defining $(\sigma_1;\sigma_2)\{u/^{\mathsf{m}}\rho^t_s\} = \sigma_1\{u/^{\mathsf{m}}\rho^t_s\};\sigma_2\{u/^{\mathsf{m}}\rho^t_s\}$ is correct but not so for $\mathfrak{r}\{u/\rho^t_s\}$. As a final observation, both of these notions of substitution are mutually recursive. Substitution of Rewrite Variables over unit rewrites is defined as:

$$
\begin{array}{rcl}
\underline{a}\{u/\rho^t_s\} & := & \underline{a} \\
\underline{v}\{u/\rho^t_s\} & := & \begin{cases} \rho, & u = v \\ \underline{v}, & u \neq v \end{cases} \\
(\lambda a.\mathfrak{s})\{u/\rho^t_s\} & := & \lambda a.\mathfrak{s}\{u/\rho^t_s\} \\
(\mathfrak{p}\,\mathfrak{q})\{u/\rho^t_s\} & := & \mathfrak{p}\{u/\rho^t_s\}\,\mathfrak{q}\{u/\rho^t_s\} \\
\langle\sigma|_p\mathfrak{q}\rangle\{u/\rho^t_s\} & := & \langle\mathfrak{p}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{p\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{q}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle \\
(let\ v \doteq \mathfrak{p}\ in\ \mathfrak{q})\{u/\rho^t_s\} & := & let\ v \doteq \mathfrak{p}\{u/\rho^t_s\}\ in\ \mathfrak{q}\{u/\rho^t_s\}
\end{array}
$$

Notice the clause for $\langle\sigma|_p\mathfrak{q}\rangle$. Substitution prepends a copy of ρ to σ (*cf.* rewrite $\mathfrak{p}\{u/\rho^t_s\}$ above) and updates σ so that all occurrences of u in σ are replaced with the target of ρ (*cf.* rewrite $\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}$ above). For the latter is relies on moded substitution defined below. Similar updates are applied to the source term p and unit rewrite $\mathfrak{q}$. Perhaps worth mentioning is that the resulting rewrite is also a unit rewrite: $\rho \in \mathbb{R}^-_1$ implies $\rho\{u/^{\mathsf{m}}\sigma^q_p\} \in \mathbb{R}^-_1$.
Moded Substitution of Rewrite Variables over rewrites is defined as:

$$
\begin{array}{rcl}
\underline{a}\{u/^{\mathsf{m}}\rho^t_s\} & := & \underline{a} \\
\underline{v}\{u/^{\mathsf{m}}\rho^t_s\} & := & \begin{cases} \mathfrak{s}, & u = v \wedge \mathsf{m} = src \\ \mathfrak{t}, & u = v \wedge \mathsf{m} = tgt \\ \underline{v}, & u \neq v \end{cases} \\
\mathbf{ba}(a.p, q)\{u/^{\mathsf{m}}\rho^t_s\} & := & \mathbf{ba}(a.p\{u/^{\mathsf{m}}\rho^t_s\}, q\{u/^{\mathsf{m}}\rho^t_s\}) \\
\mathbf{bb}(p, v.q)\{u/^{\mathsf{m}}\rho^t_s\} & := & \mathbf{bb}(p\{u/^{\mathsf{m}}\rho^t_s\}, v.q\{u/^{\mathsf{m}}\rho^t_s\}) \\
(\lambda a.\rho)\{u/^{\mathsf{m}}\rho^t_s\} & := & \lambda a.\rho\{u/^{\mathsf{m}}\rho^t_s\} \\
(\sigma\,\tau)\{u/^{\mathsf{m}}\rho^t_s\} & := & \sigma\{u/^{\mathsf{m}}\rho^t_s\}\,\tau\{u/^{\mathsf{m}}\rho^t_s\} \\
\langle\sigma|_p\tau\rangle\{u/^{\mathsf{m}}\rho^t_s\} & := & \langle\mathfrak{p}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{p\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle \\
(\sigma;\tau)\{u/^{\mathsf{m}}\rho^t_s\} & := & \sigma\{u/^{\mathsf{m}}\rho^t_s\};\tau\{u/^{\mathsf{m}}\rho^t_s\} \\
(let\ v \doteq \sigma\ in\ \tau)\{u/^{\mathsf{m}}\rho^t_s\} & := & let\ v \doteq \sigma\{u/^{\mathsf{m}}\rho^t_s\}\ in\ \tau\{u/^{\mathsf{m}}\rho^t_s\}
\end{array}
$$

Notice here how, in the clause for $\underline{v}$, it is the source s and target t that are substituted, prior to having being cast as unit rewrites. Also worthy of mention is that moded substitution still needs access to ρ itself (not just its source and target); it is used in the clause for $\langle\sigma|_p\tau\rangle$. One final comment on the above definition is that in the clause for $\sigma;\tau$ it is safe to distribute moded substitution over σ and τ.
Finally, moded Substitution of Rewrite Variables over terms is defined as:

$$
\begin{array}{rcl}
a\{u/^{\mathsf{m}}\rho^t_s\} & := & a \\
v\{u/^{\mathsf{m}}\rho^t_s\} & := & \begin{cases} s, & v = u \wedge \mathsf{m} = src \\ t, & v = u \wedge \mathsf{m} = tgt \\ v, & v \neq u \end{cases} \\
(\lambda a.r)\{u/^{\mathsf{m}}\rho^t_s\} & := & \lambda a.r\{u/^{\mathsf{m}}\rho^t_s\} \\
(p\,q)\{u/^{\mathsf{m}}\rho^t_s\} & := & p\{u/^{\mathsf{m}}\rho^t_s\}\,q\{u/^{\mathsf{m}}\rho^t_s\} \\
!(\sigma, p, q)\{u/^{\mathsf{m}}\rho^t_s\} & := & !(\mathfrak{p}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, p\{u/^{\mathsf{src}}\rho^t_s\}, q\{u/^{\mathsf{tgt}}\rho^t_s\}) \\
(let\ v \doteq p\ in\ q)\{u/^{\mathsf{m}}\rho^t_s\} & := & let\ v \doteq p\{u/^{\mathsf{m}}\rho^t_s\}\ in\ q\{u/^{\mathsf{m}}\rho^t_s\}
\end{array}
$$

Some basic, but subtle to prove, properties for substitution are presented below, after introducing structural equivalence.

Structural Equivalence and Well-Formedness. As mentioned, rewrites may not have a source and target. If it does we say it is *well-formed*. For example, if a and b are distinct variables, then $\underline{a};\underline{b}$ is not well-formed. More generally, for $\rho;\sigma$ to be well-formed the target of ρ must coincide with the source of σ. Similar requirements apply to $!(\rho, s, t)$ and $\langle\rho|_s\sigma\rangle$. This leads us to consider how terms are to be compared. Since terms may include rewrites, we need to consider rewrite comparison too.
One reasonable property is that composition be associative: rewrites $(\rho;\sigma);\tau$ and $\rho;(\sigma;\tau)$ should be considered equivalent. Similarly, $\rho;\mathfrak{t}$ should be considered equivalent to ρ, assuming that ρ and $\mathfrak{t}$ are composable (in which case t should be equivalent to the target of ρ, though it may not be identical to it). Another example of rewrite equivalence is as follows. Let I be the term $\lambda b.b$ and consider the lambda calculus reduction $\lambda a.I(\underline{Ia}) \to_\beta \lambda a.\underline{Ia} \to_\beta \lambda a.a$, where the redex being reduced in each step is underlined. It can be represented via the rewrite[1] $\lambda a.I\,\mathbf{ba}(b.b, a);\lambda a.\mathbf{ba}(b.b, a)$. However, the same reduction sequence could also have been represented as $\lambda a.(I\,\mathbf{ba}(b.b, a);\mathbf{ba}(b.b, a))$. All such minor, structural variations are absorbed through *structural equivalence*.

[1] There is an abuse of notation here since "λa" is used both as a term constructor, to build an abstraction, and as a rewrite constructor, to build a rewrite that denotes reduction under an abstraction. The context should prove sufficient to avoid confusion.

$$\dfrac{}{\underline{a} : a \rhd a}\,\text{ST-TVar} \qquad \dfrac{}{\underline{u} : u \rhd u}\,\text{ST-RVar} \qquad \dfrac{}{\mathbf{ba}(a.s,t) : (\lambda a.s)\,t \rhd s\{a/t\}}\,\text{ST-}\beta \qquad \dfrac{}{\mathbf{bb}(!(\rho,s,t),u.r) : \mathit{let}\ u \doteq !(\rho,s,t)\ \mathit{in}\ r \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\}}\,\text{ST-}\beta_\Box$$

$$\dfrac{\rho : s \rhd t}{\lambda a.\rho : \lambda a.s \rhd \lambda a.t}\,\text{ST-Abs} \qquad \dfrac{\rho : s_1 \rhd t_1 \quad \sigma : s_2 \rhd t_2}{\rho\,\sigma : s_1\,s_2 \rhd t_1\,t_2}\,\text{ST-App} \qquad \dfrac{\rho : s_1 \rhd t_1 \quad \sigma : s_2 \rhd t_2}{\mathit{let}\ u \doteq \rho\ \mathit{in}\ \sigma : \mathit{let}\ u \doteq s_1\ \mathit{in}\ s_2 \rhd \mathit{let}\ u \doteq t_1\ \mathit{in}\ t_2}\,\text{ST-Let}$$

$$\dfrac{\rho : r \rhd s \quad \sigma : s \rhd t}{\rho;\sigma : r \rhd t}\,\text{ST-Comp} \qquad \dfrac{\rho : s \rhd r \quad \sigma : r \rhd t}{\langle \rho|_s \sigma\rangle : !(\rho,s,r) \rhd !(\rho;\sigma,s,t)}\,\text{ST-Bang} \qquad \dfrac{r \simeq r' \quad \rho : r' \rhd s' \quad s' \simeq s}{\rho : r \rhd s}\,\text{ST-SEq}$$

$$\dfrac{}{\underline{a} \simeq \underline{a} : a \rhd a}\,\text{EqR-Refl-TVar} \qquad \dfrac{}{\underline{u} \simeq \underline{u} : u \rhd u}\,\text{EqR-Refl-RVar}$$

$$\dfrac{}{\mathbf{ba}(a.s,t) \simeq \mathbf{ba}(a.s,t) : (\lambda a.s)\,t \rhd s\{a/t\}}\,\text{EqR-Refl-}\beta \qquad \dfrac{}{\mathbf{bb}(!(\rho,s,t),u.r) \simeq \mathbf{bb}(!(\rho,s,t),u.r) : \mathit{let}\ u \doteq !(\rho,s,t)\ \mathit{in}\ r \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\}}\,\text{EqR-Refl-}\beta_\Box$$

$$\dfrac{\rho : p \rhd q \quad \mathfrak{t} : q \rhd r}{\rho;\mathfrak{t} \simeq \rho : p \rhd r}\,\text{EqR-IdR} \qquad \dfrac{\mathfrak{s} : p \rhd q \quad \rho : q \rhd r}{\mathfrak{s};\rho \simeq \rho : p \rhd r}\,\text{EqR-IdL}$$

$$\dfrac{\sigma : p \rhd q \quad \rho : q \rhd r \quad \tau : r \rhd s}{(\sigma;\rho);\tau \simeq \sigma;(\rho;\tau) : p \rhd s}\,\text{EqR-Ass} \qquad \dfrac{\rho : p \rhd q \quad \sigma : q \rhd r}{\lambda a.\rho;\lambda a.\sigma \simeq \lambda a.(\rho;\sigma) : \lambda a.p \rhd \lambda a.r}\,\text{EqR-Abs}$$

$$\dfrac{\rho_1 : p_1 \rhd q_1 \quad \rho_2 : q_1 \rhd r_1 \quad \sigma_1 : p_2 \rhd q_2 \quad \sigma_2 : q_2 \rhd r_2}{(\rho_1\,\sigma_1);(\rho_2\,\sigma_2) \simeq (\rho_1;\rho_2)\,(\sigma_1;\sigma_2) : p_1\,p_2 \rhd r_1\,r_2}\,\text{EqR-App}$$

$$\dfrac{\rho_1 : p_1 \rhd q_1 \quad \rho_2 : q_1 \rhd r_1 \quad \sigma_1 : p_2 \rhd q_2 \quad \sigma_2 : q_2 \rhd r_2}{\mathit{let}\ u \doteq \rho_1\ \mathit{in}\ \sigma_1; \mathit{let}\ u \doteq \rho_2\ \mathit{in}\ \sigma_2 \simeq \mathit{let}\ u \doteq \rho_1;\rho_2\ \mathit{in}\ \sigma_1;\sigma_2 : \mathit{let}\ u \doteq p_1\ \mathit{in}\ p_2 \rhd \mathit{let}\ u \doteq r_1\ \mathit{in}\ r_2}\,\text{EqR-Let}$$

$$\dfrac{\rho : p \rhd q \quad \sigma : q \rhd r \quad \tau : r \rhd s}{\langle \rho|_p \sigma\rangle;\langle \rho;\sigma|_p \tau\rangle \simeq \langle \rho|_p \sigma;\tau\rangle : !(\rho,p,q) \rhd !(\rho;\sigma;\tau,p,s)}\,\text{EqR-BangR} \qquad \dfrac{s \simeq s' \quad \rho \simeq \sigma : s' \rhd t' \quad t' \simeq t}{\rho \simeq \sigma : s \rhd t}\,\text{EqR-SEq}$$

$$\dfrac{}{a \simeq a}\,\text{EqT-TVar} \qquad \dfrac{}{u \simeq u}\,\text{EqT-RVar} \qquad \dfrac{s \simeq t}{\lambda a.s \simeq \lambda a.t}\,\text{EqT-Abs} \qquad \dfrac{s \simeq p \quad t \simeq q}{s\,t \simeq p\,q}\,\text{EqT-App}$$

$$\dfrac{s \simeq p \quad t \simeq q \quad \rho \simeq \sigma : s \rhd t}{!(\rho,s,t) \simeq !(\sigma,p,q)}\,\text{EqT-Bang} \qquad \dfrac{s \simeq p \quad t \simeq q}{\mathit{let}\ u \doteq s\ \mathit{in}\ t \simeq \mathit{let}\ u \doteq p\ \mathit{in}\ q}\,\text{EqT-Let}$$

Figure 1: Source/Target Predicate and Structural Equivalence of Rewrites and Terms

Definition 1.1 (Source/Target Predicate and Structural Equivalence). The source/target (ST) predicate $\bullet : \bullet \rhd \bullet \subseteq \mathbb{R}^- \times \mathbb{T}^- \times \mathbb{T}^-$ is defined mutually recursively with structural equivalence[2] $\simeq \subseteq \mathbb{O}^- \times \mathbb{O}^-$ via the rules in Fig. 1. There are two structural equivalence judgements:

$s \simeq t$	**Structurally equivalent terms**
$\rho \simeq \sigma : s \rhd t$	**Structurally equivalent rewrites**

If $\rho : s \rhd t$ holds then we say that ρ has source s and target t. If $s \simeq t$ holds, then we say s and t are structurally equivalent terms. Finally, if $\rho \simeq \sigma : s \rhd t$, then we say ρ and σ are structurally equivalent rewrites with source s and target t.

The rules defining the ST-predicate $\bullet : \bullet \rhd \bullet$ (those whose the names are prefixed with ST in Fig. 1) are quite expected. We comment on ST-Bang. As already mentioned, $\langle \rho|_s \sigma\rangle$ is a rewrite denoting reduction under a term of the form $!(\rho,s,r)$ and consists of the additional "work" with which ρ is extended. The additional work is represented by the rewrite σ whose source must coincide with the target of ρ (modulo structural equivalence). The source and target of $\langle \rho|_s \sigma\rangle$ are $!(\rho,s,r)$ and $!(\rho;\sigma,s,t)$.

The rules defining structural equivalence of terms (those whose names are prefixed with EqT in Fig. 1) are as expected. The rules defining structural equivalence of rewrites (those whose names are prefixed with EqR in Fig. 1) are similar to the ones one has in first-order term rewriting (*cf.* Def. 8.3.1. in [18]). Two important differences are as follows. The first is the need to rely on structural equivalence on terms to define structural equivalence on rewrites, given that terms and rewrites are mutually dependent. The other is the presence of terms as rewrites $!(\rho,s,t)$ and rewrites on such terms $\langle \rho|_p \sigma\rangle$. Also novel to this presentation is the equation $\langle \rho|_p \sigma\rangle;\langle \rho;\sigma|_p \tau\rangle \simeq \langle \rho|_p \sigma;\tau\rangle$. It states how two rewrites under a bang may be composed. Given $!(\rho,p,q)$, a rewrite σ extending ρ

[2] The congruence rules for $\simeq$ have been omitted.

must be composable with ρ and, moreover, will have $!(\rho;\sigma,p,r)$ as target. A further rewrite extending $\rho;\sigma$, say τ, will produce term $!((\rho;\sigma);\tau,p,s)$ as target.

We next mention some lemmata on structural equivalence. The first one is that the source and target are unique modulo structural equivalence. It is straightforward to prove.

Lemma 1.2 (Uniqueness of Source and Target). *If $\rho : s \rhd t$ and $\rho : p \rhd q$, then $s \simeq p$ and $t \simeq q$*

The lemma below states that a unit rewrite is a step over itself:

Lemma 1.3. *$\mathfrak{s} : p \rhd q$ implies $p \simeq q \simeq s$.*

The next result states that the rewrites related by structural equivalence have the same source and target. Its proof relies on Lem. 1.3:

Lemma 1.4. *$\rho \simeq \sigma : s \rhd t$ implies $\rho : s \rhd t$ and $\sigma : s \rhd t$.*

The term-as-a-unit-rewrite operation $\underline{\bullet}$ is compatible with structural equivalence:

Lemma 1.5. *$s \simeq t$ implies $\underline{s} \simeq \underline{t} : s \rhd s$.*

Finally, substitution is compatible with structural equivalence too. For substitution of term variables this is proved by induction $s \simeq t$:

Lemma 1.6 (Structural Equivalence is closed under substitution of term variables). *Suppose $s \simeq t$ and $p \simeq q$. Then $s\{a/p\} \simeq t\{a/q\}$.*

For substitution of rewrite variables, the result is broken down into three items all of which are proved by simultaneous induction:

Lemma 1.7 (Structural Equivalence is closed under substitution of rewrite variables). *Suppose $\tau \simeq \upsilon : p \rhd q$. Then*

- *$\rho \simeq \sigma : s \rhd t$ implies $\rho\{u/^{\mathsf{m}}\tau^q_p\} \simeq \sigma\{u/^{\mathsf{m}}\upsilon^q_p\} : s\{u/^{\mathsf{m}}\tau^q_p\} \rhd t\{u/^{\mathsf{m}}\tau^q_p\}$.*
- *$s \simeq t$ implies $s\{u/^{\mathsf{m}}\tau^q_p\} \simeq t\{u/^{\mathsf{m}}\upsilon^q_p\}$.*
- *$s \simeq t$ implies $\mathfrak{s}\{u/\tau^q_p\} \simeq \mathfrak{t}\{u/\upsilon^q_p\} : s\{u/^{\mathsf{src}}\tau^q_p\} \rhd s\{u/^{\mathsf{tgt}}\tau^q_p\}$*

Having introduced the ST-predicate and structural equivalence we can now precisely state when terms and rewrites are well-formed.

Definition 1.8 (Well-formed Terms and Rewrites).

(a) $s \in \mathbb{T}^-$ is **well-formed** if for all subexpressions of s of the form $!(\rho,p,q)$, (ρ,p,q) is well-formed.
(b) $(\rho,s,t) \in \mathbb{R}^- \times \mathbb{T}^- \times \mathbb{T}^-$ is **well-formed** iff $\rho : s \rhd t$ and s and t are well-formed.

$\rho \in \mathbb{R}^-$ is **well-formed** if there exist s and t such that (ρ,s,t) is well-formed.

For example, $\underline{a};\underline{b}$ is not well-formed, however $\mathbf{ba}(a^A.a,b)$ and $\underline{a};\underline{a}$ are. The triple $(\mathbf{ba}(a^A.!(\underline{b};\underline{c},a,a),b),(\lambda a.!(\underline{b};\underline{c},a,a))b,!(\underline{b};\underline{c},a,a))$ is not well-formed. Even though we do have $\mathbf{ba}(a^A.!(\underline{b};\underline{c},a,a),b) : (\lambda a.!(\underline{b};\underline{c},a,a))b \rhd !(\underline{b};\underline{c},a,a)$ the source term $(\lambda a.!(\underline{b};\underline{c},a,a))b$ is not well-formed (since $\underline{b};\underline{c} : a \rhd a$ does not hold).

Well-formedness is preserved by structural equivalence, a fact that relies on Lem. 1.4

Lemma 1.9 (Structural Equivalence Preserves Well-Formedness). *If s is well-formed and $s \simeq t$, then t is well-formed. Similarly, if (ρ,s,t) is well-formed and $\rho \simeq \sigma : s \rhd t$, then (σ,s,t) is well-formed.*

We conclude the section with two important results on commutation of substitutions. We assume for these results that our objects are well-formed. The first one concerns commutation of term and rewrite substitutions.

Lemma 1.10 (Commutation of Rewrite Substitution with Term Substitution). *Suppose $a \notin \mathsf{ftv}(\rho,s,t)$.*

$$p\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} = p\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\}$$

where both occurrences of m are either both src or both tgt.

The second is about commutation of rewrite substitutions and requires some care. First note that when $\bullet\{u/^{\mathsf{m}}\rho^t_s\}$ commutes "over" $\bullet\{v/^{\mathsf{m}}\mu^q_p\}$ in the expression $\mathrm{o}\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\}$, a copy of ρ has to be prefixed in front of μ. This is witnessed in item (a) of Lem. 1.11 below. We comment on item (b) of Lem. 1.11, below, after having analyzed a sample proof case for item (a) which motivates the need for it.

Lemma 1.11 (Commutation of Rewrite Substitution). *Let o be any object (i.e. term or rewrite) and suppose $v \notin \mathsf{frv}(\rho,s,t)$.*

(a) Suppose all occurrences of m below are either all src or all tgt. Then,

$$\begin{array}{ll} & \mathrm{o}\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} \\ \simeq & \mathrm{o}\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

(b) If $\mathrm{o} \in \mathbb{R}_1$, then

$$\begin{array}{ll} & \mathrm{o}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}; \\ & \mathrm{o}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} \\ \simeq & \mathrm{o}\{u/^{\mathsf{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}; \\ & \mathrm{o}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

Item (b) is motivated by analyzing the following proof case for item (a). Suppose $\mathrm{o} = !(\sigma,r_1,r_2)$ and let us introduce the following abbreviations:

$$\begin{array}{lll} \alpha^{\mathsf{m}} & := & \bullet\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\ \alpha & := & \bullet\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

We seek to prove:

$$!(\sigma,r_1,r_2)\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} \simeq !(\sigma,r_1,r_2)\{u/^{\mathsf{m}}\rho^t_s\}\alpha^{\mathsf{m}}$$

We reason as in Fig. 2 where Lem. A.5 is the property that the function that casts a term as a rewrite commutes with rewrite substitution ($\underline{p\{u/^{\mathsf{m}}\rho^t_s\}} = \underline{p}\{u/^{\mathsf{m}}\rho^t_s\}$). The topmost box signals exactly where we apply item (b) above. Consider the case where $r_1 = v$. If one just considers the left argument of the composition inside the box, namely $\underline{r_1}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}$, then the resulting term would be $\mathfrak{p}\{u/\rho^t_s\}$. If we now take the left argument of the composition in the second box, namely $\underline{r_1}\{u/^{\mathsf{src}}\rho^t_s\}\alpha$, then we have $\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}$. Clearly these rewrites are not equivalent. However, when the entire composed rewrites inside the boxes are considered, then we do obtain structurally equivalent rewrites.

2 TYPES

This section presents the type system for RC. The set of **propositions** ($\mathbb{P}$) is defined as follows:

$$A, B \quad ::= \quad P \mid A \supset B \mid [\![\rho,s,t]\!]A$$

$$
\begin{array}{rll}
= & !(\sigma, r_1, r_2)\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & !(\underline{r}_1\{v/\mu^q_p\}; \sigma\{v/^{\mathsf{tgt}}\mu^q_p\}, r_1\{v/^{\mathsf{src}}\mu^q_p\}, r_2\{v/^{\mathsf{tgt}}\mu^q_p\})\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & !(r_1\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}; (\underline{r}_1\{v/\mu^q_p\}; \sigma\{v/^{\mathsf{tgt}}\mu^q_p\})\{u/^{\mathsf{tgt}}\rho^t_s\}, q\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}, r_2\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}) & \\
= & !(\overline{\underline{r}_1\{v/^{\mathsf{src}}\mu^q_p\}}\{u/\rho^t_s\}; (\underline{r}_1\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}; \sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}), r_1\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}, r_2\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}) & \text{(Lem. A.5)} \\
\simeq & !(\boxed{(\underline{r}_1\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\}; \underline{r}_1\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\})}; \sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}, r_1\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}, r_2\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}) & \\
\simeq & !(\boxed{(\underline{r}_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha; \underline{r}_1\{u/\rho^t_s\}\alpha^{\mathsf{tgt}})}; \sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}, r_1\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}, r_2\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}) & \text{(item (b)} \\
\simeq & !((\underline{r}_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha; \underline{r}_1\{u/\rho^t_s\}\alpha^{\mathsf{tgt}}); \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}) & \text{(item (a))} \\
= & !(\underline{r}_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha; (\underline{r}_1\{u/\rho^t_s\}\alpha^{\mathsf{tgt}}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}), r_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}) & \\
= & !(\underline{r}_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha; (\underline{r}_1\{u/\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}) & \\
= & !(\overline{\underline{r}_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha}; (\underline{r}_1\{u/\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}) & \text{(Lem. A.5)} \\
= & !(\overline{\underline{r}_1\{u/\rho^t_s\}}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, r_1\{u/^{\mathsf{src}}\rho^t_s\}, r_2\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{m}} & \\
= & !(\sigma, r_1, r_2)\{u/^{\mathsf{m}}\rho^t_s\}\alpha^{\mathsf{m}} &
\end{array}
$$

Figure 2: Commutation of substitution of rewrite variables - Sample proof case

where P ranges over some set of propositional variables. We write Δ for a set of rewrite hypotheses and Γ for a set of term hypotheses. There are four typing judgements:

$\Delta; \Gamma \vdash s : A$	**Term typing judgement**
$\Delta; \Gamma \vdash \rho : s \triangleright t : A$	**Rewrite typing judgement**
$\Delta \vdash A \preceq B$	**Subtyping judgement**
$\Delta \vdash A \simeq B$	**Structural equivalence judgement**

A proposition A is well-formed if for all occurrences of $[\![\rho, s, t]\!]B$ in A, (ρ, s, t) is well-formed (*cf.* Def. 1.8). A term typing judgement $\Delta; \Gamma \vdash s : A$ is **well-formed** if, (1) for all occurrences of $[\![\rho, p, q]\!]B$ in Δ, Γ, s, A, (ρ, p, q) is well-formed; (2) $\mathrm{dom}(\Delta) \cap \mathrm{frv}(\Delta, \Gamma) = \emptyset$ and; (3) $\mathrm{dom}(\Gamma) \cap \mathrm{ftv}(\Delta, \Gamma) = \emptyset$. Similarly for the other three typing judgements. The first condition makes sure we do not use rewrites such as $\underline{a}; \underline{b}$ in propositions. The second and third conditions state that the labels of the hypothesis are fresh. In the sequel we assume type judgements to be well-formed.

Type System. The type system for RC is given by the rules of Fig. 3. A judgement is **derivable**, indicated with $\blacktriangleright\ \Delta; \Gamma \vdash s : A$, if it is provable using these rules. Moreover, we write $\blacktriangleright_\pi\ \Delta; \Gamma \vdash s : A$ if it is derivable with derivation π. This notation applies to the other typing judgements too.

We next comment on the salient typing rules. The Bang rule was motivated in the introduction. Note, however, that the type of $!(\rho, s, t)$ is $[\![\rho, s, t]\!]A$ rather than $[\![s]\!]A$. The reason for this may be understood via the Let rule. Recall that rewrites may occur in terms. Thus the term s in a type such as $[\![s]\!]A$ could have an occurrence of, say, $!(\mathfrak{u}, u, u)$. Substitution into types will require substitution into terms, which in turn will require substituting the $\mathfrak{u}$ with a rewrite and both occurrences of u with appropriate terms (an example is given below – *cf.* Ex. 2.1). Indeed, notice that the type of the conclusion of Let is $C\{u/^{\mathsf{src}}\rho^q_p\}$, where moded substitution on types is defined as follows:

$$
\begin{array}{rcl}
P\{u/^{\mathsf{m}}\rho^t_s\} & := & P \\
(A \supset B)\{u/^{\mathsf{m}}\rho^t_s\} & := & A\{u/^{\mathsf{m}}\rho^t_s\} \supset B\{u/^{\mathsf{m}}\rho^t_s\} \\
([\![\sigma, p, q]\!]A)\{u/^{\mathsf{m}}\rho^t_s\} & := & \\
\multicolumn{3}{r}{[\![\mathfrak{p}\{u/\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, p\{u/^{\mathsf{src}}\rho^t_s\}, q\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]A\{u/^{\mathsf{m}}\rho^t_s\}}
\end{array}
$$

The rule R-Bang types the rewrite that denotes reduction inside a term of the form $!(\rho, s, r)$. Reduction under such a term corresponds to extending ρ with some additional work σ. The source of $\langle\rho|_s\sigma\rangle$ is $!(\rho, s, r)$ and the target is $!(\rho; \sigma, s, t)$.

One important property of typing for rewrites is that the source and target of a typable rewrite be typable. In other words, that $\blacktriangleright\ \Delta; \Gamma \vdash \rho : s \triangleright t : A$ implies both $\Delta; \Gamma \vdash s : A$ and $\Delta; \Gamma \vdash t : A$ are typable (*cf.* Lem. 2.6). In the particular case of R-Bang, this requires our introducing a subtyping judgement. Indeed, we would require its source and target terms $!(\rho, s, r)$ and $!(\rho; \sigma, s, t)$, resp., to have type $[\![\rho, s, r]\!]A$. For the target we rely on subsumption. The intuition behind our subtyping rules are in line with our discussion in the introduction ($[\![s]\!]A$ as the type of the rewrites with source s): removing a suffix of ρ in $[\![\rho, s, t]\!]A$ should maintain typability. This is expressed via the S-Box subtyping rule. The subsumption rule Subs is actually two rules in one: the expression S or subject, denotes either a term s or an expression of the form $\rho : s \triangleright t$.

Note that $\blacktriangleright\ \Delta; \Gamma \vdash \rho : s \triangleright t : A$ implies $\rho : s \triangleright t$ (*i.e.* the triple (ρ, s, t) satisfies the ST-predicate).

Example 2.1. A sample derivation of the proposition:

$$[\![\rho, p, q]\!]A \supset [\![\langle\rho|_p\mathfrak{q}\rangle, !(\rho, p, q), !(\rho, p, q)]\!][\![\rho, p, q]\!]A$$

is presented in Fig. 4 where we omit some of the rule names to save space. Also, $\Delta := u : A$ and $\Gamma := a : [\![\rho, p, q]\!]A$. Notice that the type of the endsequent is:

$$
\begin{array}{rl}
 & ([\![\langle\mathfrak{u}|_u\mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)]\!][\![\mathfrak{u}, u, u]\!]A)\{u/^{\mathsf{src}}\rho^q_p\} \\
= & [\![\langle\mathfrak{u}|_u\mathfrak{u}\rangle\{u/\rho^q_p\}; \langle\mathfrak{u}|_u\mathfrak{u}\rangle\{u/^{\mathsf{tgt}}\rho^q_p\}, !(\mathfrak{u}, u, u)\{u/^{\mathsf{src}}\rho^q_p\}, !(\mathfrak{u}, u, u)\{u/^{\mathsf{tgt}}\rho^q_p\}]\!] \\
 & ([\![\mathfrak{u}, u, u]\!]A)\{u/^{\mathsf{src}}\rho^q_p\} \\
\simeq & [\![\langle\rho|_p\mathfrak{q}\rangle; \langle\rho|_p\mathfrak{q}\rangle, !(\rho, p, q), !(\rho, p, q)]\!]([\![\mathfrak{u}, u, u]\!]A)\{u/^{\mathsf{src}}\rho^q_p\} \\
\simeq & [\![\langle\rho|_p\mathfrak{q}\rangle, !(\rho, p, q), !(\rho, p, q)]\!]([\![\mathfrak{u}, u, u]\!]A)\{u/^{\mathsf{src}}\rho^q_p\} \\
\simeq & [\![\langle\rho|_p\mathfrak{q}\rangle, !(\rho, p, q), !(\rho, p, q)]\!][\![\rho, p, q]\!]A
\end{array}
$$

Other theorems of RC are:

- $[\![\rho, s, t]\!](A \supset B) \supset [\![\sigma, p, q]\!]A \supset [\![\rho\,\sigma, s\,p, t\,q]\!]B$
- $[\![\rho, s, t]\!]A \supset A$

These may be seen as annotated versions of the S4 theorems:

- $\Box(A \supset B) \supset \Box A \supset \Box B$

$$\frac{a : A \in \Gamma}{\Delta; \Gamma \vdash a : A}\ \text{TVar} \qquad \frac{\Delta; \Gamma, a : A \vdash s : B}{\Delta; \Gamma \vdash \lambda a.s : A \supset B}\ \text{Abs} \qquad \frac{\Delta; \Gamma \vdash s : A \supset B \quad \Delta; \Gamma \vdash t : A}{\Delta; \Gamma \vdash s\, t : B}\ \text{App}$$

$$\frac{u : A \in \Delta}{\Delta; \Gamma \vdash u : A}\ \text{RVar} \qquad \frac{\Delta; \cdot \vdash r, s : A \quad \Delta; \cdot \vdash \rho : r \triangleright s : A}{\Delta; \Gamma \vdash !(\rho, r, s) : \llbracket \rho, r, s \rrbracket A}\ \text{Bang} \qquad \frac{\Delta; \Gamma \vdash s : \llbracket \rho, p, q \rrbracket A \quad \Delta, u : A; \Gamma \vdash t : C}{\Delta; \Gamma \vdash \mathit{let}\ u \doteq s\ \mathit{in}\ t : C\{u/^{\mathsf{src}} \rho^{q}_{p}\}}\ \text{Let}$$

$$\frac{a : A \in \Gamma}{\Delta; \Gamma \vdash \underline{a} : a \triangleright a : A}\ \text{R-Refl-TVar} \qquad \frac{u : A \in \Delta}{\Delta; \Gamma \vdash \underline{u} : u \triangleright u : A}\ \text{R-Refl-RVar}$$

$$\frac{\Delta; \cdot \vdash s, r, t : A \quad \Delta; \cdot \vdash \rho : s \triangleright r : A \quad \Delta; \cdot \vdash \sigma : r \triangleright t : A}{\Delta; \Gamma \vdash \langle \rho|_{s} \sigma \rangle : !(\rho, s, r) \triangleright !(\rho; \sigma, s, t) : \llbracket \rho, s, r \rrbracket A}\ \text{R-Bang} \qquad \frac{\Delta; \Gamma \vdash \rho : r \triangleright s : A \quad \Delta; \Gamma \vdash \sigma : s \triangleright t : A}{\Delta; \Gamma \vdash \rho; \sigma : r \triangleright t : A}\ \text{R-Trans}$$

$$\frac{\Delta; \Gamma, a : A \vdash s : B \quad \Delta; \Gamma \vdash t : A}{\Delta; \Gamma \vdash \mathbf{ba}(a.s, t) : (\lambda a.s)\, t \triangleright s\{a/t\} : B}\ \text{R-}\beta$$

$$\frac{\Delta; \cdot \vdash \rho : s \triangleright t : A \quad \Delta, u : A; \Gamma \vdash r : C}{\Delta; \Gamma \vdash \mathbf{bb}(!(\rho, s, t), u.r) : \mathit{let}\ u \doteq !(\rho, s, t)\ \mathit{in}\ r \triangleright r\{u/^{\mathsf{tgt}} \rho^{t}_{s}\} : C\{u/^{\mathsf{src}} \rho^{t}_{s}\}}\ \text{R-}\beta_{\Box}$$

$$\frac{\Delta; \Gamma, a : A \vdash \rho : s \triangleright t : B}{\Delta; \Gamma \vdash \lambda a.\rho : \lambda a.s \triangleright \lambda a.t : A \supset B}\ \text{R-Abs} \qquad \frac{\Delta; \Gamma \vdash \rho : s_1 \triangleright t_1 : A \supset B \quad \Delta; \Gamma \vdash \sigma : s_2 \triangleright t_2 : A}{\Delta; \Gamma \vdash \rho\, \sigma : s_1\, s_2 \triangleright t_1\, t_2 : B}\ \text{R-App}$$

$$\frac{\Delta; \Gamma \vdash \rho : s_1 \triangleright t_1 : \llbracket \tau, p, q \rrbracket A \quad \Delta, u : A; \Gamma \vdash \sigma : s_2 \triangleright t_2 : C}{\Delta; \Gamma \vdash \mathit{let}\ u \doteq \rho\ \mathit{in}\ \sigma : \mathit{let}\ u \doteq s_1\ \mathit{in}\ s_2 \triangleright \mathit{let}\ u \doteq t_1\ \mathit{in}\ t_2 : C\{u/^{\mathsf{src}} \tau^{q}_{p}\}}\ \text{R-Let} \qquad \frac{\Delta; \Gamma \vdash S : A \quad \Delta \vdash A \preceq B}{\Delta; \Gamma \vdash S : B}\ \text{Subs}$$

$$\frac{\Delta; \Gamma \vdash s : A \quad s \simeq t \quad \Delta \vdash A \simeq B}{\Delta; \Gamma \vdash t : B}\ \text{SEq-T} \qquad \frac{\Delta; \Gamma \vdash \rho : s \triangleright t : A \quad \rho \simeq \sigma : s \triangleright t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \simeq B}{\Delta; \Gamma \vdash \sigma : p \triangleright q : B}\ \text{SEq-R}$$

$$\frac{}{\Delta \vdash P \preceq P}\ \text{S-PVar} \qquad \frac{\Delta \vdash A' \preceq A \quad \Delta \vdash B \preceq B'}{\Delta \vdash A \supset B \preceq A' \supset B'}\ \text{S-Arrow} \qquad \frac{\Delta; \cdot \vdash p, q, r : A \quad \Delta; \cdot \vdash \rho : p \triangleright q : A \quad \Delta; \cdot \vdash \sigma : q \triangleright r : A \quad \Delta \vdash A \preceq B}{\Delta \vdash \llbracket \rho; \sigma, p, r \rrbracket A \preceq \llbracket \rho, p, q \rrbracket B}\ \text{S-Box}$$

$$\frac{}{\Delta \vdash P \simeq P}\ \text{Eq-PVar} \qquad \frac{\Delta \vdash A \simeq A' \quad \Delta \vdash B \simeq B'}{\Delta \vdash A \supset B \simeq A' \supset B'}\ \text{Eq-Arrow}$$

$$\frac{\Delta; \cdot \vdash s, t, p, q : A \quad \Delta; \cdot \vdash \rho, \sigma : s \triangleright t : A \quad \rho \simeq \sigma : s \triangleright t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \simeq B}{\Delta \vdash \llbracket \rho, s, t \rrbracket A \simeq \llbracket \sigma, p, q \rrbracket B}\ \text{Eq-Bang}$$

Figure 3: Typing Rules

- $\Box A \supset A$

Remark 1. *If we drop all annotations in the modality in theorems of RC, then we obtain theorems of (minimal) S4. This follows from observing that applying this forgetful function on the typing rules, yields the system for S4 presented in [11]. Similarly, if we drop ρ and t in $\llbracket \rho, s, t \rrbracket A$ but leave the term s denoting the source of ρ, we can prove all theorems of LP. This stems from observing that by performing this transformation on the typing rules, yields the Hypothetical Logic of Proofs [8].*

Substitution Principles of RC. This section presents some basic meta-theoretic results on RC. The first states that the type rules preserve well-formedness. This may be proved by induction on the derivation and relies on Lem. 1.9.

Lemma 2.2. *The typing rules preserve well-formedness of typing judgements.*

Typable terms can be recast as typable unit rewrites.

Lemma 2.3 (Term as Unit Rewrite). ▶ $\Delta; \Gamma \vdash s : A$ *implies* ▶ $\Delta; \Gamma \vdash \underline{s} : s \triangleright s : A$.

The proof is by induction on the derivation π of $\Delta; \Gamma \vdash s : A$. We consider here one of the interesting cases, namely when $\Delta; \Gamma \vdash s : A$ is $\Delta; \Gamma \vdash !(\rho, s, t) : \llbracket \rho, s, t \rrbracket B$ and π ends in:

$$\dfrac{\dfrac{\cdot;\Gamma \vdash a : [\![\rho, p, q]\!]A \qquad \dfrac{\dfrac{\Delta;\cdot \vdash \underline{u} : u \rhd u : A}{\Delta;\cdot \vdash !(\underline{u}, u, u) : [\![\underline{u}, u, u]\!]A} \qquad \dfrac{\dfrac{}{u : A;\cdot \vdash u : A} \quad \dfrac{}{\Delta;\cdot \vdash \mathfrak{u} : u \rhd u : A} \quad \dfrac{}{\Delta;\cdot \vdash \mathfrak{u} : u \rhd u : A}}{\Delta;\cdot \vdash \langle \mathfrak{u}|_u \mathfrak{u}\rangle : !(\mathfrak{u}, u, u) \rhd !(\mathfrak{u}, u, u) : [\![\mathfrak{u}, u, u]\!]A}\,\text{R-Bang}}{\Delta;\cdot \vdash !(\langle \mathfrak{u}|_u \mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)) : [\![\langle \mathfrak{u}|_u \mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)]\!][\![\mathfrak{u}, u, u]\!]A}\,\text{Bang}}{\cdot;\Gamma \vdash let\ u \doteq a\ in\, !(\langle \mathfrak{u}|_u \mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)) : ([\![\langle \mathfrak{u}|_u \mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)]\!][\![\mathfrak{u}, u, u]\!]A)\{u/^{\mathsf{src}}\rho_p^q\}}\,\text{Let}}{\cdot;\Gamma \vdash let\ u \doteq a\ in\, !(\langle \mathfrak{u}|_u \mathfrak{u}\rangle, !(\mathfrak{u}, u, u), !(\mathfrak{u}, u, u)) : [\![\langle \rho|_p \mathfrak{q}\rangle, !(\rho, p, q), !(\rho, p, q)]\!][\![\rho, p, q]\!]A}\,\text{SEq-T}$$

Figure 4: Sample Type Derivation

$$\dfrac{\Delta;\cdot \vdash s, t : B \qquad \Delta;\cdot \vdash \rho : s \rhd t : B}{\Delta;\Gamma \vdash !(\rho, s, t) : [\![\rho, s, t]\!]B}\,\text{Bang}$$

Given ▶ $\Delta;\cdot \vdash t : B$, we may apply the i.h.[3] to obtain $\Delta;\cdot \vdash \mathfrak{t} : t \rhd t : B$. Then we deduce:

$$\dfrac{\Delta;\cdot \vdash s, t : B \qquad \Delta;\cdot \vdash \rho : s \rhd t : B \qquad \Delta;\cdot \vdash \mathfrak{t} : t \rhd t : B}{\Delta;\Gamma \vdash \langle \rho|_s \mathfrak{t}\rangle : !(\rho, s, t) \rhd !(\rho; \mathfrak{t}, s, t) : [\![\rho; \mathfrak{t}, s, t]\!]B}\,\text{R-Bang}$$

We conclude that the judgement

$$\Delta;\Gamma \vdash \langle \rho|_s \mathfrak{t}\rangle : !(\rho, s, t) \rhd !(\rho, s, t) : [\![\rho, s, t]\!]B$$

is derivable from SEq-R.

This section ends with two substitution lemmas. The first is straightforward to prove (it uses Lem. 1.6). The second (Lem. 2.5) however, is subtle and has guided the notion of substitution on rewrites that we presented in Sec. 1.

Lemma 2.4 (Term Substitution). *Suppose $\Delta;\Gamma, a : A \vdash s : B$ and $\Delta;\Gamma \vdash t : A$. Then $\Delta;\Gamma \vdash s\{a/t\} : B$.*

The second substitution lemma (Lem. 2.5) starts by assuming that ▶ $\Delta;\cdot \vdash \rho : s \rhd t : A$, ▶ $\Delta;\cdot \vdash s : A$ and ▶ $\Delta;\cdot \vdash t : A$. Note that typability of s and t from typability of ρ (upcoming Lem. 2.6) is proved with the help of Lem. 2.5 itself, so we have to assume typability of all three objects at this point.

Lemma 2.5 (Rewrite Substitution). *Suppose ▶ $\Delta;\cdot \vdash \rho : s \rhd t : A$, ▶ $\Delta;\cdot \vdash s : A$ and ▶ $\Delta;\cdot \vdash t : A$. Suppose ▶ $\Delta, u : A;\Gamma \vdash S : B$ and ▶ $\Delta, u : A \vdash C \preceq D$ and ▶ $\Delta, u : A \vdash C \simeq D$.*

(a) $S = \sigma : p \rhd q$ implies

$$▶\ \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$

(b) $S = \sigma : p \rhd q$ implies

$$▶\ \Delta, u : A;\Gamma \vdash p : B \text{ and } ▶\ \Delta, u : A;\Gamma \vdash q : B.$$

(c) $S = p$ implies

$$▶\ \Delta;\Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$

(d) $S = p$ implies

$$▶\ \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$

(e) $S = p$ implies

$$▶\ \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$

(f) ▶ $\Delta \vdash C\{u/^{\mathsf{src}}\rho_s^t\} \preceq D\{u/^{\mathsf{src}}\rho_s^t\}$.
(g) ▶ $\Delta \vdash C\{u/^{\mathsf{src}}\rho_s^t\} \simeq D\{u/^{\mathsf{src}}\rho_s^t\}$.

[3]This shows why we have included the judgement $\Delta;\cdot \vdash s, t : B$ in the hypothesis of Bang: it allows for structural induction on the derivation of a term.

The proof is by simultaneous induction on ▶ $\Delta, u : A;\Gamma \vdash S : B$ and ▶ $\Delta, u : A \vdash C \preceq D$ and ▶ $\Delta, u : A \vdash C \simeq D$.

We conclude this section with a result that states that the source and target of a typable rewrite are typable. The proof is by induction on the derivation of $\Delta;\Gamma \vdash \rho : s \rhd t : A$ and relies on subsumption, the Term Substitution Lemma (Lem. 2.4(c)), Term as a Unit Rewrite Lemma (Lem. 2.3), and the Rewrite Substitution Lemma (Lem. 2.5).

Lemma 2.6. *▶ $\Delta;\Gamma \vdash \rho : s \rhd t : A$ implies ▶ $\Delta;\Gamma \vdash s : A$ and also ▶ $\Delta;\Gamma \vdash t : A$.*

One of the key cases in the proof is when the derivation of the typing judgement $\Delta;\Gamma \vdash \rho : s \rhd t : A$ ends in an instance of the rule R-$\beta_\Box$:

$$\dfrac{\Delta;\cdot \vdash \rho_1 : p \rhd q : A \qquad \Delta, u : A;\Gamma \vdash r : C}{\Delta;\Gamma \vdash \mathbf{bb}(!(\rho_1, p, q), u.r) : let\ u \doteq !(\rho_1, p, q)\ in\ r \rhd r\{u/^{\mathsf{tgt}}\rho_{1p}^{\ q}\} : C\{u/^{\mathsf{src}}\rho_{1p}^{\ q}\}}$$

By the i.h. on $\Delta;\cdot \vdash \rho_1 : p \rhd q : A$, we deduce $\Delta;\cdot \vdash p : A$ and $\Delta;\cdot \vdash q : A$. This allows us to use the Rewrite Substitution Lemma (Lem. 2.5) for typing $r\{u/^{\mathsf{src}}\rho_{1p}^{\ q}\}$. For typing $let\ u \doteq !(\rho_1, p, q)\ in\ r$ we use Bang, then Let.

3 REWRITE EXTENSION

In the Rewrite Calculus, rather than reduction on terms we have *extension of rewrites*. Extension is similar to reduction in the lambda calculus but it is applied to terms and rewrites *modulo* structural equivalence and, also, it leaves a trail. In the case of rewrites, σ extends a rewrite ρ if σ results from appending a rewrite step to ρ, modulo structural equivalence. For example, given the rewrite $I\mathfrak{a} : Ia \rhd Ia$ one has the following extension sequence of rewrites to normal form:

$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)) : I(Ia) \rhd Ia \\ \rightarrowtail & I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array} \tag{2}$$

The rewrite $I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a)$ is in normal form since it cannot be extended further. We now define this notion formally. For that we introduce two extension judgements whose meaning is defined mutually recursively:

$$\begin{array}{rl} r \rightarrowtail s & \textbf{Term extension} \\ \rho : r \rhd s \rightarrowtail \sigma : p \rhd q & \textbf{Rewrite extension} \end{array}$$

Term r extends to s, written $r \rightarrowtail s$, iff:

$$\exists r', s' \text{ s.t. } r \simeq r' \mapsto s' \simeq s$$

Rewrite ρ extends to σ, written $\rho : r \rhd s \rightarrowtail \sigma : p \rhd q$, iff:

$$\exists \rho', \sigma' \text{ s.t. } \rho \simeq \rho' : r \rhd s \text{ and } \rho' : r \rhd s \mapsto \sigma' : r \rhd q \text{ and } \sigma' \simeq \sigma : p \rhd q$$

The judgements $r \mapsto s$ and $\rho : r \rhd s \mapsto \sigma : p \rhd q$ are defined by the rules of Fig. 5. The rules above the horizontal line apply to terms and the rules below to rewrites.

These rules are mostly self-explanatory. For example, E-β, states that if the "current" rewrite is of the form $\rho : s \rhd (\lambda a.t_1)\, t_2$, then it can be extended by adding a witness to a β-rewrite step that is sourced at its target, namely $\rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}$. Perhaps worth mentioning is that in the congruence rule for $\langle \rho|_r \sigma\rangle$, it is σ that may be extended, but not ρ.

We will only be interested in extension on well-formed terms and well-formed rewrites. Term and rewrite extension preserves well-formedness:

Lemma 3.1 (Extension preserves well-formedness). *$s \rightarrowtail t$ and s well-formed implies t well-formed. Similarly, $\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ and (ρ, s, t) well-formed implies (ρ', s, t') well-formed.*

Rewrite extension is certainly not confluent. For example, the rewrite $IIa : IIa \rhd IIa$ from above, in addition to be extended as depicted in (2), can also be extended as follows:

$$\begin{array}{ll} & I(I\underline{a}) : I(Ia) \rhd I(Ia) \\ \rightarrowtail & \mathbf{ba}(b.b, Ia) : I(Ia) \rhd Ia \\ \rightarrowtail & \mathbf{ba}(b.b, Ia); \mathbf{ba}(b.b, a) : I(Ia) \rhd a \end{array}$$

Clearly $I(\mathbf{ba}(b.b, a)); \mathbf{ba}(b.b, a) \not\simeq \mathbf{ba}(b.b, Ia); \mathbf{ba}(b.b, a)$. This is expected since structural equivalence does not include permutation of redexes as in Lévy permutation equivalence. Extension of rewrites does preserve types though. This section is dedicated to showing this result. First we need to set up some auxiliary notions and results. We begin with the definition of a **step rewrite**, a rewrite that corresponds to one reduction step. In other words, a rewrite that models the contraction of exactly on redex.

Definition 3.2 (Step Rewrite). Step rewrites are defined by the following grammar:

$$\begin{array}{lll} \xi & ::= & \mathbf{ba}(a.s, r) \mid \mathbf{bb}(s, u.r) \mid \lambda a.\xi \mid \xi\, \mathfrak{s} \mid \mathfrak{s}\, \xi \\ & & \mid \mathit{let}\ u \circeq \xi\ \mathit{in}\, \mathfrak{s} \mid \mathit{let}\ u \circeq \mathfrak{s}\ \mathit{in}\, \xi \mid \langle \rho|_{\mathfrak{s}} \xi\rangle \end{array}$$

The next result formalizes what is intuitively clear from the definition of extension, namely that extending a rewrite consists in suffixing a step:

Lemma 3.3 (Extension adds a step). *$\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ implies there exists ξ s.t. $\rho' \simeq \rho; \xi$. Moreover, (ρ, s, t) well-formed implies (ξ, t, t') well-formed.*

In our upcoming proof of Extension Reduction (Prop. 3.7) we need to extract the suffixed step from the extension of a rewrite, and analyze its form. These steps will be broken down into a step context and redex.

Definition 3.4 (Step Contexts). Step contexts are defined by the following grammar:

$$\mathsf{C} \quad ::= \quad \Box \mid \mathsf{C}\, \mathfrak{s} \mid \mathfrak{s}\, \mathsf{C} \mid \lambda a.\mathsf{C} \mid \mathit{let}\ u \circeq \mathsf{C}\ \mathit{in}\, \mathfrak{s} \mid \mathit{let}\ u \circeq \mathfrak{s}\ \mathit{in}\, \mathsf{C} \mid \langle \rho|_{\mathfrak{s}} \mathsf{C}\rangle$$

There are three notions of filling the hole of a step context. Simple replacement of a rewrite ρ for the hole is written $\mathsf{C}\langle\rho\rangle$. Such a replacement produces a rewrite. Then we have *source filling* and *target filling*. The former is denoted $\mathsf{C}[p]^{src}$ and the latter $\mathsf{C}[\rho, p, q]^{tgt}$.

These notions of filling produce terms. They are used in conjunction to denote the source and target of the rewrite $\mathsf{C}\langle\rho\rangle$ (*cf.* Lem. 3.5). Both are defined below:

$$\begin{array}{rcl} \Box[t]^{src} & ::= & t \\ (\mathsf{C}\, \mathfrak{s})[t]^{src} & ::= & \mathsf{C}[t]^{src}\, s \\ (\mathfrak{s}\, \mathsf{C})[t]^{src} & ::= & s\, \mathsf{C}[t]^{src} \\ (\lambda a.\mathsf{C})[t]^{src} & ::= & \lambda a.\mathsf{C}[t]^{src} \\ (\mathit{let}\ u \circeq \mathsf{C}\ \mathit{in}\, \mathfrak{s})[t]^{src} & ::= & \mathit{let}\ u \circeq \mathsf{C}[t]^{src}\ \mathit{in}\, s \\ (\mathit{let}\ u \circeq \mathfrak{s}\ \mathit{in}\, \mathsf{C})[t]^{src} & ::= & \mathit{let}\ u \circeq s\ \mathit{in}\, \mathsf{C}[t]^{src} \\ \langle \rho|_{\mathfrak{s}} \mathsf{C}\rangle[t]^{src} & ::= & !(\rho, s, \mathsf{C}[t]^{src}) \end{array}$$

$$\begin{array}{rcl} \Box[\rho, p, q]^{tgt} & ::= & q \\ (\mathsf{C}\, \mathfrak{s})[\rho, p, q]^{tgt} & ::= & \mathsf{C}[\rho, p, q]^{tgt}\, s \\ (\mathfrak{s}\, \mathsf{C})[\rho, p, q]^{tgt} & ::= & s\, \mathsf{C}[\rho, p, q]^{tgt} \\ (\lambda a.\mathsf{C})[\rho, p, q]^{tgt} & ::= & \lambda a.\mathsf{C}[\rho, p, q]^{tgt} \\ (\mathit{let}\ u \circeq \mathsf{C}\ \mathit{in}\, \mathfrak{s})[\rho, p, q]^{tgt} & ::= & \mathit{let}\ u \circeq \mathsf{C}[\rho, p, q]^{tgt}\ \mathit{in}\, s \\ (\mathit{let}\ u \circeq \mathfrak{s}\ \mathit{in}\, \mathsf{C})[\rho, p, q]^{tgt} & ::= & \mathit{let}\ u \circeq s\ \mathit{in}\, \mathsf{C}[\rho, p, q]^{tgt} \\ \langle \sigma|_{\mathfrak{s}} \mathsf{C}\rangle[\rho, p, q]^{tgt} & ::= & !(\sigma; \mathsf{C}\langle\rho\rangle, s, \mathsf{C}[\rho, p, q]^{tgt}) \end{array}$$

The interesting clause in the filling operations above is when the step context is $\langle \sigma|_{\mathfrak{s}} \mathsf{C}\rangle$. In particular, in the case of target filling, note how, in addition to actually inserting the target term q (as may be seen from the case for $\Box$), it suffixes a copy of the argument step itself: $\sigma; \mathsf{C}\langle\rho\rangle$.

Lemma 3.5 (Form of a Step). *Let ξ be a well-formed step rewrite. Then one of the two following hold.*

(a) $\xi = \mathsf{C}\langle \mathbf{ba}(a.s, r)\rangle$ *and*

$$\mathsf{C}\langle \mathbf{ba}(a.s, r)\rangle : \mathsf{C}[(\lambda a.s)\, t]^{src} \rhd \mathsf{C}[\mathbf{ba}(a.s, r), \lambda a.st, s\{a/t\}]^{tgt}$$

(b) $\xi = \mathsf{C}\langle \mathbf{bb}(!(\rho, p, q), u.r)\rangle$ *and*

$$\mathsf{C}\langle \mathbf{bb}(!(\rho, p, q), u.r)\rangle : \mathsf{C}[\mathit{let}\ u \circeq !(\rho, p, q)\ \mathit{in}\, r]^{src} \rhd \mathsf{C}[\mathbf{bb}(!(\rho, p, q), u.r), \mathit{let}\ u \circeq !(\rho, p, q)\ \mathit{in}\, r, r\{u/^{\mathrm{tgt}}\rho^q_p\}]^{tgt}$$

The proof is by induction on ξ. The only interesting case is when $\xi = \langle \sigma|_m \xi'\rangle$. Since ξ is well-formed we know $\sigma : m \rhd n$ and $\xi' : n \rhd o$ for some m, n, o. By the i.h. on ξ' either case (a) or (b) holds. Assume it is (a) (the case for (b) is similar and hence omitted) then $\xi' = \mathsf{C}'\langle \mathbf{ba}(a.s, r)\rangle$, for some C', a, s, r. By Lem. 1.2, $n \simeq \mathsf{C}'[(\lambda a.s)\, t]^{src}$. Then $\sigma : m \rhd \mathsf{C}'[(\lambda a.s)\, t]^{src}$. But then

$$\langle \sigma|_m \mathsf{C}'\langle \mathbf{ba}(a.s, r)\rangle\rangle : p \rhd q$$

where

$$\begin{array}{l} p := !(\sigma, m, \mathsf{C}[(\lambda a.s)\, t]^{src}) \\ q := !(\sigma; \mathsf{C}\langle \mathbf{ba}(a.s, r)\rangle, m, \mathsf{C}[\mathbf{bb}(!(\rho, p, q), u.r), \mathit{let}\ u \circeq !(\rho, p, q)\ \mathit{in}\, r, r\{u/^{\mathrm{tgt}}\rho^q_p\}]^{tgt}) \end{array}$$

Which concludes the case.

Finally, for our Subject Extension result, it will not suffice to break down a step rewrite into its components, as described above, but also to ensure typability. Typability of the source of a step suffices to type the step itself.

Lemma 3.6 (Step Typability). *$\xi : s \rhd t$ and $\Delta; \Gamma \vdash s : A$ implies $\Delta; \Gamma \vdash \xi : s \rhd t : A$.*

We are now in condition to prove the main result of this section, namely that extension preserves types for both terms and rewrites.

Proposition 3.7 (Subject Extension). *(a) $\blacktriangleright_\pi \Delta; \Gamma \vdash s : A$ and $s \rightarrowtail s'$ implies $\blacktriangleright \Delta; \Gamma \vdash s' : A$.*

$$\frac{s \mapsto s'}{\lambda a.s \mapsto \lambda a.s'}\ \text{E-AbsT} \qquad \frac{s \mapsto s'}{s\,t \mapsto s'\,t}\ \text{E-AppTL} \qquad \frac{t \mapsto t'}{s\,t \mapsto s\,t'}\ \text{E-AppTR} \qquad \frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{!(\rho, s, t) \mapsto !(\rho', s, t')}\ \text{E-BangT}$$

$$\frac{s \mapsto s'}{let\ u \doteq s\ in\ t \mapsto let\ u \doteq s'\ in\ t}\ \text{E-LetTL} \qquad \frac{t \mapsto t'}{let\ u \doteq s\ in\ t \mapsto let\ u \doteq s\ in\ t'}\ \text{E-LetTR}$$

$$\frac{}{\rho : s \rhd (\lambda a.t_1)\,t_2 \mapsto \rho; \mathbf{ba}(a.t_1, t_2) : s \rhd t_1\{a/t_2\}}\ \text{E-}\beta$$

$$\frac{}{\rho : s \rhd let\ u \doteq !(\sigma, p, q)\ in\ t \mapsto \rho; \mathbf{bb}(!(\sigma, p, q), u.t) : s \rhd t\{u/^{\mathsf{tgt}}\sigma_p^q\}}\ \text{E-}\beta_\Box$$

$$\frac{\rho : s \rhd t \mapsto \rho : s \rhd t'}{\lambda a.\rho : \lambda a.s \rhd \lambda a.t \mapsto \lambda a.\rho' : \lambda a.s \rhd \lambda a.t'}\ \text{E-AbsR}$$

$$\frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\langle \rho|_r\sigma\rangle : !(\rho, r, s) \rhd !(\rho; \sigma, r, t) \mapsto \langle \rho|_r\sigma'\rangle : !(\rho, r, s) \rhd !(\rho; \sigma', r, t')}\ \text{E-BangR} \qquad \frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\rho; \sigma : r \rhd t \mapsto \rho; \sigma' : r \rhd t'}\ \text{E-Trans}$$

$$\frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{\rho\,\sigma : s\,p \rhd t\,q \mapsto \rho'\,\sigma : s\,p \rhd t'\,q}\ \text{E-AppRL} \qquad \frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{\rho\,\sigma : p\,s \rhd q\,t \mapsto \rho\,\sigma' : p\,s \rhd q\,t'}\ \text{E-AppRR}$$

$$\frac{\rho : s \rhd t \mapsto \rho' : s \rhd t'}{let\ u \doteq \rho\ in\ \sigma : let\ u \doteq s\ in\ p \rhd let\ u \doteq t\ in\ q \mapsto let\ u \doteq \rho'\ in\ \sigma : let\ u \doteq s\ in\ p \rhd let\ u \doteq t'\ in\ q}\ \text{E-LetRL}$$

$$\frac{\sigma : s \rhd t \mapsto \sigma' : s \rhd t'}{let\ u \doteq \rho\ in\ \sigma : let\ u \doteq p\ in\ s \rhd let\ u \doteq q\ in\ t \mapsto let\ u \doteq \rho\ in\ \sigma' : let\ u \doteq p\ in\ s \rhd let\ u \doteq q\ in\ t'}\ \text{E-LetRR}$$

Figure 5: Rewrite Extension

(b) $\blacktriangleright_\pi\ \Delta; \Gamma \vdash \rho : s \rhd t : A$ *and* $\rho : s \rhd t \rightarrowtail \rho' : s \rhd t'$ *implies* $\blacktriangleright\ \Delta; \Gamma \vdash \rho' : s \rhd t' : A$.

We first prove (by induction on π) that

(a) $\blacktriangleright_\pi\ \Delta; \Gamma \vdash s : A$ and $s \mapsto s'$ implies $\blacktriangleright\ \Delta; \Gamma \vdash s' : A$.
(b) $\blacktriangleright_\pi\ \Delta; \Gamma \vdash \rho : s \rhd t : A$ and $\rho : s \rhd t \mapsto \rho' : s \rhd t'$ implies $\blacktriangleright\ \Delta; \Gamma \vdash \rho' : s \rhd t' : A$.

Then we conclude from the fact that $\blacktriangleright\ \Delta; \Gamma \vdash s : A$ and $s \simeq s'$ implies $\blacktriangleright\ \Delta; \Gamma \vdash s' : A$. Similarly, $\blacktriangleright\ \Delta; \Gamma \vdash \rho : s \rhd t : A$ and $\rho \simeq \rho' : s \rhd t$ implies $\blacktriangleright\ \Delta; \Gamma \vdash \rho' : s \rhd t : A$. We focus on three interesting cases:

- The derivation ends in:

$$\frac{\Delta; \cdot \vdash r, s : A \quad \Delta; \cdot \vdash \rho_1 : r \rhd s : A}{\Delta; \Gamma \vdash !(\rho_1, r, s) : [\![\rho_1, r, s]\!]A}\ \text{Bang}$$

Then $\rho_1 : r \rhd s \rightarrowtail \rho_1' : r \rhd s'$. By the i.h. we have

$$\Delta; \cdot \vdash \rho_1' : r \rhd s' : A \tag{3}$$

By Lem. 2.6 on (3) $\Delta; \cdot \vdash s' : A$. Thus we can use Bang to deduce $\Delta; \Gamma \vdash !(\rho_1', r, s') : [\![\rho_1', r, s']\!]A$. By Lemma 3.3, $\rho_1' \simeq \rho_1; \xi$ for some step ξ. Moreover, $\xi : s \rhd s'$. By the Typable Step Lemma (Lem. 3.6), $\Delta; \cdot \vdash \xi : s \rhd s' : A$. Then $\Delta; \Gamma \vdash !(\rho_1', r, s') : [\![\rho_1; \xi, r, s']\!]A$. Finally, by subsumption, $\Delta; \Gamma \vdash !(\rho_1', r, s') : [\![\rho_1, r, s]\!]A$.

- The derivation ends in:

$$\frac{\Delta; \cdot \vdash s, r, t : A \quad \Delta; \cdot \vdash \rho_1 : s \rhd r : A \quad \Delta; \cdot \vdash \rho_2 : r \rhd p : A}{\Delta; \Gamma \vdash \langle \rho_1|_s\rho_2\rangle : !(\rho_1, s, r) \rhd !(\rho_1; \rho_2, s, p) : [\![\rho_1, s, r]\!]A}\ \text{R-Bang}$$

Then $\rho' = \langle \rho_1|_s\rho_2'\rangle$ and $\langle \rho_1|_s\rho_2\rangle \mapsto \langle \rho_1|_s\rho_2'\rangle$ follows from $\rho_2 : r \rhd p \mapsto \rho_2' : r \rhd p'$. By the i.h. $\Delta; \cdot \vdash \rho_2' : r \rhd p' : A$. By Lem. 2.6 $\Delta; \cdot \vdash p' : A$. Using R-Bang we deduce

$$\Delta; \Gamma \vdash \langle \rho_1|_s\rho_2'\rangle : !(\rho_1, s, r) \rhd !(\rho_1; \rho_2', s, p') : [\![\rho_1, s, r]\!]A$$

- The derivation ends in:

$$\frac{\Delta; \Gamma, a : B \vdash p : A \quad \Delta; \Gamma \vdash q : B}{\Delta; \Gamma \vdash \mathbf{ba}(a.p, q) : (\lambda a^B.p)\,q \rhd p\{a/q\} : A}\ \text{R-}\beta$$

Suppose $\mathbf{ba}(a.p, q) : (\lambda a^B.p)\,q \rhd p\{a/q\} \mapsto \rho'$. Then by Lem. 3.3, $\rho' \simeq \mathbf{ba}(a.p, q); \xi$ for some step ξ. By Lem. 3.5, one of the two following hold.

(a) $\xi = \mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle$ and
$$\mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle : \mathsf{C}[(\lambda b.m)\,n]^{src} \rhd \mathsf{C}[\mathbf{ba}(b.m, n), \lambda b.m\,n, m\{b/n\}]^{tgt}$$
(b) or $\xi = \mathsf{C}\langle \mathbf{bb}(m, u.n)\rangle$ and
$$\mathsf{C}\langle \mathbf{bb}(m, u.n)\rangle : \mathsf{C}[let\ u^C \doteq !(\sigma, p', q')\ in\ r]^{src} \rhd \mathsf{C}[\mathbf{bb}(m, u.n), let\ u^C \doteq !(\sigma, p', q')\ in\ r, r\{u/^{\mathsf{src}}\sigma_{p'}^{q'}\}]^{tgt}$$

By Lem. 2.6 $p\{a/q\}$ is typable. That is, $\Delta; \Gamma \vdash p\{a/q\} : A$. By Lem. 3.6, in the first case above we obtain

$$\Delta; \Gamma \vdash \mathsf{C}\langle \mathbf{ba}(b.m, n)\rangle : r_1 \rhd r_2 : A$$

where

$$
\begin{array}{lll}
r_1 & := & \mathrm{C}[(\lambda b.m)\, n]^{src} \\
r_2 & := & \mathrm{C}[\mathbf{ba}(a.s, r), \lambda b.m\, n, m\{b/n\}]^{tgt}
\end{array}
$$

Similarly, we obtain

$$\Delta; \Gamma \vdash \mathrm{C}\langle \mathbf{bb}(m, u.n)\rangle : r_1 \rhd r_2 : A$$

in the second case above, where

$$
\begin{array}{lll}
r_1 & := & \mathrm{C}[let\ u^C \doteq !(\rho', p', q')\ in\, r]^{src} \\
r_2 & := & \mathrm{C}[\mathbf{bb}(!(\rho', p', q'), u.r), let\ u^C \doteq !(\rho', p', q')\ in\, r, r\{u/^{\mathrm{src}}\rho'^{q'}_{p'}\}]^{tgt}
\end{array}
$$

4 RELATED WORK AND CONCLUSION

Related Work. Propositions-as-types for modal logic has an extensive body of literature which would be impossible to summarize here. We focus on Justification Logic and the Logic of Proofs. Artemov introduced LP in [1, 2]. It was presented as the missing link between the provability interpretation of classical S4 and provability in PA. The more general setting of Justification Logic was presented in [4]. A recent survey is [12] and recent texts [3, 14]. For Natural Deduction and Sequent Calculus presentations consult [2, 5, 9]. Computational interpretation of proofs in JL is studied in [5–7, 15, 16]. The first-order logic of proofs is studied in [17].
Also related to this work is the literature on rewrites. Rewrites are called proof terms in [18]. They are used as a tool to prove various properties of first-order term rewriting systems (such as that various notions of equivalence of reductions coincide). An extension to higher-order term rewriting was given by Bruggin in [10]. Note, however, that he is forced to deal with the composition operator ";" in an ad-hoc manner, the reason being that his HOAS approach to proof terms has no obvious means of coping with the problem discussed above on properly substituting in $\underline{u}; \underline{u}$. A theory of proof terms for the typed lambda calculus was developed by Hilken [13]; however proof terms themselves are not reified as terms.

Conclusions. We present a novel propositions-as-types interpretation of the Logic of Proofs in which reductions between terms are reified as terms. We dub the system the Rewrite Calculus or RC. The resulting set of objects consists of terms and rewrites, both of which are mutually dependent. The salient term is $!(\rho, s, t)$ denoting a reduction from source term s to target term t. We assign it a modal type. The expression ρ is a rewrite. An example of a rewrite is $\lambda a.\sigma$ that denotes reduction taking place under an abstraction. Reduction under a "!" is understood as extending the rewrite ρ with further work σ, leading to the rewrite $\langle \rho|_\sigma s\rangle$. We devise a notion of structural equivalence for our rewrites that includes composition of rewrites such as $\langle \rho|_\sigma s\rangle$. We then introduce a type system for RC and a notion of "reduction" on rewrites that we call extension. Extension is proved to preserve types. One avenue we intend to pursue is an analysis of Lévy permutation equivalence formulated in terms of rewrites and projection equivalence also formulated in terms of the rewrites presented here. We would like to prove equivalence of both these notions. Also of interest, once that is in place, is to prove a notion of algebraic confluence: any two reductions to normal form are Lévy permutation.

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A TERMS AND REWRITES

A.1 Terms and Rewrites

Definition A.1 (Free Variables). The set of free truth and validity variables of a preobject o are defined as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{ftv}(a) & := & \{a\} & \mathsf{frv}(a) & := & \emptyset \\
\mathsf{ftv}(u) & := & \emptyset & \mathsf{frv}(u) & := & \{u\} \\
\mathsf{ftv}(\lambda a.s) & := & \mathsf{ftv}(s) \setminus \{a\} & \mathsf{frv}(\lambda a.s) & := & \mathsf{frv}(s) \\
\mathsf{ftv}(s\,t) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(t) & \mathsf{frv}(s\,t) & := & \mathsf{frv}(s) \cup \mathsf{frv}(t) \\
\mathsf{ftv}(!(\rho,s,t)) & := & \emptyset & \mathsf{frv}(!(\rho,s,t)) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(s) \cup \mathsf{frv}(t) \\
\mathsf{ftv}(\mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(t) & \mathsf{frv}(\mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\ t) & := & \mathsf{frv}(s) \cup \mathsf{frv}(t) \setminus \{u\}
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{ftv}(\underline{a}) & := & \{a\} & \mathsf{frv}(\underline{a}) & := & \emptyset \\
\mathsf{ftv}(\underline{u}) & := & \emptyset & \mathsf{frv}(\underline{u}) & := & \{u\} \\
\mathsf{ftv}(\mathbf{ba}(a.s,r)) & := & \mathsf{ftv}(s) \setminus \{a\} \cup \mathsf{ftv}(r) & \mathsf{frv}(\mathbf{ba}(a.s,r)) & := & \mathsf{frv}(s) \cup \mathsf{frv}(r) \\
\mathsf{ftv}(\mathbf{bb}(s,u.r)) & := & \mathsf{ftv}(s) \cup \mathsf{ftv}(r) & \mathsf{frv}(\mathbf{bb}(s,u.r)) & := & \mathsf{frv}(s) \cup \mathsf{frv}(r) \setminus \{u\} \\
\mathsf{ftv}(\rho;\sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\rho;\sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(\lambda a.\rho) & := & \mathsf{ftv}(\rho) \setminus \{a\} & \mathsf{frv}(\lambda a.\rho) & := & \mathsf{frv}(\rho) \\
\mathsf{ftv}(\rho\,\sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\rho\,\sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(\langle \rho|_s \sigma\rangle) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(s) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\langle \rho|_s \sigma\rangle) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \\
\mathsf{ftv}(\mathit{let}\ u \mathrel{\hat{=}} \rho\ \mathit{in}\ \sigma) & := & \mathsf{ftv}(\rho) \cup \mathsf{ftv}(\sigma) & \mathsf{frv}(\mathit{let}\ u \mathrel{\hat{=}} \rho\ \mathit{in}\ \sigma) & := & \mathsf{frv}(\rho) \cup \mathsf{frv}(\sigma) \setminus \{u\}
\end{array}
$$

A.2 Structural Equivalence

Lemma A.2 (Generation for Source/Target). *If $\blacktriangleright_\pi \rho : s \rhd t$, then there exist s', t', π' s.t. $s \simeq s'$, $t' \simeq t$ and $\blacktriangleright_{\pi'} \rho : s' \rhd t'$ and, moreover, exactly one of the following cases holds. Rewrite $\rho : s' \rhd t'$ has the form:*

(a) $\underline{a} : a \rhd a$.
(b) $\underline{u} : u \rhd u$.
(c) $\mathbf{ba}(a.p,q) : (\lambda a.p)\,q \rhd p\{a/q\}$.
(d) $\mathbf{bb}(!(\rho,p,q),u.r) : \mathit{let}\ u \mathrel{\hat{=}} !(\rho,p,q)\ \mathit{in}\ r \rhd r\{u/{}^{\mathsf{tgt}}\rho^{q}_{p}\}$.
(e) $\lambda a.\rho : \lambda a.p \rhd \lambda a.q$ *and* $\rho : p \rhd q$.
(f) $\sigma\,\tau : p_1\,p_2 \rhd q_1\,q_2$ *and* $\sigma : p_1 \rhd q_1$ *and* $\tau : p_2 \rhd q_2$.
(g) $\mathit{let}\ u \mathrel{\hat{=}} \rho\ \mathit{in}\ \sigma : \mathit{let}\ u \mathrel{\hat{=}} p_1\ \mathit{in}\,p_2 \rhd \mathit{let}\ u \mathrel{\hat{=}} q_1\ \mathit{in}\,q_2$ *and and* $\sigma : p_1 \rhd q_1$ *and* $\tau : p_2 \rhd q_2$.
(h) $\sigma;\tau : s' \rhd t'$ *and there exists* r *s.t.* $\sigma : s' \rhd r$ *and* $\tau : r \rhd t'$.
(i) $\langle\sigma|_p\tau\rangle : !(\sigma,p,r) \rhd !(\sigma;\tau,p,q)$ *and* $\sigma : p \rhd r$ *and* $\tau : r \rhd q$.

Proof. By induction on π using the i.h. and transitivity of $\simeq$ in the ST-Eq case. □

Lemma. (Lem 1.3) $\blacktriangleright_\pi \mathfrak{s} : p \rhd q$ implies $p \simeq q \simeq s$.

Proof. By induction on s.

- $s = a$ and $\underline{a} : p \rhd q$. We conclude from Lem. A.2.
- $s = u$. Same as in the previous case.
- $s = \lambda a.s_1$ and $\mathfrak{s} : p \rhd q$. Note first that $\mathfrak{s} = \underline{\lambda a.s_1} = \lambda a.\underline{s_1}$. By Lem. A.2, there exist p', q', π' s.t. $p \simeq p'$, $q' \simeq q$ and $\blacktriangleright_{\pi'} \mathfrak{s} : p' \rhd q'$ and, moreover, $p' = \lambda a.s''$ and $q' = \lambda a.t''$ (for some s'', t'') and π' proves $\blacktriangleright \mathfrak{s}_1 : s'' \rhd t''$. By the i.h. $s_1 \simeq s'' \simeq t''$. Then $p \simeq p' = \lambda a.s'' \simeq \lambda a.t'' = q' \simeq q$.
- $s = s\,t$ and $s = \mathit{let}\ u \mathrel{\hat{=}} s\ \mathit{in}\,t$. Similar to the previous case.
- $s = !(\sigma,m,n)$. First note that $\underline{!(\sigma,m,n)} = \langle\sigma|_m\mathfrak{n}\rangle$. By Lem. A.2 there exist p', q', π' s.t. $p \simeq p'$, $q' \simeq q$ and $\blacktriangleright_{\pi'} \langle\sigma|_m\mathfrak{n}\rangle : p' \rhd q'$. Moreover, $p' = !(\sigma,m,n)$ and $q' = !(\sigma;\mathfrak{n},m,r)$ and π' ends in an instance of ST-BangR. Thus $\mathfrak{n} : n \rhd r$ implies $r = n$. Then $p \simeq p' = !(\sigma,m,n) \simeq !(\sigma;\mathfrak{n},m,n) = q' \simeq q$.

□

Lemma. [Lem. 1.4] $\rho \simeq \sigma : s \rhd t$ implies $\rho : s \rhd t$ and $\sigma : s \rhd t$.

Proof. By induction on the derivation of $\rho \simeq \sigma : s \rhd t$. For the cases EqR-IdR and EqR-IdL we use Lem. 1.3. For the case EqR-SEq we use ST-SEq. □

Lemma. [Lem. 1.5] $s \simeq t$ implies $\underline{s} \simeq \underline{t} : s \rhd s$.

Proof. By induction on the derivation of $s \simeq t$. □

Lemma. [Structural Equivalence is closed under substitution of term variables – Lem. 1.6] Suppose $s \simeq t$ and $p \simeq q$. Then $s\{a/p\} \simeq t\{a/q\}$.

Proof. By induction on the derivation of $s \simeq t$. □

Lemma. [Structural Equivalence is closed under substitution of rewrite variables – Lem. 1.7] Suppose $\tau \simeq \upsilon : p \rhd q$. Then

- $\rho \simeq \sigma : s \rhd t$ implies $\rho\{u/^{\mathsf{m}}\tau_p^q\} \simeq \sigma\{u/^{\mathsf{m}}\upsilon_p^q\} : s\{u/^{\mathsf{m}}\tau_p^q\} \rhd t\{u/^{\mathsf{m}}\tau_p^q\}$.
- $s \simeq t$ implies $s\{u/^{\mathsf{m}}\tau_p^q\} \simeq t\{u/^{\mathsf{m}}\upsilon_p^q\}$.
- $\mathfrak{s}\{u/\tau_p^q\} \simeq \mathfrak{s}\{u/\upsilon_p^q\} : s\{u/^{\mathsf{src}}\tau_p^q\} \rhd s\{u/^{\mathsf{tgt}}\tau_p^q\}$

Proof. By simultaneous induction on $\rho \simeq \sigma : s \rhd t$ and $s \simeq t$ for the first two items. We use induction on s for the third item. □

Lemma. [Structural Equivalence Preserves Well-Formedness – Lem. 1.9] If s is well-formed and $s \simeq t$, then t is well-formed. Similarly, if (ρ, s, t) is well-formed and $\rho \simeq \sigma : s \rhd t$, then (σ, s, t) is well-formed.

Proof. By induction on the derivation of $s \simeq t$ and $\rho \simeq \sigma : s \rhd t$. It uses Lem. 1.4. □

A.3 Substitution Commutation Results

All objects (*i.e.* terms and rewrites) are assumed well-formed.

Lemma A.3. *Suppose* $u \notin \mathsf{fv}(\mathsf{o})$ *and* $a \notin \mathsf{fv}(r)$.

(a) $\mathsf{o}\{u/^{\mathsf{m}}\rho_s^t\} \simeq \mathsf{o}$.
(b) $\mathsf{o} \in \mathbb{R}_1^-$ *implies* $\mathsf{o}\{u/\rho_s^t\} \simeq \mathsf{o}$.
(c) $r\{a/s\} = r$.

Proof. The third item is by induction on r. We focus on the other two. We prove them simultaneously by induction on the size of o. For the first six cases below, the second item holds trivially since $\mathsf{o} \notin \mathbb{R}_1$.

- $\mathsf{o} = a$. Then $LHS = a = RHS$.
- $\mathsf{o} = w$. If $w \neq v, u$, then $LHS = w = RHS$. Otherwise, if $w = v$, then $LHS = v = RHS$. The case where $w = u$ is not possible by hypothesis.
- $\mathsf{o} = \lambda a.p_1$.

$$\begin{array}{lll} & LHS & \\ = & (\lambda a.p_1)\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} & \\ \simeq & \lambda a.p_1 & \text{(i.h./1)} \\ = & RHS & \end{array}$$

- $\mathsf{o} = p_1\, p_2$. By the i.h..
- $\mathsf{o} = \mathit{let}\ v \,\hat{=}\, p_1\ \mathit{in}\, p_2$. By the i.h..
- $\mathsf{o} = !(\sigma, p, q)$. We reason as follows

$$\begin{array}{lll} & LHS & \\ = & !(\sigma, p, q)\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(\mathfrak{p}\{u/\rho_s^t\}; \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}, p\{u/^{\mathsf{src}}\rho_s^t\}, q\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !(\mathfrak{p}; \sigma, p, q) & \text{(i.h./2,i.h./1 3-times)} \\ \simeq & !(\sigma, p, q) & \\ = & RHS & \end{array}$$

  Note that the i.h./2 is applied to $\mathfrak{p}$ rather than p. Hence the reason why we perform induction over the size of o (both $\mathfrak{p}$ and p have the same size) and not the structure. Note also that in order to determine that $\mathfrak{p}; \sigma \simeq \sigma : p \rhd q$ we rely on well-formedness.
- $\mathsf{o} = \underline{a}$. For both items we have $LHS = \underline{a} = RHS$.
- $\mathsf{o} = \underline{w}$. Then $w \neq v$ by hypothesis and $LHS = \underline{v} = RHS$ for both items.
- $\mathsf{o} = \mathbf{ba}(a.p_1, p_2)$. The second item holds trivially since $\mathsf{o} \notin \mathbb{R}_1$. For the first item

$$\begin{array}{lll} & LHS & \\ = & (\mathbf{ba}(a.p_1, p_2))\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & \mathbf{ba}(a.p_1\{u/^{\mathsf{m}}\rho_s^t\}, p_2\{u/^{\mathsf{m}}\rho_s^t\}) & \\ \simeq & \mathbf{ba}(a.p_1, p_2) & \text{(i.h./1 twice)} \\ = & RHS & \end{array}$$

- $\mathsf{o} = \mathbf{bb}(p_1, w.p_2)$. Similar to the previous case.
- $\mathsf{o} = \langle \sigma|_p \tau \rangle$. For item 1 we reason as follows.

$$\begin{array}{rll} & LHS & \\ = & \langle\sigma|_p\tau\rangle\{u/^{\mathsf{m}}\rho^t_s\} & \\ = & \langle\mathfrak{p}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{q\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\ \simeq & \langle\mathfrak{p};\sigma|_q\tau\rangle & \text{(i.h./2,i.h./1 3-times)} \\ = & RHS & \end{array}$$

For item 2 $\langle\sigma|_p\tau\rangle = \langle\sigma|_p\mathfrak{r}\rangle$ for some term r and we reason as for item 1.

- $\mathfrak{o} = \lambda a.\sigma$. For the first item we have:

$$\begin{array}{rll} & LHS & \\ = & (\lambda a.\sigma)\{u/^{\mathsf{m}}\rho^t_s\} & \\ = & \lambda a.\sigma\{u/^{\mathsf{m}}\rho^t_s\} & \\ \simeq & \lambda a.\sigma & \text{(i.h./1)} \\ = & RHS & \end{array}$$

For the second item $\sigma = \mathfrak{p}$ for some pre-term p. We reason as above but use i.h./2.

- $\mathfrak{o} = \sigma;\tau$. The second item holds trivially since $\mathfrak{o} \notin \mathbb{R}_1$. The first item follows from the i.h..
- $\mathfrak{o} = \sigma\,\tau$. We use the i.h. for both items.
- $\mathfrak{o} = let\ v \mathrel{\hat{=}} \sigma\ in\,\tau$. We use the i.h. for both items.

□

Lemma A.4. *$\rho \in \mathbb{R}^-_1$ implies $\rho\{u/^{\mathsf{m}}\sigma^q_p\} \in \mathbb{R}^-_1$.*

Proof. Suppose $\rho = \mathfrak{r}$. We proceed by induction on $\mathfrak{r}$:

- $\mathfrak{r} = \underline{a}$. Then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma^q_p\} = \underline{a}$ and we conclude.
- $\mathfrak{r} = \underline{w}$. If $w \neq u$, then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma^q_p\} = \underline{w}$. If $w = u$ and $\mathsf{m} = src$ then $\mathfrak{r}\{u/^{\mathsf{src}}\sigma^q_p\} = \mathfrak{p} \in \mathbb{R}^-_1$. The case where $\mathsf{m} = tgt$ is similar.
- $\mathfrak{r} = \lambda a.\mathfrak{s}$. Then $\mathfrak{r}\{u/^{\mathsf{m}}\sigma^q_p\} = \lambda a.\mathfrak{s}\{u/^{\mathsf{m}}\sigma^q_p\}$. By the i.h. $\mathfrak{s}\{u/^{\mathsf{m}}\sigma^q_p\} \in \mathbb{R}^-_1$, say $\mathfrak{s}\{u/^{\mathsf{m}}\sigma^q_p\} = \mathfrak{s}'$, and hence $\lambda a.\mathfrak{s}' \in \mathbb{R}^-_1$ too.
- $\mathfrak{r} = \mathfrak{s}\,\mathfrak{t}$. We use the i.h..
- $\mathfrak{r} = \langle\sigma|_s\mathfrak{t}\rangle$. We reason as follows:

$$\begin{array}{rll} & \langle\sigma|_s\mathfrak{t}\rangle\{u/^{\mathsf{m}}\sigma^q_p\} & \\ = & \langle\mathfrak{s}\{u/\sigma^q_p\};\sigma\{u/^{\mathsf{tgt}}\sigma^q_p\}|_{s\{u/^{\mathsf{src}}\sigma^q_p\}}\mathfrak{t}\{u/^{\mathsf{tgt}}\sigma^q_p\}\rangle & \\ = & \langle\mathfrak{s}\{u/\sigma^q_p\};\sigma\{u/^{\mathsf{tgt}}\sigma^q_p\}|_{s\{u/^{\mathsf{src}}\sigma^q_p\}}\mathfrak{t}'\rangle & \text{(i.h.)} \\ \in & \mathbb{R}^-_1 & \end{array}$$

- $\mathfrak{r} = let\ u \mathrel{\hat{=}} \mathfrak{s}\ in\,\mathfrak{t}$. We use the i.h..

□

Lemma. (Commutation of Rewrite Substitution with Term Substitution – Lem. 1.10) Suppose $a \notin \mathsf{fv}(\rho, s, t)$.

$$p\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} = p\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\}$$

Proof. By induction on p.

- $p = b$. If $a \neq b$, then $LHS = b = RHS$. Otherwise,

$$\begin{array}{rl} & LHS \\ = & a\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\ = & a\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\ = & q\{u/^{\mathsf{m}}\rho^t_s\} \\ = & a\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\} \\ = & RHS \end{array}$$

- $p = v$. If $u \neq v$, then $LHS = v = RHS$. Otherwise, if $u = v$ and $\mathsf{m} = src$ (the case where $\mathsf{m} = tgt$ is similar and omitted), we have

$$\begin{array}{rll} & LHS & \\ = & s\{a/q\{u/^{\mathsf{src}}\rho^t_s\}\} & \\ = & s & \text{(Lem. A.3(c))} \\ = & u\{a/q\}\{u/^{\mathsf{src}}\rho^t_s\} & \\ = & RHS & \end{array}$$

- $p = \lambda a.p_1$. By the i.h..
- $p = p_1\,p_2$. By the i.h..
- $p = !(\sigma, o, r)$. We reason as follows:

$$
\begin{array}{ll}
& LHS \\
= & !(\sigma, o, r)\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\
= & !(\mathfrak{o}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\})\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\
= & !(\mathfrak{o}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \\
= & !(\sigma, o, r)\{u/^{\mathsf{m}}\rho^t_s\} \\
= & !(\sigma, o, r)\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\} \\
= & RHS
\end{array}
$$

- $p = let\ v \circeq p_1\ in\, p_2$.

$$
\begin{array}{ll}
& LHS \\
= & (let\ v \circeq p_1\ in\, p_2)\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\
= & let\ v \circeq p_1\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\}\ in\, p_2\{u/^{\mathsf{m}}\rho^t_s\}\{a/q\{u/^{\mathsf{m}}\rho^t_s\}\} \\
= & let\ v \circeq p_1\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\}\ in\, p_2\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\} \\
= & (let\ v \circeq p_1\ in\, p_2)\{a/q\}\{u/^{\mathsf{m}}\rho^t_s\} \\
= & RHS
\end{array}
$$

□

Lemma A.5. *Let $p \in \mathbb{T}^-$. Then $\underline{p\{u/^{\mathsf{m}}\rho^t_s\}} = \underline{p}\{u/^{\mathsf{m}}\rho^t_s\}$.*

Proof. By induction on p.

- $p = a$. Then $LHS = a = RHS$.
- $p = v$. If $u \neq v$, then $LHS = v = RHS$. Otherwise, if $u = v$ and $\mathsf{m} = src$ (the case where $\mathsf{m} = tgt$ is similar and omitted), we have
$$LHS = \underline{s} = \underline{u}\{u/^{\mathsf{src}}\rho^t_s\}$$
- $p = \lambda a.p_1$. We use the i.h. and $\underline{\lambda a.p_1} := \lambda a.\underline{p_1}$
- $p = p_1\, p_2$. We use the i.h. and $\underline{p_1\, p_2} := \underline{p_1}\,\underline{p_2}$.
- $p = !(\sigma, o, r)$. We reason as follows.

$$
\begin{array}{ll}
& LHS \\
= & \underline{!(\sigma, o, r)\{u/^{\mathsf{m}}\rho^t_s\}} \\
= & \underline{!(\mathfrak{o}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\})} \\
= & \langle \mathfrak{o}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\underline{r\{u/^{\mathsf{tgt}}\rho^t_s\}}\rangle
\end{array}
$$

$$
\begin{array}{ll}
& RHS \\
= & \underline{!(\sigma, o, r)}\{u/^{\mathsf{m}}\rho^t_s\} \\
= & \langle \sigma|_o \mathfrak{r}\rangle\{u/^{\mathsf{m}}\rho^t_s\} \\
= & \langle \mathfrak{o}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{o\{u/^{\mathsf{src}}\rho^t_s\}}\underline{r}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle
\end{array}
$$

We conclude from the i.h. that $\underline{r\{u/^{\mathsf{tgt}}\rho^t_s\}} = \underline{r}\{u/^{\mathsf{tgt}}\rho^t_s\}$ and hence $LHS{=}RHS$.
- $p = let\ v \circeq p_1\ in\, p_2$.

$$
\begin{array}{lll}
& LHS & \\
= & \underline{(let\ v \circeq p_1\ in\, p_2)\{u/^{\mathsf{m}}\rho^t_s\}} & \\
= & \underline{let\ v \circeq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ in\, p_2\{u/^{\mathsf{m}}\rho^t_s\}} & \\
= & let\ v \circeq \underline{p_1\{u/^{\mathsf{m}}\rho^t_s\}}\ in\, \underline{p_2\{u/^{\mathsf{m}}\rho^t_s\}} & \\
= & let\ v \circeq \underline{p_1}\{u/^{\mathsf{m}}\rho^t_s\}\ in\, \underline{p_2}\{u/^{\mathsf{m}}\rho^t_s\} & (i.h.) \\
= & (let\ v \circeq \underline{p_1}\ in\, \underline{p_2})\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & (\underline{let\ v \circeq p_1\ in\, p_2})\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & RHS &
\end{array}
$$

□

Lemma. [Commutation of Rewrite Substitution – Lem. 1.11] Let o be any object (*i.e.* term or rewrite) and suppose $v \notin \mathrm{fv}(\rho, s, t)$.

(a) Suppose all occurrences of m below are either all src or all tgt. Then,
$$\mathrm{o}\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} \simeq \mathrm{o}\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}$$

(b) If $\mathrm{o} \in \mathbb{R}_1$, then
$$\mathrm{o}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\mathrm{o}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}$$
$$\simeq$$
$$\mathrm{o}\{u/^{\mathsf{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\};\mathrm{o}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}$$

Proof. We prove both items simultaneously by induction on the size of o. For the first six cases below, the second item holds trivially since $\mathrm{o} \notin \mathbb{R}_1$.

- $\mathrm{o} = a$. Then $LHS = a = RHS$.
- $\mathrm{o} = w$. If $w \neq v, u$, then $LHS = w = RHS$. Otherwise, if $w = v$ and $\mathsf{m} = src$ (the case where $\mathsf{m} = tgt$ is similar and omitted) we have

$$\begin{array}{ll} & LHS \\ = & v\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\} \\ = & p\{u/^{\mathsf{src}}\rho_s^t\} \\ = & v\{v/^{\mathsf{src}}\mathfrak{o}\{u/\rho_s^t\}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}_{p\{u/^{\mathsf{src}}\rho_s^t\}}\} \\ = & v\{u/^{\mathsf{src}}\rho_s^t\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}_{p\{u/^{\mathsf{src}}\rho_s^t\}}\} \\ = & RHS \end{array}$$

  if $w = u$, we have

$$\begin{array}{lll} & LHS & \\ = & u\{u/^{\mathsf{m}}\rho_s^t\} & \\ \simeq & u\{u/^{\mathsf{m}}\rho_s^t\}\{v/^{\mathsf{m}}\mathfrak{o}\{u/\rho_s^t\}^{o\{u/^{\mathsf{tgt}}\rho_s^t\}}_{o\{u/^{\mathsf{src}}\rho_s^t\}}\} & \text{(Lem. A.3)} \\ = & RHS & \end{array}$$

- $\mathrm{o} = \lambda a.r_1$. We use the i.h..
- $\mathrm{o} = r_1\, r_2$. We use the i.h..
- $\mathrm{o} = !(\sigma, r_1, r_2)$.

  Below we write α^{m} to abbreviate the substitution $\bullet\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}^{q\{u/^{\mathsf{tgt}}\rho_s^t\}}_{p\{u/^{\mathsf{src}}\rho_s^t\}}\}$ and α to abbreviate the substitution $\bullet\{v/\mathfrak{p}\{u/\rho_s^t\};\mu\{u/^{\mathsf{tgt}}\rho_s^t\}^{q\{u/^{\mathsf{t}}}_{p\{u/^{\mathsf{s}}}$

$$\begin{array}{lll} & LHS & \\ = & !(\sigma, r_1, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(\mathfrak{r}_1\{v/\mu_p^q\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{m}}\rho_s^t\} & \\ = & !(r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};(\underline{r_1}\{v/\mu_p^q\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\})\{u/^{\mathsf{tgt}}\rho_s^t\}, q\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ = & !(\overline{\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}}\{u/\rho_s^t\};(\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\};\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}), r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !((\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\});\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \\ \simeq & !((\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}});\sigma\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{v/^{\mathsf{src}}\mu_p^q\}\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{v/^{\mathsf{tgt}}\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) & \text{(i.h./2)} \\ \simeq & !((\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}});\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(i.h./1 three times)} \\ = & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;(\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}), r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\ = & !(\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;(\underline{r_1}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \\ = & !(\overline{\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}}\alpha;(\underline{r_1}\{u/\rho_s^t\};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{tgt}}, r_1\{u/^{\mathsf{src}}\rho_s^t\}\alpha^{\mathsf{src}}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\}\alpha^{\mathsf{tgt}}) & \text{(Lem. A.5)} \\ = & !(\overline{\underline{r_1}\{u/\rho_s^t\}};\sigma\{u/^{\mathsf{tgt}}\rho_s^t\}, r_1\{u/^{\mathsf{src}}\rho_s^t\}, r_2\{u/^{\mathsf{tgt}}\rho_s^t\})\alpha^{\mathsf{m}} & \\ = & !(\sigma, r_1, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} & \\ = & RHS & \end{array}$$

- $\mathrm{o} = let\ v \mathrel{\dot{=}} r_1\ in\, r_2$.

$$\begin{array}{ll} & LHS \\ = & (let\ w \mathrel{\dot{=}} r_1\ in\, r_2)\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} \\ = & let\ w \mathrel{\dot{=}} r_1\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\}\ in\, r_2\{v/^{\mathsf{m}}\mu_p^q\}\{u/^{\mathsf{m}}\rho_s^t\} \\ = & let\ w \mathrel{\dot{=}} r_1\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}}\ in\, r_2\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} \\ = & (let\ w \mathrel{\dot{=}} r_1\ in\, r_2)\{u/^{\mathsf{m}}\rho_s^t\}\alpha^{\mathsf{m}} \\ = & RHS \end{array}$$

- $\mathrm{o} = \underline{a}$. Then for item 1 we have $LHS = \underline{a} = RHS$. For item 2

$$\begin{array}{ll} & \underline{a}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{a}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} \\ \simeq & \underline{a};\underline{a} \\ \simeq & \underline{a}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{a}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} \end{array}$$

- $\mathrm{o} = \underline{w}$. Item 1. If $w \neq v, u$, then $LHS = \underline{w} = RHS$. Suppose $w = v$ and $\mathsf{m} = src$ (the case $\mathsf{m} = tgt$ is similar and omitted). Then we have

$$
\begin{array}{ll}
 & \mathit{LHS} \\
= & \underline{v}\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\} \\
= & \mathfrak{p}\{u/^{\mathsf{src}}\rho^t_s\} \\
= & \underline{p\{u/^{\mathsf{src}}\rho^t_s\}} \\
= & \underline{v}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\
= & \underline{v}\{u/^{\mathsf{src}}\rho^t_s\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\
= & \mathit{RHS}
\end{array}
$$

if $w = u$ and $\mathsf{m} = src$ (the case $\mathsf{m} = tgt$ is similar and omitted), we have

$$
\begin{array}{lll}
 & \mathit{LHS} & \\
= & \underline{u}\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\} & \\
= & \underline{u}\{u/^{\mathsf{src}}\rho^t_s\} & \\
= & \mathfrak{s} & \\
\simeq & \mathfrak{s}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} & \text{(Lem. A.3)} \\
= & \underline{u}\{u/^{\mathsf{src}}\rho^t_s\}\{v/^{\mathsf{src}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} & \\
= & \mathit{RHS} &
\end{array}
$$

For item 2 we consider two cases. Suppose $w = u$:

$$
\begin{array}{lll}
 & \mathit{LHS} & \\
= & \underline{u}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\underline{u}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} & \\
= & \underline{u}\{u/\rho^t_s\};\underline{u}\{u/^{\mathsf{tgt}}\rho^t_s\} & \\
= & \rho;\mathfrak{t} & \\
\simeq & \rho & \\
\simeq & \mathfrak{s};\rho & \\
\simeq & \underline{u}\{u/^{\mathsf{src}}\rho^t_s\};\underline{u}\{u/\rho^t_s\} & \\
\simeq & \underline{u}\{u/^{\mathsf{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\};\underline{u}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} & \text{(Lem. A.3)} \\
= & \mathit{RHS} &
\end{array}
$$

Then $w = v$:

$$
\begin{array}{ll}
 & \mathit{LHS} \\
= & \underline{v}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\underline{v}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} \\
= & \mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\} \\
\simeq & (\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\});\underline{q\{u/^{\mathsf{tgt}}\rho^t_s\}} \\
= & \underline{v}\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\};\underline{v}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\
= & \underline{v}\{u/^{\mathsf{src}}\rho^t_s\}\{v/\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\};\underline{v}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} \\
= & \mathit{RHS}
\end{array}
$$

- $\mathsf{o} = \mathbf{ba}(a.r_1, r_2)$. Item 2 holds trivially since $\mathsf{o} \notin \mathbb{R}_1$. For item 1 we use the i.h..

$$
\begin{array}{lll}
 & \mathit{LHS} & \\
= & \mathbf{ba}(a.r_1, r_2)\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & \mathbf{ba}(a.r_1\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\}, r_2\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\}) & \\
\simeq & \mathbf{ba}(a.r_1\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}, r_2\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\}) & \text{(i.h./1 twice)} \\
\simeq & \mathbf{ba}(a.r_1, r_2)\{u/^{\mathsf{m}}\rho^t_s\}\{v/^{\mathsf{m}}\mathfrak{p}\{u/\rho^t_s\};\mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{q\{u/^{\mathsf{tgt}}\rho^t_s\}}_{p\{u/^{\mathsf{src}}\rho^t_s\}}\} & \\
= & \mathit{RHS} &
\end{array}
$$

- $\mathsf{o} = \mathbf{bb}(r_1, w.r_2)$. Item 2 holds trivially since $\mathsf{o} \notin \mathbb{R}_1$. For item 1 we use the i.h., as in the previous case.
- $\mathsf{o} = \langle \sigma |_r \tau \rangle$. For the first item we reason as follows:

$$
\begin{array}{cll}
 & \mathit{LHS} & \\
= & \langle\sigma|_r\tau\rangle\{v/^{\mathsf{m}}\mu^q_p\}\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & \langle\mathfrak{r}\{v/\mu^q_p\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}}\tau\{v/^{\mathsf{tgt}}\mu^q_p\}\rangle\{u/^{\mathsf{m}}\rho^t_s\} & \\
= & \langle r\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};(\mathfrak{r}\{v/\mu^q_p\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\})\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\
= & \langle\overline{\underline{r}\{v/^{\mathsf{src}}\mu^q_p\}}\{u/\rho^t_s\};(\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\})|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \text{(Lem. A.5)} \\
\simeq & \langle(\underline{r}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\});\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\
\simeq & \langle(\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\underline{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}});\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{r}\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \text{(i.h./2)} \\
\simeq & \langle(\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\underline{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}});\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \text{(i.h./1 three times)} \\
= & \langle\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}})|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle r\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \text{(Lem. A.5)} \\
= & \langle\overline{\underline{r}\{u/\rho^t_s\}};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{u/^{\mathsf{src}}\rho^t_s\}}\tau\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle\alpha^{\mathsf{m}} & \\
= & \langle\sigma|_r\tau\rangle\{u/^{\mathsf{m}}\rho^t_s\}\alpha^{\mathsf{m}} & \\
= & \mathit{RHS} &
\end{array}
$$

For item 2, $\langle\sigma|_r\tau\rangle = \langle\sigma|_r\mathfrak{o}\rangle$ for term o the target of σ. We reason as follows:

$$
\begin{array}{cll}
 & \mathit{LHS} & \\
= & \langle\sigma|_r\mathfrak{o}\rangle\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\langle\sigma|_r\mathfrak{o}\rangle\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} & \\
\simeq & \langle\sigma|_r\mathfrak{o}\rangle\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\langle\sigma|_r\mathfrak{o}\rangle\{u/\rho^t_s\}\alpha^{\mathsf{tgt}} & (\star) \\
= & \mathit{RHS} &
\end{array}
$$

Step $(\star)$ follows from proving

$$
\langle\sigma|_r\mathfrak{o}\rangle\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\} \simeq \langle\sigma|_r\mathfrak{o}\rangle\{u/^{\mathsf{src}}\rho^t_s\}\alpha \tag{4}
$$

and

$$
\langle\sigma|_r\mathfrak{o}\rangle\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} \simeq \langle\sigma|_r\mathfrak{o}\rangle\{u/\rho^t_s\}\alpha^{\mathsf{tgt}} \tag{5}
$$

separately. For (4) we reason as follows (the case (5) is similar).

$$
\begin{array}{cll}
= & \langle\sigma|_r\mathfrak{o}\rangle\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\} & \\
= & \langle\mathfrak{r}\{v/\mu^q_p\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}}\mathfrak{o}\{v/^{\mathsf{tgt}}\mu^q_p\}\rangle\{u/\rho^t_s\} & \\
= & \langle r\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};(\mathfrak{r}\{v/\mu^q_p\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\})\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{o}\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\
= & \langle\overline{\underline{r}\{v/^{\mathsf{src}}\mu^q_p\}}\{u/\rho^t_s\};(\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\};\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\})|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{o}\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\
= & \langle(\underline{r}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\});\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{o}\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \\
\simeq & \langle(\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\underline{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}});\sigma\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{v/^{\mathsf{src}}\mu^q_p\}\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{o}\{v/^{\mathsf{tgt}}\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle & \text{(i.h./2)} \\
= & \langle(\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\underline{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}});\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\mathfrak{o}\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \text{(i.h./1 three times)} \\
= & \langle\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\}\{v/^{\mathsf{tgt}}\mathfrak{o}\{u/\rho^t_s\}^{o\{u/^{\mathsf{tgt}}\rho^t_s\}}_{o\{u/^{\mathsf{src}}\rho^t_s\}}\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}})|_{q\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\mathfrak{o}\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle\underline{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{src}}}\mathfrak{o}\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \\
= & \langle r\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\underline{r}\{u/\rho^t_s\};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\})\alpha^{\mathsf{tgt}}|_{r\{u/^{\mathsf{src}}\rho^t_s\}\alpha^{\mathsf{tgt}}}\mathfrak{o}\{u/^{\mathsf{tgt}}\rho^t_s\}\alpha^{\mathsf{tgt}}\rangle & \text{(Lem. A.5)} \\
= & \langle\overline{\underline{r}\{u/\rho^t_s\}};\sigma\{u/^{\mathsf{tgt}}\rho^t_s\}|_{r\{u/^{\mathsf{src}}\rho^t_s\}}\mathfrak{o}\{u/^{\mathsf{tgt}}\rho^t_s\}\rangle\alpha & \\
= & \langle\sigma|_r\mathfrak{o}\rangle\{u/^{\mathsf{src}}\rho^t_s\}\alpha &
\end{array}
$$

- $\mathrm{o} = \lambda a.\sigma$. Item 1 follows from the i.h./1. In the case of item 2, $\lambda a.\sigma = \lambda a.\mathfrak{r}$ for some term r. We reason as follows:

$$
\begin{array}{cll}
 & \mathit{LHS} & \\
= & (\lambda a.\mathfrak{r})\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};(\lambda a.\mathfrak{r})\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} & \\
= & \lambda a.\mathfrak{r}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\lambda a.\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\} & \\
\simeq & \lambda a.(\mathfrak{r}\{v/^{\mathsf{src}}\mu^q_p\}\{u/\rho^t_s\};\mathfrak{r}\{v/\mu^q_p\}\{u/^{\mathsf{tgt}}\rho^t_s\}) & \\
\simeq & \lambda a.(\mathfrak{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\mathfrak{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}}) & \text{(i.h./2 4 times)} \\
\simeq & \lambda a.\mathfrak{r}\{u/^{\mathsf{src}}\rho^t_s\}\alpha;\lambda a.\mathfrak{r}\{u/\rho^t_s\}\alpha^{\mathsf{tgt}} & \\
= & (\lambda a.\mathfrak{r})\{u/^{\mathsf{src}}\rho^t_s\}\alpha;(\lambda a.\mathfrak{r})\{u/\rho^t_s\}\alpha^{\mathsf{tgt}} & \\
= & \mathit{RHS} &
\end{array}
$$

- $\mathrm{o} = \sigma;\tau$. Item 1 follows from the i.h.. Item 2 is immediate since $\mathrm{o} \notin \mathbb{R}_1$.
- $\mathrm{o} = \sigma\,\tau$. Item 1 follows from the i.h.. For item 2 we reason as follows, where $\sigma\,\tau = \underline{r_1}\,\mathfrak{r}_y$ for some term r_1 and r_2.

$$
\begin{array}{cl}
 & LHS \\
= & (\underline{r_1}\,\mathfrak{r}_y)\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};(\underline{r_1}\,\mathfrak{r}_y)\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} \\
= & \underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\}\,\mathfrak{r}_y\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}\,\mathfrak{r}_y\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\} \\
\simeq & (\underline{r_1}\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\underline{r_1}\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\})\,(\mathfrak{r}_y\{v/^{\mathsf{src}}\mu_p^q\}\{u/\rho_s^t\};\mathfrak{r}_y\{v/\mu_p^q\}\{u/^{\mathsf{tgt}}\rho_s^t\}) \\
\simeq & (\underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}})\,(\mathfrak{r}_y\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\mathfrak{r}_y\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}) \qquad \text{(i.h./2 4 times)} \\
\simeq & \underline{r_1}\{u/^{\mathsf{src}}\rho_s^t\}\alpha\,\mathfrak{r}_y\{u/^{\mathsf{src}}\rho_s^t\}\alpha;\underline{r_1}\{u/\rho_s^t\}\alpha^{\mathsf{tgt}}\,\mathfrak{r}_y\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} \\
= & (\underline{r_1}\,\mathfrak{r}_y)\{u/^{\mathsf{src}}\rho_s^t\}\alpha;(\underline{r_1}\,\mathfrak{r}_y)\{u/\rho_s^t\}\alpha^{\mathsf{tgt}} \\
= & RHS
\end{array}
$$

- $\mathfrak{o} = let\ w \doteq \sigma\ in\ \tau$. Item 1 follows from the i.h.. For item 2, $let\ w \doteq \sigma\ in\ \tau = let\ w \doteq \underline{r_1}\ in\ \mathfrak{r}_y$ for some r_1 and r_2. We reason as in the previous item, using the i.h./2.

□

B TYPES

Sample proof of
$[\![\rho, s, t]\!](A \supset B) \supset [\![\sigma, p, q]\!]A \supset [\![\rho\,\sigma, s\,p, t\,q]\!]B$

$$
\dfrac{\cdot;\Gamma \vdash b : [\![\sigma, s, t]\!]A \qquad \dfrac{\cdot;\Gamma \vdash a : [\![\rho, p, q]\!](A \supset B) \qquad \dfrac{\dfrac{\dfrac{}{\Delta;\cdot \vdash \mathfrak{u} : u \rhd u : A \supset B} \qquad \dfrac{}{\Delta;\cdot \vdash \mathfrak{v} : v \rhd v : A}}{\Delta;\cdot \vdash \mathfrak{u}\,\mathfrak{v} : u\,v \rhd u\,v : B}\ \text{R-App}}{\Delta;\cdot \vdash !(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : [\![\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v]\!]B}\ \text{Bang}}{\cdot;\Gamma \vdash let\ u^C \doteq a\ in\ !(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : ([\![\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v]\!]B)\{u/^{\mathsf{src}}\rho_p^q\}}\ \text{Let}}{\cdot;\Gamma \vdash let\ v^D \doteq b\ in\ let\ u^C \doteq a\ in\ !(\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v) : (([\![\mathfrak{u}\,\mathfrak{v}, u\,v, u\,v]\!]B)\{u/^{\mathsf{src}}\rho_p^q\})\{v/^{\mathsf{src}}\sigma_s^t\}}\ \text{Let}
$$

Lemma B.1 (Weakening). *(a)* $\Delta;\Gamma \vdash s : B$ *and* $u \notin dom(\Delta)$, *implies* $\Delta, u : A;\Gamma \vdash s : B$
(b) $\Delta;\Gamma \vdash \rho : s \rhd t : B$ *and* $u \notin dom(\Delta)$, *implies* $\Delta, u : A;\Gamma \vdash \rho : s \rhd t : B$
(c) $\Delta;\Gamma \vdash s : B$ *and* $a \notin dom(\Gamma)$, *implies* $\Delta;\Gamma, a : A \vdash s : B$
(d) $\Delta;\Gamma \vdash \rho : s \rhd t : B$ *and* $a \notin dom(\Gamma)$, *implies* $\Delta;\Gamma, a : A \vdash \rho : s \rhd t : B$

Proof. By induction on the corresponding derivations of each item. □

Lemma. [Term as Unit Rewrite – Lem. 2.3] $\blacktriangleright\ \Delta;\Gamma \vdash s : A$ implies $\blacktriangleright\ \Delta;\Gamma \vdash \mathfrak{s} : s \rhd s : A$.

Proof. By induction on the derivation of $\Delta;\Gamma \vdash s : A$. □

Lemma. [Term Substitution – Lem. 2.4] Suppose $\Delta;\Gamma, a : A \vdash s : B$ and $\Delta;\Gamma \vdash t : A$. Then $\Delta;\Gamma \vdash s\{a/t\} : B$.

Proof. By induction on the derivation of $\Delta;\Gamma, a : A \vdash s : B$. It relies on Lem. 1.6. □

Lemma [Rewrite Substitution Lemma – Lem. 2.5] Suppose $\blacktriangleright\ \Delta;\cdot \vdash \rho : s \rhd t : A$, $\blacktriangleright\ \Delta;\cdot \vdash s : A$ and $\blacktriangleright\ \Delta;\cdot \vdash t : A$. Suppose $\blacktriangleright\ \Delta, u : A;\Gamma \vdash S : B$ and $\blacktriangleright\ \Delta, u : A \vdash C \preceq D$ and $\blacktriangleright\ \Delta, u : A \vdash C \simeq D$.

(a) $S = \sigma : p \rhd q$ implies
$$\blacktriangleright\ \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$
(b) $S = \sigma : p \rhd q$ implies
$$\blacktriangleright\ \Delta, u : A;\Gamma \vdash p : B \text{ and } \blacktriangleright\ \Delta, u : A;\Gamma \vdash q : B.$$
(c) $S = p$ implies
$$\blacktriangleright\ \Delta;\Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$
(d) $S = p$ implies
$$\blacktriangleright\ \Delta;\Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$
(e) $S = p$ implies
$$\blacktriangleright\ \Delta;\Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}.$$
(f) $\blacktriangleright\ \Delta \vdash C\{u/^{\mathsf{src}}\rho_s^t\} \preceq D\{u/^{\mathsf{src}}\rho_s^t\}$.
(g) $\blacktriangleright\ \Delta \vdash C\{u/^{\mathsf{src}}\rho_s^t\} \simeq D\{u/^{\mathsf{src}}\rho_s^t\}$.

Proof. By simultaneous structural induction on the derivations of $\blacktriangleright\ \Delta, u : A;\Gamma \vdash S : B$ and $\blacktriangleright\ \Delta, u : A \vdash C \preceq D$.

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:
$$\frac{a : B \in \Gamma}{\Delta, u : A; \Gamma \vdash a : B}\ \mathsf{TVar}$$
Then $u \notin \mathsf{frv}(B)$. The first two items holds trivially. For the other three we reason as follows:
  - $\Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash a : B$.
  - $\Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash \underline{a} : a \rhd a : B$.
  - $\Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash \underline{a} : a \rhd a : B$.
- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:
$$\frac{\Delta, u : A; \Gamma, a : B_1 \vdash p_1 : B_2}{\Delta, u : A; \Gamma \vdash \lambda a.p_1 : B_1 \supset B_2}\ \mathsf{Abs}$$
Then $u \notin \mathsf{frv}(B_1)$. The first two items holds trivially. For item (c) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1\{u/^{\mathsf{src}}\rho_s^t\} \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1 \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from
$$\blacktriangleright\ \Delta; \Gamma, a : B_1 \vdash p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_2\{u/^{\mathsf{src}}\rho_s^t\}$$
which we obtain from the i.h. w.r.t. (c).
For item (d) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_g\{u/\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_1\{u/^{\mathsf{src}}\rho_s^t\} \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_g\{u/\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_1 \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from
$$\blacktriangleright\ \Delta; \Gamma, a : B_1 \vdash \mathfrak{p}_g\{u/\rho_s^t\} : p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B_2\{u/^{\mathsf{src}}\rho_s^t\}$$
which we obtain from the i.h. w.r.t. (d).
For (e) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1\{u/^{\mathsf{src}}\rho_s^t\} \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \lambda a.\mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\} : \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd \lambda a.p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_1 \supset B_2\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from
$$\blacktriangleright\ \Delta; \Gamma, a : B_1 \vdash \mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\} : p_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{m}}\rho_s^t\} : B_2\{u/^{\mathsf{src}}\rho_s^t\}$$
which we obtain from the i.h. w.r.t. (e).
- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:
$$\frac{\Delta, u : A; \Gamma \vdash p_1 : C \supset B \quad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash p_1\, p_2 : B}\ \mathsf{App}$$
The first two items holds trivially. For item (c) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho_s^t\}\, p_2\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from
$$\begin{array}{l} \blacktriangleright\ \Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho_s^t\} : (C \supset B)\{u/^{\mathsf{src}}\rho_s^t\} \\ \blacktriangleright\ \Delta; \Gamma \vdash p_2\{u/^{\mathsf{m}}\rho_s^t\} : C\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
which we obtain from the i.h. w.r.t. (c).
For item (d) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\} \rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \mathfrak{p}_g\{u/\rho_s^t\}\, \mathfrak{p}_g\{u/\rho_s^t\} : p_1\{u/^{\mathsf{src}}\rho_s^t\}\, p_2\{u/^{\mathsf{src}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{tgt}}\rho_s^t\}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from
$$\begin{array}{l} \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_g\{u/\rho_s^t\} : p_1\{u/^{\mathsf{src}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : (C \supset B)\{u/^{\mathsf{src}}\rho_s^t\} \\ \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_y\{u/\rho_s^t\} : p_2\{u/^{\mathsf{src}}\rho_s^t\} \rhd p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
which we obtain from the i.h. w.r.t. (d).
For item (e) we reason as follows:
$$\begin{array}{rl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\} \rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\}\, \mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\} : p_1\{u/^{\mathsf{m}}\rho_s^t\}\, p_2\{u/^{\mathsf{m}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{m}}\rho_s^t\}\, p_2\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \end{array}$$
The latter is derivable from

$$\blacktriangleright\ \Delta;\Gamma\vdash \mathfrak{p}_g\{u/^{\mathsf{m}}\rho_s^t\} : p_1\{u/^{\mathsf{m}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{m}}\rho_s^t\} : (C\supset B)\{u/^{\mathsf{src}}\rho_s^t\}$$
$$\blacktriangleright\ \Delta;\Gamma\vdash \mathfrak{p}_y\{u/^{\mathsf{m}}\rho_s^t\} : p_2\{u/^{\mathsf{m}}\rho_s^t\} \rhd p_2\{u/^{\mathsf{m}}\rho_s^t\} : C\{u/^{\mathsf{src}}\rho_s^t\}$$

which we obtain from the i.h. w.r.t. (e).

- The derivation of $\Delta,u:A;\Gamma\vdash p:B$ ends in:

$$\frac{v:B\in(\Delta,u:A)}{\Delta,u:A;\Gamma\vdash v:B}\ \mathsf{VVar}$$

Note that $u\notin \mathsf{frv}(B)$. The first two items holds trivially. If $u\neq v$ then, for items (c), (d) and (e) we reason as follows:
  - $\Delta;\Gamma\vdash p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash v:B$.
  - $\Delta;\Gamma\vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\}\rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash \underline{v} : v\rhd v : B$.
  - $\Delta;\Gamma\vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\}\rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}$
    $= \Delta;\Gamma\vdash \underline{v} : v\rhd v : B$

  If $u=v$, then
  - For item (c) we consider two cases. If m=src, then
    $$\Delta;\Gamma\vdash p\{u/^{\mathsf{src}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash s:B$$
    Moreover, the latter is derivable from the hypothesis and Weakening (Lem. B.1). If m=tgt, then
    $$\Delta;\Gamma\vdash p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash t:B$$
    and the latter is derivable from the hypothesis and Weakening (Lem. B.1) too.
  - For item (d) we have: $\Delta;\Gamma\vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\}\rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash \rho : s\rhd t : B$. The latter is derivable from the hypothesis and Weakening (Lem. B.1).
  - For (e) and $\mathsf{m}=src$ (the case $\mathsf{m}=tgt$ is similar and omitted) we have: $\Delta;\Gamma\vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho_s^t\} : p\{u/^{\mathsf{m}}\rho_s^t\}\rhd p\{u/^{\mathsf{m}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta;\Gamma\vdash \mathfrak{s} : s\rhd s : B$. The latter follows from the hypothesis and Lem. 2.3.

- The derivation of $\Delta,u:A;\Gamma\vdash p:B$ ends in:

$$\frac{\Delta,u:A;\cdot\vdash o,r:D\qquad \Delta,u:A;\cdot\vdash\tau:o\rhd r:D}{\Delta,u:A;\Gamma\vdash\, !(\tau,o,r):[\![\tau,o,r]\!]D}\ \mathsf{Bang}$$

Items (a) and (b) hold trivially. For item (c) we reason as follows. By the hypothesis we deduce $\Delta,u:A;\cdot\vdash o:D$ and $\Delta,u:A;\cdot\vdash r:D$. This allows us to apply the i.h. w.r.t. (d), to deduce

$$\blacktriangleright\ \Delta;\cdot\vdash \mathfrak{o}\{u/\rho_s^t\} : o\{u/^{\mathsf{src}}\rho_s^t\}\rhd o\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\} \tag{6}$$

By the i.h. w.r.t (a),

$$\blacktriangleright\ \Delta;\cdot\vdash \tau\{u/^{\mathsf{tgt}}\rho_s^t\} : o\{u/^{\mathsf{tgt}}\rho_s^t\}\rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\} \tag{7}$$

By the i.h. w.r.t. (c) twice we have:

$$\Delta;\cdot\vdash o\{u/^{\mathsf{src}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}\Delta;\cdot\vdash r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\} \tag{8}$$

We can derive the following:

$$\frac{\begin{array}{c}\Delta;\cdot\vdash o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}\\ \blacktriangleright_\pi\ \Delta;\cdot\vdash \mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\} : o\{u/^{\mathsf{src}}\rho_s^t\}\rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}\end{array}}{\Delta;\Gamma\vdash\, !(\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) : [\![\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}]\!]D\{u/^{\mathsf{src}}\rho_s^t\}}\ \mathsf{Bang}$$

where π is the derivation:

$$\frac{\begin{array}{c}\Delta;\cdot\vdash \mathfrak{o}\{u/\rho_s^t\} : o\{u/^{\mathsf{src}}\rho_s^t\}\rhd o\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}\\ \Delta;\cdot\vdash \tau\{u/^{\mathsf{tgt}}\rho_s^t\} : o\{u/^{\mathsf{tgt}}\rho_s^t\}\rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}\end{array}}{\Delta;\cdot\vdash \mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\} : o\{u/^{\mathsf{src}}\rho_s^t\}\rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D\{u/^{\mathsf{src}}\rho_s^t\}}\ \mathsf{R\text{-}Trans}$$

Note that

$$!(\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) =\, !(\tau,o,r)\{u/^{\mathsf{m}}\rho_s^t\}$$

and

$$\begin{array}{rl} & [\![\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}]\!]D\{u/^{\mathsf{src}}\rho_s^t\}\\ = & ([\![\tau,o,r]\!]D)\{u/^{\mathsf{src}}\rho_s^t\}\end{array}$$

This concludes the proof of item (c).
For item (d) we reason as follows. Note that $\underline{!(\tau,o,r)} = \langle\tau|_o\mathfrak{r}\rangle$.

$$\begin{array}{rl} & \Delta;\Gamma\vdash \mathfrak{p}\{u/\rho_s^t\} : p\{u/^{\mathsf{src}}\rho_s^t\}\rhd p\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\tau,o,r]\!]D)\{u/^{\mathsf{src}}\rho_s^t\}\\ = & \Delta;\Gamma\vdash \langle\tau|_o\mathfrak{r}\rangle\{u/\rho_s^t\} : !(\tau,o,r)\{u/^{\mathsf{src}}\rho_s^t\}\rhd\, !(\tau,o,r)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\tau,o,r]\!]D)\{u/^{\mathsf{src}}\rho_s^t\}\\ = & \Delta;\Gamma\vdash \langle\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}|_{o\{u/^{\mathsf{src}}\rho_s^t\}}\mathfrak{r}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle : !(\tau,o,r)\{u/^{\mathsf{src}}\rho_s^t\}\rhd\, !(\tau,o,r)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\tau,o,r]\!]D)\{u/^{\mathsf{src}}\rho_s^t\}\\ = & \Delta;\Gamma\vdash \langle\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}|_{o\{u/^{\mathsf{src}}\rho_s^t\}}\mathfrak{r}\{u/^{\mathsf{tgt}}\rho_s^t\}\rangle : !(\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\})\rhd\, !(\mathfrak{o}\{u/\rho_s^t\};\tau\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\})\end{array}$$

Recall from above that $\Delta, u : A; \cdot \vdash r : D$. This allows us to apply the i.h. w.r.t. (e), to deduce

$$\blacktriangleright\ \Delta; \cdot \vdash \mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} : o\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd o\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/^{\mathsf{src}}\rho^t_s\} \tag{9}$$

Consider the following abbreviations:

$$\begin{array}{rcl} \mathfrak{o}_u & := & \mathfrak{o}\{u/\rho^t_s\} \\ \tau^{\mathsf{tgt}}_u & := & \tau\{u/^{\mathsf{tgt}}\rho^t_s\} \\ \mathfrak{r}^{\mathsf{tgt}}_u & := & \mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} \end{array}$$

Then, for example, $\mathfrak{o}\{u/\rho^t_s\}; \tau\{u/^{\mathsf{tgt}}\rho^t_s\}$ is just $\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u$. We can derive:

$$\dfrac{\begin{array}{c}\blacktriangleright_\pi\ \Delta; \Gamma \vdash \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u : o\{u/^{\mathsf{src}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/^{\mathsf{src}}\rho^t_s\} \\ \Delta; \Gamma \vdash \mathfrak{r}^{\mathsf{tgt}}_u : r\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : D\{u/^{\mathsf{src}}\rho^t_s\}\end{array}}{\Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u |_{o\{u/^{\mathsf{src}}\rho^t_s\}} \mathfrak{r}^{\mathsf{tgt}}_u \rangle : !\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u \rhd !(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u); \mathfrak{r}^{\mathsf{tgt}}_u : [\![(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u); \mathfrak{r}^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}]\!] D\{u/^{\mathsf{src}}\rho^t_s\}}\ \text{R-Bang}$$

Moreover, from Lem. A.4, substitution preserves membership in $\mathbb{R}_1$. Thus $\mathfrak{r}\{u/^{\mathsf{tgt}}\rho^t_s\} \in \mathbb{R}_1$ and hence $(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u); \mathfrak{r}^{\mathsf{tgt}}_u \simeq \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u$. For item (e) we reason as follows.

$$\begin{array}{cl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : ([\![\tau, o, r]\!]D)\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash \langle \tau |_o \mathfrak{r} \rangle \{u/^{\mathsf{m}}\rho^t_s\} : !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} : ([\![\tau, o, r]\!]D)\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u |_{o\{u/^{\mathsf{src}}\rho^t_s\}} \mathfrak{r}^{\mathsf{tgt}}_u \rangle : !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} \rhd !(\tau, o, r)\{u/^{\mathsf{m}}\rho^t_s\} : ([\![\tau, o, r]\!]D)\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u |_{o\{u/^{\mathsf{src}}\rho^t_s\}} \mathfrak{r}^{\mathsf{tgt}}_u \rangle : !(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : ([\![\tau, o, r]\!]D)\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

We have already shown that the judgment

$$\Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau^{\mathsf{tgt}}_u |_{o\{u/^{\mathsf{src}}\rho^t_s\}} \mathfrak{r}^{\mathsf{tgt}}_u \rangle : !(\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) \rhd !((\mathfrak{o}_u; \tau^{\mathsf{tgt}}_u); \mathfrak{r}^{\mathsf{tgt}}_u, o\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}) : ([\![\tau; v, o, r']\!]D)\{u/^{\mathsf{src}}\rho^t_s\}$$

is derivable.

- The derivation of $\Delta, u : A; \Gamma \vdash p : B$ ends in:

$$\dfrac{\Delta, u : A; \Gamma \vdash p_1 : [\![\mu, m, n]\!]D \qquad \Delta, u : A, v : D; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash let\ v \doteq p_1\ in\ p_2 : C\{v/^{\mathsf{src}}\mu^n_m\}}\ \text{Let}$$

where $u \notin \mathrm{frv}(D)$. The first two items holds trivially. For item (c) we reason as follows:

$$\begin{array}{cl} & \Delta; \Gamma \vdash p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash let\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{v/^{\mathsf{src}}\mu^n_m\}\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \blacktriangleright\ \Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\} : ([\![\mu, m, n]\!]D)\{u/^{\mathsf{src}}\rho^t_s\} \\ \blacktriangleright\ \Delta, v : D; \Gamma \vdash p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

which we obtain from the i.h. w.r.t. (c), an application of Let leading to (note $D\{u/^{\mathsf{src}}\rho^t_s\} = D$):

$$\dfrac{\begin{array}{c}\Delta; \Gamma \vdash p_1\{u/^{\mathsf{m}}\rho^t_s\} : [\![\mathfrak{m}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}, m\{u/^{\mathsf{src}}\rho^t_s\}, n\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]D \\ \Delta, v : D; \Gamma \vdash p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/^{\mathsf{src}}\rho^t_s\}\end{array}}{\Delta; \Gamma \vdash let\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/^{\mathsf{src}}\rho^t_s\}\{v/^{\mathsf{src}}\mathfrak{m}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{n\{u/^{\mathsf{tgt}}\rho^t_s\}}_{m\{u/^{\mathsf{src}}\rho^t_s\}}\}}\ \text{Let}$$

We conclude from the substitution lemma Lem. 1.11 that

$$\begin{array}{cl} & C\{v/^{\mathsf{src}}\mu^n_m\}\{u/^{\mathsf{src}}\rho^t_s\} \\ = & C\{u/^{\mathsf{src}}\rho^t_s\}\{v/^{\mathsf{src}}\mathfrak{m}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}^{n\{u/^{\mathsf{tgt}}\rho^t_s\}}_{m\{u/^{\mathsf{src}}\rho^t_s\}}\} \end{array}$$

For item (d) we reason as follows:

$$\begin{array}{cl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash let\ v \doteq \mathfrak{p}_g\{u/\rho^t_s\}\ in\ \mathfrak{p}_y\{u/\rho^t_s\} : let\ v \doteq p_1\{u/^{\mathsf{src}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd let\ v \doteq p_1\{u/^{\mathsf{tgt}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_g\{u/\rho^t_s\} : p_1\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{tgt}}\rho^t_s\} : [\![\mathfrak{m}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}, m\{u/^{\mathsf{src}}\rho^t_s\}, n\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]D \\ \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_y\{u/\rho^t_s\} : p_2\{u/^{\mathsf{src}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{tgt}}\rho^t_s\} : C\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

which we obtain from the i.h. w.r.t. (d), an application of R-Let and Lem. 1.11.

For item (e) we reason as follows:

$$\begin{array}{cl} & \Delta; \Gamma \vdash \mathfrak{p}\{u/^{\mathsf{m}}\rho^t_s\} : p\{u/^{\mathsf{m}}\rho^t_s\} \rhd p\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta; \Gamma \vdash let\ v \doteq \mathfrak{p}_g\{u/^{\mathsf{m}}\rho^t_s\}\ in\ \mathfrak{p}_y\{u/^{\mathsf{m}}\rho^t_s\} : let\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd let\ v \doteq p_1\{u/^{\mathsf{m}}\rho^t_s\}\ in\ p_2\{u/^{\mathsf{m}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

The latter is derivable from

$$\begin{array}{l} \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_g\{u/^{\mathsf{m}}\rho^t_s\} : p_1\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_1\{u/^{\mathsf{m}}\rho^t_s\} : [\![\mathfrak{m}\{u/\rho^t_s\}; \mu\{u/^{\mathsf{tgt}}\rho^t_s\}, m\{u/^{\mathsf{src}}\rho^t_s\}, n\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]D \\ \blacktriangleright\ \Delta; \Gamma \vdash \mathfrak{p}_y\{u/^{\mathsf{m}}\rho^t_s\} : p_2\{u/^{\mathsf{m}}\rho^t_s\} \rhd p_2\{u/^{\mathsf{m}}\rho^t_s\} : C\{u/^{\mathsf{src}}\rho^t_s\} \end{array}$$

which we obtain from the i.h. w.r.t. (e), an application of R-Let and Lem. 1.11.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{a : B \in \Gamma}{\Delta, u : A; \Gamma \vdash \underline{a} : a \rhd a : B}\ \text{R-Refl-TVar}$$

The last three items are immediate. For (a) and (b) we have ($u \notin \mathsf{frv}(B)$):
  - $\Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash \underline{a} : a \rhd a : B$.
  - $\Delta, u : A; \Gamma \vdash a : B$.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{v : B \in (\Delta, u : A)}{\Delta, u : A; \Gamma \vdash \underline{v} : v \rhd v : B}\ \text{R-Refl-VVar}$$

The last three items are immediate. For (a) ($u \notin \mathsf{frv}(B)$):
  - Suppose $u \neq v$. Then $\Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash \underline{v} : v \rhd v : B$.
  - Suppose $u = v$. Then $\Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash \mathsf{t} : t \rhd t : B$. The latter is derivable from the Term as Unit Rewrite Lemma (Lem. 2.3), the hypothesis and Weakening (Lem. B.1).

For item (b) we have $\Delta, u : A; \Gamma \vdash v : B$.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash o, r, n : C \quad \Delta, u : A; \cdot \vdash \tau : o \rhd r : C \quad \Delta, u : A; \cdot \vdash \mu : r \rhd n : C}{\Delta, u : A; \Gamma \vdash \langle \tau |_o \mu \rangle : !(\tau, o, r) \rhd !(\tau; \mu, o, n) : [\![\tau, o, r]\!]C}\ \text{R-Bang}$$

The last four items are immediate. For (b) we have:

$$\frac{\Delta, u : A; \cdot \vdash o, r : C \quad \Delta, u : A; \cdot \vdash \tau : o \rhd r : C}{\Delta, u : A; \Gamma \vdash !(\tau, o, r) : [\![\tau, o, r]\!]C}\ \text{Bang}$$

and since

$$\frac{\Delta, u : A; \cdot \vdash \tau : o \rhd r : C \quad \Delta, u : A; \cdot \vdash \mu : r \rhd n : C}{\Delta, u : A; \cdot \vdash \tau; \mu : o \rhd n : C}\ \text{R-Trans}$$

then

$$\frac{\dfrac{\Delta, u : A; \cdot \vdash o, n : C \quad \Delta, u : A; \cdot \vdash \tau; \mu : o \rhd n : C}{\Delta, u : A; \Gamma \vdash !(\tau; \mu, o, n) : [\![\tau; \mu, o, n]\!]C}\ \text{Bang} \qquad \dfrac{\begin{array}{c}\Delta, u : A; \Gamma \vdash o, r, n : C \\ \Delta, u : A; \cdot \vdash \tau : o \rhd r : C \\ \Delta, u : A; \cdot \vdash \mu : r \rhd n : C \\ \Delta, u : A \vdash C \preceq C\end{array}}{\Delta, u : A \vdash [\![\tau; \mu, o, n]\!]C \preceq [\![\tau, o, r]\!]C}}{\Delta, u : A; \Gamma \vdash !(\tau; \mu, o, n) : [\![\tau, o, r]\!]C}\ \text{Subs}$$

We now address items (a) and (b). Consider the following abbreviations:

$$\begin{array}{rcl}\mathfrak{o}_u & := & \mathfrak{o}\{u/\rho_s^t\} \\ \tau_u^{\mathsf{tgt}} & := & \tau\{u/^{\mathsf{tgt}}\rho_s^t\} \\ \mu_u^{\mathsf{tgt}} & := & \mu\{u/^{\mathsf{tgt}}\rho_s^t\}\end{array}$$

For item (a) we reason as follows:

$$\begin{array}{rl} & \Delta; \Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \langle \tau |_o \mu \rangle\{u/^{\mathsf{tgt}}\rho_s^t\} : !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd !(\tau; \mu, o, n)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\tau; \mu, o, n]\!]C)\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau_u^{\mathsf{tgt}} |_{o\{u/^{\mathsf{src}}\rho_s^t\}} \mu_u^{\mathsf{tgt}} \rangle : !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd !(\tau; \mu, o, n)\{u/^{\mathsf{tgt}}\rho_s^t\} : ([\![\tau; \mu, o, n]\!]C)\{u/^{\mathsf{src}}\rho_s^t\} \\ = & \Delta; \Gamma \vdash \langle \mathfrak{o}_u; \tau_u^{\mathsf{tgt}} |_{o\{u/^{\mathsf{src}}\rho_s^t\}} \mu_u^{\mathsf{tgt}} \rangle : !(\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) \rhd !(\mathfrak{o}_u; (\tau; \mu)\{u/^{\mathsf{tgt}}\rho_s^t\}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}) : ([\![o]\!]D)\{u/^{\mathsf{src}}\rho_s^t\}\end{array}$$

For (b) is similar.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash \sigma_1 : p \rhd r : B \quad \Delta, u : A; \Gamma \vdash \sigma_2 : r \rhd q : B}{\Delta, u : A; \Gamma \vdash \sigma_1; \sigma_2 : p \rhd q : B}\ \text{R-Trans}$$

The last four items are immediate. Item (b) follows from the i.h.. For item (a) we reason as above.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma, a : C \vdash p_1 : B \quad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathbf{ba}(a^C.p_1, p_2) : (\lambda a^{p_1}.C)\, p_2 \rhd p_1\{a/p_2\} : B}\ \text{R-}\beta$$

The last four items are immediate. Also, $u \notin \mathsf{frv}(C)$. For (b) we have:

$$\dfrac{\dfrac{\Delta, u : A; \Gamma, a : C \vdash p_1 : B}{\Delta, u : A; \Gamma \vdash \lambda a^{s}.C : C \supset B}\,\mathsf{Abs} \qquad \Delta, u : A; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash (\lambda a^{p_1}.C)\, p_2 : B}\,\mathsf{App}$$

Also, ► $\Delta, u : A; \Gamma \vdash p_1\{a/p_2\} : B$ follows from ► $\Delta, u : A; \Gamma, a : C \vdash p_1 : B$, ► $\Delta, u : A; \Gamma \vdash p_2 : C$ and the Truth Substitution Lemma (Lem. 2.4).
For item (a) we reason as follows:

$$\begin{array}{rl}
 & \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{ba}(a.p_1, p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} : ((\lambda a.p_1)\, p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\}, p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : ((\lambda a.p_1)\, p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\}, p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : (\lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\})\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_1\{a/p_2\}\{u/^{\mathsf{src}}\rho_s^t\} : r\{u/^{\mathsf{tgt}}\rho_s^t\}!(\mathfrak{o}_u; p_1\{a/p_2\}\{u/^{\mathsf{tgt}}\rho_s^t\}, p_1\{a/p_2\}\{u/^{\mathsf{src}}\rho_s^t\},
\end{array}$$

By the i.h. w.r.t (c):
- ► $\Delta; \Gamma, a : C \vdash p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}$,
- ► $\Delta; \Gamma \vdash p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C\{u/^{\mathsf{src}}\rho_s^t\} = \Delta; \Gamma \vdash p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C$

$$\dfrac{\dfrac{\Delta; \Gamma, a : C \vdash p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\}}{\Delta; \Gamma \vdash \lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\} : C \supset B\{u/^{\mathsf{src}}\rho_s^t\}}\,\mathsf{Abs} \qquad \Delta; \Gamma \vdash p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C}{\Delta; \Gamma \vdash \mathbf{ba}(a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\}, p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : (\lambda a.p_1\{u/^{\mathsf{tgt}}\rho_s^t\})\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_1\{u/^{\mathsf{tgt}}\rho_s^t\}\{a/p_2\{u/^{\mathsf{tgt}}\rho_s^t\}\} : B}\,\mathsf{R}\text{-}\beta$$

Note that, $p_1\{u/^{\mathsf{tgt}}\rho_s^t\}\{a/p_2\{u/^{\mathsf{tgt}}\rho_s^t\}\} = p_1\{a/p_2\}\{u/^{\mathsf{tgt}}\rho_s^t\}$ follows from the Commutation of Validity Substitution with Truth Substitution Lemma (Lem. 1.10).

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\dfrac{\Delta, u : A; \cdot \vdash \tau : o \rhd r : D \quad \Delta, u : A, v : D; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathbf{bb}(!(\tau, o, r), v.p_2) : \mathit{let}\ v \doteq !(\tau, o, r)\ \mathit{in}\, p_2 \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\} : C\{v/^{\mathsf{src}}\tau_o^r\}}\,\mathsf{R}\text{-}\beta_\Box$$

The last four items are immediate. Also, $u \notin \mathsf{frv}(D)$. Also, by i.h. w.r.t to (b) applied to $\Delta, u : A; \cdot \vdash \tau : o \rhd r : D$ we know:

$$\Delta, u : A; \cdot \vdash o, r : D \tag{10}$$

For (b) we have:

$$\dfrac{\dfrac{\Delta, u : A; \cdot \vdash o, r : D \quad \Delta, u : A; \cdot \vdash \tau : o \rhd r : D}{\Delta, u : A; \Gamma \vdash !(\tau, o, r) : [\![\tau, o, r]\!]D}\,\mathsf{Bang} \qquad \Delta, u : A, v : D; \Gamma \vdash p_2 : C}{\Delta, u : A; \Gamma \vdash \mathit{let}\ v \doteq !(\tau, o, r)\ \mathit{in}\, p_2 : C\{v/^{\mathsf{src}}\tau_o^r\}}\,\mathsf{Let}$$

To deduce that the following judgement is derivable:

$$\Delta, u : A; \Gamma \vdash p_2\{v/^{\mathsf{tgt}}\tau_o^r\} : C\{v/^{\mathsf{src}}\tau_o^r\}$$

we resort to the i.h. w.r.t. (c).
Consider the following abbreviations:

$$\begin{array}{l}
\mathfrak{o}_u := \mathfrak{o}\{u/\rho_s^t\} \\
\tau_u^{\mathsf{tgt}} := \tau\{u/^{\mathsf{tgt}}\rho_s^t\} \\
\mu := \mathbf{bb}(!(\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}), v.p_2\{u/^{\mathsf{tgt}}\rho_s^t\})
\end{array}$$

For (a) we reason as follows:

$$\begin{array}{rl}
 & \Delta;\Gamma \vdash \sigma\{u/^{\mathsf{tgt}}\rho_s^t\} : p\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd q\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{bb}(!(\tau, o, r), v.p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} : (\mathit{let}\ v \doteq !(\tau, o, r)\ \mathit{in}\, p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{bb}(!(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\}, v.p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : (\mathit{let}\ v \doteq !(\tau, o, r)\ \mathit{in}\, p_2)\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{tgt}}\rho_s^t\} : B\{u/^{\mathsf{src}}\rho_s^t\} \\
= & \Delta;\Gamma \vdash \mathbf{bb}(!(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\}, v.p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : \mathit{let}\ v \doteq !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\}\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{src}}\rho_s^t\} : r\{u/^{\mathsf{tgt}}\rho_s^t\}!(\mathfrak{o}_u; p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{tgt}}\rho_s^t \\
= & \Delta;\Gamma \vdash \mathbf{bb}(!(\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\}), v.p_2\{u/^{\mathsf{tgt}}\rho_s^t\}) : \mathit{let}\ v \doteq !(\tau, o, r)\{u/^{\mathsf{tgt}}\rho_s^t\}\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{src}}\rho_s^t\} : r\{u/^{\mathsf{tgt}}\rho_s^t\}!(\mathfrak{o}_u; p_2\{v \\
= & \Delta;\Gamma \vdash \mu : \mathit{let}\ v \doteq !(\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\})\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{src}}\rho_s^t\} : r\{u/^{\mathsf{tgt}}\rho_s^t\}!(\mathfrak{o}_u; p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{tgt}}\rho_s^t\}, p_2\{v/^{\mathsf{tgt}}\tau_o^r\}\{u/^{\mathsf{src}}
\end{array}$$

$$\dfrac{\begin{array}{c}\Delta; \Gamma \vdash \mathfrak{o}_u; \tau_u^{\mathsf{tgt}} : o\{u/^{\mathsf{src}}\rho_s^t\} \rhd r\{u/^{\mathsf{tgt}}\rho_s^t\} : D \\ \Delta, v : D; \Gamma \vdash p_2\{u/^{\mathsf{tgt}}\rho_s^t\} : C\{u/^{\mathsf{src}}\rho_s^t\}\end{array}}{\Delta, u : A; \Gamma \vdash \mu : \mathit{let}\ v \doteq !(\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}, o\{u/^{\mathsf{src}}\rho_s^t\}, r\{u/^{\mathsf{tgt}}\rho_s^t\})\ \mathit{in}\, p_2\{u/^{\mathsf{tgt}}\rho_s^t\} \rhd p_2\{u/^{\mathsf{tgt}}\rho_s^t\}\{v/^{\mathsf{tgt}}\tau_o^r\} : C\{u/^{\mathsf{src}}\rho_s^t\}\{v/^{\mathsf{src}}\mathfrak{o}_u; \tau_u^{\mathsf{tgt}}{}^{r\{u/^{\mathsf{tgt}}\rho_s^t\}}_{o\{u/^{\mathsf{src}}\rho_s^t\}}\}}\,\mathsf{R}\text{-}\beta_\Box$$

We use Lem. 1.11 to conclude.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma, a : B_1 \vdash \sigma_1 : p_1 \rhd q_1 : B_2}{\Delta, u : A; \Gamma \vdash \lambda a^{\sigma_1}.B_1 : \lambda a^{p_1}.B_1 \rhd \lambda a^{q_1}.B_1 : B_1 \supset B_2}\ \text{R-Abs}$$

The last four items are immediate. For item (b) we conclude from the i.h. and an application of Abs that $\blacktriangleright\ \Delta, u : A; \Gamma \vdash \lambda a^{p_1}.B_1 : B_1 \supset B_2$. Likewise for $\blacktriangleright\ \Delta, u : A; \Gamma \vdash \lambda a^{q_1}.B_1 : B_1 \supset B_2$. For item (a), we proceed as in the previous cases.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash \sigma_1 : p_1 \rhd q_1 : C \supset B \quad \Delta, u : A; \Gamma \vdash \sigma_2 : p_2 \rhd q_2 : C}{\Delta, u : A; \Gamma \vdash \sigma_1\, \sigma_2 : p_1\, p_2 \rhd q_1\, q_2 : B}\ \text{R-App}$$

The last four items are immediate. For item (b) we conclude from the i.h. (twice) and an application of App that $\blacktriangleright\ \Delta, u : A; \Gamma \vdash p_1\, p_2 : B$. Likewise for $\blacktriangleright\ \Delta, u : A; \Gamma \vdash q_1\, q_2 : B$. For item (a), we proceed as in the previous cases.

- The derivation of $\Delta, u : A; \Gamma \vdash \sigma : p \rhd q : B$ ends in:

$$\frac{\Delta, u : A; \Gamma \vdash \sigma_1 : p_1 \rhd q_1 : [\![\mu, m, n]\!]D \quad \Delta, u : A, v : D; \Gamma \vdash \sigma_2 : p_2 \rhd q_2 : C}{\Delta, u : A; \Gamma \vdash let\ v \doteq \sigma_1\ in\, \sigma_2 : let\ v \doteq p_1\ in\, p_2 \rhd let\ v \doteq q_1\ in\, q_2 : C\{v/^{\mathsf{src}}\mu^n_m\}}\ \text{R-Let}$$

The last three items are immediate. For item (b) we conclude from the i.h. (twice) and an application of Let that $\blacktriangleright\ \Delta, u : A; \Gamma \vdash let\ v \doteq p_1\ in\, p_2 : C\{v/^{\mathsf{src}}\mu^n_m\}$. Likewise for $\blacktriangleright\ \Delta, u : A; \Gamma \vdash let\ v \doteq q_1\ in\, q_2 : C\{v/^{\mathsf{src}}\mu^n_m\}$. For item (a), we proceed as in the previous cases.

- The derivation ends in

$$\frac{\Delta, u : A; \Gamma \vdash S : B \quad \Delta, u : A \vdash B \preceq C}{\Delta, u : A; \Gamma \vdash S : C}\ \text{Subs}$$

We use the i.h. w.r.t. (a)-(e) on the one hand and (f) on the other; then we apply Subs.

- The derivation ends in

$$\frac{}{\Delta, u : A \vdash P \preceq P}\ \text{S-PVar}$$

Immediate.

- The derivation ends in

$$\frac{\Delta, u : A \vdash A' \preceq A \quad \Delta, u : A \vdash B \preceq B'}{\Delta, u : A \vdash A \supset B \preceq A' \supset B'}\ \text{S-Arrow}$$

We use the i.h. w.r.t. (f) and then S-Arrow.

- The derivation ends in

$$\frac{\begin{array}{c}\Delta, u : A; \cdot \vdash p, q, r : B \\ \Delta, u : A; \cdot \vdash \rho : p \rhd q : B \\ \Delta, u : A; \cdot \vdash \sigma : q \rhd r : B \\ \Delta, u : A \vdash B \preceq C\end{array}}{\Delta, u : A \vdash [\![\rho; \sigma, p, r]\!]B \preceq [\![\rho, p, q]\!]C}\ \text{S-Box}$$

By the i.h. w.r.t. (c)

$$\blacktriangleright\ \Delta; \cdot \vdash p\{u/^{\mathsf{tgt}}\rho^t_s\}, q\{u/^{\mathsf{tgt}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \tag{11}$$

By the i.h. w.r.t. (a)

$$\begin{array}{l}\blacktriangleright\ \Delta; \cdot \vdash \rho\{u/^{\mathsf{tgt}}\rho^t_s\} : p\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \\ \blacktriangleright\ \Delta; \cdot \vdash \sigma\{u/^{\mathsf{tgt}}\rho^t_s\} : q\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\}\end{array} \tag{12}$$

By the i.h. w.r.t. (d):

$$\blacktriangleright\ \Delta; \cdot \vdash \mathfrak{p}\{u/\rho^t_s\} : p\{u/^{\mathsf{src}}\rho^t_s\} \rhd p\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \tag{13}$$

By the i.h. w.r.t. (f):

$$\blacktriangleright\ \Delta \vdash B\{u/^{\mathsf{src}}\rho^t_s\} \preceq C\{u/^{\mathsf{src}}\rho^t_s\} \tag{14}$$

By (12) and (14) we deduce:

$$\begin{array}{l}\blacktriangleright\ \Delta; \cdot \vdash \rho\{u/^{\mathsf{tgt}}\rho^t_s\} : p\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd q\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\} \\ \blacktriangleright\ \Delta; \cdot \vdash \sigma\{u/^{\mathsf{tgt}}\rho^t_s\} : q\{u/^{\mathsf{tgt}}\rho^t_s\} \rhd r\{u/^{\mathsf{tgt}}\rho^t_s\} : B\{u/^{\mathsf{src}}\rho^t_s\}\end{array} \tag{15}$$

We have to prove:

$$\begin{array}{rl} & \Delta \vdash ([\![\rho; \sigma, p, r]\!]B)\{u/^{\mathsf{src}}\rho^t_s\} \preceq ([\![\rho, p, q]\!]C)\{u/^{\mathsf{src}}\rho^t_s\} \\ = & \Delta \vdash [\![\mathfrak{p}\{u/\rho^t_s\}; \rho\{u/^{\mathsf{tgt}}\rho^t_s\}; \sigma\{u/^{\mathsf{tgt}}\rho^t_s\}, p\{u/^{\mathsf{src}}\rho^t_s\}, r\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]B\{u/^{\mathsf{src}}\rho^t_s\} \preceq [\![\mathfrak{p}\{u/\rho^t_s\}; \rho\{u/^{\mathsf{tgt}}\rho^t_s\}, p\{u/^{\mathsf{src}}\rho^t_s\}, q\{u/^{\mathsf{tgt}}\rho^t_s\}]\!]C\{u/^{\mathsf{src}}\rho^t_s\}\end{array}$$

We conclude using S-Box (where $\rho := \mathfrak{p}\{u/\rho_s^t\};\rho\{u/^{\mathsf{tgt}}\rho_s^t\}$ and $\sigma := \sigma\{u/^{\mathsf{tgt}}\rho_s^t\}$).

- If the derivation ends in Eq-PVar or Eq-Arrow we conclude imemdiately from the i.h.. If the derivation ends in:

$$\frac{\Delta;\cdot \vdash s,t,p,q : A \quad \Delta;\cdot \vdash \rho,\sigma : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \simeq B}{\Delta \vdash [\![\rho,s,t]\!]A \simeq [\![\sigma,p,q]\!]B}\text{Eq-Bang}$$

We resort to the i.h. w.r.t item (d) and Lem. 1.7 (Structural Equivalence is closed under substitution of rewrite variables).

□

Lemma. [Lem. 2.6] ► $\Delta;\Gamma \vdash \rho : s \rhd t : A$ implies ► $\Delta;\Gamma \vdash s : A$ and ► $\Delta;\Gamma \vdash t : A$.

Proof. By induction on the derivation of $\Delta;\Gamma \vdash \rho : s \rhd t : A$.

- R-Refl-TVar and R-Refl-VVar. We conclude immediately from TVar and VVar.
- The derivation ends in:

$$\frac{\Delta;\cdot \vdash r,s,t : A \quad \Delta;\cdot \vdash \rho_1 : r \rhd s : A \quad \Delta;\cdot \vdash \rho_2 : s \rhd t : A}{\Delta;\Gamma \vdash \langle \rho_1|_r \rho_2\rangle : !(\rho_1,r,s) \rhd !(\rho_1;\rho_1,r,t) : [\![\rho_1,r,s]\!]A}\text{R-Bang}$$

Then for $!(\rho_1,r,s)$ we derive:

$$\frac{\Delta;\cdot \vdash r,s : A \quad \Delta;\cdot \vdash \rho_1 : r \rhd s : A}{\Delta;\Gamma \vdash !(\rho_1,r,s) : [\![\rho_1,r,s]\!]A}\text{Bang}$$

For $!(\rho_1;\rho_2,r,t)$ we obtain the following derivation π:

$$\frac{\Delta;\cdot \vdash r,t : A \qquad \dfrac{\Delta;\cdot \vdash \rho_1 : r \rhd s : A \quad \Delta;\cdot \vdash \rho_2 : s \rhd t : A}{\Delta;\cdot \vdash \rho_1;\rho_2 : r \rhd t : A}\text{R-Trans}}{\Delta;\Gamma \vdash !(\rho_1;\rho_2,r,t) : [\![\rho_1;\rho_2,r,t]\!]A}\text{Bang}$$

We conclude from π and subsumption:

$$\frac{\blacktriangleright_\pi\ \Delta;\Gamma \vdash !(\rho_1;\rho_2,r,t) : [\![\rho_1;\rho_2,r,t]\!]A \quad \Delta \vdash [\![\rho_1;\rho_2,r,t]\!]A \preceq [\![\rho_1,r,s]\!]A}{\Delta;\Gamma \vdash !(\rho_1;\rho_2,r,t) : [\![\rho_1,r,s]\!]A}\text{Subs}$$

- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s \rhd r : A \quad \Delta;\Gamma \vdash \rho_2 : r \rhd t : A}{\Delta;\Gamma \vdash \rho_1;\rho_2 : s \rhd t : A}\text{R-Trans}$$

We conclude from the i.h..

- The derivation ends in:

$$\frac{\Delta;\Gamma,a:A \vdash p : B \quad \Delta;\Gamma \vdash q : A}{\Delta;\Gamma \vdash \mathbf{ba}(a.p,q) : (\lambda a.p)\,q \rhd p\{a/q\} : B}\text{R-}\beta$$

For $(\lambda a.p)\,q$ we use Abs and then App. For $p\{a/q\}$ we use the Truth Substitution Lemma (Lem. 2.4(c)).

- The derivation ends in:

$$\frac{\Delta;\cdot \vdash \rho_1 : p \rhd q : A \quad \Delta,u:A;\Gamma \vdash r : C}{\Delta;\Gamma \vdash \mathbf{bb}(!(\rho_1,p,q),u.r) : \mathit{let}\ u \doteq !(\rho_1,p,q)\ \mathit{in}\, r \rhd r\{u/^{\mathsf{tgt}}\rho_{1p}^{\ q}\} : C\{u/^{\mathsf{src}}\rho_{1p}^{\ q}\}}\text{R-}\beta_\Box$$

By the i.h. on $\Delta;\cdot \vdash \rho_1 : p \rhd q : A$, we deduce $\Delta;\cdot \vdash p : A$ and $\Delta;\cdot \vdash q : A$. For $\mathit{let}\ u \doteq !(\rho_1,p,q)\ \mathit{in}\, r$ we use Bang, then Let. For $r\{u/^{\mathsf{src}}\rho_{1p}^{\ q}\}$ we use the Validity Substitution Lemma (Lem. 2.5).

- The derivation ends in:

$$\frac{\Delta;\Gamma,a:A \vdash \rho_1 : p \rhd q : B}{\Delta;\Gamma \vdash \lambda a.\rho_1 : \lambda a.p \rhd \lambda a.q : A \supset B}\text{R-Abs}$$

We conclude from the i.h. and an application of Abs.

- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s_1 \rhd t_1 : A \supset B \quad \Delta;\Gamma \vdash \rho_2 : s_2 \rhd t_2 : A}{\Delta;\Gamma \vdash \rho_1\,\rho_2 : s_1\,s_2 \rhd t_1\,t_2 : B}\text{R-App}$$

We conclude from the i.h. (twice) and an application of App.

- The derivation ends in:

$$\frac{\Delta;\Gamma \vdash \rho_1 : s_1 \rhd t_1 : [\![\tau,p,q]\!]A \quad \Delta,u:A;\Gamma \vdash \rho_2 : s_2 \rhd t_2 : C}{\Delta;\Gamma \vdash \mathit{let}\ u \doteq \rho_1\ \mathit{in}\,\rho_2 : \mathit{let}\ u \doteq s_1\ \mathit{in}\,s_2 \rhd \mathit{let}\ u \doteq t_1\ \mathit{in}\,t_2 : C\{u/^{\mathsf{src}}\tau_p^q\}}\text{R-Let}$$

We conclude from the i.h. (twice) and an application of Let.

- The derivation ends in
$$\frac{\Delta; \Gamma \vdash \tau : m \rhd n : A \quad \Delta \vdash A \leq B}{\Delta; \Gamma \vdash \tau : m \rhd n : B} \text{ Subs}$$
By the i.h. on $\Delta; \Gamma \vdash \tau : m \rhd n : A$ we deduce $\Delta; \Gamma \vdash m : A$ and $\Delta; \Gamma \vdash n : A$. We conclude from an application of Subs.
- The derivation ends in:
$$\frac{\Delta; \Gamma \vdash \rho : s \rhd t : A \qquad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad A \simeq B}{\Delta; \Gamma \vdash \sigma : p \rhd q : B} \text{ SEq-R}$$
From the i.h. on $\Delta; \Gamma \vdash \rho : s \rhd t : A$, we deduce $\Delta; \Gamma \vdash s : A$ and $\Delta; \Gamma \vdash t : A$. We conclude from SEq-T that $\Delta; \Gamma \vdash p : B$ and $\Delta; \Gamma \vdash q : B$.

$\square$

$$\frac{\Delta;\Gamma \vdash S : A \quad \Delta \vdash A \preceq B}{\Delta;\Gamma \vdash S : B}\ \mathsf{Subs}$$

$$\frac{\Delta;\Gamma \vdash s : A \quad s \simeq t \quad \Delta \vdash A \simeq B}{\Delta;\Gamma \vdash t : B}\ \mathsf{SEq\text{-}T} \qquad \frac{\Delta;\Gamma \vdash \rho : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \simeq B}{\Delta;\Gamma \vdash \sigma : p \rhd q : B}\ \mathsf{SEq\text{-}R}$$

$$\frac{}{\Delta \vdash P \preceq P}\ \mathsf{S\text{-}PVar} \qquad \frac{\Delta \vdash A' \preceq A \quad \Delta \vdash B \preceq B'}{\Delta \vdash A \supset B \preceq A' \supset B'}\ \mathsf{S\text{-}Arrow} \qquad \frac{\Delta;\cdot \vdash p,q,r : A \quad \Delta;\cdot \vdash \rho : p \rhd q : A \quad \Delta;\cdot \vdash \sigma : q \rhd r : A \quad \Delta \vdash A \preceq B}{\Delta \vdash [\![\rho;\sigma,p,r]\!]A \preceq [\![\rho,p,q]\!]B}\ \mathsf{S\text{-}Box}$$

$$\frac{}{\Delta \vdash P \simeq P}\ \mathsf{Eq\text{-}PVar} \qquad \frac{\Delta \vdash A \simeq A' \quad \Delta \vdash B \simeq B'}{\Delta \vdash A \supset B \simeq A' \supset B'}\ \mathsf{Eq\text{-}Arrow}$$

$$\frac{\Delta;\cdot \vdash s,t,p,q : A \quad \Delta;\cdot \vdash \rho,\sigma : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \simeq B}{\Delta \vdash [\![\rho,s,t]\!]A \simeq [\![\sigma,p,q]\!]B}\ \mathsf{Eq\text{-}Bang}$$

Figure 6: Subtyping and Equivalence Rules

$$\frac{\Delta;\Gamma \vdash s : A \quad s \simeq t \quad \Delta \vdash A \precsim B}{\Delta;\Gamma \vdash t : B}\ \mathsf{SEq\text{-}T'} \qquad \frac{\Delta;\Gamma \vdash \rho : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \precsim B}{\Delta;\Gamma \vdash \sigma : p \rhd q : B}\ \mathsf{SEq\text{-}R'}$$

$$\frac{}{\Delta \vdash P \precsim P}\ \mathsf{SEq\text{-}PVar} \qquad \frac{\Delta \vdash A' \precsim A \quad \Delta \vdash B \precsim B'}{\Delta \vdash A \supset B \precsim A' \supset B'}\ \mathsf{SEq\text{-}Arrow}$$

$$\frac{\Delta;\cdot \vdash p,p',q,q',r : A \quad \Delta;\cdot \vdash \rho : p \rhd q : A \quad \Delta;\cdot \vdash \sigma : q \rhd r : A \quad \Delta;\cdot \vdash \tau : q \rhd r : A \quad \rho \simeq \tau;\mu : p \rhd r \quad p \simeq p' \quad q \simeq q' \quad \Delta \vdash A \precsim B}{\Delta \vdash [\![\rho,p,r]\!]A \precsim [\![\tau,p',q']\!]B}\ \mathsf{SEq\text{-}Box}$$

Figure 7: Alternative Subtyping and Equivalence Rules

Let us call the subset of rules of RC of Fig. 6 R_1. Consider also the rules R_2 of Fig. 7.

Lemma B.2. *(a) $\Delta \vdash A \preceq B$ implies $\Delta \vdash A \precsim B$*
(b) $\Delta \vdash A \simeq B$ implies $\Delta \vdash A \precsim B$

Proof. Both items are by induction on the derivation of corresponding judgement. □

Lemma B.3. *$\Delta \vdash A \precsim B$ implies there exists $n \geq 2$ and $C_1, \ldots C_n$ s.t. $C_1 = A$ and $C_n = B$ and $\Delta \vdash C_1 \bowtie_1 C_2, \Delta \vdash C_2 \bowtie_2 C_3, \ldots, \Delta \vdash C_{n-1} \bowtie_{n-1} C_n$, where $\Delta \vdash D \bowtie_i E$ means either $\Delta \vdash D \preceq E$ or $\Delta \vdash D \simeq E$.*

Proof. By induction on the derivation of $\Delta \vdash A \precsim B$.

- The derivation ends in:
$$\frac{}{\Delta \vdash P \precsim P}\ \mathsf{SEq\text{-}PVar}$$
Then we have $\Delta \vdash P \preceq P$.
- The derivation ends in:
$$\frac{\Delta \vdash A' \precsim A \quad \Delta \vdash B \precsim B'}{\Delta \vdash A \supset B \precsim A' \supset B'}\ \mathsf{SEq\text{-}Arrow}$$
By i.h. there exists:
  - $m \geq 2$ and $C_1, \ldots C_m$ s.t. $C_1 = A'$ and $C_m = A$ and $\Delta \vdash C_1 \bowtie_1 C_2, \Delta \vdash C_2 \bowtie_2 C_3, \ldots, \Delta \vdash C_{m-1} \bowtie_{m-1} C_m$,
  - $n \geq 2$ and $D_1, \ldots D_n$ s.t. $D_1 = B$ and $D_n = B'$ and $\Delta \vdash D_1 \bowtie_m D_2, \Delta \vdash D_2 \bowtie_{m+1} D_3, \ldots, \Delta \vdash D_{n-1} \bowtie_{m+n-2} D_n$.

  We proceed as follows. First we perform induction on m to show that
$$\Delta \vdash C_m \supset B \bowtie C_{m-1} \supset B, \ldots, \Delta \vdash C_{m-1} \supset B \bowtie C_{m-2} \supset B, \Delta \vdash C_2 \supset B \bowtie C_1 \supset B \tag{16}$$
Suppose by the i.h. w.r.t. m we have:

$$\Delta \vdash C_m \supset B \bowtie C_{m-1} \supset B, \Delta \vdash C_{m-1} \supset B \bowtie C_{m-2} \supset B, \ldots, \Delta \vdash C_3 \supset B \bowtie C_2 \supset B$$

We reason as follows:

- If $\bowtie_1 = \preceq$, we have

$$\frac{\Delta \vdash C_1 \preceq C_2 \quad \Delta \vdash B \preceq B}{\Delta \vdash C_2 \supset B \preceq C_1 \supset B}\ \text{SEq-Arrow}$$

- If $\bowtie_1 = \simeq$, we have

$$\frac{\dfrac{\Delta \vdash C_1 \simeq C_2}{\Delta \vdash C_2 \simeq C_1}\ \text{Eq-Symm} \quad \Delta \vdash B \simeq B}{\Delta \vdash C_2 \supset B \simeq C_1 \supset B}\ \text{Eq-Arrow}$$

Now we perform induction on n to show that

$$\Delta \vdash A' \supset D_1 \bowtie A' \supset D_2, \ldots, \Delta \vdash A' \supset D_{n-2} \bowtie A' \supset D_{n-1}, \Delta \vdash A' \supset D_{n-1} \bowtie A' \supset D_n \tag{17}$$

Suppose by the i.h. w.r.t. n we have:

$$\Delta \vdash A' \supset D_1 \bowtie A' \supset D_2, \ldots, \Delta \vdash A' \supset D_{n-3} \bowtie A' \supset D_{n-2}, \Delta \vdash A' \supset D_{n-2} \bowtie A' \supset D_{n-1}$$

We reason as follows:

- If $\bowtie_{m+n-2} = \preceq$, we have

$$\frac{\Delta \vdash A' \preceq A' \quad \Delta \vdash D_{n-1} \preceq D_n}{\Delta \vdash A' \supset D_{n-1} \preceq A' \supset D_n}\ \text{SEq-Arrow}$$

- If $\bowtie_{m+n-2} = \simeq$, we have

$$\frac{\Delta \vdash A' \simeq A' \quad \Delta \vdash D_{n-1} \simeq D_n}{\Delta \vdash A' \supset D_{n-1} \simeq A' \supset D_n}\ \text{Eq-Arrow}$$

We conclude from (16) and (17).

- The derivation ends in:

$$\frac{\Delta; \cdot \vdash p, p', q, q', r : A \quad \Delta; \cdot \vdash \rho : p \rhd r : A \quad \Delta; \cdot \vdash \tau : p \rhd q : A \quad \Delta; \cdot \vdash \mu : q \rhd r : A \quad \rho \simeq \tau; \mu : p \rhd r \quad p \simeq p' \quad q \simeq q' \quad \Delta \vdash A \precsim B}{\Delta \vdash [\![\rho, p, r]\!]A \precsim [\![\tau, p', q']\!]B}\ \text{SEq-Box}$$

Notice that $\Delta; \cdot \vdash \tau; \mu : p \rhd r : A$. Then we can resort to Eq-Bang and infer:

$$\frac{\Delta; \cdot \vdash p, r : A \quad \Delta; \cdot \vdash \rho, (\tau; \mu) : p \rhd r : A \quad \rho \simeq \tau; \mu : p \rhd r \quad p \simeq p \quad r \simeq r \quad \Delta \vdash A \simeq A}{\Delta \vdash [\![\rho, p, r]\!]A \simeq [\![\tau; \mu, p, r]\!]A}\ \text{Eq-Bang}$$

Then S-Box and infer:

$$\frac{\Delta; \cdot \vdash p, q, r : A \quad \Delta; \cdot \vdash \tau : p \rhd q : A \quad \Delta; \cdot \vdash \mu : q \rhd r : A \quad \Delta \vdash A \preceq A}{\Delta \vdash [\![\tau; \mu, p, r]\!]A \preceq [\![\tau, p, q]\!]A}\ \text{S-Box}$$

Then we can resort to Eq-Bang and infer:

$$\frac{\Delta; \cdot \vdash p, q, p', q' : A \quad \Delta; \cdot \vdash \tau, (\tau; \mathfrak{q}) : p \rhd q : A \quad \tau \simeq \tau; \mathfrak{q} : p \rhd q \quad p \simeq p' \quad q \simeq q' \quad \Delta \vdash A \simeq A}{\Delta \vdash [\![\tau, p, q]\!]A \simeq [\![\tau, p', q']\!]A}\ \text{Eq-Bang}$$

By the i.h. on $\Delta \vdash A \precsim B$ we obtain:

$$\Delta \vdash C_1 \bowtie C_2, \ldots, \Delta \vdash C_2 \bowtie C_3, \Delta \vdash C_{n-1} \bowtie C_n \tag{18}$$

where $C_1 = A$ and $C_n = B$. Then, for some $m \geq n$, we can obtain:

$$\begin{array}{l} \Delta \vdash [\![\rho, p, r]\!]A \simeq [\![\tau; \mu, p, r]\!]A \\ \Delta \vdash [\![\tau; \mu, p, r]\!]A \preceq [\![\tau, p, q]\!]A \\ \Delta \vdash [\![\tau, p, q]\!]A \simeq [\![\tau, p', q']\!]A \\ \Delta \vdash [\![\tau, p', q']\!]C_1 \bowtie [\![\tau, p', q']\!]C_2, \ldots, \Delta \vdash [\![\tau, p', q']\!]C_2 \bowtie [\![\tau, p', q']\!]C_3, \Delta \vdash [\![\tau, p', q']\!]C_{m-1} \bowtie [\![\tau, p', q']\!]C_m \end{array} \tag{19}$$

The reason that m might be greater than n is that if $\Delta \vdash E \preceq F$, then we can't prove $\Delta \vdash [\![\rho, s, t]\!]E \preceq [\![\rho, s, t]\!]F$ directly, but rather using Eq-Bang as $\Delta \vdash [\![\rho, s, t]\!]E \simeq [\![\rho; \mathrm{t}, s, t]\!]E$ and then $\Delta \vdash [\![\rho; \mathrm{t}, s, t]\!]E \preceq [\![\rho, s, t]\!]F$

□

Lemma B.4. *(a) Consider $\Delta; \Gamma \vdash \rho : s \rhd t : A$. It is derivable in R_1 iff it is derivable in R_2.*
(b) Consider $\Delta; \Gamma \vdash s : A$. It is derivable in R_1 iff it is derivable in R_2.

Proof. For the first item we reason as follows. For an instance of Subs with S a term s, we use SEq-T' and rely on $s \simeq s$ and Lem. B.2(a). For an instance of Subs with S a rewrite $\rho : s \rhd t$, we use SEq-R' and rely on $s \simeq s$, $t \simeq t$, $\rho \simeq \rho : s \rhd t$ and Lem. B.2(a).
For an instance of SEq-T we use SEq-T' and rely on Lem. B.2(b). For an instance of SEq-R, we use SEq-R' and rely on Lem. B.2(b).
We now address item two. Suppose the derivation ends in:

$$\frac{\Delta; \Gamma \vdash s : A \quad s \simeq t \quad \Delta \vdash A \precsim B}{\Delta; \Gamma \vdash t : B}\ \text{SEq-T'}$$

First we apply Lem. B.3 to $\Delta \vdash A \precsim B$ to deduce that there exists $n \geq 2$ and $C_1, \ldots C_n$ s.t. $C_1 = A$ and $C_n = B$ and $\Delta \vdash C_1 \bowtie_1 C_2$, $\Delta \vdash C_2 \bowtie_2 C_3, \ldots, \Delta \vdash C_{n-1} \bowtie_{n-1} C_n$, where $\Delta \vdash D \bowtie_i E$ means either $\Delta \vdash D \preceq E$ or $\Delta \vdash D \simeq E$. We then resort to SEq-T (with $s \simeq s$) and Subs, to deduce $\Delta; \Gamma \vdash s : B$. Finally, we conclude with SEq-T using $s \simeq t$.
Next suppose the derivation ends in:

$$\frac{\Delta; \Gamma \vdash \rho : s \rhd t : A \quad \rho \simeq \sigma : s \rhd t \quad s \simeq p \quad t \simeq q \quad \Delta \vdash A \precsim B}{\Delta; \Gamma \vdash \sigma : p \rhd q : B}\ \text{SEq-R'}$$

First we apply Lem. B.3 to $\Delta \vdash A \precsim B$ to deduce that there exists $n \geq 2$ and $C_1, \ldots C_n$ s.t. $C_1 = A$ and $C_n = B$ and $\Delta \vdash C_1 \bowtie_1 C_2$, $\Delta \vdash C_2 \bowtie_2 C_3, \ldots, \Delta \vdash C_{n-1} \bowtie_{n-1} C_n$, where $\Delta \vdash D \bowtie_i E$ means either $\Delta \vdash D \preceq E$ or $\Delta \vdash D \simeq E$. We then resort to SEq-R (with $\rho \simeq \rho : s \rhd t$, $s \simeq s$ and $t \simeq t$) and Subs, to deduce $\Delta; \Gamma \vdash \rho : s \rhd t : B$. Finally, we conclude with SEq-R using $\rho \simeq \sigma : s \rhd t$, $s \simeq p$ and $t \simeq q$. □

Lemma B.5 (Generation for Structural Equivalence on Terms). *(a) $s \simeq \lambda a.t$ implies $s = \lambda a.r$ and $r \simeq t$.*
(b) $\lambda a.t \simeq s$ implies $s = \lambda a.r$ and $t \simeq r$.
(c) $s \simeq t_1\, t_2$ implies $s = s_1\, s_2$ and $s_1 \simeq t_1$ and $s_2 \simeq t_2$.
(d) $t_1\, t_2 \simeq s$ implies $s = s_1\, s_2$ and $t_1 \simeq s_1$ and $t_2 \simeq s_2$.
(e) $s \simeq \text{let } u \doteq t_1 \text{ in } t_2$ implies $s = \text{let } u \doteq s_1 \text{ in } s_2$ and $s_1 \simeq t_1$ and $s_2 \simeq t_2$.
(f) $\text{let } u \doteq t_1 \text{ in } t_2 \simeq s$ implies $s = \text{let } u \doteq s_1 \text{ in } s_2$ and $t_1 \simeq s_1$ and $t_2 \simeq s_2$.
(g) $s \simeq !(\rho, t_1, t_2)$ implies $s = !(\sigma, s_1, s_2)$ and $s_1 \simeq t_1$ and $s_2 \simeq t_2$ and $\rho \simeq \sigma : s_1 \rhd s_2$.
(h) $!(\rho, t_1, t_2) \simeq s$ implies $s = !(\sigma, s_1, s_2)$ and $t_1 \simeq s_1$ and $t_2 \simeq s_2$ and $\rho \simeq \sigma : t_1 \rhd t_2$.

Proof. By induction on the derivation of $p \simeq q$. The derivation can clearly not end in EqT-TVar or EqT-RVar. Also, the cases for symmetry and transitivity are immediate. The remaining cases are:

- The derivation ends in:
$$\frac{s \simeq t}{\lambda a.s \simeq \lambda a.t}\ \text{EqT-Abs}$$
Then only case (a) or (b) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q}{s\, t \simeq p\, q}\ \text{EqT-App}$$
Then only case (c) or (d) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q}{\text{let } u \doteq s \text{ in } t \simeq \text{let } u \doteq p \text{ in } q}\ \text{EqT-Let}$$
Then only case (e) or (f) applies and we conclude immediately.
- The derivation ends in:
$$\frac{s \simeq p \quad t \simeq q \quad \rho \simeq \sigma : s \rhd t}{!(\rho, s, t) \simeq !(\sigma, p, q)}\ \text{EqT-Bang}$$
Then only case (g) or (h) applies and we conclude immediately.

□

Let RC' be $(\mathsf{RC} \setminus R_1) \cup R_2$.

Lemma B.6 (Generation for Typability on Terms for RC'). *(a) $\Delta; \Gamma \vdash \lambda a.s : A$ implies there exists $s', B \supset A'$ s.t. $\Delta; \Gamma, a : B \vdash s' : A'$ and $s \simeq s'$ and $\Delta \vdash B \supset A' \precsim A$.*
(b) $\Delta; \Gamma \vdash s\, t : A$ implies there exists s', t', B, A' s.t. $\Delta; \Gamma \vdash s' : B \supset A'$ and $\Delta; \Gamma \vdash t' : B$ and $s \simeq s'$ and $t \simeq t'$ and $\Delta \vdash A' \precsim A$.
(c) $\Delta; \Gamma \vdash \text{let } u \doteq s \text{ in } t : A$ implies there exists $s', t', B, A', \rho, p, q$ s.t. $\Delta; \Gamma \vdash s' : [\![\rho, p, q]\!]B$ and $\Delta, u : B; \Gamma \vdash t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $\Delta \vdash A'\{u/^{\mathsf{src}}\rho_p^q\} \precsim A$.
(d) $\Delta; \Gamma \vdash !(\rho, s, t) : A$ implies there exists s', t', A', ρ' s.t. $\Delta; \cdot \vdash s', t' : A'$ and $\Delta; \Gamma \vdash \rho' : s' \rhd t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $\Delta \vdash [\![\rho', s', t']\!]A' \precsim A$.

Proof. Consider the first item. The derivation can end in one of two cases:

- Abs and $A = A_1 \supset A_2$:
$$\frac{\Delta; \Gamma, a : A_1 \vdash s : A_2}{\Delta; \Gamma \vdash \lambda a.s : A_1 \supset A_2}\ \mathsf{Abs}$$
We take $s' := s$, $B := A_1$ and conclude.
- SEq-T'
$$\frac{\Delta; \Gamma \vdash t : C \quad t \simeq \lambda a.s \quad \Delta \vdash C \lesssim A}{\Delta; \Gamma \vdash \lambda a.s : A}\ \text{SEq-T'}$$
By Lem. B.5 on $t \simeq \lambda a.s$, there exists t' s.t. $t = \lambda a.t'$ and, moreover, $t' \simeq s$. By the i.h. on $\Delta; \Gamma \vdash t : C$ there exists $t'', B' \supset A''$ s.t. $\Delta; \Gamma, a : B' \vdash t'' : A''$ and $t'' \simeq t'$ and $\Delta \vdash B' \supset A'' \lesssim C$. We set $s' := t''$, $B := B'$ and notice that $t'' \simeq s$, and $\Delta \vdash B' \supset A'' \lesssim A$ from $\Delta \vdash B' \supset A'' \lesssim C$ and $\Delta \vdash C \lesssim A$ and transitivity of subtyping.

Consider the second item. The derivation can end in one of two cases:
- App:
$$\frac{\Delta; \Gamma \vdash s : B \supset A \quad \Delta; \Gamma \vdash t : B}{\Delta; \Gamma \vdash s\, t : A}\ \mathsf{App}$$
We take $s' := s$, $t' := t$, $A' := A$ and conclude.
- SEq-T'
$$\frac{\Delta; \Gamma \vdash r : C \quad r \simeq s\, t \quad \Delta \vdash C \lesssim A}{\Delta; \Gamma \vdash s\, t : A}\ \text{SEq-T'}$$
By Lem. B.5 on $r \simeq s\,t$, there exists r_1, r_2 s.t. $r = r_1\, r_2$ and, moreover, $r_1 \simeq s$ and $r_2 \simeq t$. By the i.h. on $\Delta; \Gamma \vdash r : C$ there exists r'_1, r'_2, B', A'' s.t. $\Delta; \Gamma \vdash r'_1 : B' \supset A''$ and $\Delta; \Gamma \vdash r'_2 : B'$ and $r'_1 \simeq r_1$ and $r'_2 \simeq r_2$ and $\Delta \vdash A'' \lesssim C$. We set $s' := r'_1$, $t' := r'_2$, $B := B'$, $A' := A''$ and notice that $s \simeq r'_1$, $t \simeq r'_2$, and $\Delta \vdash A'' \lesssim A$ from $\Delta \vdash A'' \lesssim C$ and $\Delta \vdash C \lesssim A$ and transitivity of subtyping.

Consider the third item. The derivation can end in one of two cases:
- Let and $A = C\{u/^{\mathsf{src}}\rho^q_p\}$:
$$\frac{\Delta; \Gamma \vdash s : [\![\rho, p, q]\!]D \quad \Delta, u : D; \Gamma \vdash t : C}{\Delta; \Gamma \vdash \mathit{let}\ u \triangleq s\ \mathit{in}\, t : C\{u/^{\mathsf{src}}\rho^q_p\}}\ \mathsf{Let}$$
We take $s' := s$, $t' := t$, $B := D$, $A' := C$.
- SEq-T'
$$\frac{\Delta; \Gamma \vdash r : C \quad r \simeq \mathit{let}\ u \triangleq s\ \mathit{in}\, t \quad \Delta \vdash C \lesssim A}{\Delta; \Gamma \vdash \mathit{let}\ u \triangleq s\ \mathit{in}\, t : A}\ \text{SEq-T'}$$
By Lem. B.5 on $r \simeq \mathit{let}\ u \triangleq s\ \mathit{in}\, t$, there exists r_1, r_2 s.t. $r = \mathit{let}\ u \triangleq r_1\ \mathit{in}\, r_2$ and, moreover, $r_1 \simeq s$ and $r_2 \simeq t$. By the i.h. on $\Delta; \Gamma \vdash r : C$ there exists $r'_1, r'_2, B, A'\rho, p, q$ s.t. $\Delta; \Gamma \vdash s' : [\![\rho, p, q]\!]B$ and $\Delta, u : B; \Gamma \vdash t' : A'$ and $s \simeq s'$ and $t \simeq t'$ and $\Delta \vdash A'\{u/^{\mathsf{src}}\rho^q_p\} \lesssim A$. We set $s' := r'_1$, $t' := r'_2$, $B := B'$ and notice that $s \simeq r'_1$, $t \simeq r'_2$ and $\Delta \vdash A'\{u/^{\mathsf{src}}\rho^q_p\} \lesssim A$ from $\Delta \vdash A'\{u/^{\mathsf{src}}\rho^q_p\} \lesssim C$ and $\Delta \vdash C \lesssim A$ and transitivity of subtyping.

Consider the fourth item. The derivation can end in one of two cases:
- Bang and $A = [\![\rho, s, t]\!]B$:
$$\frac{\Delta; \cdot \vdash s, t : B \quad \Delta; \cdot \vdash \rho : s \rhd t : B}{\Delta; \Gamma \vdash !(\rho, s, t) : [\![\rho, s, t]\!]B}\ \mathsf{Bang}$$
We take $s' := s$, $t' := t$, $A' := B$.
- SEq-T'
$$\frac{\Delta; \Gamma \vdash r : C \quad r \simeq !(\rho, s, t) \quad \Delta \vdash C \lesssim A}{\Delta; \Gamma \vdash !(\rho, s, t) : A}\ \text{SEq-T'}$$
By Lem. B.5 on $r \simeq !(\rho, s, t)$, there exists s_1, t_1, ρ_1 s.t. $r = !(\rho_1, s_1, t_1)$ and, moreover, $s_1 \simeq s$ and $t_1 \simeq t$ and $\rho_1 \simeq \rho : s_1 \rhd t_1$. By the i.h. on $\Delta; \Gamma \vdash r : C$ there exists s_2, t_2, A'', ρ_2 s.t. $\Delta; \cdot \vdash s_2, t_2 : A''$ and $\Delta; \cdot \vdash \rho_2 : s_2 \rhd t_2 : A''$ and $s_2 \simeq s_1$ and $t_2 \simeq t_1$ and $\Delta \vdash [\![\rho_2, s_2, t_2]\!]A'' \lesssim C$. We set $s' := r'_1$, $t' := r'_2$, $B := B'$ and notice that $s \simeq s_2$, $t \simeq t_2$ and $\Delta \vdash [\![\rho_2, s_2, t_2]\!]A'' \lesssim A$ from $\Delta \vdash [\![\rho_2, s_2, t_2]\!]A'' \lesssim C$ and $\Delta \vdash C \lesssim A$ and transitivity of subtyping.

□

Lemma. [Step Typability – Lem. 3.6] $\xi : s \rhd t$ and $\Delta; \Gamma \vdash s : A$ implies $\Delta; \Gamma \vdash \xi : s \rhd t : A$.

Proof. By induction on the derivation of $\xi : s \rhd t$. □