Transient growth in stable collisionless plasma

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The first kinetic study of transient growth for a collisionless homogeneous Maxwellian plasma in a uniform magnetic field is presented. A system which is linearly stable may display transient growth if the linear operator describing its evolution is non-normal so that its eigenvectors are nonorthogonal. In order to include plasma kinetic effects, a Landau fluid model is employed. The linear operator of the model is shown to be non-normal and the results suggest that the non-normality of a collisionless plasma is intrinsically related to its kinetic nature, with the transient growth being more accentuated for smaller scales and higher plasma beta. The results based on linear spectral theory have been confirmed with nonlinear simulations. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3094759]

The stability theory of hot magnetized plasma has been historically based on normal mode analysis, which has led over time to the identification of a great variety of instabilities and waves. The normal mode approach is usually applied to the linear approximation of the dynamical equations of the system, where a small perturbation, that is assumed to have the form of a plane wave $\sim \exp(i\mathbf{k} \cdot x + \omega t)$, is imposed on an initial equilibrium configuration. This procedure generally leads to the formulation of the stability problem as a linear eigenvalue equation, from which a dispersion relation $\omega = \omega(\mathbf{k})$, which relates the wavenumber $\mathbf{k}$ of the perturbation to its complex frequency $\omega = \gamma + i\omega_i$, can be obtained. Normal mode stability analysis reduces ultimately to the study of the sign of γ, and the general statement is that the system is stable for small perturbations when $\gamma \leq 0$ for every $\mathbf{k}$, and unstable otherwise.

However, it is now acknowledged that normal mode analysis can dramatically fail to predict the short-term behavior of a system if the linear operator is non-normal, producing a result which is only valid in the large time asymptotic limit.[1] A non-normal operator $\mathbf{A}$ is one that does not commute with its adjoint: $\mathbf{AA}^* \neq \mathbf{A}^*\mathbf{A}$. If $\mathbf{A}$ is a matrix, this is equivalent to saying that it does not have a complete set of orthogonal eigenvectors. A fundamental characteristic of a system described by a non-normal operator is the possible presence of *transient growth*: an initial perturbation can be amplified by a very large factor in a short-time period, even if the system is stable, i.e., all the normal modes are predicted to damp exponentially. Non-normal linear operators have been extensively studied in hydrodynamics,[1] especially for shear flow configurations. The role of non-normality has been also considered in plasmas, for drift waves,[2] for the resistive Alfven paradox,[3] and for shear flows,[4] but so far only in the fluid plasma scenario. In this paper we present the first investigation of non-normal effects in a model of a stable plasma which includes kinetic effects. We will show that the linear operator that describes the evolution of the plasma is non-normal, and that the non-normality is an intrinsic characteristic of the kinetic treatment of the plasma. Transient growth is more accentuated for smaller scales and higher plasma beta, that is when the magnetohydrodynamics (MHD) description becomes more and more inadequate, and the plasma can be correctly described only with the inclusion of kinetic effects. Moreover, while previous studies focused on MHD drift instabilities, we will study a Maxwellian plasma in a homogeneous magnetic field, i.e., we will focus on a stable plasma. In this case the departure from the evolution predicted by normal mode analysis is most evident since perturbations will grow instead of decay for a certain period of time. A Maxwellian plasma in a homogeneous magnetic field does not dispose of any source of free energy. Therefore, the amplification of a small fluctuation is energetically driven solely by the particular initial condition of the disturbance that perturbing the initial equilibrium, provides a small input of free energy to the system. The inclusion of the short-time dynamics described in the following can completely distort the physics of some phenomena, where the validity of the linear theory is generally accepted. For instance, some recent models have been developed to understand the role of kinetic waves for the small-scale dissipation of turbulent energy in the solar wind.[5] Numerical simulations have shown that the linear approximation is a valid ansatz in this case, and therefore it is a scenario where the results of the present letter might be applicable.

A full kinetic treatment of a plasma would require in principle the use of the Vlasov–Maxwell equations. The standard linear theory for those equations is usually formulated as an initial value problem, and the dispersion relation is found via a nonlinear eigenvalue equation $\mathbf{D} \cdot \mathbf{E} = 0$ (with $\mathbf{D}$ a 3×3 complex matrix, and $\mathbf{E}$ the amplitude of the perturbed electric field, assumed to vary as a normal mode). The Vlasov–Maxwell equation is not yet amenable as a linear eigenvalue problem, and for this reason we will use instead

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the linear equations of the Landau fluid model.[6] This is a hierarchical set of fluid equations for the dynamics of both protons and electrons, neglecting electron inertia, truncated at the fourth-order moments, with a closure relation that evaluates higher-order moments using linear kinetic theory. The Landau fluid model includes linear Landau damping and finite Larmor radius corrections, and it has been shown to correctly reproduce the linear dynamics of kinetic Alfven waves,[7] and of the mirror instability.[8] We have used a nonlinear Landau fluid code to confirm all the predictions of the linear theory, including the non-normal effects we describe here. We study the linear evolution of an electron-proton Maxwellian plasma in a homogeneous magnetic field B_0. The temperature and the density are chosen to be equal for electrons and protons. We follow the formalism for the study of non-normal operators given in the monograph by Trefethen and Embree.[9] The advantage of using a fluid model instead of a fully kinetic one is that, in the version used here, it leads to a linear eigenvalue problem with 16 equations for the following physical quantities: density, velocity, magnetic field, pressure, heat flux, and the gyrotropic part of the fourth-order cumulant tensor. The coefficient matrix is therefore a 16×16 (sparse) complex matrix, which makes the problem computationally affordable without the use of any particular method used for large matrix eigenvalue problems. The entries of the matrix are generated with a symbolic algebra software, and the 16 linear equations of the model will not be reported here, due to lack of space. A detailed description of the model can be found in Ref. 7 and references therein.

Although we do not make use of the normal mode ansatz, we use the Fourier transform in space of all quantities: $\Phi(x,t) = \Phi(t)\exp(i\mathbf{k}\cdot\mathbf{x})$. The linear set of equations can be formulated as $dy(t)/dt = \mathbf{A}y(t)$, where y is the state vector composed of the amplitude of the 16 variables, and the matrix $\mathbf{A}$ is a function of $\mathbf{k}$, and of the plasma beta, $\beta = 8\pi nT/B_0^2$. The solution of the linear equation is given by $y(t) = e^{\mathbf{A}t}y(0)$. We define $G(t) = \|y(t)\|/\|y(0)\|$, where $\|\cdot\|$ is the Euclidean two norm. The quantity $G(t)$ gives the amplification (or reduction) of a perturbation, i.e., the amplitude of that perturbation in time, relative to its initial value: $G(t) = \|e^{\mathbf{A}t}y(0)\|/\|y(0)\|$. The supremum of $G(t)$ over all nonzero vectors $y(0)$ is the standard definition of the norm $\|e^{\mathbf{A}t}\|$. This quantity defines the envelope curve which bounds from above the evolution of $G(t)$ for all possible perturbations. Although in general the amplification of a perturbation could stay well below $\|e^{\mathbf{A}t}\|$, and one single perturbation will not reach maximum amplification for all times, we here use the same assumption that is always implicitly made in linear plasma theory. That is, we assume that all the possible perturbations of the system are excited, and we will focus on the particular one that reaches the maximum possible amplification $M = \max\|e^{\mathbf{A}t}\|$. We notice that the 2-norm of the state vector is not strictly related to the perturbed energy of the system, as it is usually done in works dealing with hydrodynamics nonmodal theory.[1] This is because in our model the state vector y contains variables (such as high order moments) that do not enter in the expression for the energy. The norm of the state vector has therefore to be considered only

as a measure of the perturbation applied to the system. It is obvious that a large (norm of the) perturbation implies a deviation from the assumption of linearity, which could result in the triggering of nonlinear effects. In other words, transient growth of the Euclidian norm of the state vector is physically equivalent to the development of an instability.

The key aspect of non-normal operators is that the spectrum may be highly sensitive to small perturbations. As a consequence it is difficult, or rather improbable, for an initial perturbation to excite only a single mode of the system. This is due to the nonorthogonality of the eigenvectors: a state vector that slightly deviates from lying on a single eigenvector can have large projections on other eigenvectors, thus resulting in the excitement of other modes. This is not the case if the eigenvectors are all mutually orthogonal, as for normal operators. A mathematical tool to characterize this behavior is given by the concept of pseudospectrum, which is a generalization of the standard spectrum. The spectrum $\sigma(\mathbf{A})$ is defined as the set of points $z \in \mathbb{C}$ for which the resolvent matrix $(z\mathbf{I}-\mathbf{A})^{-1}$ does not exist or, conventionally $\|(z\mathbf{I}-\mathbf{A})^{-1}\| = \infty$. The ε-pseudospectrum $\sigma_\varepsilon(\mathbf{A})$ of $\mathbf{A}$ is the set of $z \in \mathbb{C}$ such that $\|(z\mathbf{I}-\mathbf{A})^{-1}\| > \varepsilon^{-1}$ or equivalently, z is an eigenvalue of the matrix $(\mathbf{A}+\mathbf{E})$ for some matrix $\mathbf{E}$ with $\|\mathbf{E}\| < \varepsilon$.[9] So the ε-pseudospectrum gives a measure of how the spectrum is distorted due to a perturbation of the operator of size ε. Physically one can think that a perturbation on the evolution linear operator is equivalent to perturbations or inhomogeneities of quantities such as density or magnetic field.

In passing we note that the concept of pseudospectra has never been emphasized in the analysis of numerical plasma simulations, even though it is a common experience to see "transient effects" at the beginning of simulations (which could also, of course, have other causes). In particular, works that address the decay of a single normal mode should be interpreted within this context since numerical fluctuations (especially in particle-in-cell codes where they are unavoidable) approximately play the role of perturber of the linear operator.

Pseudospectra are a convenient graphical tool for understanding the behavior of an operator. For a normal matrix, the ε-pseudospectrum is just the union of the open ε-balls about the point of the spectrum: $\|(z\mathbf{I}-\mathbf{A})^{-1}\| = 1/\text{dist}[z, \sigma(\mathbf{A})]$, where dist indicates the distance of a point to a set in the complex plane.[9] We plot in Fig. 1 an example of the contours of the ε-pseudospectrum of our Landau fluid operator (not all the eigenvalues are shown). The interpretation of the contours is that a perturbation ε will move the spectrum within the region bounded by the ε-contour [ε-pseudospectra are nested sets so that $\sigma_{\varepsilon_1}(\mathbf{A}) \subseteq \sigma_{\varepsilon_2}(\mathbf{A})$ for $\varepsilon_1 \leq \varepsilon_2$]. It is clear that small perturbations make some normal modes become connected to each other, and result in a distortion of the spectrum. For instance, the contour for $\varepsilon = 10^{-2.9}$ encloses all the nine eigenvalues, which means that the system has completely lost the information about its exactly unperturbed solutions since the solutions for the lightly perturbed system could lie anywhere within the ε-contour. We also plot with a dotted line the ε-contour for $\varepsilon = 0.5$, as it would be if the operator was normal. In order to obtain the
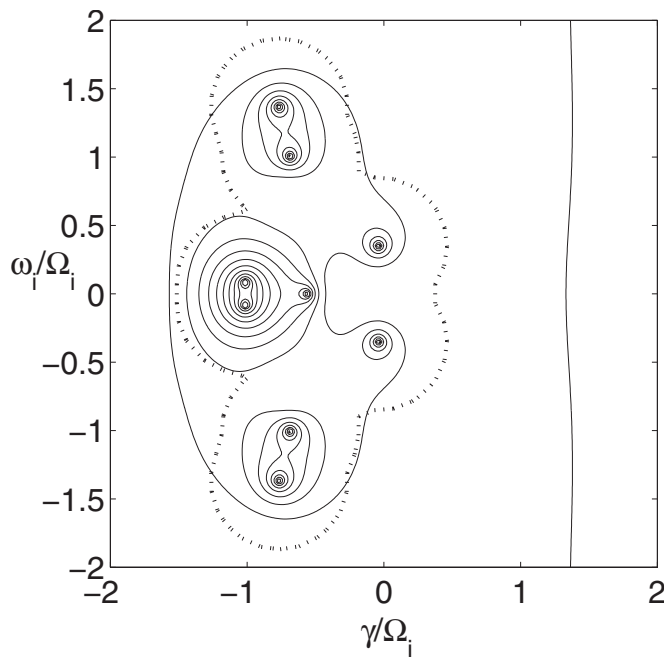
FIG. 1. Contours of the ε-pseudospectrum for a plasma with $\beta=10$ and $k=1$. Contours are plotted for $\log_{10} \varepsilon = -4.9, -4.7, \ldots, -2.7$. The dotted line is how the ε-contour would appear if the operator were normal, for $\varepsilon=0.5$.

FIG. 2. Numerical abscissa η (solid) and departure from normality D (dashed), as a function of k (normalized to the ion Larmor radius). The four curves are for $\beta=0.1, 1, 5,$ and 10 from below to above.

same kind of distortion of the spectrum for a normal operator, the perturbation has to be 400 times larger.

The damping rate of the least damped mode $\alpha(\mathbf{A}) = \max\{\Re[\sigma(\mathbf{A})]\}$ is the object of the normal mode stability analysis, and dictates the behavior at large times: $\lim_{t\to\infty} t^{-1} \log \|e^{\mathbf{A}t}\| = \alpha(\mathbf{A})$. If the linear operator were normal this would be also the damping rate of $\|e^{\mathbf{A}t}\|$ for any initial perturbation for any time $t \geq 0$, and there would be no transient growth. In general however the initial growth of $\|e^{\mathbf{A}t}\|$ is defined as the numerical abscissa $\eta(\mathbf{A}) = \frac{d}{dt}\|e^{\mathbf{A}t}\|\big|_{t=0}$, which is given by the following formula:[9] $\eta(\mathbf{A}) = \sup \sigma\left[\frac{1}{2}(\mathbf{A}+\mathbf{A}^*)\right]$, from which it is evident that $\eta(\mathbf{A}) = \alpha(\mathbf{A})$ for a normal matrix.[10]

The final definition we provide is the "departure from normality" $D(\mathbf{A})$, which is a scalar that defines "how non-normal" an operator is. There are several different ways to characterize $D(\mathbf{A})$, and we refer again to Ref. 9 for more details. In the following we will use the definition due to Henrici: $\mathbf{A}$ can be Schur decomposed $\mathbf{A} = \mathbf{U}(\mathbf{\Lambda}+\mathbf{R})\mathbf{U}^*$, where $\mathbf{U}$ is unitary, $\mathbf{\Lambda}$ is diagonal, and $\mathbf{R}$ is strictly upper triangular. When $\mathbf{R}$ is zero, $\mathbf{A}$ is normal, hence we define $D(\mathbf{A}) = \|\mathbf{R}\|_F$, where $\|\mathbf{A}\|_F = \sqrt{\text{Tr}(\mathbf{A}\mathbf{A}^*)}$ is the Frobenius norm.

We present now a parametric study for a plasma subject to an oblique perturbation with an angle $\theta=70°$ between the wavevector and the background magnetic field. We notice that the Landau fluid model has a domain of validity that extends to small scales only for oblique wavevectors. We span the range $[0.1,10]$ for the values of k and β. In Fig. 2 we show the numerical abscissa $\eta(\mathbf{A})$ (solid line), and the departure from normality $D(\mathbf{A})$ (dashed line) as functions of the wavenumber k. Here and in the following figures, k is normalized to the ion Larmor radius. The four curves are for different values of $\beta=0.1, 1, 5,$ and 10 from the lower to the
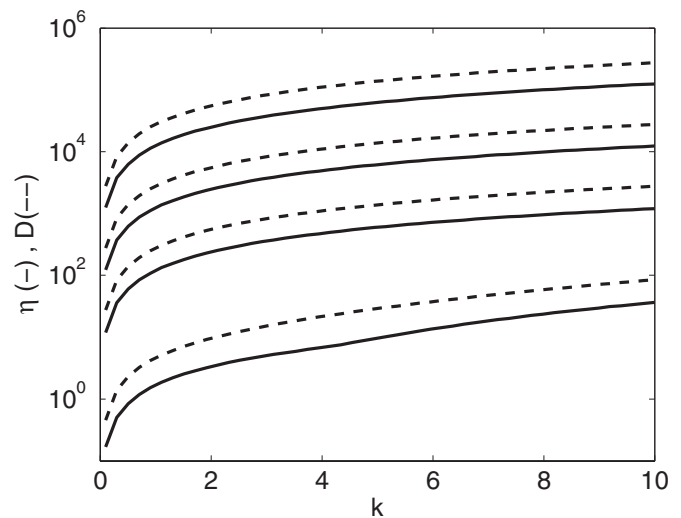
upper curve. All the curves are monotonically increasing denoting that the degree of non-normality increases with k and β.

In order to understand how large the transients can grow, we show in Fig. 3 the value of $G(t)$ for four different cases, with $\beta=1, 10,$ and $k=1, 10$. Figure 3 was produced with values computed via a nonlinear Landau fluid code, and the predictions from the linear theory (not shown) agree perfectly with the nonlinear simulations (any differences would not be noticeable, if plotted on the same figure). The small initial perturbations have been chosen so that max $G(t)=M$. We emphasize that the time scales involved in Fig. 3 are a central point of our argument. One could indeed reasonably ignore all transient effects that take place on times much smaller than the typical time scales of the plasma. However
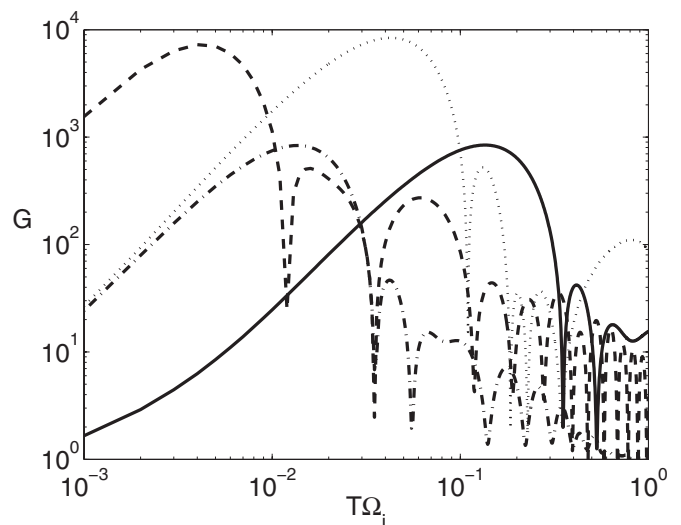

FIG. 3. Amplification G as a function of time T (normalized to the ion cyclotron frequency Ω_i), as computed via a nonlinear Landau fluid code. The initial perturbation is the one that maximizes the amplification. The four curves are for the following parameters: $\beta=1$, $k=1$ (solid line); $\beta=10$, $k=1$ (dotted); $\beta=1$, $k=10$ (dashed-dotted); $\beta=10$, $k=10$ (dashed).
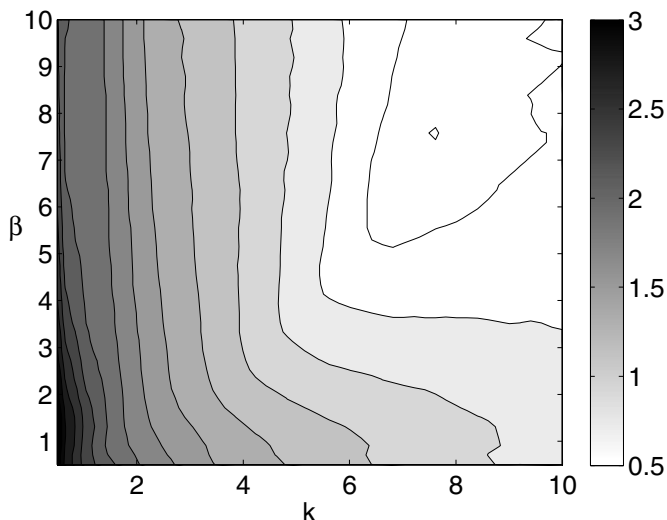
FIG. 4. Contour plot of time Θ for which $\|e^{\mathbf{A}\Theta}\| \leq 1$, that is the time for which transient growth is exhausted, as a function of k and β. Color bar is given in $\log_{10}\Theta$, and normalized to the ion cyclotron frequency Ω_i.

one can see that depending on the case considered, the amplification $G(t)$ can reach a value $\sim 10^3$ for a time equal to 1/10 of a ion gyroperiod, or even 10^4 for $T\Omega_i \sim 5 \times 10^{-2}$. Those are times scales where electron dynamics are important. Also, looking at the time $T\Omega_i = 1$, all the four curves are within the interval [1,100]. This suggests that in some cases the protons also could be influenced by transient behavior.

The dependence on the values of β and k of the maximum amplification M and of the time τ at which this amplification is reached cannot be easily inferred from Fig. 3. It turns out that while τ is a monotonic decreasing function of k for any β, M changes its behavior as β varies. For small β, M increases for increasing k, while at high β it decreases when k increases.

Having seen that transient growth can reach large values of amplification we address now the point of how long the transient effect can last. We plot in Fig. 4 contours of the time Θ for which $\|e^{\mathbf{A}\Theta}\| \leq 1$, as a function of k and β. This is the time when $G < 1$ for any initial perturbation and we consider it as the time when transient effects lose importance, and the system starts to damp according to the normal mode analysis. It appears that transient growth takes a longer time to disappear at small k. Also, there is a large portion of parameter space where $10 < \Theta\Omega_i < 100$, and therefore the transient time is comparable with typical plasma time scales.

We have shown that a collisionless plasma modeled through a Landau fluid model is a non-normal system where transient growth can take place for short periods of time. By studying how the transients behave as functions of k and β, we argue that the non-normality of the system is connected with small scales and high plasma β. For this reason, although one is not permitted to generalize *tout court* the results of the present letter to a Vlasov plasma, we conjecture that the Vlasov equation should also present non-normal effects. Hence, the key point is to understand if the transients can affect the plasma dynamics, and therefore whether a generalized nonmodal plasma instability theory should supersede in some cases the more traditional normal mode analysis. This question is intuitively related to how large the transients can grow, and how long they last. We have shown that for the Landau fluid model, the transients are surprisingly large, with possible amplification of the order of 10^3–10^4; they can also last impressively long, for times of 10–100 ion gyroperiods.

We believe that the results of the present letter will have important and immediate applications in at least two areas. The first is the study of kinetic turbulence in space plasma, where the linear approximation is universally accepted and where models have so far neglected any nonmodal prediction. The second deals with the interpretation of computer simulations, especially those that study damping waves in a stable plasma. In the light of these results, the idea that a single normal mode can be numerically excited without triggering transient effects should be revisited.

To conclude, the results presented here point toward a revision of kinetic plasma theory from a nonmodal perspective: a route that has already successfully been followed in hydrodynamics.

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