

1. Suppose  $U$  is a subspace of a finite-dimensional vector space  $V$  over some field  $F$ . Recall that the annihilator  $U^\circ$  of  $U$  consists of all linear maps  $\ell \in V^*$  which are zero on the vectors in  $U$ .

$$\dim U^\circ + \dim U = \dim V$$

Since  $V$  is finite-dimensional,  $U \subseteq V$  is also finite-dimensional. So, construct a basis  $\{u_1, \dots, u_n\}$  of  $U$  and extend it to a basis  $\mathcal{B} = \{u_1, \dots, u_n, v_1, \dots, v_m\}$  of  $V$ .

$\mathcal{B}$  induces a dual basis  $\mathcal{B}^* = \{u_1^*, \dots, u_n^*, v_1^*, \dots, v_m^*\}$  of  $V^*$ , where  $b_i^*(b_j) = \delta_{ij}$  for all  $b_i^* \in \mathcal{B}^*, b_j \in \mathcal{B}$ . It suffices to show that  $\dim U^\circ = m$ , since then  $\dim U^\circ + \dim U = m + n = \dim V$ .

**Claim.**  $\{v_1^*, \dots, v_m^*\}$  is a basis of  $U^\circ$ .

Since  $\{v_1^*, \dots, v_m^*\} \subseteq \mathcal{B}^*$ , we have linear independence.

Let  $\ell \in U^\circ$  and consider the map  $\ell' : V \rightarrow \mathbb{R}$  given by  $\ell'(v) := \sum \ell(v)v_i^*(v)$ . Then, for any  $b \in \mathcal{B}$ , we have two cases:

- If  $b = u_i$  for some  $i$ , then  $\ell'(u_i) = 0 = \ell(u_i)$ .
- If  $b = v_i$  for some  $i$ , then  $\ell'(v_i) = \sum \ell(v_i)v_j^*(v_i) = 0 + \dots + 0 + \ell(v_i)v_i^*(v_i) + 0 + \dots + 0 = \ell(v_i)$ .

Since a linear map is uniquely characterized by its behavior on a basis, we conclude that  $\ell' = \ell$ , so  $\{v_1^*, \dots, v_m^*\}$  spans  $U^\circ$ .

2. Suppose  $T : V \rightarrow W$  is a linear map between two finite-dimensional vector spaces and  $T^* : W^* \rightarrow V^*$  is the dual map. Prove that

(a)

$$N(T^*) = (R(T))^\circ$$

Let  $\ell \in N(T^*)$ , so  $\ell \circ T = 0$ . Fix  $w \in R(T)$  and pick  $v \in V$  such that  $T(v) = w$ . Then,  $\ell(w) = \ell(T(v)) = (\ell \circ T)(v) = 0$ . Hence,  $\ell \in (R(T))^\circ$ .

Let  $\ell \in (R(T))^\circ$ , so  $\ell(w) = 0$  for all  $w \in R(T)$ . Fix  $v \in V$  and set  $w = T(v)$ . Then,  $(T^*(\ell))(v) = (\ell \circ T)(v) = \ell(w) = 0$ . Hence,  $\ell \in N(T^*)$ .

(b)

$$\dim R(T^*) = \dim R(T)$$

Since  $W$  is finite-dimensional,  $R(T) \subseteq W$  is finite-dimensional. So, we can apply the result from problem 1 to get  $\dim(R(T))^\circ + \dim R(T) = \dim W$ . It then follows from (a) that  $\dim N(T^*) + \dim R(T) = \dim W$ .

Recall from Rank/Nullity that  $\dim R(T^*) = \dim W^* - \dim N(T^*)$ . So, by some algebra, we conclude that  $\dim W^* - \dim R(T^*) + \dim R(T) = \dim W$ .

Then, since  $\dim W = \dim W^*$ , we conclude that  $\dim R(T) = \dim R(T^*)$ .

(c) for any  $m \times n$  matrix  $A$ , the rank of  $A$  equals the rank of its transpose  $A^T$ .

Let  $A = (a_{ij}) \in M_{m,n}(F)$ . Then, by Lemma 16.3 from lecture, we have  $L_{A^T} = L_A^*$ . So, by (b), we conclude that  $\dim R(L_A) = \dim R(L_A^*) = \dim R(L_{A^T})$ .

**3.** Assume that  $V, W$  are finite-dimensional vector spaces. Prove that if  $T^* : W^* \rightarrow V^*$  is injective, then  $T : V \rightarrow W$  is surjective.

Since  $W$  is finite-dimensional,  $R(T) \subseteq W$  is also finite-dimensional. So, we apply the result from problem 1 to get  $(\dim R(T))^\circ + \dim R(T) = \dim W$ . Moreover, 2(a) implies that  $(R(T))^\circ = N(T^*)$ , so we rewrite the left-hand side of the equation to get  $\dim N(T^*) + \dim R(T) = \dim W$ . But, since  $T^*$  is injective,  $\dim N(T^*) = 0$ , so we conclude that  $\dim R(T) = \dim W$ . Thus,  $T$  is surjective.

**4.** Let  $V, W$  be two vector spaces. Prove that the space of bilinear maps

$$\text{Bilin}(V, W; \mathbb{R}) := \{ b : V \times W \rightarrow \mathbb{R} \mid b \text{ is bilinear} \}$$

is a vector space with scalar multiplication and vector addition defined by

$$(\lambda b)(v, w) := \lambda b(v, w) \text{ for all } \lambda \in \mathbb{R}, v \in V, w \in W$$

and

$$(b_1 + b_2)(v, w) := b_1(v, w) + b_2(v, w) \text{ for all } v \in V, w \in W.$$

Let  $b_1, b_2 \in \text{Bilin}(V, W; \mathbb{R})$ ,  $\lambda, \mu \in \mathbb{R}$ ,  $v \in V$ , and  $w \in W$ .

1. Addition is commutative:

$$(b_1 + b_2)(v, w) = b_1(v, w) + b_2(v, w) = b_2(v, w) + b_1(v, w) = (b_2 + b_1)(v, w)$$

2. Addition is associative:

$$\begin{aligned} (b_1 + (b_2 + b_3))(v, w) &= b_1(v, w) + (b_2 + b_3)(v, w) = b_1(v, w) + (b_2(v, w) + b_3(v, w)) \\ &= (b_1(v, w) + b_2(v, w)) + b_3(v, w) = (b_1 + b_2)(v, w) + b_3(v, w) = ((b_1 + b_2) + b_3)(v, w) \end{aligned}$$

3. The map  $0 : V \times W \rightarrow \mathbb{R}$ ,  $(v, w) \mapsto 0_{\mathbb{R}}$  behaves like a zero:

$$b_1(v, w) + 0(v, w) = b_1(v, w) + 0_{\mathbb{R}} = b_1(v, w) = 0_{\mathbb{R}} + b_1(v, w) = 0(v, w) + b_1(v, w)$$

4.  $-b_1 : V \times W \rightarrow \mathbb{R}$ ,  $(v, w) \mapsto -b_1(v, w)$  is the additive inverse of  $b_1$ :

$$(b_1 + -b_1)(v, w) = b_1(v, w) + -b_1(v, w) = b_1(v, w) - b_1(v, w) = 0_{\mathbb{R}} = 0(v, w)$$

5.  $1 \in \mathbb{R}$  is the multiplicative identity:

$$(1 \cdot b_1)(v, w) = 1 \cdot b_1(v, w) = b_1(v, w)$$

6. Scalar multiplication is associative:

$$(\lambda \cdot \mu b_1)(v, w) = \lambda(\mu b_1)(v, w) = (\lambda \mu) b_1(v, w) = ((\lambda \mu) b_1)(v, w)$$

7. Scalar multiplication distributes over vector addition:

$$(\lambda(b_1 + b_2))(v, w) = \lambda(b_1 + b_2)(v, w) = \lambda(b_1(v, w) + b_2(v, w)) = \lambda b_1(v, w) + \lambda b_2(v, w)$$

8. Scalar multiplication distributes over field addition:

$$((\lambda + \mu) b_1)(v, w) = (\lambda + \mu) b_1(v, w) = \lambda b_1(v, w) + \mu b_1(v, w)$$

Hence,  $\text{Bilin}(V, W; \mathbb{R})$  is a vector space.

5. Let  $V, W$  be two vector spaces over  $\mathbb{R}$ . Let  $b : V \times W \rightarrow \mathbb{R}$  be a bilinear map.

(a) Prove that for any  $w \in W$ , the function  $w^\# : V \rightarrow \mathbb{R}$  defined by

$$w^\#(v) := b(v, w) \quad \text{for all } v \in V \quad \text{i.e., } w^\#(\cdot) = b(\cdot, w)$$

is linear.

For all  $v, v' \in V, \lambda, \mu \in \mathbb{R}$ ,

$$\begin{aligned} w^\#(\lambda v + \mu v') &= b(\lambda v + \mu v', w) \\ &= \lambda b(v, w) + \mu b(v', w) && (b \text{ is bilinear}) \\ &= \lambda w^\#(v) + \mu w^\#(v') \end{aligned}$$

So,  $w^\#$  is linear.

(b) Prove that the map

$$\# : W \rightarrow V^*, \quad w \mapsto w^\#$$

is linear.

For all  $w, w' \in W, \lambda, \mu \in \mathbb{R}$ , we have for every  $v \in V$  that

$$\begin{aligned} \#(\lambda w + \mu w')(v) &= (\lambda w + \mu w')^\#(v) \\ &= b(v, \lambda w + \mu w') \\ &= \lambda b(v, w) + \mu b(v, w') && (b \text{ is bilinear}) \\ &= \lambda \#(w)(v) + \mu \#(w')(v) \end{aligned}$$

So,  $\#$  is linear.

(c) Prove that the null space  $N(\#)$  of  $\#$  is

$$S = \{ w \in W \mid b(v, w) = 0 \text{ for all } v \in V \}$$

Let  $w \in W$  be such that  $b(v, w) = 0$  for all  $v \in V$ . Then, for all  $v \in V$ ,

$$\begin{aligned} \#(w)(v) &= w^\#(v) \\ &= b(v, w) \\ &= 0 \end{aligned}$$

Hence,  $\#(w)$  is the zero vector in  $V^*$ , so  $w \in N(\#)$ .

Let  $w \in N(\#)$ . Then,  $\#(w) = w^\# = b(\cdot, w) = 0_{V^*}$ . Hence,  $(b(\cdot, w))(v) = b(v, w) = 0$ , for all  $v \in V$ , so  $w \in S$ .

- (d) Now suppose that  $W$  is finite-dimensional,  $V = W^*$ , and  $b : W^* \times W \rightarrow \mathbb{R}$  is given by

$$b(\ell, w) := \ell(w)$$

Prove that, in this case,  $\# : W \rightarrow (W^*)^*$  is injective.

Let  $w \in N(\#)$  and  $\ell \in W^*$ . Then, we have  $\#(w)(\ell) = w^\#(\ell) = b(\ell, w) = 0$ . We want to show that  $w = 0$ .

Since  $W$  is finite-dimensional, it has a basis  $\mathcal{B} = \{w_1, \dots, w_n\}$ . So, there are  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  so that  $\sum \lambda_i w_i = w$ .

For each  $w_i$ , consider the map  $w_i^* : W \rightarrow \mathbb{R}$  given by  $w_i^*(w_j) := \delta_{ij}$  for each  $w_j \in \mathcal{B}$ . Then,  $w_i^*(w) = w_i^*(\sum \lambda_j w_j) = 0 + \dots + 0 + w_i^*(\lambda_i w_i) + 0 + \dots + 0 = \lambda_i = 0$  by assumption. Hence,  $\lambda_1 = \dots = \lambda_n = 0$ , so  $w = 0$ , implying  $\#$  is injective.

6. (a) Write the permutation  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$  as a product of transpositions.

$$(12)(23)(45) = \begin{cases} 1 \mapsto 1 \mapsto 1 \mapsto 2 \\ 2 \mapsto 2 \mapsto 3 \mapsto 3 \\ 3 \mapsto 3 \mapsto 2 \mapsto 1 \\ 4 \mapsto 5 \mapsto 5 \mapsto 5 \\ 5 \mapsto 4 \mapsto 4 \mapsto 4 \end{cases} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

- (b) Compute the sign of the permutation  $\sigma$ .

We count the number of inversions in  $\sigma$ .

$$\left( \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & \diagdown & \diagup & & \\ & 1 & 2 & 3 & \\ & \diagup & \diagdown & & \\ 1 & 2 & 3 & 4 & 5 \end{array} \right)$$

Thus, the sign of  $\sigma$  is  $(-1)^3 = -1$ .

7. Recall that an inversion of a permutation  $\sigma$  is a pair of indices  $i, j$  so that  $i < j$  and  $\sigma(i) > \sigma(j)$ . The inversion number of  $\sigma$  is the number of all inversions of  $\sigma$ . What is the inversion number of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

We count the crossings in  $\sigma$ .

$$\left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & \diagdown & \diagup & \diagdown \\ & 1 & 2 & 3 \\ & \diagup & \diagdown & \diagup \\ 1 & 2 & 3 & 4 \end{array} \right)$$

So, the inversion number of  $\sigma$  is 3.

8. Let  $T : V \rightarrow W$  be a linear map and  $\alpha : \overbrace{W \times \cdots \times W}^k \rightarrow \mathbb{R}$  be  $k$ -linear and alternating. Define

$$T^*\alpha : \overbrace{V \times \cdots \times V}^k \rightarrow \mathbb{R}$$

by

$$(T^*\alpha)(v_1, \dots, v_k) := \alpha(T(v_1), \dots, T(v_k)) \quad \text{for all } v_1, \dots, v_k \in V.$$

Prove that  $T^*\alpha$  is  $k$ -linear and alternating.

Let  $\lambda, \mu \in \mathbb{R}$ ,  $u, v \in V$ , and  $1 \leq i \leq k$ . Then,

$$\begin{aligned} T^*\alpha(v_1, \dots, \overbrace{\lambda u + \mu v}^{\text{ith slot}}, \dots, v_k) &= \alpha(T(v_1), \dots, \lambda T(u) + \mu T(v), \dots, T(v_k)) \\ &= \lambda \alpha(T(v_1), \dots, T(u), \dots, T(v_k)) + \mu \alpha(T(v_1), \dots, T(v), \dots, T(v_k)) \\ &\quad (\alpha \text{ is } k\text{-linear}) \\ &= \lambda T^*\alpha(v_1, \dots, u, \dots, v_k) + \mu T^*\alpha(v_1, \dots, v, \dots, v_k) \end{aligned}$$

Hence,  $T^*\alpha$  is  $k$ -linear.

Similarly, for  $1 \leq i < j \leq k$ , we have

$$\begin{aligned} T^*\alpha(v_1, \dots, \overbrace{v_j}^{\text{ith slot}}, \dots, \overbrace{v_i}^{\text{jth slot}}, \dots, v_k) &= \alpha(T(v_1), \dots, T(v_j), \dots, T(v_i), \dots, T(v_k)) \\ &= -\alpha(T(v_1), \dots, T(v_i), \dots, T(v_j), \dots, T(v_k)) \\ &\quad (\alpha \text{ is alternating}) \\ &= -T^*\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

So,  $T^*\alpha$  is alternating.