1. Suppose U is a subspace of a finite-dimensional vector space V over some field F. Recall that the *annihilator* U° of U consists of all linear maps $\ell \in V$ which are zero on the vectors in U.

$$\dim U^\circ + \dim U = \dim V$$

> Since V is finite-dimensional, $U \subseteq V$ is also finite-dimensional. So, construct a basis $\{ u_1, \dots, u_n \}$ of U and extend it to a basis $\mathcal{B} = \{ u_1, \dots, u_n, v_1, \dots, v_m \}$ of V.
>
> $\mathcal{B}$ induces a dual basis $\mathcal{B}^* = \{ u_1^*, \dots, u_n^*, v_1^*, \dots v_m^* \}$ of V^*, where $b_i^*(b_j) = \delta_{ij}$ for all $b_i^* \in \mathcal{B}^*, b_j \in \mathcal{B}$. It suffices to show that $\dim U^\circ = m$, since then $\dim U^\circ + \dim U = m + n = \dim V$.
>
> **Claim.** $\{ v_1^*, \dots, v_m^* \}$ is a basis of U°.
>
> Since $\{ v_1^*, \dots, v_m^* \} \subseteq \mathcal{B}^*$, we have linear independence.
>
> Let $\ell \in U^\circ$ and consider the map $\ell' : V \to \mathbb{R}$ given by $\ell'(v) := \sum \ell(v) v_i^*(v)$. Then, for any $b \in \mathcal{B}$, we have two cases:
>
> - If $b = u_i$ for some i, then $\ell'(u_i) = 0 = \ell(u_i)$.
> - If $b = v_i$ for some i, then $\ell'(v_i) = \sum \ell(v_i) v_j^*(v_i) = 0 + \dots + 0 + \ell(v_i) v_i^*(v_i) + 0 + \dots + 0 = \ell(v_i)$.
>
> Since a linear map is uniquely characterized by its behavior on a basis, we conclude that $\ell' = \ell$, so $\{ v_1^*, \dots, v_m^* \}$ spans U°.

2. Suppose $T : V \to W$ is a linear map between two finite-dimensional vector spaces and $T^* : W^* \to V^*$ is the dual map. Prove that

(a)

$$N(T^*) = (R(T))^\circ$$

> Let $\ell \in N(T^*)$, so $\ell \circ T = 0$. Fix $w \in R(T)$ and pick $v \in V$ such that $T(v) = w$. Then, $\ell(w) = \ell(T(v)) = (\ell \circ T)(v) = 0$. Hence, $\ell \in (R(T))^\circ$.
>
> Let $\ell \in (R(T))^\circ$, so $\ell(w) = 0$ for all $w \in R(T)$. Fix $v \in V$ and set $w = T(v)$. Then, $(T^*(\ell))(v) = (\ell \circ T)(v) = \ell(w) = 0$. Hence, $\ell \in N(T^*)$.

(b)

$$\dim R(T^*) = \dim R(T)$$

> Since W is finite-dimensional, $R(T) \subseteq W$ is finite-dimensional. So, we can apply the result from problem 1 to get $\dim (R(T))^\circ + \dim R(T) = \dim W$. It then follows from (a) that $\dim N(T^*) + \dim R(T) = \dim W$.
>
> Recall from Rank/Nullity that $\dim R(T^*) = \dim W^* - \dim N(T^*)$. So, by some algebra, we conclude that $\dim W^* - \dim R(T^*) + \dim R(T) = \dim W$.
>
> Then, since $\dim W = \dim W^*$, we conclude that $\dim R(T) = \dim R(T^*)$.

(c) for any $m \times n$ matrix A, the rank of A equals the rank of its transpose A^T.

> Let $A = (a_{ij}) \in M_{m,n}(F)$. Then, by Lemma 16.3 from lecture, we have $L_{A^T} = L_A^*$. So, by (b), we conclude that $\dim R(L_A) = \dim R(L_A^*) = \dim R(L_{A^T})$.

3. Assume that V, W are finite-dimensional vector spaces. Prove that if $T^* : W^* \to V^*$ is injective, then $T : V \to W$ is surjective.

> Since W is finite-dimensional, $R(T) \subseteq W$ is also finite-dimensional. So, we apply the result from problem 1 to get $(\dim R(T))^\circ + \dim R(T) = \dim W$. Moreover, 2(a) implies that $(R(T))^\circ = N(T^*)$, so we rewrite the left-hand side of the equation to get $\dim N(T^*) + \dim R(T) = \dim W$. But, since T^* is injective, $\dim N(T^*) = 0$, so we conclude that $\dim R(T) = \dim W$. Thus, T is surjective.

4. Let V, W be two vector spaces. Prove that the space of bilinear maps

$$\mathrm{Bilin}(V, W; \mathbb{R}) \coloneqq \{\, b : V \times W \to \mathbb{R} \mid b \text{ is bilinear} \,\}$$

is a vector space with scalar multiplication and vector addition defined by

$$(\lambda b)(v, w) \coloneqq \lambda b(v, w) \text{ for all } \lambda \in \mathbb{R}, v \in V, w \in W$$

and

$$(b_1 + b_2)(v, w) \coloneqq b_1(v, w) + b_2(v, w) \text{ for all } v \in V, w \in W.$$

Let $b_1, b_2 \in \mathrm{Bilin}(V, W; \mathbb{R})$, $\lambda, \mu \in \mathbb{R}$, $v \in V$, and $w \in W$.

1. Addition is commutative:
$$(b_1 + b_2)(v, w) = b_1(v, w) + b_2(v, w) = b_2(v, w) + b_1(v, w) = (b_2 + b_1)(v, w)$$
2. Addition is associative:
$$\begin{aligned}(b_1 + (b_2 + b_3))(v, w) &= b_1(v, w) + (b_2 + b_3)(v, w) = b_1(v, w) + (b_2(v, w) + b_3(v, w)) \\ &= (b_1(v, w) + b_2(v, w)) + b_3(v, w) = (b_1 + b_2)(v, w) + b_3(v, w) = ((b_1 + b_2) + b_3)(v, w)\end{aligned}$$
3. The map $0 : V \times W \to \mathbb{R}$, $(v, w) \mapsto 0_{\mathbb{R}}$ behaves like a zero:
$$b_1(v, w) + 0(v, w) = b_1(v, w) + 0_{\mathbb{R}} = b_1(v, w) = 0_{\mathbb{R}} + b_1(v, w) = 0(v, w) + b_1(v, w)$$
4. $-b_1 : V \times W \to \mathbb{R}$, $(v, w) \mapsto -b(v, w)$ is the additive inverse of b_1:
$$(b_1 + -b_1)(v, w) = b_1(v, w) + -b_1(v, w) = b_1(v, w) - b_1(v, w) = 0_{\mathbb{R}} = 0(v, w)$$
5. $1 \in \mathbb{R}$ is the multiplicative identity:
$$(1 \cdot b_1)(v, w) = 1 \cdot b_1(v, w) = b_1(v, w)$$
6. Scalar multiplication is associative:
$$(\lambda \cdot \mu b_1)(v, w) = \lambda(\mu b_1)(v, w) = (\lambda\mu) b_1(v, w) = ((\lambda\mu) b_1)(v, w)$$
7. Scalar multiplication distributes over vector addition:
$$(\lambda(b_1 + b_2))(v, w) = \lambda(b_1 + b_2)(v, w) = \lambda(b_1(v, w) + b_2(v, w)) = \lambda b_1(v, w) + \lambda b_2(v, w)$$
8. Scalar multiplication distributes over field addition:
$$((\lambda + \mu) b_1)(v, w) = (\lambda + \mu) b_1(v, w) = \lambda b_1(v, w) + \mu b_1(v, w)$$

Hence, $\mathrm{Bilin}(V, W; \mathbb{R})$ is a vector space.

5. Let V, W be two vector spaces over $\mathbb{R}$. Let $b : V \times W \to \mathbb{R}$ be a bilinear map.

(a) Prove that for any $w \in W$, the function $w^{\#} : V \to \mathbb{R}$ defined by

$$w^{\#}(v) := b(v, w) \qquad \text{for all } v \in V \qquad \text{i.e., } w^{\#}(\cdot) = b(\cdot, w)$$

is linear.

> For all $v, v' \in V, \lambda, \mu \in \mathbb{R}$,
>
> $$\begin{aligned} w^{\#}(\lambda v + \mu v') &= b(\lambda v + \mu v', w) \\ &= \lambda b(v, w) + \mu b(v', w) && (b \text{ is bilinear}) \\ &= \lambda w^{\#}(v) + \mu w^{\#}(v') \end{aligned}$$
>
> So, $w^{\#}$ is linear.

(b) Prove that the map

$$\# : W \to V^*, \qquad w \mapsto w^{\#}$$

is linear.

> For all $w, w' \in W, \lambda, \mu \in \mathbb{R}$, we have for every $v \in V$ that
>
> $$\begin{aligned} \#(\lambda w + \mu w')(v) &= (\lambda w + \mu w')^{\#}(v) \\ &= b(v, \lambda w + \mu w') \\ &= \lambda b(v, w) + \mu b(v, w') && (b \text{ is bilinear}) \\ &= \lambda \#(w)(v) + \mu \#(w')(v) \end{aligned}$$
>
> So, $\#$ is linear.

(c) Prove that the null space $N(\#)$ of $\#$ is

$$S = \{ \, w \in W \mid b(v, w) = 0 \text{ for all } v \in V \, \}$$

> Let $w \in W$ be such that $b(v, w) = 0$ for all $v \in V$. Then, for all $v \in V$,
>
> $$\begin{aligned} \#(w)(v) &= w^{\#}(v) \\ &= b(v, w) \\ &= 0 \end{aligned}$$
>
> Hence, $\#(w)$ is the zero vector in V^*, so $w \in N(\#)$.
>
> Let $w \in N(\#)$. Then, $\#(w) = w^{\#} = b(\cdot, w) = 0_{V^*}$. Hence, $(b(\cdot, w))(v) = b(v, w) = 0$, for all $v \in V$, so $w \in S$.

(d) Now suppose that W is finite-dimensional, $V = W^*$, and $b : W^* \times W \to \mathbb{R}$ is given by

$$b(\ell, w) := \ell(w)$$

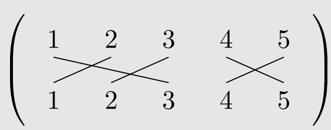
Prove that, in this case, $\# : W \to (W^*)^*$ is injective.

> Let $w \in N(\#)$ and $\ell \in W^*$. Then, we have $\#(w)(\ell) = w^\#(\ell) = b(\ell, w) = 0$. We want to show that $w = 0$.
>
> Since W is finite-dimensional, it has a basis $\mathcal{B} = \{ w_1, \dots, w_n \}$. So, there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ so that $\sum \lambda_i w_i = w$.
>
> For each w_i, consider the map $w_i^* : W \to \mathbb{R}$ given by $w_i(w_j) := \delta_{ij}$ for each $w_j \in \mathcal{B}$. Then, $w_i^*(w) = w_i^*(\sum \lambda_j w_j) = 0 + \cdots + 0 + w_i^*(\lambda_i w_i) + 0 + \cdots + 0 = \lambda_i = 0$ by assumption. Hence, $\lambda_1 = \cdots = \lambda_n = 0$, so $w = 0$, implying $\#$ is injective.

6. **(a)** Write the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ as a product of transpositions.

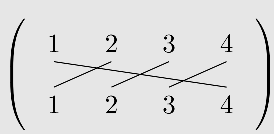
> $$(12)(23)(45) = \begin{cases} 1 \mapsto 1 \mapsto 1 \mapsto 2 \\ 2 \mapsto 2 \mapsto 3 \mapsto 3 \\ 3 \mapsto 3 \mapsto 2 \mapsto 1 \\ 4 \mapsto 5 \mapsto 5 \mapsto 5 \\ 5 \mapsto 4 \mapsto 4 \mapsto 4 \end{cases} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

(b) Compute the sign of the permutation σ.

> We count the number of inversions in σ.
>
> 
>
>
> Thus, the sign of σ is $(-1)^3 = -1$. It also suffices to observe that σ is the product of an odd number of permutations.

7. Recall that an inversion of a permutation σ is a pair of indices i, j so that $i < j$ and $\sigma(i) > \sigma(j)$. The inversion number of σ is the number of all inversions of σ. What is the inversion number of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

> We count the crossings in σ.
>
> 
>
>
> So, the inversion number of σ is 3.

8. Let $T : V \to W$ be a linear map and $\alpha : \overbrace{W \times \cdots \times W}^{k} \to \mathbb{R}$ be k-linear and alternating. Define

$$T^*\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$$

by

$$(T^*\alpha)(v_1, \ldots, v_k) \coloneqq \alpha(T(v_1), \ldots, T(v_k)) \qquad \text{for all } v_1, \ldots, v_k \in V.$$

Prove that $T^*\alpha$ is k-linear and alternating.

Let $\lambda, \mu \in \mathbb{R}$, $u, v \in V$, and $1 \leq i \leq k$. Then,

$$\begin{aligned} T^*\alpha(v_1, \ldots, \overbrace{\lambda u + \mu v}^{i\text{th slot}}, \ldots, v_k) &= \alpha(T(v_1), \ldots, \lambda T(u) + \mu T(v), \ldots, T(v_k)) \\ &= \lambda\alpha(T(v_1), \ldots, T(u), \ldots, T(v_k)) + \mu\alpha(T(v_1), \ldots, T(v), \ldots, T(v_k)) \\ &\qquad (\alpha \text{ is } k\text{-linear}) \\ &= \lambda T^*\alpha(v_1, \ldots, u, \ldots, v_k) + \mu T^*\alpha(v_1, \ldots, v, \ldots, v_k) \end{aligned}$$

Hence, $T^*\alpha$ is k-linear.

Similarly, for $1 \leq i < j \leq k$, we have

$$\begin{aligned} T^*\alpha(v_1, \ldots, \overbrace{v_j}^{i\text{th slot}}, \ldots, \overbrace{v_i}^{j\text{th slot}}, \ldots, v_k) &= \alpha(T(v_1), \ldots, T(v_j), \ldots, T(v_i), \ldots, T(v_k)) \\ &= -\alpha(T(v_1), \ldots, T(v_i), \ldots, T(v_j), \ldots, T(v_k)) \\ &\qquad (\alpha \text{ is alternating}) \\ &= -T^*\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) \end{aligned}$$

So, $T^*\alpha$ is alternating.