1. Let s be a Cauchy sequence in E. Prove that if a subsequence of s converges to $L \in E$, then the whole sequence converges to L.

> Let $\epsilon > 0$ and $\{ s_{n_i} \}$ denote the subsequence that converges to L. Since $\{ s_n \}$ is Cauchy, there is an $N \in \mathbb{N}$ so that for all $n, m > N$, we have $d(s_n, s_m) < \frac{\epsilon}{2}$. Moreover, since $\{ s_{n_i} \}$ converges to L, there is an $M \in \mathbb{N}$ so that for all $i \geq M$, we have $d(s_{n_i}, L) < \frac{\epsilon}{2}$. Thus, for all $n > K := \max(N, M)$, we have
> $$d(s_n, L) \leq d(s_n, s_{n_K}) + d(s_{n_K}, L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$
> i.e. s_n converges to L.

2. Prove that in any metric space, compact sets are closed.

> Let K be compact. We prove that K is closed by showing that K^C is open.
>
> Let $p \in K^C$ and consider the collection of nested open sets $O := \{ U_n := \overline{B\left(p, \frac{1}{n}\right)}^C : n \in \mathbb{N} \}$. Observe that $\bigcup U_n \supset E \setminus \{ p \} \supset K$. Thus, since K is compact, we obtain a finite subcover $U_{n_1}, \dots, U_{n_k}$ of K. Without loss of generality, assume that $n_1 > \cdots > n_k$. Then, since $U_1 \subset U_2 \subset \cdots$, we have $\bigcup U_{n_i} = U_{n_1} \supset K$. Hence, $B\left(p, \frac{1}{n_1}\right) \subset \overline{B\left(p, \frac{1}{n_1}\right)} = U_{n_1}^C \subset K^C$, as desired.

3. Prove that in any metric space, compact sets are sequentially compact.

> Let K be compact and suppose for contradiction that K is not sequentially compact. Then, there is a sequence $\{ s_n \} \subset K$ that does not have a convergent subsequence. Equivalently, $\{ s_n \}$ has no cluster points, so for every $x \in K$, there is an $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ contains only finitely many sequence members $\{ s_n \}$. We therefore construct the open cover $O := \{ U_x := B(x, \epsilon_x) : x \in K \}$ of K. Since K is compact, we obtain a finite subcover $U_{x_1}, \dots, U_{x_k}$ of K. But since we have finitely many U_{x_i} that each contain only finitely many s_n, the union $\bigcup U_{x_i}$ contains only finitely many sequence members s_n, a contradiction since $\bigcup U_{x_i} \supset K$ contains infinitely many!

4. Prove that if $f_n : E \to E'$ are continuous and f_n converges to $f : E \to E'$ uniformly on E, then f is continuous.

> Let $\epsilon > 0$ and fix $p \in E$. Since $f_n \to f$ uniformly, there is an $N \in \mathbb{N}$ so that $d(f_N(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in E$. Moreover, since f_N is continuous, there is a $\delta > 0$ so that, for every $x \in E$ with $d(x, p) < \delta$, we have $d(f_N(x), f_N(p)) < \frac{\epsilon}{3}$. Hence,
> $$d(f(x), f(p)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(p)) + d(f_N(p), f(p)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$
> i.e. f is continuous at p.

5. State and prove Rolle's Theorem and the Mean Value Theorem.

> **Theorem** (Rolle)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b), and have $f(a) = f(b)$. Then, there is some $c \in (a, b)$ so that $f'(c) = 0$.

Proof. Since f is continuous on $[a,b]$ compact, it attains its max and min, say M and m respectively. If $M = m = f(a) = f(b)$, then f must be constant of $[a,b]$. Hence, any $c \in (a,b)$ satisfies $f'(c) = 0$. Otherwise, one of M or m (say M for definiteness) is not equal to $f(a)$. We therefore have a $c \in (a,b)$ so that $f(c) = M$, which implies $f'(c) = 0$ by the Max-Min Test. □

Theorem. Mean Value Theorem Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there is some $c \in (a,b)$ so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Define the map $g : [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

By continuity rules, g is continuous on $[a,b]$. Similarly, by differentiability rules, g is differentiable on (a,b) with derivative $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

Observe, too, that $g(a) = f(a)$ and $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. Thus, by Rolle's Theorem, there is some $c \in (a,b)$ so that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$, i.e. $f'(c) = \frac{f(b)-f(a)}{b-a}$, as desired. □

6. State and prove F.T.C. 1.

Theorem (F.T.C. 1)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f' is bounded and integrable on $[a,b]$, then $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

Proof. Fix a partition $P \coloneqq t_0 < t_1 < \cdots < t_n$ and apply the Mean Value Theorem on each interval of the partition: for every $1 \le k \le n$, there is an $x_k \in [t_{k-1}, t_k]$ such that

$$m(f', [t_{k-1}, t_k]) \le f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \le M(f', [t_{k-1}, t_k]).$$

Thus,

$$L(f', P) \le \sum f(t_k) - f(t_{k-1}) \le U(f', P)$$

for every partition P, so $\int_a^b f' = L(f') \le f(b) - f(a) \le U(f') = \int_a^b f'$, i.e. $\int_a^b f' = f(b) - f(a)$. □

7. State and prove F.T.C. 2.

Theorem (F.T.C. 2)**.** Let f be bounded and integrable on $[a,b]$. Then, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$. Moreover, if f is continuous at x_0, then F is differentiable at x_0 with derivative $F'(x_0) = f(x_0)$.

Proof. Since f is bounded on $[a,b]$, there is an $M \ge 0$ so that $|f(x)| \le M$ for all $x \in [a,b]$. Thus,

$$|F(x) - F(y)| = \left| \int_y^k f(t)\,\mathrm{d}t \right| \le \left| \int_y^x |f(t)|\,\mathrm{d}t \right| \le \left| \int_y^x M\,\mathrm{d}t \right| = M|y - x|,$$

i.e. F is Lipschitz (and thus continuous) on $[a,b]$.

Now, suppose that f is continuous at x_0 and let $\epsilon > 0$. Then, there is a $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta)$, we have $|f(x) - f(x_0)| < \epsilon$. Thus,

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^{x} f(t)\,\mathrm{d}t - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^{x} f(t) - f(x_0)\,\mathrm{d}t\right|.$$

Since t is between x and x_0 and $|x - x_0| < \delta$, we have $|t - x_0| < \delta$ and thus, $|f(t) - f(x)| < \epsilon$. Hence,

$$\left|\frac{1}{x - x_0}\int_{x_0}^{x} f(t) - f(x_0)\,\mathrm{d}t\right| \le \left|\frac{1}{x - x_0}\int_{x_0}^{x} |f(t) - f(x_0)|\,\mathrm{d}t\right| < \left|\frac{1}{x - x_0}\int_{x_0}^{x} \epsilon\,\mathrm{d}t\right| = \epsilon.$$

Thus, the derivative quotient exists and is equal to $f(x_0)$. □

8. State and prove the Cauchy criterion for Darboux integration.

Theorem (Cauchy Criterion)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded. Then, f is integrable on $[a, b]$ if and only if, for all $\epsilon > 0$, there is a partition P of $[a, b]$ so that $U(f, P) - L(f, P) < \epsilon$.

Proof. ($\Longrightarrow$). Suppose that f is integrable on $[a, b]$. Then, $U(f) = L(f)$. Fix $\epsilon > 0$. Then, by the definition of $U(f) = \inf_P U(f, P)$, we have some partition P so that $U(f, P) < U(f) + \frac{\epsilon}{2}$. Similarly, since $L(f) = \sup_P L(f, P)$, we have some partition Q so that $L(f, P) > L(f) - \frac{\epsilon}{2}$. Thus, for the partition $R := P \cup Q$, we have

$$U(f, R) - L(f, R) < U(f) - L(f) + \epsilon = \epsilon,$$

as desired.

($\Longleftarrow$). Now, suppose that, for every $\epsilon > 0$, there is a partition P so that $U(f, P) - L(f, P) < \epsilon$. Then,

$$U(f) \le U(f, P) - L(f, P) + L(f, P) \le \epsilon + L(f, P) \le L(f) + \epsilon,$$

so $U(f) - L(f) \le \epsilon$. Since this inequality holds for all $\epsilon > 0$, we thus conclude that $U(f) \le L(f)$ (and hence, $U(f) = L(f)$). □

9. Prove that every continuous function $f : [0, 1] \to \mathbb{R}$ is Darboux integrable.

Fix $\epsilon > 0$. Since f is continuous on $[0, 1]$ compact, f is uniformly continuous. Thus, there is some $\delta > 0$ so that for all $x, y \in [0, 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let P be a partition of $[0, 1]$ with $\mathrm{mesh}(P) < \delta$. Then,

$$U(f, P) - L(f, P) = \sum (M(f, [t_{i-1}, t_i]) - m(f, [t_{i-1}, t_i]))(t_i - t_{i-1}).$$

Since $|t_i - t_{i-1}| < \delta$, we must have $M - m < \epsilon$. Thus,

$$U(f, P) - L(f, P) < \epsilon \sum t_i - t_{i-1} = \epsilon,$$

so f is integrable by the Cauchy Criterion.

10. Prove that if f_n are bounded and integrable on $[0,1]$ and f_n converges to f uniformly on $[0,1]$, then f is integrable on $[0,1]$ and $\int_0^1 f_n \to \int_0^1 f$ as $n \to \infty$.

> Given $\epsilon > 0$, we can pick an $N \in \mathbb{N}$ so that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a,b]$ (since $f_n \to f$ uniformly). By the Cauchy Criterion, we can also pick a partition P so that $U(f_N, P) - L(f_N, P) < \epsilon$. Now, note that
>
> $$U(f,P) \leq U(f_N,P) + U(f - f_N, P) \leq U(f_N, P) + \epsilon(b-a).$$
>
> Similarly,
>
> $$L(f,P) \geq L(f_N,P) - L(f - f_N, P) \geq L(f_N, P) - \epsilon(b-a).$$
>
> Thus,
>
> $$U(f,P) - L(f,P) \leq U(f_N,P) - L(f_N,P) + 2\epsilon(b-a) < \epsilon + 2\epsilon(b-a).$$
>
> So, by the Cauchy Criterion, f is integrable on $[a,b]$.
>
> Now, observe that
>
> $$\left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \leq \left| \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \right| \leq \epsilon(b-a).$$
>
> Thus, $\int_a^b f_n \to \int_a^b f$.

11. Prove that any power series with radius of convergence R and center x_0 converges uniformly on any compact subset of the interval $(x_0 - R, x_0 + R)$.

> It suffices to prove that for any $0 \leq R_1 < R$, the series converges on $[x_0 - R_1, x_0 + R_1]$. Recall that the series
>
> $$S \coloneqq \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
>
> converges absolutely for $x \in (x_0 - R, x_0 + R)$.
>
> Thus, if $x \in [x_0 - R_1, x_0 + R_1] \subset (x_0 - R, x_0 + R)$, the series $\sum |a_n| R_1^n$ converges, too. But since $|a_n(x - x_0)^n| \leq |a_n| R_1^n$ here, S converges uniformly on $[x_0 - R_1, x_0 + R_1]$ by the Weierstrass M-Test.

12. Prove Banach's Fixed Point Theorem: Let (E, d) be a complete metric space and let $T : E \to E$ be a contraction (i.e. $\exists c \in [0,1)$, $\forall x, y \in E$, $d(T(x), T(y)) \leq c d(x,y)$), then there exists a unique $p \in E$ so that $T(p) = p$.

> (Uniqueness). Suppose that $x, y \in E$ are fixed points of T. Then,
>
> $$0 \leq d(x,y) = d(T(x), T(y)) \leq c d(x,y).$$
>
> Since $c < 1$, we thus conclude that $d(x,y) = 0$, i.e. $x = y$.
>
> (Existence). Take $p_1 \in S$ and define a sequence by the recurrence $p_{n+1} = T(p_n)$. Note that for $n \geq 2$, we have
>
> $$d(p_n, p_{n+1}) = d(T(p_{n-1}), T(p_n)) \leq c^{n-1} d(p_1, p_2).$$

Thus, for $n < m$, we get

$$d(p_n, p_m) \leq d(p_n, p_{n+1}) + \cdots + d(p_{m-1}, p_m) \leq \sum_{k=n+1}^{m} c^{k-1} d(p_1, p_2) \leq \sum_{k=n+1}^{\infty} c^{k-1} d(p_1, p_2).$$

Since $c \in [0, 1)$, this geometric series converges, and thus, for any $\epsilon > 0$, we can find a tail $\sum_{k=N}^{\infty} c^{k-1} d(p_1, p_2) < \epsilon$. So, for $m > n > N$, we therefore get

$$d(p_n, p_m) \leq \sum_{k=n+1}^{\infty} c^{k-1} d(p_1, p_2) \leq \sum_{k=N}^{\infty} c^{k-1} d(p_1, p_2) < \epsilon,$$

i.e. $\{ p_n \}$ is Cauchy. Since (E, d) is complete, $p_n \to p \in E$. Moreover, since T is Lipschitz, it's continuous. So,

$$p = \lim T(p_n) = T(\lim p_n) = T(p),$$

i.e. p is indeed a fixed point.