T-501-FMAL Programming languages, Practice class 9
Spring 2021

1. Normalize the following terms:

   - $(\lambda x.\ x)\,(\lambda y.\ y\,y)\,(\lambda z.\ w\,z)$
   - $(\lambda x.\ \lambda y.\ x\,y\,y)\,(\lambda z.\ z)\,x$
   - $(\lambda x.\ \lambda y.\ x\,y\,y)\,(\lambda x.\ y)\,x$ (is there a potential for capture?)
   - $(\lambda x.\ x\,x)\,(\lambda y.\ y)\,(\lambda y.\ y)$

2. Derive the following typing judgements:

   - $\lambda f.\ \lambda x.\ \lambda y.\ f\,y\,x : (A \to B \to C) \to B \to A \to C$
   - $\lambda f.\ \lambda x.\ \lambda y.\ \lambda y'.f\,(f\,x\,y)\,y' : (A \to B \to A) \to A \to B \to B \to A$
   - $\lambda f.\ \lambda g.\ \lambda x.\ \lambda x'.\ g\,(f\,x)\,(f\,x') : (A \to B) \to (B \to B \to C) \to A \to A \to C$

3. The pair type is Church-encoded as follows:

$$
\begin{aligned}
A \times B &= \forall X.\,(A \to B \to X) \to X \\
\mathsf{pair} &= \lambda a.\ \lambda b.\ \lambda m.\ m\,a\,b \\
\mathsf{fst} &= \lambda p.\ p\,(\lambda a.\ \lambda b.\ a) \\
\mathsf{snd} &= \lambda p.\ p\,(\lambda a.\ \lambda b.\ b)
\end{aligned}
$$

   Show that $\mathsf{pair} : A \to B \to A \times B$, $\mathsf{fst} : A \times B \to A$, $\mathsf{snd} : A \times B \to B$, $\mathsf{fst}\,(\mathsf{pair}\,u\,v) \to^* u$, $\mathsf{snd}\,(\mathsf{pair}\,u\,v) \to^* v$.

4. Write a term for the function that swaps the first and second component of a pair. (Write it in terms of pair, fst, snd and then see if the result can be simplified.)

5. Let $\Theta = (\lambda x.\ \lambda f.\ f\,(x\,x\,f))\,(\lambda x.\ \lambda f.\ f\,(x\,x\,f))$.

   Θ is called the Turing fixed-point combinator.

   Show that it has the fixed-point property $\Theta\,f \to^* f\,(\Theta\,f)$.