T-501-FMAL Programming languages, Practice class 9
Spring 2021
Model solutions

1.

$$\begin{array}{ll} \underline{(\lambda x.\, x)\,(\lambda y.\, y\, y)\,(\lambda z.\, w\, z)} \\ \quad \to & \underline{(\lambda y.\, y\, y)\,(\lambda z.\, w\, z)} \\ \quad \to & \underline{(\lambda z.\, w\, z)\,(\lambda z.\, w\, z)} \\ \quad \to & w\,(\lambda z.\, w\, z) \end{array}$$

$$\begin{array}{ll} \underline{(\lambda x.\, \lambda y.\, x\, y\, y)\,(\lambda z.\, z)\, x} \\ \quad \to & \underline{(\lambda y.\, (\lambda z.\, z)\, y\, y)\, x} \\ \quad \to & \underline{(\lambda z.\, z)\, x\, x} \\ \quad \to & x\, x \end{array}$$

$$\begin{array}{ll} \underline{(\lambda x.\, \lambda y.\, x\, y\, y)\,(\lambda x.\, y)\, x} \\ \quad = & \underline{(\lambda x.\, \lambda z.\, x\, z\, z)\,(\lambda x.\, y)\, x} \\ \quad \to & \underline{(\lambda z.\, (\lambda x.\, y)\, z\, z)\, x} \\ \quad \to & \underline{(\lambda x.\, y)\, x\, x} \\ \quad \to & y\, x \end{array}$$

$$\begin{array}{ll} \underline{(\lambda x.\, x\, x)\,(\lambda y.\, y)\,(\lambda y.\, y)} \\ \quad \to & \underline{(\lambda y.\, y)\,(\lambda y.\, y)\,(\lambda y.\, y)} \\ \quad \to & \underline{(\lambda y.\, y)\,(\lambda y.\, y)} \\ \quad \to & \lambda y.\, y \end{array}$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma \vdash f : A \to B \to C} \quad \overline{\Gamma \vdash y : A}}{\Gamma \vdash f\, y : B \to C} \quad \overline{\Gamma \vdash x : B}}{f : A \to B \to C, x : B, y : A \vdash f\, y\, x : C}}{f : A \to B \to C, x : B \vdash \lambda y.\, f\, y\, x : A \to C}}{f : A \to B \to C \vdash \lambda x.\, \lambda y.\, f\, y\, x : B \to A \to C}}{\vdash \lambda f.\, \lambda x.\, \lambda y.\, f\, y\, x : (A \to B \to C) \to B \to A \to C}$$

where $\Gamma = f : A \to B \to C, x : B, y : A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma \vdash f : A \to B \to A} \quad \dfrac{\dfrac{\overline{\Gamma \vdash f : A \to B \to A} \quad \overline{\Gamma \vdash x : A}}{\Gamma \vdash f\, x : B \to A} \quad \overline{\Gamma \vdash y : B}}{\Gamma \vdash f\, x\, y : A}}{\Gamma \vdash f\, (f\, x\, y) : B \to A} \quad \overline{\Gamma \vdash y' : B}}{\Gamma \vdash f\, (f\, x\, y)\, y' : A}}{f : A \to B \to A, x : A, y : B, y' : B \vdash f\, (f\, x\, y)\, y' : A}}{f : A \to B \to A, x : A, y : B \vdash \lambda y'.\, f\, (f\, x\, y)\, y' : B \to A}}{f : A \to B \to A, x : A \vdash \lambda y.\, \lambda y'.\, f\, (f\, x\, y)\, y' : B \to B \to A}}{f : A \to B \to A \vdash \lambda x.\, \lambda y.\, \lambda y'.\, f\, (f\, x\, y)\, y' : A \to B \to B \to A}}{\vdash \lambda f.\, \lambda x.\, \lambda y.\, \lambda y'.\, f\, (f\, x\, y)\, y' : (A \to B \to A) \to A \to B \to B \to A}$$

where $\Gamma = f : A \to B \to A, x : A, y : B, y' : B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to B \to C} \quad \dfrac{\dfrac{}{\Gamma \vdash f : A \to B} \quad \dfrac{}{\Gamma \vdash x : A}}{\Gamma \vdash f\, x : B}}{\Gamma \vdash g\,(f\, x) : B \to C} \quad \dfrac{\dfrac{}{\Gamma \vdash f : A \to B} \quad \dfrac{}{\Gamma \vdash x' : A}}{\Gamma \vdash f\, x' : B}}{f : A \to B, g : B \to B \to C, x : A, x' : A \vdash g\,(f\, x)\,(f\, x') : C}}{f : A \to B, g : B \to B \to C, x : A \vdash \lambda x'.\, g\,(f\, x)\,(f\, x') : A \to C}}{f : A \to B, g : B \to B \to C \vdash \lambda x.\, \lambda x'.\, g\,(f\, x)\,(f\, x') : A \to A \to C}}{f : A \to B \vdash \lambda g.\, \lambda x.\, \lambda x'.\, g\,(f\, x)\,(f\, x') : (B \to B \to C) \to A \to A \to C}}{\vdash \lambda f.\, \lambda g.\, \lambda x.\, \lambda x'.\, g\,(f\, x)\,(f\, x') : (A \to B) \to (B \to B \to C) \to A \to A \to C}$$

where $\Gamma = f : A \to B, g : B \to B \to C, x : A, x' : A$.

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash m : A \to B \to X} \quad \dfrac{}{\Gamma \vdash a : A}}{\Gamma \vdash m\, a : B \to X} \quad \dfrac{}{\Gamma \vdash b : B}}{a : A, b : B, m : A \to B \to X \vdash m\, a\, b : X}}{a : A, b : B \vdash \lambda m.\, m\, a\, b : (A \to B \to X) \to X}}{a : A, b : B \vdash \lambda m.\, m\, a\, b : A \times B}}{a : A, b : B \vdash \lambda m.\, m\, a\, b : A \times B}}{a : A \vdash \lambda b.\, \lambda m.\, m\, a\, b : B \to A \times B}}{\vdash \lambda a.\, \lambda b.\, \lambda m.\, m\, a\, b : A \to B \to A \times B}$$

where $\Gamma = a : A, b : B, m : A \to B \to X$.

$$\dfrac{\dfrac{\dfrac{}{p : A \times B \vdash p : A \times B}}{p : A \times B \vdash p : (A \to B \to A) \to A} \quad \dfrac{\dfrac{\dfrac{}{p : A \times B, a : A, b : B \vdash a : A}}{p : A \times B, a : A \vdash \lambda b.\, a : B \to A}}{p : A \times B \vdash \lambda a.\, \lambda b.\, a : A \to B \to A}}{\dfrac{p : A \times B \vdash p\,(\lambda a.\, \lambda b.\, a) : A}{\vdash \lambda p.\, p\,(\lambda a.\, \lambda b.\, a) : A \times B \to A}}$$

$$\dfrac{\dfrac{\dfrac{}{p : A \times B \vdash p : A \times B}}{p : A \times B \vdash p : (A \to B \to B) \to B} \quad \dfrac{\dfrac{\dfrac{}{p : A \times B, a : A, b : B \vdash b : B}}{p : A \times B, a : A \vdash \lambda b.\, b : B \to B}}{p : A \times B \vdash \lambda a.\, \lambda b.\, b : A \to B \to B}}{\dfrac{p : A \times B \vdash p\,(\lambda a.\, \lambda b.\, b) : B}{\vdash \lambda p.\, p\,(\lambda a.\, \lambda b.\, b) : A \times B \to B}}$$

$$\begin{aligned}
\mathsf{fst}\,(\mathsf{pair}\, u\, v) &= (\lambda p.\, p\,(\lambda a.\, \lambda b.\, a))\,(\mathsf{pair}\, u\, v) \\
&\to \mathsf{pair}\, u\, v\,(\lambda a.\, \lambda b.\, a) \\
&= (\lambda a.\, \lambda b.\, \lambda m.\, m\, a\, b)\, u\, v\,(\lambda a.\, \lambda b.\, a) \\
&\to (\lambda b.\, \lambda m.\, m\, u\, b)\, v\,(\lambda a.\, \lambda b.\, a) \\
&\to (\lambda m.\, m\, u\, v)\,(\lambda a.\, \lambda b.\, a) \\
&\to (\lambda a.\, \lambda b.\, a)\, u\, v \\
&\to (\lambda b.\, u)\, v \\
&\to u
\end{aligned}$$

$$\begin{aligned}
\mathsf{snd}\,(\mathsf{pair}\, u\, v) &= (\lambda p.\, p\,(\lambda a.\, \lambda b.\, b))\,(\mathsf{pair}\, u\, v) \\
&\to \mathsf{pair}\, u\, v\,(\lambda a.\, \lambda b.\, b)
\end{aligned}$$

$$
\begin{array}{rcl}
 & = & (\lambda a.\, \lambda b.\, \lambda m.\, m\, a\, b)\, u\, v\, (\lambda a.\, \lambda b.\, b) \\
 & \to & (\lambda b.\, \lambda m.\, m\, u\, b)\, v\, (\lambda a.\, \lambda b.\, b) \\
 & \to & (\lambda m.\, m\, u\, v)\, (\lambda a.\, \lambda b.\, b) \\
 & \to & (\lambda a.\, \lambda b.\, v)\, u\, v \\
 & \to & (\lambda b.\, b)\, v \\
 & \to & v
\end{array}
$$

4.

$$
\begin{array}{l}
\lambda p.\, \mathsf{pair}\, (\mathsf{snd}\, p)\, (\mathsf{fst}\, p) \\
\quad = \quad \lambda p.\, (\lambda a.\, \lambda b.\, \lambda m.\, m\, a\, b)\, (\mathsf{snd}\, p)\, (\mathsf{fst}\, p) \\
\quad \to \quad \lambda p.\, (\lambda b.\, \lambda m.\, m\, (\mathsf{snd}\, p),\, b)\, (\mathsf{fst}\, p) \\
\quad \to \quad \lambda p.\, \lambda m.\, m\, (\mathsf{snd}\, p)\, (\mathsf{fst}\, p) \\
\quad = \quad \lambda p.\, \lambda m.\, m\, ((\lambda p.\, p\, (\lambda a.\, \lambda b.\, b))\, p)\, ((\lambda p.\, p\, (\lambda a.\, \lambda b.\, a))\, p) \\
\quad \to \quad \lambda p.\, \lambda m.\, m\, (p\, (\lambda a.\, \lambda b.\, b))\, (p\, (\lambda a.\, \lambda b.\, a))
\end{array}
$$

5.

$$
\begin{array}{rcl}
\Theta\, f & = & (\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, (\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, f \\
 & \to & (\lambda f.\, f\, ((\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, (\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, f)\, f \\
 & \to & (f\, ((\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, (\lambda x.\, \lambda f.\, f\, (x\, x\, f))\, f) \\
 & = & f\, (\Theta\, f)
\end{array}
$$