For *degree* $l \geq 0$ and *order* $-l \leq m \leq l$, we have *spherical harmonics*

$$Y_l^m(\lambda, \theta) := \begin{cases} \sqrt{2} a_l^m P_l^m(cos\theta) cos(m\lambda) & m > 0 \\ a_l^0 P_l(cos\theta) & m = 0 \\ \sqrt{2} a_l^{|m|} P_l^{|m|}(cos\theta) sin(m\lambda) & m < 0 \end{cases}$$

- where P_l^k, for $0 \leq k \leq l$ is the associated Legendre function
- where a_l^k is the normalization factor:

$$a_l^k := \sqrt{\frac{(2l+1)(l-k)!}{4\pi(l+k)!}}$$

Then for $f \in L^2(S^2)$, we have

$$f(\lambda, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_l^m Y_l^m(\lambda, \theta)$$

where our coefficients c_l^m can be defined as:

$$c_l^m = \int_{S^2} f Y_l^m dS = \int_0^{2\pi} \int_0^{\pi} f(\lambda, \theta) Y_l^m(\lambda, \theta) d\theta d\lambda$$

In fact, this holds for any $n \geq 3$, where for a function $f \in L^2(S^n)$:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(\theta, \phi)$$

where our coefficients $f_{l,m}$ are calculated by:

$$f_{l,m} = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y_{l,m}^*(\theta, \phi) sin(\phi) d\theta d\phi$$

[*Note:*]

$$Y_{l,m}^*(\theta, \phi) = a_l^m P_l^m(cos\phi) e^{-m\phi}$$

In **Zonal Spherical Harmonics**, we say that:

$$Z^{(l)}(\theta, \phi) := P_l(cos\theta)$$

where again we have P^l a Legendre polynomial of degree l. We can then say we have general zonal harmonic with fixed axis x and variable y given by $Z_x^{(l)}(y)$.

Thus for any $Y \in H_l$, a finite dimensional Hilbert space of spherical harmonics degree l:

$$Y(x) = \int_{S^{n-1}} Z_x^{(l)}(y) Y(y) d\Omega(y)$$

Finally, we provide an alternate view of our spherical harmonics: again we define our harmonics as:

$$Y_l^m(\theta, \varphi) = N_l^m P_l^m(cos\theta)e^{im\varphi}$$

for normalization factor

$$N_l^m := \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}$$

and associated Legendre polynomial P_l^m.

Now define $L^2(S^n)$ as the space of (real) square-integrable functions on the sphere S^n. Then, defining

$$\langle f, g\rangle = \int_{S^n} fg\Omega_n$$

means that $L^2(S^n)$ is indeed a Hilbert space. In fact, we have

$$L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^n)$$

and so for every $f \in L^2(S^n)$, we know that

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} Y_l^m$$

where we can find $c_{l,m}$ by:

$$c_{l,m} = \langle f, Y_l^m\rangle$$