

**MAST31005 Algebra II exercise 4 (14.02.2024)**

1. Define ‘revlex’ order for  $\alpha, \beta \in \mathbb{N}^n$  by

$$\alpha > \beta \iff \text{the right-most nonzero entry of } \alpha - \beta \text{ is negative}$$

Show that revlex is not a monomial order.

2. Consider on  $K[x_1, \dots, x_8]$  the wdegrevlex order with weights  $(1, 1, 2, 3, 3, 4, 4, 4)$ . List all of the monomials of weighted degree 4 in increasing order.

3. Let  $>_x$  be a monomial order on  $K[x_1, \dots, x_n]$  and let  $>_y$  be a monomial order on  $K[y_1, \dots, y_m]$ . Define the *product order*  $>_{xy}$  on  $K[x_1, \dots, x_n, y_1, \dots, y_m]$  by

$$x^\alpha y^\beta >_{xy} x^\gamma y^\delta \iff x^\alpha >_x x^\gamma \quad \text{or} \quad \begin{cases} \alpha = \gamma, \text{ and} \\ y^\beta >_y y^\delta \end{cases}$$

- (a) Show that  $>_{xy}$  is a monomial order.
- (b) Is the lex order on  $K[x, y]$  the product order of the canonical orders on  $K[x]$  and  $K[y]$ ?
4. Compute the remainder of polynomial division for  $f = x^4 y^2 + x^2 y^2 - y + 1$  divided by the ordered tuple  $P$  with respect to the lex order on  $\mathbb{Q}[x, y]$  when
- (a)  $P = (xy^2 - x, x - y^3)$
- (b)  $P = (x - y^3, xy^2 - x)$
5. Let  $I \subset K[x_1, \dots, x_n]$  be an ideal with the following property: for every polynomial  $p = \sum c_\alpha x^\alpha \in I$ , if  $c_\alpha \neq 0$  then  $x^\alpha \in I$ . Show that  $I$  is a monomial ideal.
6. Let  $I = \langle x^\alpha : \alpha \in A \rangle \subset K[x_1, \dots, x_n]$  be a monomial ideal and let  $>$  be a monomial order on  $K[x_1, \dots, x_n]$ . Let  $\beta = \min\{\alpha : x^\alpha \in I\}$ . Show that  $\beta \in A$ .
7. Let  $I = \langle x^6, x^2 y^3, y^7 \rangle \subset \mathbb{Q}[x, y]$ .
- (a) Draw a visualization of the exponents  $(m, n) \in \mathbb{N}^2$  of the monomials  $x^m y^n \in I$ .
- (b) If we divide  $f \in \mathbb{Q}[x, y]$  by the tuple  $(x^6, x^2 y^3, y^7)$ , which monomials can appear in the remainder?
- (c) Compute the dimension of the quotient  $\mathbb{Q}[x, y]/I$  as a vector space over  $\mathbb{Q}$ .