INFINITE GEODESICS AND ISOMETRIC EMBEDDINGS IN CARNOT GROUPS OF STEP 2

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Abstract. In the setting of step 2 sub-Finsler Carnot groups with strictly convex norms, we prove that all infinite geodesics are lines. It follows that for any other homogeneous distance, all geodesics are lines exactly when the induced norm on the horizontal space is strictly convex. As a further consequence, we show that all isometric embeddings between such homogeneous groups are affine.

Contents

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1. Introduction

Carnot groups have rich algebraic and metric structures, and share many properties with normed spaces. Recently several articles have generalized classical regularity results of isometric embeddings in normed spaces into the setting of Carnot groups. In real normed spaces, there are two simple criteria for an isometric embedding to be affine: surjectivity or strict convexity of the norm on the target. Both regularity criteria have analogues for isometric embeddings of Carnot groups.

Surjective isometric embeddings behave in the Carnot group case similarly as they do in the normed-space case. Namely, isometries between arbitrary (open subsets of) Carnot groups are affine [LDO16], i.e., compositions of left translations and a group homomorphisms. For globally defined isometries, there is an even more general result that isometries between connected nilpotent metric Lie groups are affine [KLD17].

For non-surjective isometric embeddings, it was proved in [Kis03] that if G is a sub-Riemannian Carnot group of step 2, then all isometric embeddings $\mathbb{R} \hookrightarrow G$, i.e., all *infinite geodesics*, are affine. This property was coined the *geodesic linearity property* in [BFS18], and was used as an alternative to the strict convexity criterion as the two conditions are equivalent in normed spaces. More precisely, it was shown in [BFS18] that if $\mathbb{H}^n$ is a Heisenberg group with a homogeneous distance satisfying the geodesic linearity property, then all isometric embeddings $\mathbb{R}^m \hookrightarrow \mathbb{H}^n$ and $\mathbb{H}^m \hookrightarrow \mathbb{H}^n$ are affine.

It was conjectured in [BFS18] and subsequently proved in [BC18] that for Heisenberg groups the geodesic linearity property is equivalent to strict convexity of the projection norm (see Definition 2.4). The main result of this paper is to generalize the same characterization to arbitrary Carnot groups of step 2:

Theorem 1.1. *In every sub-Finsler Carnot group of step 2 with a strictly convex norm, every infinite geodesic is affine.*

Corollary 1.2. *Let G be a stratified group of step 2 equipped with a homogeneous distance d such that the projection norm of d is strictly convex. Then every infinite geodesic in (G, d) is affine.*

The necessity of the strict convexity assumption is a direct consequence of the necessity of strict convexity for linearity of geodesics in the normed-space case, see Proposition 5.2. The restriction to step 2 is motivated by the known counterexample in the simplest Carnot group of step 3, the sub-Riemannian Engel group. The complete study of geodesics in the sub-Riemannian Engel group in [AS15] gives the first (and to date essentially only) known example of a non-affine infinite geodesic in a sub-Riemannian Carnot group.

The proof for Heisenberg groups in [BFS18] that the geodesic linearity property of the target implies that all isometric embeddings are affine works also more generally for stratified groups. Consequently, Corollary 1.2 leads to the corresponding regularity for arbitrary isometric embeddings:

Theorem 1.3. *Let (H, d_H) and (G, d_G) be stratified groups with homogeneous distances such that G has step 2 and the projection norm of d_G is strictly convex. Then every isometric embedding $(H, d_H) \hookrightarrow (G, d_G)$ is affine.*

It is worth remarking that although there are no explicit restrictions on the domain (H, d_H) in Theorem 1.3, the mere existence of an isometric embedding $(H, d_H) \hookrightarrow (G, d_G)$ implies some restrictions. In particular, Pansu's Rademacher theorem [Pan89] implies that there must exist an injective homogeneous homomorphism $H \to G$. It follows that H has step at most 2 and rank at most the rank of G.

1.1. **Structure of the paper.** Section 2 presents the relevant definitions that will be used throughout the rest of the paper and some basic lemmas. The main points of interest are properties of blowdowns of geodesics, i.e., geodesics "viewed from afar", and the collection of observations about subdifferentials of convex functions.

Sections 3–5 are devoted to the proofs of Theorem 1.1 and Corollary 1.2 about infinite geodesics. Section 3 rephrases the classical first order optimality condition of the Pontryagin Maximum Principle in the setting of a step 2 sub-Finsler Carnot group. In the sub-Riemannian case the PMP reduces to a linear ODE for the controls. This is no longer true in the sub-Finsler case, making explicit solution of the system unfeasible. Nonetheless, the PMP has a form (Proposition 3.1) that is well suited to the study of asymptotic behavior of optimal controls. The key object is the bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$.

Section 4 covers the aforementioned asymptotic study. The goal of the section is to study blowdowns of infinite geodesics through the behavior of their controls. Using integral averages of controls, it is shown that any blowdown control must in fact be contained in the kernel of the bilinear form B.

Section 5 wraps up the proof of Theorem 1.1 using the conclusions of the previous sections. This section is the only place where strict convexity appears. The importance of the assumption is that any linear map has a unique maximum on the ball. By observing that any element of $\ker B$ defines an invariant along the corresponding optimal control, the uniqueness is exploited to prove that infinite geodesics must be invariant under blowdowns. Corollary 1.2 follows from the sub-Finsler case by the observation that the length metric associated to a homogeneous norm is always a sub-Finsler metric.

Section 6 covers the proof of Theorem 1.3 about isometric embeddings as a consequence of Corollary 1.2. The link between geodesics and general isometric embeddings arises from considering a foliation by horizontal lines in the domain and studying the induced foliation by infinite geodesics in the image. The affinity of isometric embeddings follows from the observation that two lines are at a sublinear distance from each other if and only if they are parallel.

2. Preliminaries

2.1. **Stratified groups and homogeneous distances.**

Definition 2.1. A *stratified group* is a Lie group G whose Lie algebra has a decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ such that $V_s \neq \{0\}$ and $[V_1, V_k] = V_{k+1}$ for all $k = 1, \dots, s$, with the convention that $V_{s+1} = \{0\}$. The *rank* and *step* of the stratified group G are the integers $r = \dim V_1$ and s respectively.

Definition 2.2. A *dilation* by a factor $h \in \mathbb{R}$ on a stratified group G is the Lie group automorphism $\delta_h \colon G \to G$ defined for any $X = X_1 + \cdots + X_s \in V_1 \oplus \cdots \oplus V_s$ by

$$\delta_h \exp(X_1 + X_2 + \cdots + X_s) = \exp(hX_1 + h^2X_2 + \cdots + h^sX_s).$$

Definition 2.3. A *homogeneous distance* on a stratified group G is a left-invariant distance d, which is one-homogeneous with respect to the dilations, i.e., which satisfies

$$d(\delta_h(g), \delta_h(h)) = hd(g, h) \quad \forall h > 0, \forall g, h \in G.$$

2.2. **The projection norm.**

Definition 2.4. Let G be a stratified group and let d be a homogeneous distance on G. The *projection norm* associated to the homogeneous distance d is the function

$$\|\cdot\|_d : V_1 \to \mathbb{R}, \quad \|X\|_d = d(e, \exp(X)),$$

where e is the identity element of the group G.

It is not immediate that $\|\cdot\|_d$ defines a norm. In the setting of the Heisenberg groups, this is proved in [BFS18, Proposition 2.8]. Their proof works with minor modification for any homogeneous distances in arbitrary stratified groups and is captured in the following lemmas.

The triangle inequality of $\|\cdot\|_d$ is the only non-trivial part. In order to make use of the triangle inequality of the distance d, the following distance estimate is required. The estimate relies on the existence of a dilation, and may fail for non-homogeneous left-invariant distances.

Lemma 2.5. *Let $\pi_{V_1} \colon \mathfrak{g} = V_1 \oplus \cdots \oplus V_s \to V_1$ be the projection with respect to the direct sum decomposition. Then*

$$\|X\|_d \leq d(e, \exp(X + Y)) \quad \forall X \in V_1, \forall Y \in [\mathfrak{g}, \mathfrak{g}],$$

so the horizontal projection $\pi = \pi_{V_1} \circ \log \colon (G, d) \to (V_1, \|\cdot\|_d)$ is a submetry.

Proof. Observe first that for any $X \in V_1$ and $Y = Y_2 + \cdots + Y_s \in V_2 \oplus \cdots \oplus V_s = [\mathfrak{g}, \mathfrak{g}]$, and any $n \in \mathbb{N}$, homogeneity and the triangle inequality imply that

$$\begin{aligned} nd(e, \exp(X + \frac{1}{n}Y_2 + \cdots + \frac{1}{n^{s-1}}Y_s)) &= d(e, \exp(nX + nY)) \\ &\leq nd(e, \exp(X + Y)). \end{aligned}$$

Continuity of the distance then gives the bound

$$\begin{aligned} d(e, \exp(X)) &= \lim_{n\to\infty} d(e, \exp(X + \frac{1}{n}Y_2 + \cdots + \frac{1}{n^{s-1}}Y_s)) \\ &\leq d(e, \exp(X + Y)) \end{aligned}$$

for any $X \in V_1$ and $Y \in [\mathfrak{g}, \mathfrak{g}]$ as claimed.

The previous estimate implies the containment $\pi(B(e, r)) \subset B_{\|\cdot\|_d}(0, r)$ for the projection of any ball $B(e, r) \subset G$. On the other hand, Definition 2.4 of the projection norm directly implies the opposite containment

$$B_{\|\cdot\|_d}(0, r) = V_1 \cap \log B(e, r) \subset \pi(B(e, r)).$$

By left-invariance of the distance d it follows that the map π is a submetry. □

Lemma 2.6. *The projection norm is a norm.*

Proof. Positivity and homogeneity of the projection norm $\|\cdot\|_d$ follow immediately from positivity and homogeneity of the homogeneous distance d. For the triangle inequality, let $X, X' \in V_1$ and let $Y \in [\mathfrak{g}, \mathfrak{g}]$ be the element given by the Baker-Campbell-Hausdorff formula such that

$$\exp(X) \exp(X') = \exp(X + X' + Y).$$

Lemma 2.5 gives the bound $\|X + X'\|_d \leq d(e, \exp(X + X' + Y))$. By the choice of Y, the left-invariance and triangle inequality of d conclude the claim:

$$d(e, \exp(X + X' + Y)) = d(e, \exp(X) \exp(X')) \leq \|X\|_d + \|X'\|_d. \quad \square$$

2.3. **Length structures and sub-Finsler Carnot groups.**

Definition 2.7. Let (X, d) be a metric space. Let Ω be the space of rectifiable curves of X and let $\ell_d \colon \Omega \to \mathbb{R}$ be the length functional. For points $x, y \in X$, denote by $\Omega(x, y) \subset \Omega$ the space of all rectifiable curves connecting the points x and y. The *length metric associated to the metric* d is the map $d_\ell \colon X \times X \to \mathbb{R} \cup \{\infty\}$ defined by

$$d_\ell(x, y) := \inf\{\ell_d(\gamma) : \gamma \in \Omega(x, y)\}.$$

If $d = d_\ell$, then the metric d is called a *length metric*.

See [BBI01, Section 2.3] for further information about length structures induced by metrics. For the purposes of this paper, only the special case of the length metric associated to a homogeneous distance will be relevant. Such a length metric always determines a sub-Finsler Carnot group, see Definition 2.9 and Lemma 5.1.

Definition 2.8. Let G be a stratified group. Denote by $L_g \colon G \to G$ the left-translation $L_g(h) = gh$. An absolutely continuous curve $\gamma\colon [0,T] \to G$ is a *horizontal curve* if $(L_{\gamma(t)^{-1}})_*\dot\gamma(t) \in V_1$ for almost every $t \in [0,T]$. The *control* of a horizontal curve γ is its left-trivialized derivative, i.e., the map

$$u\colon [0,T] \to V_1, \quad u(t) = (L_{\gamma(t)^{-1}})_*\dot\gamma(t).$$

Definition 2.9. A *sub-Finsler Carnot group* is a stratified group G equipped with a norm $\|\cdot\| : V_1 \to \mathbb{R}$. The norm induces a homogeneous distance d_{SF} via the length structure induced by $\|\cdot\|$ over horizontal curves.

More explicitly, for a horizontal curve $\gamma\colon [0,T] \to G$ with control $u\colon [0,T] \to V_1$, define the length

$$\ell_{\|\cdot\|}(\gamma) = \int_0^T \|u(t)\| \, dt.$$

For $g, h \in G$, let $\Omega(g,h)$ be the family of all horizontal curves connecting g and h. The sub-Finsler distance d_{SF} is defined as

$$d_{SF}(g,h) := \inf\{\ell_{\|\cdot\|}(\gamma) : \gamma \in \Omega(g,h)\}.$$

2.4. **Geodesics and blowdowns.**

Definition 2.10. Let G be a stratified group equipped with a homogeneous distance d. A *geodesic* is an isometric embedding $\gamma\colon [0,T] \to (G,d)$. That is, a geodesic satisfies

$$d(\gamma(t),\gamma(s)) = |t-s| \quad \forall t,s \in [0,T].$$

In the proof of Theorem 1.3 it will be convenient to consider also curves which preserve distances up to a constant factor. A curve $\gamma\colon [0,T] \to (G,d)$ for which there exists some constant $C > 0$ such that

$$d(\gamma(t),\gamma(s)) = C\,|t-s| \quad \forall t,s \in [0,T]$$

will be called a *geodesic with speed C*.

Lemma 2.11. *Let $\gamma\colon [0,\infty) \to G$ be a horizontal curve with control $u\colon [0,\infty) \to V_1$ and let $h > 0$ be a dilating factor. Then the dilated and reparametrized curve*

$$\gamma_h\colon [0,\infty) \to G, \quad \gamma_h(t) := \delta_{1/h}\gamma(ht),$$

has the control

$$u_h\colon [0,\infty) \to V_1, \quad u_h(t) := u(ht).$$

Proof. Since the dilations are group homomorphisms, the claim follows directly by the chain rule and Definition 2.8 of a control:

$$\frac{d}{dt}\gamma_h(t) = (\delta_{1/h})_* \frac{d}{dt}\gamma(ht) = (\delta_{1/h})_*(L_{\gamma(ht)})_* u(ht)h = (L_{\gamma_h(t)})_* u_h(t). \quad \square$$

Definition 2.12. Let $\gamma\colon [0,\infty) \to G$ be a horizontal curve. Suppose for some sequence of scales $h_k \to \infty$ the pointwise limit

$$\tilde{\gamma}\colon [0,\infty) \to G, \quad \tilde{\gamma}(t) = \lim_{k\to\infty} \gamma_{h_k}(t) = \lim_{k\to\infty} \delta_{1/h_k}\gamma(h_k t)$$

exists. Such a curve $\tilde{\gamma}$ is called a *blowdown* of the curve γ along the sequence of scales h_k.

Remark 2.13. If the curve γ is L-Lipschitz, then the curves γ_h are also all L-Lipschitz. Hence by Arzelà-Ascoli, up to taking a subsequence a blowdown along a sequence of scales will always exist.

Lemma 2.14. *Let $\gamma\colon [0,\infty) \to G$ be an infinite geodesic and let $\tilde{\gamma} = \lim_{k\to\infty} \gamma_{h_k}$ be any blowdown of the curve γ. Let u and $\tilde{u}$ be the controls of the curves γ and $\tilde{\gamma}$ respectively. Then*

(i) The curve $\tilde{\gamma}$ is an infinite geodesic.

(ii) Up to taking a subsequence, the dilated controls u_{h_k} converge to the control $\tilde{u}$ in $\mathrm{L}^2_{loc}([0,\infty); V_1)$.

Proof. **(i).** The curve $\tilde{\gamma}$ is a geodesic as the pointwise limit of geodesics.

(ii). The claim follows from [MPV18, Remark 3.13]. The point is that by weak compactness of closed balls in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty); V_1)$ there exists a weakly convergent subsequence $u_h \rightharpoonup v$ to some $v \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); V_1)$. The definitions of control and weak compactness imply that v is a control for $\tilde{\gamma}$, so in particular $\tilde{u}(t) = v(t)$ for almost every t. Finally, the geodesic assumption implies that $\|u(t)\| \equiv 1 \equiv \|\tilde{u}(t)\|$, so the weak convergence is upgraded to strong convergence $u_h \to \tilde{u}$ in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty); V_1)$. $\square$

Lemma 2.15. *Let G be a sub-Finsler Carnot group with a strictly convex norm and let $\gamma\colon [0,\infty) \to G$ be an infinite geodesic. Then there exists a sequence $h_k \to \infty$ such that the blowdown $\tilde{\gamma} = \lim_{k\to\infty} \gamma_{h_k}$ is affine.*

Proof. If the geodesic γ is itself affine, then the claim is immediate. Otherwise, consider the horizontal projection $\pi\circ\gamma\colon [0,\infty) \to G/[G,G]$. Since $G/[G,G]$ is a normed space with a strictly convex norm, and the geodesic γ is not affine, the projection curve $\pi\circ\gamma$ is not affine and hence not a geodesic. Then [HLD18, Theorem 1.4] states that there exists a Carnot subgroup $H < G$ of lower rank such that all blowdowns of the geodesic γ are contained in H.

Let the curve $\beta\colon [0,\infty) \to H < G$ be any blowdown. By Lemma 2.14(i), β is a geodesic. If β is also not affine, then iterating the above there exists a Carnot subgroup $K < H < G$ of even lower rank such that all blowdowns of β are in K. Blowdowns of the geodesic β are also blowdowns of the geodesic γ by a diagonal argument, so the claim follows by induction, since a Carnot subgroup of rank 1 is just a one parameter subgroup. $\square$

2.5. **Subdifferentials.** In this section, let V be some fixed finite dimensional vector space and let $E\colon V \to \mathbb{R}$ be a convex continuous function. In the application in Section 5, the space V will be the horizontal layer $V_1 \subset \mathfrak{g}$, and the convex function of interest will be a squared norm $\frac{1}{2}\left\|\cdot\right\|^2$.

Definition 2.16. A linear function $a\colon V \to \mathbb{R}$ is a *subdifferential* of the function E at a point $Y \in V$ if

$$a(X - Y) \leq E(X) - E(Y) \quad \forall X \in V.$$

The collection of all subdifferentials a at a point $Y \in V$ is denoted $\partial E(Y) \subset V^*$.

The following lemmas are all simple properties of convex functions and their subdifferentials. They will be utilized in the proof of Theorem 1.1 in Section 5. The first lemma is the continuity of subdifferentials as a set valued map $V \to \mathcal{P}(V^*)$, $Y \mapsto \partial E(Y)$.

Lemma 2.17. *Let $Y_k \to Y \in V$ be a converging sequence and let $a_k \in \partial E(Y_k)$. Then there exists a converging subsequence $a_k \to a \in \partial E(Y)$.*

Proof. [Roc70, Theorem 24.7] shows (among other things) that since the set of points $\mathcal{S} = \{Y_i : i \in \mathbb{N}\} \cup \{Y\}$ is closed and bounded, the family of subdifferentials

$$\partial E(\mathcal{S}) := \bigcup_{X\in\mathcal{S}} \{a \in \partial E(X)\}$$

is also closed and bounded, and the subdifferentials $a \in \partial E(\mathcal{S})$ are equicontinuous. Hence the existence of a converging subsequence $a_k \to a$ to some linear map $a\colon V \to \mathbb{R}$ is a consequence of Arzelà-Ascoli.

The claim is concluded by [Roc70, Theorem 24.4], which shows that the convergences $Y_k \to Y$ and $a_k \to a$ with $a_k \in \partial E(Y_k)$ imply that $a \in \partial E(Y)$. $\square$

The next two lemmas contain the maximality argument that will eventually be used to allow a blowdown argument to conclude that all infinite geodesics are lines.

Lemma 2.18. *Suppose the map E is strictly convex and let $a \in \partial E(Y)$. Then the point Y is the unique maximizer of the linear function a in the sublevel set $\{X \in V : E(X) \leq E(Y)\}$.*

Proof. For $E(X) \leq E(Y)$, the subdifferential condition $a \in \partial E(Y)$ gives the bound

$$a(X) - a(Y) = a(X - Y) \leq E(X) - E(Y) \leq 0, \tag{1}$$

proving maximality of Y.

Let $X \in V$ be another maximum. That is, suppose $a(X) = a(Y)$ and $E(X) \leq E(Y)$, so the bound (1) implies that necessarily $E(X) = E(Y)$. By linearity also $a((X+Y)/2) = a(Y)$, so the bound (1) further implies that also $E(X) = E((X+Y)/2) = E(Y)$. By strict convexity, this is only possible if $X = Y$, proving uniqueness of the maximizer Y. ☐

Lemma 2.19. *Let $\|\cdot\|$ be a norm on V and let $a\colon V \to \mathbb{R}$ be a subdifferential of the map $E = \frac{1}{2}\|\cdot\|^2$ at a point $Y \in V$. Then $|a(X)| \leq \|X\|\,\|Y\|$ for all $X \in V$, and $a(Y) = \|Y\|^2$.*

Proof. For any points $X, Y \in V$ and any $\epsilon > 0$, the subdifferential condition $a \in \partial E(Y)$ implies that

$$\begin{aligned}
\epsilon a(X) &= a(Y + \epsilon X - Y) \leq E(Y + \epsilon X) - E(Y) \\
&\leq \frac{1}{2}\Big((\|Y\| + \epsilon\|X\|)^2 - \|Y\|^2\Big) = \epsilon\|X\|\,\|Y\| + \frac{1}{2}\epsilon^2\|X\|^2.
\end{aligned}$$

Letting $\epsilon \to 0$ proves the bound $a(X) \leq \|X\|\,\|Y\|$. Repeating the same consideration for $-X$, gives the opposite bound $-a(X) \leq \|X\|\,\|Y\|$.

For the equality $a(Y) = \|Y\|^2$, let $\epsilon > 0$, and observe that a similar computation as before shows that

$$\begin{aligned}
-\epsilon a(Y) &= a((1-\epsilon)Y - Y) \leq E((1-\epsilon)Y) - E(Y) \\
&= \frac{1}{2}((1-\epsilon)^2 - 1)\|Y\|^2 = (-\epsilon + \frac{1}{2}\epsilon^2)\|Y\|^2.
\end{aligned}$$

That is, $a(Y) \geq (1 - \frac{1}{2}\epsilon)\|Y\|^2$. The limit as $\epsilon \to 0$ and the previous upper bound prove the claim. ☐

3. Step 2 sub-Finsler Pontryagin Maximum Principle

In this section, the Pontryagin Maximum Principle will be rephrased in a convenient form for the purposes of Theorem 1.1. The precise statement to be proved is the following:

Proposition 3.1 (Step 2 sub-Finsler PMP.)**.** *Let G be a step 2 sub-Finsler Carnot group with an arbitrary norm $\|\cdot\| : V_1 \to \mathbb{R}$ and let $0 \leq T \leq \infty$. If $u\colon [0,T] \to V_1$ is the control of a geodesic, then there exists an absolutely continuous curve $a\colon [0,T] \to V_1^*$ and a skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$ such that* *(i) At almost every $t \in [0,T]$, the curve a has the derivative*

$$\frac{d}{dt}a(t)Y = B(u(t),Y) \quad \forall Y \in V_1.$$

(ii) At almost every $t \in [0,T]$, the linear map $a(t)\colon V_1 \to \mathbb{R}$ is a subdifferential of the squared norm $\frac{1}{2}\|\cdot\|^2$ at the point $u(t) \in V_1$.

Remark 3.2. In the sub-Riemannian case, the the squared norm $\frac{1}{2}\|\cdot\|^2$ is differentiable at every point, and the unique subdifferential is the inner product $a(t) = \langle u(t), \cdot\rangle$. The derivative condition (i) then gives the usual linear ODE of controls in implicit form

$$\langle \dot{u}(t), Y\rangle = \frac{d}{dt}\langle u(t), Y\rangle = B(u(t),Y) \quad \forall Y \in V_1.$$

3.1. **General statement of the PMP.** For the rest of Section 3, let G be a fixed sub-Finsler Carnot groupof step 2 with a generic norm $\|\cdot\| : V_1 \to \mathbb{R}$, and let $u\colon [0,T] \to V_1$ be the control of a geodesic $\gamma\colon [0,T] \to G$.

Consider first the finite time $T < \infty$ case. By Definition 2.9 of the sub-Finsler distance, the control u minimizes the length functional $\int_0^T \|u(t)\|\, dt$ among all controls defining curves with the same endpoints as γ. Since a geodesic has by definition constant speed, it follows that u is also a minimizer of the energy functional $\frac{1}{2}\int_0^T \|u(t)\|^2\, dt$.

Define the left-trivialized Hamiltonian

$$h\colon V_1 \times \mathbb{R} \times \mathfrak{g}^* \to \mathbb{R}, \quad h(u,\xi,\lambda) = \lambda(u) + \frac{1}{2}\xi\|u\|^2. \tag{2}$$

By the Pontryagin Maximum Principle as presented in [AS04, Theorem 12.10], the control $u\colon [0,T] \to V_1$ can minimize the energy $\frac{1}{2}\int_0^T \|u(t)\|^2\, dt$ only if there is an almost everywhere non-zero absolutely continuous dual curve $t \mapsto (\xi, \lambda(t)) \in \mathbb{R} \times T^*_{\gamma(t)}G$ such that

$$\xi \le 0 \tag{3}$$

$$\dot{\lambda} = \vec{h}_{u(t),\xi}(\lambda) \quad \text{a.e. } t \in [0,T], \tag{4}$$

$$h_{u(t),\xi}(\lambda(t)) \ge h_{v,\xi}(\lambda(t)) \quad \forall v \in V_1 \quad \text{a.e. } t \in [0,T]. \tag{5}$$

Here $h_{v,\xi}$ and $\vec{h}_{v,\xi}$, for $v \in V_1$, are the left-invariant Hamiltonian and the associated Hamiltonian vector field respectively.

More explicitly, $h_{v,\xi}\colon T^*G \to \mathbb{R}$ is the function defined from the left-trivialized Hamiltonian (2) in the natural way by

$$h_{v,\xi}(\lambda) = h(v,\xi,L_g^*\lambda), \quad \forall \lambda \in T_g^*G, \tag{6}$$

and $\vec{h}_{v,\xi}$ is the Hamiltonian vector field associated to the left-invariant Hamiltonian $h_{v,\xi}$ and the canonical symplectic form ω on the cotangent bundle T^*G by the duality

$$\omega(w, \vec{h}_{v,\xi}(\lambda)) = dh_{v,\xi}(w) \quad \forall w \in T_\lambda(T^*G). \tag{7}$$

Observe that if $(\xi, \lambda(t))$ is a dual curve satisfying the conditions (3)–(5) of the PMP, then also any scalar multiple $(C\xi, C\lambda(t))$ for any $C > 0$ satisfies the conditions (3)–(5) of the PMP. This observation allows the infinite time case $T = \infty$ to be handled as a limit of the finite time case. Namely, if $u\colon [0,\infty) \to V_1$ is the control of a geodesic, then all its finite restrictions $u|_{[0,k]} : [0,k] \to V_1$ for $k \in \mathbb{N}$ are also controls of geodesics, so by the above they have corresponding dual curves $t \mapsto (\xi_k, \lambda_k(t))$. By taking suitable rescalings of the (ξ_k, λ_k), there exists a non-zero limit $(\xi_\infty, \lambda_\infty)$, which then satisfies the conditions (3)–(5) of the PMP on the entire interval $[0, \infty)$.

3.2. **The normality/abnormality condition.** Condition (3) is a binary condition $\xi = 0$ or $\xi \neq 0$. The case $\xi = 0$ is the case of an abnormal control u, and may be ignored in the step 2 setting.

Indeed, suppose to the contrary that there does not exist a dual curve (ξ, λ) with $\xi \neq 0$, but only some dual curve with $\xi = 0$. In this case, the maximality condition (5) states that

$$\lambda(t)((L_{\gamma(t)})_* u(t)) \geq \lambda(t)((L_{\gamma(t)})_* v) \quad \forall v \in V_1 \quad \text{a.e. } t \in [0,T],$$

which is only possible when $(L_{\gamma(t)})_* V_1 \subset \ker \lambda(t)$. Moreover, second order optimality conditions from [AS04, Section 20] further imply that (possibly changing the dual curve λ) the Goh condition

$$\lambda(t)((L_{\gamma(t)})_*[v,w]) = 0, \quad \forall v, w \in V_1 \quad \text{a.e. } t \in [0,T]$$

is also satisfied. Since the group G has step 2, its Lie algebra has the decomposition $\mathfrak{g} = V_1 \oplus [V_1, V_1]$. The above would then imply that $\lambda(t) = 0$ almost everywhere, which would contradict the assumption that (ξ, λ) is almost everywhere non-zero.

Therefore without loss of generality it suffices to consider the normal case $\xi < 0$. By rescaling (ξ, λ) it further suffices to consider the case $\xi = -1$.

3.3. **The Hamiltonian ODE in left-trivialized coordinates.** The normal Hamiltonian vector field $\vec{h}_{u(t),-1}(\lambda)$ appearing in the ODE (4) is straight-forward to compute in left-trivialized coordinates on T^*G. The explicit expression will be given in Lemma 3.3.

Let $X_1, \dots, X_r$ be a basis of V_1. Fix a basis $X_{r+1}, \dots, X_n$ for $V_2 = [V_1, V_1]$ by choosing a maximal linearly independent subset of the Lie brackets $\{[X_i, X_j] : 1 \leq i < j \leq r\}$. By an abuse of notation, denote also by $X_1, \dots, X_n$, the corresponding left-invariant frame of TG. Let $\theta_1, \dots, \theta_n$ be the dual left-invariant frame of T^*G. Any covector $\lambda \in T^*_g G$ can be written in the frame as

$$\lambda = \sum_{i=1}^{n} a_i(\lambda) \theta_i(g).$$

The functions $a_i\colon T^*G \to \mathbb{R}$ together with coordinates $g \in G$ define left-trivialized coordinates on T^*G.

Lemma 3.3. *For any vector $v \in V_1$, the Hamiltonian vector field of the normal left-invariant Hamiltonian $h_{v,-1}$ has in left-trivialized coordinates the expression*

$$\vec{h}_{v,-1}(\lambda) = \sum_{1\leq i\leq r} \lambda((L_{\pi(\lambda)})_*[v, X_i])\partial a_i + (L_{\pi(\lambda)})_* v \in T_\lambda(T^*G).$$

Proof. Let

$$F(v, \lambda) := \sum_{1\leq i\leq r} \lambda((L_{\pi(\lambda)})_*[v, X_i])\partial a_i + (L_{\pi(\lambda)})_* v \tag{8}$$

be the expression on the right-hand side of the claimed formula for the normal Hamiltonian $\vec{h}_{v,-1}$. By the definition of a Hamiltonian vector field, it suffices to verify that the vector field $\lambda \mapsto F(v, \lambda)$ satisfies the duality (7), i.e., that

$$\omega(w, F(v, \lambda)) = dh_{v,-1}(w) \quad \forall w \in T_\lambda(T^*G). \tag{9}$$

The differential on the right-hand side of (9) is easily computed. The normal Hamiltonian $h_{v,-1}\colon T^*G \to \mathbb{R}$ defined by (6) is left-invariant and linear on fibers. Therefore in left-trivialized coordinates, the differential has the expression

$$dh_{v,-1} = \sum_{i=1}^{n} v_i da_i. \tag{10}$$

The expression for the symplectic form is more intricate. By definition the canonical symplectic form ω on the cotangent bundle T^*G is the differential of the tautological one-form $\sum_{i=1}^{n} a_i\theta_i$. That is, in left-trivialized coordinates, the symplectic form has the expression (see e.g. [ABB19, Section 4.2] for more details)

$$\omega = \sum_{i=1}^{n} da_i \wedge \theta_i + \sum_{i=1}^{n} a_i d\theta_i. \tag{11}$$

The differentials $d\theta_i$ can be evaluated along vector fields $X, Y \in \Gamma(TG)$ by the classical formula

$$d\theta_i(X, Y) = X\theta_i(Y) - Y\theta_i(X) - \theta_i([X, Y]).$$

For left-invariant vector fields this reduces to $d\theta_i(X, Y) = -\theta_i([X, Y])$.

Let W and Z be vector fields on the cotangent T^*G such that the projections $X := \pi_* W$ and $Y := \pi_* Z$ are left invariant. Writing the vector fields in left-trivialized coordinates as $W = \sum_{i=1}^{n} w_i\partial_{a_i} + X$ and $Z = \sum_{i=1}^{n} z_i\partial_{a_i} + Y$, the identity (11) gives the expression

$$\omega(W, Z) = \sum_{i=1}^{n} w_i\theta_i(Y) - \sum_{i=1}^{n} z_i\theta_i(X) - \sum_{i=1}^{n} a_i\theta_i([X, Y]). \tag{12}$$

The duality (9) will now be deduced by comparing the expressions (10) and (12). For an arbitrary vector $w \in T_\lambda(T^*G)$, let W be the left-invariant extension to a vector field on the cotangent T^*G. That is, the vector field W has a constant coefficient coordinate expression

$$W = \sum_{i=1}^{n} w_i \partial a_i + \sum_{i=1}^{n} x_i X_i. \tag{13}$$

Denote by $X := \pi_* W$ and $Y := \pi_* F(v, \cdot)$ the projection vector fields on the group G of the vector fields W and $\lambda \mapsto F(v, \lambda)$. Substituting the expressions (13) of W and (8) of $F(v, \cdot)$ into the expression (12) for $\omega(W, Z)$ with $Z = F(v, \lambda)$ gives the three sums

$$\sum_{i=1}^{n} w_i \theta_i((L_{\pi(\lambda)})_* v) = \sum_{i=1}^{n} w_i v_i,$$

$$\sum_{i=1}^{n} \lambda((L_{\pi(\lambda)})_*[v, X_i])\theta_i(X) = \lambda([V, X]) \quad \text{and}$$

$$\sum_{i=1}^{n} a_i \theta_i([X, V]) = \lambda([X, V]).$$

By anti-commutativity of the Lie bracket, the last two sums cancel out, so $\omega(W, F(v, \lambda)) = \sum_{i=1}^{n} w_i v_i$. Since the expression (10) of the differential also gives $dh_{v,-1}(W) = \sum_{i=1}^{n} w_i v_i$, the vector field $\lambda \mapsto F(v, \lambda)$ indeed satisfies the duality (9). □

3.4. **Conclusion of the step 2 sub-Finsler PMP.**

Proof of Proposition 3.1. The curve $a\colon [0, T] \to V_1^*$ will be given by restricting the linear map

$$a(t) := (L_{\gamma(t)})^* \lambda(t) \colon \mathfrak{g} \to \mathbb{R} \tag{14}$$

to V_1. The skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$ will be given by

$$B(X, Y) = a(t)[X, Y]. \tag{15}$$

In order to see that the above expressions are well defined and have the desired properties, express the curve $\lambda(t)$ in left-trivialized coordinates as

$$\lambda(t) = \sum_{i=1}^{n} a_i(t) \theta_i(\gamma(t)).$$

The curve $a(t)$ of (14) has the same coefficients as the curve $\lambda(t)$, i.e., it is given by $a(t) = \sum_{i=1}^{n} a_i(t)\theta_i(e)$. Using the explicit expression for the normal Hamiltonian vector field from Lemma 3.3, the normal

Hamiltonian ODE $\dot{\lambda} = \vec{h}_{u(t),-1}(\lambda)$ implies that the components of the curve a have the derivatives

$$\begin{cases} \dot{a}_i(t) = a(t)[u(t), X_i], & i \in \{1, \dots, r\} \\ \dot{a}_i(t) = 0, & i \in \{r+1, \dots, n\} \end{cases} \tag{16}$$

Observe that the vertical coefficients $a_{r+1}, \dots, a_n$ are all constant, and that $\theta_i([X, Y]) = 0$ for $i = 1, \dots, r$. Therefore $a(t)[X, Y] = \sum_{i=r+1}^{n} a_i \theta_i([X, Y])$ is constant in t, so (15) defines a unique bilinear form B independent from t. Moreover, the non-trivial equations of the system (16) are then exactly

$$\dot{a}_i(t) = a(t)[u(t), X_i] = B(u(t), X_i), \quad i \in \{1, \dots, r\}.$$

Writing an arbitrary vector $Y \in V_1$ in the basis $X_1, \dots, X_r$ as $Y = y_1 X_1 + \dots + y_r X_r$, the derivative condition 3.1(i) follows by linearity:

$$\frac{d}{dt} a(t) Y = \frac{d}{dt} \sum_{i=1}^{n} a_i(t) y_i = \sum_{i=1}^{n} B(u(t), X_i) y_i = B(u(t), Y).$$

The subdifferential condition 3.1(ii) for the linear functions $a(t)$ follows from rephrasing the maximality condition (5). Namely, expanding out the explicit expressions of the normal Hamiltonians $h_{u(t),-1}$ and $h_{v,-1}$ from (2) and reorganizing terms, the maximality condition (5) is equivalently stated as

$$a(t)v - a(t)u(t) \le \frac{1}{2} \|v\|^2 - \frac{1}{2} \|u(t)\|^2 \quad \forall v \in V_1 \quad \text{a.e. } t \in [0, T].$$

This is exactly Definition 2.16 for the linear function $a(t)$ being a subdifferential of the squared norm $\frac{1}{2} \|\cdot\|^2$ at the point $u(t) \in V_1$. □

4. Asymptotic behavior of controls

In this section, let $u \colon [0, \infty) \to V_1$ be a fixed control satisfying the PMP 3.1. Let $a \colon [0, \infty) \to V_1^*$ be the associated curve of subdifferentials and let $B \colon V_1 \times V_1 \to \mathbb{R}$ be the associated bilinear form.

Lemma 4.1. *For every vector $X \in V_1$,*

$$\lim_{T \to \infty} B \left(\fint_0^T u(t)\, dt, X \right) = 0.$$

Proof. Fix an arbitrary vector $X \in V_1$. Bilinearity of the map B implies that

$$B \left(\fint_0^T u(t)\, dt, X \right) = \frac{1}{T} \int_0^T B\left(u(t), X\right)\, dt. \tag{17}$$

Since the curve a is absolutely continuous, the derivative condition PMP 3.1(i) implies that

$$\int_0^T B(u(t), X) = \int_0^T \frac{d}{dt} a(t) X = a(T)X - a(0)X. \tag{18}$$

For almost every t, the linear map $a(t)$ is a subdifferential of the squared norm $\frac{1}{2}\|\cdot\|^2$ at the point $u(t)$. Since $\|u(t)\| \equiv 1$ is constant, continuity of the curve a and Lemma 2.19 imply the bound $|a(t)X| \leq \|X\|$ for every $t \in [0,T]$. The identities (17) and (18) then imply the desired conclusion that

$$\lim_{T\to\infty} \left| B\left(\fint_0^T u(t)\,dt, X \right) \right| \leq \lim_{T\to\infty} \frac{2}{T}\|X\| = 0. \qquad \square$$

Lemma 4.2. *Let $h_k \to \infty$ be a diverging sequence and let $u_{h_k}(t) = u(h_k t)$ be the corresponding dilated controls. If $u_{h_k} \to \tilde{u}$ in $\mathrm{L}^2_{loc}([0,\infty); V_1)$, then $\tilde{u}(t) \in \ker B$ for almost every $t \in [0,\infty)$.*

Proof. By the Lebesgue differentiation theorem it suffices to prove that $\fint_a^b \tilde{u}(t)\,dt \in \ker B$ for any $0 \leq a < b < \infty$.

Fix $0 \leq a < b < \infty$. By assumption $u_{h_k} \to \tilde{u}$ in $\mathrm{L}^2([a,b]; V_1)$, so there exists some error term $\epsilon\colon \mathbb{N} \to V_1$ with $\lim\limits_{k\to\infty} \epsilon(k) = 0$ such that

$$\fint_a^b \tilde{u}(t)\,dt = \fint_a^b u(h_k t)\,dt + \epsilon(k) = \fint_{ah_k}^{bh_k} u(t)\,dt + \epsilon(k). \tag{19}$$

The right-hand integral average can further be expressed as a difference of integral averages as

$$\fint_{ah_k}^{bh_k} u(t)\,dt = \frac{b}{b-a} \cdot \fint_0^{bh_k} u(t)\,dt - \frac{a}{b-a} \cdot \fint_0^{ah_k} u(t)\,dt. \tag{20}$$

Lemma 4.1 implies that for any $X \in V_1$

$$\lim_{k\to\infty} B\left(\fint_0^{bh_k} u(t)\,dt, X \right) = \lim_{k\to\infty} B\left(\fint_0^{ah_k} u(t)\,dt, X \right) = 0$$

Combining the identities (19) and (20) and using bilinearity of B then implies that $B\left(\fint_a^b \tilde{u}(t)\,dt, X \right) = 0$. Since the vector $X \in V_1$ was arbitrary, this proves the desired claim that $\fint_a^b \tilde{u}(t)\,dt \in \ker B$. $\square$

5. Affinity of infinite geodesics

5.1. **Sub-Finsler Carnot groups.** The proof of Theorem 1.1 will now be concluded. The key ingredients are the sub-Finsler PMP 3.1, the knowledge of asymptotic behavior of blowdown controls from Lemma 4.2, and the convex analysis arguments from Subsection 2.5.

Proof of Theorem 1.1. Let $\gamma\colon [0,\infty) \to G$ be an infinite geodesic and let $u\colon [0,\infty) \to V_1$ be its control. Let $a\colon [0,\infty) \to V_1^*$ be the curve of subdifferentials of the squared norm $\frac{1}{2}\|\cdot\|^2$ and let $B\colon V_1 \times V_1 \to \mathbb{R}$ be the skew-symmetric bilinear form given by the PMP 3.1.

By Lemma 2.15, there exists a sequence $h_k \to \infty$ such that the blowdown $\tilde{\gamma} = \lim\limits_{k\to\infty} \delta_{1/h_k} \circ \gamma \circ \delta_{h_k}\colon [0,\infty) \to G$ is affine. By Lemma 2.14, taking a subsequence if necessary, the dilated controls $u_{h_k}(t) = u(h_k t)$

converge in $L^2_{\mathrm{loc}}([0,\infty);V_1)$ to the control $\tilde{u}$ of the curve $\tilde{\gamma}$. Since the curve $\tilde{\gamma}$ is affine, the control $\tilde{u}$ is constant. That is, there exists a constant vector $Y \in V_1$, which for almost every $t \in [0,\infty)$ is the limit

$$Y = \tilde{u}(t) = \lim_{k\to\infty} u(h_k t). \tag{21}$$

By Lemma 4.2, $Y \in \ker B$, so the derivative condition PMP 3.1(i) implies that the curve $t \mapsto a(t)Y$ is constant $a(t)Y \equiv: C$.

Fix any $t \in [0,\infty)$ such that the limit (21) holds. By Lemma 2.17, up to taking a further subsequence, the subdifferentials $a(h_k t)$ of the squared norm $\frac{1}{2}\|\cdot\|^2$ at the points $u(h_k t)$ converge to a subdifferential $\tilde{a}\colon V_1 \to \mathbb{R}$ of the squared norm $\frac{1}{2}\|\cdot\|^2$ at the point Y. Moreover, since $a(t)Y \equiv C$ is constant, also the limit evaluates to $\tilde{a}Y = C$. Applying Lemma 2.19 for the subdifferential $\tilde{a}$ shows that $C = \tilde{a}Y = \|Y\|^2$. Similarly applying Lemma 2.19 for the subdifferential $a(t)$ shows that $a(t)u(t) = \|u(t)\|^2$. Since the curves γ and $\tilde{\gamma}$ are both geodesics, $\|u(t)\| = 1 = \|Y\|$, so combining all of the above gives the equality

$$a(t)Y = 1 = a(t)u(t). \tag{22}$$

The norm $\|\cdot\|$ is by assumption strictly convex and the map $x \mapsto \frac{1}{2}x^2$ is strictly increasing and convex on $[0,\infty)$, so also the squared norm $\frac{1}{2}\|\cdot\|^2$ is strictly convex. Then by Lemma 2.18, the point $u(t)$ is the unique maximizer for the linear map $a(t)$ in the corresponding sublevel set, so the equality (22) implies that $u(t) = Y$. Repeating the same argument at all the times t satisfying the limit (21), it follows that $u(t) = Y$ for almost every $t \in [0,\infty)$, so the geodesic γ is itself affine. □

5.2. **Arbitrary homogeneous distances.** The proof of Corollary 1.2 about infinite geodesics for arbitrary homogeneous norms follows from the sub-Finsler case by passing to the induced length metric. The relevant properties are captured in the next lemma.

Lemma 5.1. *Let (G,d) be a stratified group of step 2 equipped with a homogeneous distance d and let d_ℓ be the length metric of d. Then*

(i) (G,d_ℓ) is a sub-Finsler Carnot group.
(ii) All geodesics of (G,d) are also geodesics of (G,d_ℓ).
(iii) The projection norm of d is the sub-Finsler norm of d_ℓ.

Proof. **(i).** In [LD15, Theorem 1.1] sub-Finsler Carnot groups are characterized as the only geodesic metric spaces that are locally compact, isometrically homogeneous, and admit a dilation. Therefore it suffices to verify that the length metric associated to a homogeneous distance satisfies these properties.

The claims of isometric homogeneity and admitting a dilation follow directly from the corresponding properties of the metric d. Namely, since left-translations are isometries of the metric d, they preserve the length of curves, and hence are also isometries of the length metric d_ℓ. Similarly since dilations scale the length of curves linearly, they are dilations for the length metric d_ℓ.

Finiteness of the length metric d_ℓ follows from the stratification assumption: each element $g \in G$ can be written as a product of elements in $\exp(V_1)$ and the horizontal lines $t \mapsto \exp(tX)$ are all geodesics. Therefore concatenation of suitable horizontal line segments defines a finite length curve from the identity e to any desired point g. It follows that the length metric d_ℓ determines a well defined homogeneous distance on G, so by [LD17, Proposition 3.5] it induces the manifold topology of G. In particular (G, d_ℓ) is a boundedly compact length space, so it is a geodesic metric space (see [BBI01, Corollary 2.5.20]). Applying [LD15, Theorem 1.1] shows that (G, d_ℓ) is a sub-Finsler Carnot group.

(ii). The lengths of all rectifiable curves in the original metric d and its associated length metric d_ℓ always agree (see [BBI01, Proposition 2.3.12]). In particular, the claim that the geodesics of (G, d) are geodesics of (G, d_ℓ) follows.

(iii). The horizontal projection $\pi\colon (G, d) \to V_1$ is a submetry both for the sub-Finsler norm $\|\cdot\|_{SF}$ (by definition) and for the projection norm $\|\cdot\|_d$ (by Lemma 2.5). Hence the norms $\|\cdot\|_{SF}$ and $\|\cdot\|_d$ have exactly the same balls, so $\|\cdot\|_{SF} = \|\cdot\|_d$. □

Proof of Corollary 1.2. Let (G, d) be a stratified group of step 2 equipped with a homogeneous distance d whose projection norm is strictly convex, and let $\gamma\colon [0, \infty) \to (G, d)$ be an infinite geodesic.

Let d_ℓ be the length-metric associated to d. By Lemma 5.1(i) and (ii), the curve γ is also a geodesic of $(G, \|\cdot\|)$, where $\|\cdot\| : V_1 \to \mathbb{R}$ is the sub-Finsler norm of the sub-Finsler metric d_ℓ. Moreover by Lemma 5.1(iii) the norm $\|\cdot\| = \|\cdot\|_d$ is by assumption strictly convex.

Consequently by Theorem 1.1, the geodesic γ is affine. □

The necessity of the strict convexity assumption is an immediate consequence of the classical case of normed spaces by the following simple lifting argument.

Proposition 5.2. *Let G be a stratified group of step 2 equipped with an arbitrary homogeneous distance d. If the projection norm of d is not strictly convex, then there exist an infinite geodesic $\gamma\colon \mathbb{R} \to G$ which is not affine.*

Proof. If the projection norm $\|\cdot\|_d : V_1 \to \mathbb{R}$ is not strictly convex, then there exists a non-linear geodesic $\beta\colon \mathbb{R} \to V_1$. For example, if the norm $\|X + cY\|_d$ is constant for $-\epsilon \le c \le \epsilon$, then the curve $\beta(t) = tX + \epsilon \sin(t) Y$ is an infinite geodesic.

By Lemma 2.5, the projection $\pi\colon (G, d) \to (V_1, \|\cdot\|)$ is a submetry, so the geodesic $\beta\colon \mathbb{R} \to V_1$ lifts to an infinite geodesic $\gamma\colon \mathbb{R} \to G$. Since the projection is a homomorphism and the geodesic β is not affine, neither is the geodesic γ. □

6. Affinity of isometric embeddings

Theorem 1.3 about isometric embeddings being affine follows from Corollary 1.2 by an abstraction of the argument of [BFS18, Theorem 4.1]. The key link between the metric and algebraic properties is the following simple lemma stating that the distance between two lines grows sublinearly if and only if the lines are parallel.

Lemma 6.1. *Let (G, d) be a stratified group with a homogeneous distance. Then for all points $g, h \in G$ and all vectors $X, Y \in V_1$*

$$d(g\exp(tX), h\exp(tY)) = o(t) \text{ as } t \to \infty \iff X = Y.$$

Proof. Consider dilations by $1/t$. Since dilations are homomorphisms, continuity of the distance gives the limit

$$\begin{aligned} \lim_{t\to\infty} \frac{d(g\exp(tX), h\exp(tY))}{t} &= \lim_{t\to\infty} d(\delta_{1/t}(g)\exp(X), \delta_{1/t}(h)\exp(Y)) \\ &= d(\exp(X), \exp(Y)). \end{aligned}$$

□

Proof of Theorem 1.3. Let $\varphi\colon (H, d_H) \hookrightarrow (G, d_G)$ be an isometric embedding. Since left-translations are isometries, it suffices to consider the case when the map φ preserves the identity element, and prove that such an isometric embedding is a homomorphism.

Consider an arbitrary point $h \in H$ and a horizontal vector $X \in V_1^H$. The horizontal line $t \mapsto h\exp(tX)$ is an infinite geodesic with speed $\|X\|_H$ through the point $h \in H$. The image of the line under the isometric embedding φ is an infinite geodesic in the group G through the point $\varphi(h)$ with exactly the same speed. By Corollary 1.2 all infinite geodesics in the group G are horizontal lines, so there exists some vector $Y \in V_1^G$ (a priori depending on the point h and the vector X) with $\|X\|_H = \|Y\|_G$ such that

$$\varphi(h\exp(tX)) = \varphi(h)\exp(tY) \quad \forall t \in \mathbb{R}.$$

Consider then the two parallel infinite geodesics $t \mapsto \exp(tX)$ and $t \mapsto h\exp(tX)$ with speed $\|X\|_H$. Repeating the previous consideration, since the map φ was assumed to preserve the identity, there exists another horizontal direction $Z \in V_1^G$ such that $\varphi(\exp(tX)) = \exp(tZ)$. By Lemma 6.1, the distance between the two lines in the group H grows sublinearly. Since the map φ is an isometric embedding, also the distance between the image lines in the group G grows sublinearly. Hence applying Lemma 6.1 in the converse direction implies that $Y = Z$. That is, the vector $Y \in V_1^G$ does not depend on the point $h \in H$, only on the vector $X \in V_1^H$.

The above shows that there is a well defined map $\varphi_*\colon V_1^H \to V_1^G$ such that $\varphi(h\exp(X)) = \varphi(h)\exp(\varphi_* X)$. In particular,

$$\varphi(h_1 h_2) = \varphi(h_1)\varphi(h_2) \quad \forall h_1 \in H\ \forall h_2 \in \exp(V_1^H). \tag{23}$$

Since the group H is stratified, the subset $\exp(V_1^H)$ generates the entire group H. That is, any element $h \in H$ can be written as a finite product of elements in $\exp(V_1^H)$. Applying the identity (23) repeatedly using such decompositions shows that the map φ is a homomorphism. □

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