Carnot groups and abnormal dynamics

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Sub-Riemannian manifolds

A sub-Riemannian manifold consists of

- a smooth manifold M
- a bracket-generating distribution $\Delta \subset TM$
- a smoothly varying inner product on Δ

Assume (for simplicity):

- Δ has a global frame $X_1, \ldots, X_r$
- the vector fields $X_1, \ldots, X_r$ are complete

The endpoint map

Fix a base point $p \in M$.

Definition (Endpoint map)

The *endpoint map* is the map

$$\mathrm{End}\colon L^2([0,1];\mathbb{R}^r) \to M, \quad u \mapsto \gamma_u(1),$$

where $\gamma_u\colon [0,1] \to M$ is the curve

$$\dot{\gamma}_u(t) = \sum u_i(t) X_i(\gamma_u(t))$$
$$\gamma_u(0) = p$$

Assumptions $\implies$ endpoint map well defined and surjective.

The endpoint map

Abnormal $\leftrightarrow$ critical points and values of the endpoint map.

Abnormal control $=$ critical point $u \in L^2$ of the endpoint map

Abnormal curve $=$ integral curve γ_u of an abnormal control u

Abnormal set $=$ the set of critical values of the endpoint map
$=$ the subset of M that can be reached from the basepoint with an abnormal curve.

Open problems

Conjecture (Sard)

The abnormal set has zero measure.

Conjecture (Regularity)

All length-minimizing curves are smooth.

There are two types of length-minimizing curves.

1. normal: satisfy a geodesic equation $\implies$ are smooth
2. abnormal: unknown regularity

A dynamical approach

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

In sub-Riemannian manifolds of rank 2 and step 4, abnormal minimizers have C^1 regularity.

Theorem (Boarotto and Vittone 2020)

In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set of codimension at least one.

rank = rank of the distribution Δ
step = length of Lie brackets needed to span TM

step 1	step 2	step 3	step 4
X_k	$[X_j, X_k]$	$[X_i, [X_j, X_k]]$	$[X_h, [X_i, [X_j, X_k]]]$

The homogeneous setting

- $M = G$ is a Carnot group: a nilpotent Lie group whose Lie algebra is stratified

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}$$

- The basepoint p is the identity element e.
- Δ is the left-invariant distribution with $\Delta_e = \mathfrak{g}^{[1]}$.
- $X_1, \ldots, X_r$ are left-invariant.

Characterization of abnormal curves

$$\begin{tikzcd} & T^*M \arrow[d] \\ {[0,1]} \arrow[ur, "\lambda"] \arrow[r, "\gamma_u"] & M \end{tikzcd}$$

$\gamma_u \colon [0,1] \to M$ abnormal $\iff$ λ is a characteristic curve of the symplectic form restricted to $\Delta^\perp$

Characterization of abnormal curves

$T^*G \simeq G \times \mathfrak{g}^*$ by right-trivialization

$$\begin{tikzcd} & G \times \mathfrak{g}^* \arrow[d] \\ {[0,1]} \arrow[ur, "(\gamma_u,\lambda)"] \arrow[r, "\gamma_u"] & G \end{tikzcd}$$

$\gamma_u \colon [0,1] \to M$ abnormal $\iff$ $\lambda \in \mathfrak{g}^*$ constant with $\lambda(\mathrm{Ad}_{\gamma_u(t)}\, \mathfrak{g}^{[1]}) = 0$

$$\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g}), \quad \mathrm{Ad}_\gamma X = \frac{d}{ds} \gamma \cdot \exp(sX) \cdot \gamma^{-1} \Big|_{s=0}$$

A dynamical approach

For $X \in \mathfrak{g}^{[1]}$, define the *abnormal polynomial*

$$P_X \colon G \to \mathbb{R}, \quad P_X(g) = \lambda(\mathrm{Ad}_g X)$$

- γ abnormal $\iff P_X(\gamma(t)) = 0$ for all $X \in \mathfrak{g}^{[1]}$.

Idea: consider the (singular) foliation tangent to $\Delta \cap T\{P_X = 0\}$.

A dynamical approach

Rank 2: for $P = P_X$

$$0 = \frac{d}{dt} P(\gamma_u(t)) = u_1(t) X_1 P(\gamma_u(t)) + u_2(t) X_2 P(\gamma_u(t)).$$

When $(X_1 P, X_2 P) \neq 0$, up to reparametrization

$$\begin{aligned} u_1(t) &= -X_2 P(\gamma_u(t)) \\ u_2(t) &= X_1 P(\gamma_u(t)) \end{aligned}$$

$\implies$ ODE for γ_u.

A dynamical approach

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

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In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set of codimension at least one.

Proof strategy:

1. The dynamics is linear.
2. Separate cases by the Jordan form of the linear part.
3. Study the dynamics explicitly in the normal forms.

Abnormal dynamics is complicated

Theorem (H. 2020)

Let $\dot{x} = P(x)$ be a polynomial ODE system in $\mathbb{R}^r$.
There exists a Carnot group of rank r such that all trajectories of the ODE lift to abnormal curves.

For $x = (x_1, \dots, x_r)$, a lift is γ_u where $u_i = \dot{x}_i$.

Proof idea:

1. Every polynomial ODE has a polynomial first integral in a lift.
2. Curves contained in an algebraic variety are abnormal in a lift.

Construction of a first integral

Theorem (H. 2020)

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Horizontal gradients

Lemma

Every polynomial vector field $P\colon \mathbb{R}^r \to \mathbb{R}^r$ is the horizontal gradient of some polynomial in a Carnot group of high enough step.

For the frame $X_1, \ldots, X_r$ the horizontal gradient of $Q\colon G \to \mathbb{R}$ is

$$\nabla_{\mathrm{hor}} Q = \sum (X_i Q) X_i \colon G \to TG.$$

In coordinates, lift $P\colon \mathbb{R}^r \to \mathbb{R}^r$ to the horizontal vector field

$$P\colon G \to TG, \quad P(x_1, \ldots, x_r, \ldots, x_n) = \sum_{i=1}^{r} P_i(x_1, \ldots, x_r) X_i(x)$$

Gradients in $\mathbb{R}^r$

$$P = (P_1, \ldots, P_r) = \nabla Q \text{ for some } Q\colon \mathbb{R}^r \to \mathbb{R} \iff \partial_i P_j = \partial_j P_i$$

Recursion for Q:

$$\begin{aligned} Q_1 &= \int P_1 \, dx_1 \\ Q_2 &= Q_1 + \int (P_2 - \partial_2 Q_1) \, dx_2 \\ &\vdots \\ Q = Q_r &= Q_{r-1} + \int (P_r - \partial_r Q_{r-1}) \, dx_r \end{aligned}$$

A non-gradient vector field in $\mathbb{R}^r$

$P(x) = (x_1 - x_2, x_1 + x_2) \neq \nabla Q$ for any $Q \colon \mathbb{R}^2 \to \mathbb{R}$.
Lift to a horizontal vector field in the Heisenberg group.

$$\begin{aligned} X_1(x) &= \partial_1 \\ X_2(x) &= \partial_2 + x_1 \partial_3 \\ X_3(x) &= [X_1, X_2](x) = \partial_3 \end{aligned}$$

$$P \colon H \to TH, \quad P(x) = (x_1 - x_2) X_1(x) + (x_1 + x_2) X_2(x)$$

Then $P = \nabla_{\mathrm{hor}} Q$ for the polynomial

$$Q(x) = \frac{1}{2} x_1^2 - x_1 x_2 + \frac{1}{2} x_2^2 + 2 x_3$$

Recursion for horizontal gradient integration

$$X_1 Q = x_1 - x_2$$
$$X_2 Q = x_1 + x_2$$

Compute commutators:

$$X_3 Q = [X_1, X_2]Q = X_1(X_2 Q) - X_2(X_1 Q) = 2$$

Integrate backwards:

$$Q_3 = \int X_3 Q \, dx_3$$
$$Q_2 = Q_3 + \int (X_2 Q - X_2 Q_3) \, dx_2$$
$$Q = Q_1 = Q_2 + \int (X_1 Q - X_1 Q_2) \, dx_1$$
$$= \frac{1}{2}x_1^2 - x_1 x_2 + \frac{1}{2}x_2^2 + 2x_3$$

Recursion for horizontal gradient integration

Why it works:

- As *weighted* differential operators, $[X_1, X_2]$ is a degree 2 operator, $[X_1, [X_1, X_2]]$ is degree 3, etc.
  $\implies$ partial derivatives of a polynomial eventually vanish
- There exist coordinates such that $X_i = \partial_i + \sum_{j>i} c_{ij}\partial_j$.
  $\implies$ integration variable by variable is possible

A horizontal first integral

For an ODE

$$\dot{x}_i = P_i(x), \quad x \in \mathbb{R}^r, \quad i = 1, \ldots, n$$

integrate any nonzero orthogonal vector field.
E.g. if $P_1 \neq 0$, integrate

$$X_1 Q = -P_2, \quad X_2 Q = P_1 \quad X_3 Q = X_4 Q = \cdots = X_r Q = 0.$$

Then for a trajectory $x\colon [0,1] \to G$ of $\dot{x} = \sum P_i(x) X_i(x)$

$$\frac{d}{dt} Q(x) = P_1(x) X_1 Q(x) + \cdots + P_r(x) X_r Q(x) = 0.$$

Abnormal factors

Theorem (H. 2020)

Let $\dot{x} = P(x)$ be a polynomial ODE system in $\mathbb{R}^r$.
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Proof idea:

1. Every polynomial ODE has a polynomial first integral in a lift.
2. Curves contained in an algebraic variety are abnormal in a lift.

Higher order abnormality

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}.$$

Definition

$\gamma\colon [0,1] \to G$ abnormal $\iff \lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[1]}) = 0$

Definition

γ abnormal of order k $\iff \lambda(\mathrm{Ad}_{\gamma(t)}(\mathfrak{g}^{[1]} \oplus \cdots \oplus \mathfrak{g}^{[k]})) = 0$

Lemma

If $\gamma(0) = e$ and $\lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[k]}) = 0$, then γ is abnormal of order k.

Abnormal factors

Proposition

For any polynomial $Q\colon H \to \mathbb{R}$*, there exists*

- *a Carnot group* G *with a projection* $\pi\colon G \to H$
- $\lambda \in \mathfrak{g}^*$
- $k \in \mathbb{N}$

such that $Q \circ \pi\colon G \to \mathbb{R}$ *is a factor of the polynomial* $x \mapsto \lambda(\mathrm{Ad}_x Y)$ *for every* $Y \in \mathfrak{g}^{[k]}$.

Abnormal factors proof

Consider a linear system

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \ldots, m$$

in the variables (λ, ν), where

- $P_i^\lambda(x) = \lambda(\mathrm{Ad}_x Y_i)$ for a basis $Y_1, \ldots, Y_m$ of $\mathfrak{g}^{[k]}$
- S_i^ν are generic polynomials of the form

$$S^\nu = \nu_0 + \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 + \nu_4 x_1^2 + \nu_5 x_1 x_2 + \nu_6 x_2^2 + \ldots$$

such that $\deg(S_i^\nu) + \deg(Q) = \deg(P_i)$.

Abnormal factors proof

Let

- $k = \deg Q + 1$
- G_s a free Carnot group of step s

Lemma

The linear system

$$P_i^{\lambda} = Q \cdot S_i^{\nu}, \quad i = 1, \dots, m$$

has a non-trivial solution (λ, ν) *in* G_s *for large* s.

Monomial counting

Proof of Lemma:

1. Hall basis argument $\implies \exists\lambda = \lambda(\nu)$ such that $P_1^{\lambda(\nu)} = Q \cdot S_1^{\nu}$
   Consider the remaining system
   $$P_i^{\lambda(\nu)} = Q \cdot S_i^{\nu}, \quad i = 2, \dots, m$$
2. In step s, $\deg(P_i^{\lambda}) \leq s - k$. The number of equations is
   $$(m-1) \cdot \#\{\text{monomials of degree up to } s-k\}$$
   and the number of variables is
   $$m \cdot \#\{\text{monomials of degree up to } s-k-\deg(Q)\}$$
3. Poincaré series asymptotics for $s \to \infty$
   $\implies$ #variables $\gg$ #equations.

The entire proof

Theorem (H. 2020)

Let $\dot{x} = P(x)$ be a polynomial ODE system in $\mathbb{R}^r$.
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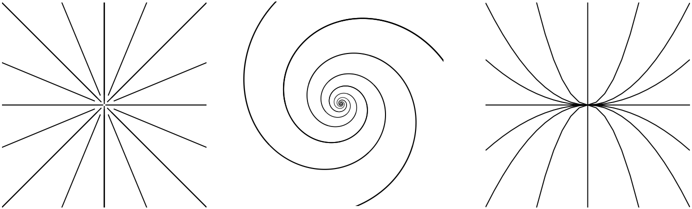
Proof:

1. Every polynomial ODE has a polynomial first integral in a lift.
   - Consider an orthogonal vector field.
   - Every polynomial vector field is a horizontal gradient.
2. Curves contained in an algebraic variety are abnormal in a lift.
   - Common factors of abnormal polynomials = linear system.
   - Monomial counting $\implies$ the system is underdetermined.

Linear ODEs

Abnormals in the free Carnot group of rank 2 and step 7



$$\dot{x} = x \qquad \dot{x} = -\tfrac{1}{4}x - y \qquad \dot{x} = x$$
$$\dot{y} = y \qquad \dot{y} = x - \tfrac{1}{4}y \qquad \dot{y} = 2y$$

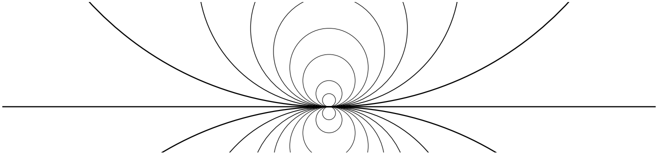
$\exists\lambda\colon \mathbb{R}^6 \to \mathfrak{g}^*$ semi-algebraic such that trajectories of

$$\dot{x} = ax + by + c \quad \dot{y} = dx + ey + f$$

are abnormal with covector $\lambda(a, b, c, d, e, f)$.

Concatenations of trajectories

Abnormals in the free Carnot group of rank 2 and step 13



$$\dot{z} = z^2, \quad z \in \mathbb{C}$$

Let $E \subset [0,1]$ be nowhere dense. $\exists$ abnormal curve that is

- injective
- parametrized by arc length
- C^1
- not C^2 at any point $x \in E$

An inefficient formula

Let $P\colon \mathbb{R}^r \to \mathbb{R}^r$ be a polynomial vector field.

Let $d(r,k) = \dim \mathfrak{f}_r^{[k]} = \dfrac{1}{k}\displaystyle\sum_{d|k} \mu(d) r^{k/d}$.

Consider the rational function

$$\sum_{k=0}^{\infty} C_k t^k = \frac{\Big(1-(d(r,\deg(P)+1))(1-t^{\deg(P)})\Big)\, t^{\deg(P)+1}}{\prod_{k=1}^{\deg(P)}(1-t^k)^{d(r,k)}}.$$

If $\sum_{k=0}^{s} C_k > 0$, then trajectories of P are abnormal in step s.

Inefficient numbers from an inefficient formula

P a polynomial vector field in $\mathbb{R}^r$. Trajectories abnormal in step:

$r\backslash\deg(P)$	1	2	3	4	5
2	11	38	172	577	2372
3	89	724	6034	46036	365813
4	386	5322	73109	983505	13529000

Example

A polynomial vector field in $\mathbb{R}^4$ of degree 5 has abnormal lifts in the free Carnot group G of rank 4 and step 13529000.

$$\dim G \approx 4.1338 \cdot 10^{8145262}$$

Inefficient numbers from an inefficient formula

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Example

A polynomial vector field in $\mathbb{R}^4$ of degree 5 has abnormal lifts in the free Carnot group G of rank 4 and step 2372?

$$\dim G \approx 6.857 \cdot 10^{1425}$$

Conjecture

The abnormality step only depends on $\deg(P)$ *and not the rank* r.

Thank you for your attention!