Group Assignment 1

Lemma 1.14

Prove $\mathbb{P}(\cdot \mid A): \mathcal{F} \to [0,1]$
is a probability measure.

As such prove:

1. $\mathbb{P}(\Omega) = 1$
2. Addition rule holds. $A \cap B = 0 \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Let $B \in \mathcal{F}$ & definition 1.13 gives:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

1. $B = \Omega$ & $A \in \Omega$:

$$\mathbb{P}(\Omega \mid A) \overset{1.13}{=} \frac{\mathbb{P}(\overbrace{\Omega \cap A}^{\text{the intersection of the entire sample space with any set the set itself.}})}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1$$

2. Let $B, C \in \mathcal{F}$ be an arbitrary sets that are mutually exclusive such that $B \cap C = \emptyset$

$$\mathbb{P}(B \cap C \mid A) \overset{1.13}{=} \frac{\mathbb{P}((B \cap C) \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}((A \cap B) \cup (A \cap C))}{\mathbb{P}(A)} \overset{1.10}{=}$$

$$= \frac{\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} + \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(A)} \overset{1.13}{=}$$

$$= \mathbb{P}(B \mid A) + \mathbb{P}(C \mid A)$$

Eftersom $\mathbb{P}(B \cap C \mid A) = 0$ (då $\mathbb{P}(B \cap C) = 0$ pga mutually exclusive) är $VL = 0$. Om detta bevis ska hålla måste $HL = 0$.

$\mathbb{P}(B \mid A) = 0$ då A, B, C alla är mutually exclusive
&
$\mathbb{P}(C \mid A) = 0$

Lemma 2.8

1. Completetion behaves like the probability

Prove $\mathbb{1}_A = 1 - \mathbb{1}_{A^c}$

Case 1: if $\omega \in A$, then $\mathbb{1}_{A^c(\omega)} = 0$ & as such:

$$\mathbb{1}_{A(\omega)} = 1 - \mathbb{1}_{A^c(\omega)} = 1 - 0 = 1.$$

eller $\mathbb{1}_{A(\omega)} = 1 = 1 - 0 = 1 - \mathbb{1}_{A^c(\omega)}$

Case 2: if $\omega \notin A$ then $\mathbb{1}_{A(\omega)} = 0$ & as such:

$$\mathbb{1}_{A(\omega)} = 1 - \underbrace{\mathbb{1}_{A^c(\omega)}} = 1 - 1 = 0.$$

eller $\mathbb{1}_A(\omega) = 0 = 1 - 1 = 1 - \mathbb{1}_{A^c(\omega)}$ $\left(\mathbb{1}_{A^c(\omega)} = 1\right)$

2. Intersection becomes product.

Prove $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$

Case 1: if $\omega \in A \cap B$ then:

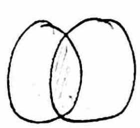
$$\mathbb{1}_{A(\omega)} = 1 \ \& \ \mathbb{1}_{B(\omega)} = 1$$

Thus: $\mathbb{1}_{A(\omega)} \cdot \mathbb{1}_{B(\omega)} = 1 \cdot 1 = 1 = \mathbb{1}_{A \cap B}$

eller: $\mathbb{1}_{A \cap B(\omega)} = 1 = 1 \cdot 1 = \mathbb{1}_A(\omega) \cdot \mathbb{1}_B(\omega)$

3. Union becomes addition − intersection.

Prove: $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$



Case 1a: If $\omega \in A \cup B$ and $\omega \notin A \cap B$ then $\mathbb{1}_{A(\omega)} = 1$ or $\mathbb{1}_{B(\omega)} = 1$

and $\mathbb{1}_A \cdot \mathbb{1}_B = 0$ since one of the factors is 0.

Thus $1 + 0 - 0 = 1 = \mathbb{1}_{A \cup B(\omega)}$

Case 1b: $\omega \in A \cup B$ and $\omega \in A \cap B$ then $\mathbb{1}_{A(\omega)} = \mathbb{1}_{B(\omega)} = 1$ &

$\mathbb{1}_A \cdot \mathbb{1}_B = 1$. Thus $1 \cdot 1 - 1 = 1 = \mathbb{1}_{A \cup B(\omega)}$

Case 2: If $\omega \notin A \cup B$ then $\mathbb{1}_A = \mathbb{1}_B = 0 = \mathbb{1}_{A \cup B}$

Property 4. Theorem 2.18

Prove $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Proof. By definition of the improper integral we have:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f(x)\,dx$$

F is the cumulative distribution function of f.

Since $\int_a^b f(x)\,dx = F(b) - F(a)$ for $a, b \in \mathbb{R}$ we get:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} (F(b) - F(a)) = \lim_{b \to \infty} F(b) - \lim_{a \to -\infty} F(a)$$

Proof that $\lim_{b \to +\infty} F(b) = 1$: B_n is an unbounded strictly increasing sequence.

$$1 = \Omega = \overbrace{\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{i=0}^{\infty} \mathbb{P}(B_n)}^{\text{Countably additive axiom}} = \lim_{n \to \infty} F(b_n)$$

Proof that $\lim_{a \to -\infty} F(a) = 0$: a_n is an unbounded strict decreasing sequence.

Samma typ.

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