week 45?

Intro to DS: Markov Chains.

det. pseudorandom

$\mathcal{M} = \{0, 1, \dots \mathcal{M}-1\}$

$u_i \in \mathcal{M}$

$\frac{N_n(a)}{n} \longrightarrow \frac{1}{M}$ for any $a \in \mathcal{M}$

$N_n(a)$ ← how many times a number appears

n ← how many numbers you see

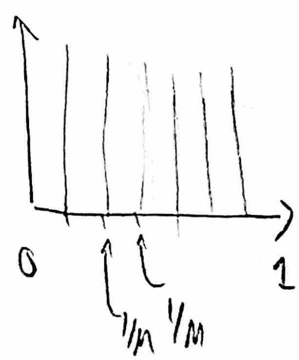
seed = 1
$a = 2$ $\mathcal{M} = 10$
$b = 1$
$u_{i+1} = 3 \bmod 10 = 3.$

LCG

$u_{i+1} = a \cdot u_i + b \bmod \mathcal{M}$

The correct choice of $a, b, \mathcal{M}$ gives u_i is pseudorandom sequence on $\mathcal{M}$

$u_i \to \frac{u_i}{\mathcal{M}} \in [0, 1]$



as $\mathcal{M} \to$ infinity

Def Markov Chain

A sequence of RVs $X_1, \dots,$ is called a Markov chain if the distribution of X_t only depends on the previous state X_{t-1}.

Def.
Stochastic process

Stochastic process — Sequence of RVs

$$(X_\alpha)_{\alpha \in \mathbb{N}}$$

think of this as time. Time 1, Time 2 etc. . . .

$\alpha = \mathbb{R}$ — continuous time

Def.
Finite Markov Chain

$$(X_n)_{n \in \mathbb{N}}$$

Markov property (Assumption)

$$\mathbb{P}(X_{t+1} = x \mid X_0, X_1, \dots, X_t) =$$

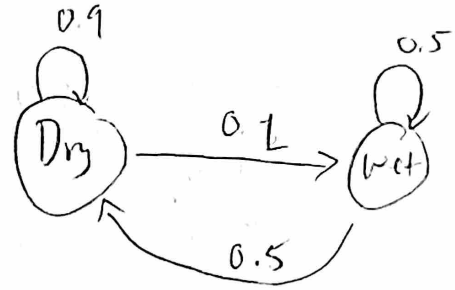
$$\mathbb{P}(X_{t+1} = x \mid X_t)$$

$$\underbrace{X_t \in \mathcal{X}} = \{0, 1, \dots, K\}$$

state space

finite as the state space is finite

Simple weather example



transition matrix

We can write this as a matrix:

$$P = \begin{bmatrix} P(X_t = \text{"dry"} \mid X_{t-1} = \text{"dry"}) & P(X_t = \text{"wet"} \mid X_{t-1} = \text{"dry"}) \\ P(X_t = \text{"dry"} \mid X_{t-1} = \text{"wet"}) & P(X_t = \text{"wet"} \mid X_{t-1} = \text{"wet"}) \end{bmatrix}$$

the diagonal is always transitioning to itself.

Transition matrix

$$(P^t)_{xy} = P(X_t = y \mid X_{t-1} = x)$$

$$= \left[\text{probability of going from } x \text{ to } y \text{ at time } t. \right]$$

the Ex

$$p_{t-1} = q_{t-1} = [\mathbb{P}(X_{t-1} = \text{"dry"}), \mathbb{P}(X_{t-1} = \text{"wet"})]$$

We want to compute $[\mathbb{P}(X_t = \text{"dry"}), \mathbb{P}(X_t = \text{"wet"})]$

$$\mathbb{P}(X_t = \text{"dry"}) = \mathbb{P}(X_t = \text{"dry"} \mid X_{t-1} = \text{"dry"}) \cdot P(X_t = \text{"d}$$

$$+$$

$$\mathbb{P}(X_t = \text{"dry"} \mid X_{t-1} = \text{"wet"}) \cdot P(X_t = \text{"w}$$

= se nästa sida

the example (continued...

$= (P^t)_{dry,dry} (q_{t-1})_{dry} + (P^t)_{wet,dry} (q_{t-1})_{wet}$

$= (q_t)_{dry}$ column

$(P^t)_{*dry}$ ∘ q_{t-1}

dotproduct

$q_{t-1} P^t = \left(\begin{array}{l} \text{row of } q_{t-1} \text{ dot product of } P^t \\ \text{is the first instance,} \\ \text{osv, ...} \end{array} \right)$

$= q_t$

Repeat & get

index

$q_0 P^{(1)} P^{(2)} P^{(3)} \ldots P^{(t)} = q_t$

In this case, the "wet-dry" case:

$P^t = P \quad \forall t$

$q_0 (P)^t = q_t$

~~np.log~~

-np.sum (Y * np.log (λ) - (lam)

correlation threshold > 0.3

correlate Y with X.

concept

Import np

P = np.matrix ([[0.9 , 0.1]
[0.5 , 0.5]])

p0 = np.array ([1,0]) ← initial state is dry

p0 · P

p0 * (P ** 10)

q0 = np.array([[0,1]]) initial state is wet

Stationary distribution

$$p_0 P^t \longrightarrow p_* \quad \forall\, p_0$$

$$p_* P = p_*$$

p_* is a left eigenvector.
eigenvalue $\boxed{1}$

$$P^T p_*^T = p_*^T$$

$$(p_* P)^T = (p_*)^T$$

evals, evecs = np.linalg.eig (P.T)

first_evec = evecs[:, 0]

np.array(first_evec) / np.sum (first_evec) homogenous markov chain = the markov chain is not dependent on time

Homogenous: we need to observe for a long time only one chain

Goal: estimate P. K^2 parameters.

inhomogenous: we need to observe multiple chains.

Goal: estimate $P^{(0)}, P^{(1)}, P^{(2)}, \ldots$ index

$0 \leq t \leq T$ $\quad T \cdot K^2$ parameters

Property of P:

1) $P_{x\cdot} = P(X_{t+1} = y \mid X_t = x)$

transition from state x to any other state

$= \sum_y P_{xy} = 1$

Skapad med Tiny Scanner

Example wet-dry continued

$$E[\ln(p_n(X_1, \dots X_n))]$$

start small

$$E[\ln P_2(x_1, x_2)] =$$

$$= E[\ln P(x_2 | x_1) P(x_1)] =$$

$$E[\ln P(x_2 | x_1)] + E[\ln P(x_1)]$$

$$\hat{R}(p) = -\sum_{i=0}^{n} \ln(p_{i,i})\, n_{i,i} + \ln(1 - p_{0,1}) n_{0,1} + \ln(1 - p_{1,1}) n_{1,0})$$

$$\frac{\partial \hat{R}}{\partial p_{0,0}} = 0$$

$$\frac{\partial \hat{R}}{\partial p_{1,1}} = 0$$

Minima

$$p_{0,0} = \frac{n_{0,0}}{n_{0,1} + n_{0,0}}$$

number of times we've seen this transition

all transitions from 0

Tokit: # of times we've seen transition $n_{0,0}$ out of all transitions

$$p_{1,1} = \frac{n_{1,1}}{n_{1,0} + n_{1,1}}$$

RMR

$P(x, \omega) \to y$

$x = x_{t-1}$ $P(x_t \dots$

Example wet-dry continued

$$E[\ln(p_n(X_1, \dots X_n))]$$

start small

$$E[\ln P_2(x_1, x_2)] =$$

$$= E[\ln \mathbb{P}(x_2 | x_1) P(x_1)] =$$

$$E[\ln P(x_2 | x_1)] + E[\ln P(x_1)]$$

$$\hat{R}(p) = -\sum_{i=0}^{n} \ln(p_{i,i})\, n_{i,i} + \ln(1 - p_{0,0})\, n_{0,1}$$

$$+ \ln(1 - p_{1,1})\, n_{1,0})$$

$$\frac{\partial \hat{R}}{\partial p_{0,0}} = 0$$

$$\frac{\partial \hat{R}}{\partial p_{1,1}} = 0$$

number of times we've seen this transition

Minima

$$p_{0,0} = \frac{n_{0,0}}{n_{0,1} + n_{0,0}}$$

all transitions from 0

Tokit: # of times we've seen transition $n_{0,0}$ out of all transit

$$p_{1,1} = \frac{n_{1,1}}{n_{1,0} + n_{1,1}}$$

RMR

$$\varphi(x, w) \to y$$

$$x_t = x_{t-1} \quad \varphi(x_{t-1}, w) \to x_t$$

w is random variable

$$\mathbb{P}(\varphi(x, W) = y) = P_{xy}$$

$w_1, w_2 \dots w_t$ i.i.d uniform(0,1)