Rank = 2

$UDV^T = M$

$U = M \cdot D^T V$

$V = M^T U D$

## How to calculate singular vectors?

Rank = 2

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

2×2

$$M^T M = \underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}}_{2\times3} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}}_{3\times2} = \begin{pmatrix} 1\cdot1+0\cdot0+3\cdot3 & 1\cdot1+0\cdot3+3\cdot0 \\ 1\cdot1+3\cdot0+0\cdot3 & \end{pmatrix}$$

$$= \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$$

höger singulärvektorer $= V$.

$MM^T$ ger vänster singulärvektorer $= U$

$$MM^T = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}}_{3\times2} \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}}_{2\times3} = \begin{bmatrix} 1\cdot1+1\cdot1 & 1\cdot0+1\cdot3 & 1\cdot3+1\cdot0 \\ 0\cdot1+3\cdot1 & 0\cdot0+3\cdot3 & 0\cdot3+3\cdot0 \\ 3\cdot1+0\cdot1 & 3\cdot0+0\cdot3 & 3\cdot3+0\cdot0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 9 & 0 \\ 3 & 0 & 9 \end{bmatrix}$$

$D$ = rötterna ur $\underbrace{MM^T}_{\sqrt{0},\sqrt{9},\sqrt{11}}$ eller $\underbrace{M^TM}_{\sqrt{9}\ \sqrt{11}}$ = roten ur egenvärdena

positiv förutsätts.

ordna decreasing



$$D = \begin{pmatrix} \sqrt{11} & 0 \\ 0 & 3 \end{pmatrix}$$

eigenvalues to $M^TM$ : $\det(A - \lambda I) = 0$

$$0 = \begin{vmatrix} 10-\lambda & 1 \\ 1 & 10-\lambda \end{vmatrix} = (10-\lambda)(10-\lambda) - 1 \cdot 1 = 10^2 - 10\lambda - 10\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 20\lambda + 99 \Rightarrow \lambda = -\frac{(-20)}{2} \pm \sqrt{\left(\frac{-20}{2}\right)^2 - 99}$$

$$\lambda = 10 \pm \sqrt{100 - 99}$$

$$\lambda = 10 \pm 1 \Rightarrow \begin{cases} \lambda_1 = 11 \\ \lambda_2 = 9. \end{cases}$$ ✓ eigenvalues

eigenvalues for $MM^T$:

$$0 = \begin{vmatrix} 2-\lambda & 3 & 3 \\ 3 & 9-\lambda & 0 \\ 3 & 0 & 9-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 9-\lambda & 0 \\ 0 & 9-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 3 & 9-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & 9-\lambda \\ 3 & 0 \end{vmatrix}$$

$$= (2-\lambda)(9-\lambda)^2 - 3(3(9-\lambda) - 3 \cdot 0) + 3(3 \cdot 0 - (9-\lambda) \cdot 3)$$

$$= (2-\lambda)(9-\lambda)^2 - 9(9-\lambda) - 9(9-\lambda)$$

$$= (2-\lambda)(9-\lambda)^2 - 18(9-\lambda)$$

$$= (9-\lambda)\underbrace{((2-\lambda)(9-\lambda) - 18)}_{\cancel{18} - 2\lambda - 9\lambda + \lambda^2 - \cancel{18} = 0} \qquad \lambda_1 = 9.$$

$$\lambda^2 - 11\lambda = 0$$

$$\lambda(\lambda - 11)$$

$$\lambda_2 = 0$$

$$\lambda_3 = 11$$

$\Rightarrow$ eigenvalues $0, 9, 11$

eigenvectors for $U$: $\begin{pmatrix} 2-\lambda & 3 & 3 \\ 3 & 9-\lambda & 0 \\ 3 & 0 & 9-\lambda \end{pmatrix}$

$\lambda_1 = 11$ gives:

$$\underbrace{\begin{pmatrix} -9 & 3 & 3 \\ 3 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix}}_{\text{gausser}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 3 & 3 \\ 3 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 3 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 0 & -2 & 2 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 0 & -1 & 1 \\ 3 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -2/3 \end{pmatrix}$$

ordna om...

$$\begin{pmatrix} 1 & 0 & -2/3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} x - 2/3 z = 0 \Rightarrow x = -\frac{2}{3} z \\ -y + z = 0 \Rightarrow y = z \end{array}$$

eigenvector: (for $z = 1$) $\cancel{\begin{bmatrix} -2/3 \\ 1 \\ 1 \end{bmatrix} = v_1} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$

$$v_1 \text{ normalized} = \frac{\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}^{\leftarrow v_1}}{\left\| \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \right\|} = \frac{\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}}{\sqrt{2^2 + 3^2 + 3^2}} = \frac{(2, 3, 3)}{\sqrt{22}} = \frac{2}{\sqrt{22}}, \frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}}$$

$\lambda = 9$ gives for U:

$$\begin{pmatrix} 2-9 & 3 & 3 \\ 3 & 9-9 & 0 \\ 3 & 0 & 9-9 \end{pmatrix} = \begin{pmatrix} -7 & 3 & 3 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{matrix} \\ 1/3 \\ 1/3 \end{matrix}$$

$$\begin{pmatrix} -7 & 3 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ 7 \end{matrix} \sim \begin{pmatrix} 0 & 3 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} 1/3 \\ \\ \\ \end{matrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} \\ (-1) \\ \\ \end{matrix}$$

then rearrange:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x = 0 \\ y + z = 0 \Rightarrow y = (-z) \end{matrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \rightarrow \frac{0,-1,1}{\sqrt{(-1)^2+1^2}} = \sqrt{2}$$

$\lambda = 0$ gives:

$$\begin{matrix} \\ 3 \\ 2 \end{matrix}\begin{pmatrix} 2 & 3 & 3 \\ 3 & 9 & 0 \\ 3 & 0 & 9 \end{pmatrix} \sim \begin{pmatrix} 6 & 9 & 9 \\ 6 & 18 & 0 \\ 3 & 0 & 9 \end{pmatrix} \begin{matrix} (-1) \\ \\ \\ \end{matrix} \sim \begin{pmatrix} 6 & 9 & 9 \\ 0 & 9 & -9 \\ 3 & 0 & 9 \end{pmatrix} \begin{matrix} 1/3 \\ 1/9 \\ 1/3 \end{matrix}$$

$$\begin{pmatrix} 2 & 3 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & -3 \end{pmatrix} \begin{matrix} \\ \\ (-2) \end{matrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix} \begin{matrix} \\ (-3) \\ \\ \end{matrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x + 3z = 0 \Rightarrow x = -3z \\ y + (-z) = 0 \Rightarrow y = z \\ z = z \end{matrix} \Rightarrow \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

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$$U = \begin{pmatrix} \frac{2}{\sqrt{22}} & 0 & \frac{-3}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} \end{pmatrix}$$

eigenvalues for "V" is 11 & 9:

$\lambda_1 = 11$ yields

$$\begin{pmatrix} 10-11 & 1 \\ 1 & 10-11 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$x = y \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad v_{1\,norm} = \dfrac{(1,1)}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \;\; \dfrac{1}{\sqrt{2}}$

$\lambda_2 = 9$ yields:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$x + y = 0$

$x = -y$

$y = 1$

$x = -y \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad v_{2\,norm} = \left(\dfrac{-1}{\sqrt{2}}\right), \dfrac{1}{\sqrt{2}}$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \left(\frac{-1}{\sqrt{2}}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$$V = \begin{matrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \left(\frac{-1}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}} \end{matrix}$$