NUMERICAL COMPUTATION OF REAL OR COMPLEX ELLIPTIC INTEGRALS

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Abstract. Algorithms for numerical computation of symmetric elliptic integrals of all three kinds are improved in several ways and extended to complex values of the variables (with some restrictions in the case of the integral of the third kind). Numerical check values, consistency checks, and relations to Legendre's integrals and Bulirsch's integrals are included.

Key words. elliptic integral, algorithm, numerical computation

AMS(MOS) subject classifications. primary 33A25, 33-04, 65D20; secondary 33A10, 30-04, 32-04

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Running title

COMPUTATION OF ELLIPTIC INTEGRALS

1 Introduction

Let f(x) be a real function that is rational except for the square root of a cubic or quartic polynomial with at least one pair of conjugate complex zeros. Then $\int f(x)dx$ can be expressed in terms of standard elliptic integrals with complex variables, which are subsequently changed to real variables by using quadratic transformations [10][11]. Since the transformations complicate the formulas, it is desirable to have algorithms for numerical computation of standard elliptic integrals with complex variables. Such integrals are met in other contexts also, for example in conformal mapping, and the complex variables might not occur in conjugate pairs.

This paper contains algorithms for numerical computation of complete and incomplete elliptic integrals of all three kinds when the variables are complex (with some restrictions for integrals of the third kind). They are similar to algorithms published earlier [5] for real variables, but several improvements that apply to the real case have been made in the course of extending them to complex variables. The integrals computed are the symmetric integrals of the first and third kinds and two degenerate cases, of which one is an elementary function and the other is an elliptic integral of the second kind. Other integrals can be obtained from these by using the formulas and references in Section 4. The method of computation is to iterate the duplication theorem and then sum a power series up to terms of degree five; the error ultimately decreases by a factor of $4^6 = 4096$ with each duplication. Since this method is slowest when the integrals are complete, we add a faster algorithm for computing complete integrals of the first and second kinds, including Legendre's K(k) and E(k) for complex k, by the method of arithmetic and geometric means.

The symmetric integral of the first kind is

$$R_F(x,y,z) = \frac{1}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} dt, \qquad (1)$$

where the square root is taken real and positive if x, y, z are positive and varies continuously when x, y, z become complex. The integral is well defined if x, y, z lie in the complex plane cut along the nonpositive real axis (henceforth called the "cut plane"),

with the exception that at most one of x, y, z may be 0. The same requirements apply to the symmetric integral of the third kind,

$$R_J(x,y,z,p) = \frac{3}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} (t+p)^{-1} dt, \qquad (2)$$

where $p \neq 0$ and the Cauchy principal value is to be taken if p is real and negative. A degenerate case of R_F that embraces the inverse circular and inverse hyperbolic functions (see Section 4) is

$$R_C(x,y) = R_F(x,y,y) = \frac{1}{2} \int_0^\infty (t+x)^{-1/2} (t+y)^{-1} dt.$$
 (3)

It is well defined if x lies in the cut plane or is 0 and if $y \neq 0$; the Cauchy principal value is to be taken if y is real and negative. Professor Luigi Gatteschi pointed out to me that Fubini, while still a student at Pisa, proposed the use of a function equivalent to $1/R_C$ in his first published paper [12]. A degenerate case of R_J that is an elliptic integral of the second kind is

$$R_D(x,y,z) = R_J(x,y,z,z) = \frac{3}{2} \int_0^\infty [(t+x)(t+y)]^{-1/2} (t+z)^{-3/2} dt, \qquad (4)$$

which is well defined under the same conditions as R_F except that z must not be 0.

Because R_D is symmetric only in x and y, it is sometimes convenient to use a completely symmetric integral of the second kind,

$$R_G(x,y,z) = \frac{1}{4} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} \left(\frac{x}{t+x} + \frac{y}{t+y} + \frac{z}{t+z}\right) t dt,$$
 (5)

where any or all of x, y, z may be 0 and those that are nonzero lie in the cut plane. If the closed convex hull of $\{x, y, z\}$ lies in the union of 0 and the cut plane, R_G is represented by a double integral that accounts for its usefulness in problems connected with ellipsoids,

$$R_G(x, y, z) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} (x \sin^2 \theta \cos^2 \phi + y \sin^2 \theta \sin^2 \phi + z \cos^2 \theta)^{1/2} \sin \theta \, d\theta \, d\phi. \tag{6}$$

Except that at most one of x, y, z may be 0, R_F has the same representation with exponent -1/2 instead of 1/2. With the same exception and with x, y, z permuted so that $z \neq 0$, R_G can be obtained from R_F and R_D by using the relation

$$2R_G(x,y,z) = zR_F(x,y,z) - \frac{1}{3}(x-z)(y-z)R_D(x,y,z) + \sqrt{\frac{xy}{z}}.$$
 (7)

Algorithms for direct computation of R_G and R_F with real variables by successive Landen transformations are given in [3]. In Section 2 of the present paper, one of those algorithms is extended to complex variables, but only in the complete case. For each of the functions defined above, the complete case is the case in which one of x, y, z is 0.

The functions R_F and R_C are homogeneous of degree -1/2 in their variables, R_J and R_D of degree -3/2, and R_G of degree +1/2. Their homogeneity and symmetry replace a set of linear transformations of Legendre's integrals, simplify their quadratic transformations and other properties, and make it possible (see [7]-[11]) to unify many of the formulas in customary integral tables.

The earlier versions of the algorithms in Section 2 (except the last one) were published in [5], modified in [6, (4), (5)] to avoid underflows, and implemented by Fortran codes in several major software libraries, in the Supplements to [7] and [8], and in [13] and $[14, \S 6.11]$. Codes in C can be found in $[15, \S 6.11]$. The algorithms in the present paper are preferable to those in [5] in the following four respects (in addition to incorporating the modification in [6]):

- (1) All complex values of the variables are admissible for which R_F, R_D , and R_C were defined above, with the exception that computing the Cauchy principal value of $R_C(x,y)$ when y < 0 requires the preliminary transformation (21). The variables of R_J are restricted by conditions that are sufficient but not necessary to keep the fourth variable from being transformed to 0 by the duplication theorem.
- (2) The algorithms have been rearranged to reduce substantially the number of arithmetic operations.
 - (3) A bound on the fractional error of the result can be specified directly.
- (4) Because R_C is used repeatedly in the algorithm for R_J , its computation has been speeded up a little by including terms up to degree seven (instead of five) in the truncated power series.

Section 3 contains information for checking codes based on the algorithms of Section 2, and Section 4 relates other integrals to the integrals used here.

2 Algorithms

We shall summarize briefly the method of computation; details are given in [5] for the case of real variables, and a full discussion of the complex case with proof of error bounds will appear eventually in a book now in preparation. Validity of algorithms can be tested in the complex domain by the consistency checks in Section 3 and sometimes by comparison of the complete case with the last algorithm in the present Section.

If we obtain x_{m+1} , y_{m+1} , z_{m+1} from x_m , y_m , z_m as prescribed by (11) below, the duplication theorem implies

$$R_F(x_0, y_0, z_0) = R_F(x_m, y_m, z_m), \qquad m = 1, 2, 3, \dots$$
 (8)

Although x_m , y_m , z_m converge to a common limit in the cut plane as $m \to \infty$, differences like $x_m - y_m$ decrease by a factor of only 4 when m increases by 1. When the fractional differences between x_m , y_m , z_m and their arithmetic average become smaller than an amount determined by the desired accuracy of the final result, R_F is expanded in a multiple series of powers of these fractional differences, denoted by X, Y, Z. Because of the symmetry of R_F the series can be rewritten in terms of the elementary symmetric functions E_1 , E_2 , E_3 of X, Y, Z; and because $E_1 = X + Y + Z = 0$ the terms up to degree 5 are very simple. The truncation error, being of degree 6, ultimately decreases by a factor of 4^6 with each duplication. An estimate of truncation error is provided by [5, (A.10)].

ALGORITHM FOR R_F . We suppose that at most one of x, y, z is 0 and those that are nonzero have phase less in magnitude than π . The function $R_F(x, y, z)$ defined by (1) is to be computed with relative error less in magnitude than r. (We assume $r < 3 \times 10^{-4}$.)

Let
$$x_0 = x$$
, $y_0 = y$, $z_0 = z$, and

$$A_0 = \frac{x+y+z}{3}, \qquad Q = (3r)^{-1/6} \max\{|A_0 - x|, |A_0 - y|, |A_0 - z|\}.$$
 (9)

For $m = 0, 1, 2, \ldots$, define

$$\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}, \qquad A_{m+1} = \frac{A_m + \lambda_m}{4}, \qquad (10)$$

$$x_{m+1} = \frac{x_m + \lambda_m}{4}, \qquad y_{m+1} = \frac{y_m + \lambda_m}{4}, \qquad z_{m+1} = \frac{z_m + \lambda_m}{4},$$
 (11)

where each square root has nonnegative real part.

Compute A_m for m = 0, 1, ..., n, where $4^{-n}Q < |A_n|$. Define

$$X = \frac{A_0 - x}{4^n A_n}, \qquad Y = \frac{A_0 - y}{4^n A_n}, \qquad Z = -X - Y,$$
 (12)

$$E_2 = XY - Z^2$$
, $E_3 = XYZ$. (13)

Then

$$R_F(x,y,z) \approx A_n^{-1/2} \left(1 - \frac{1}{10} E_2 + \frac{1}{14} E_3 + \frac{1}{24} E_2^2 - \frac{3}{44} E_2 E_3 \right)$$
 (14)

with relative error less in magnitude than r.

The statement that the relative error is less in magnitude than r means that the true value of the function lies inside a circle in the complex plane with center at the computed value and radius equal to r times the distance of the center from the origin. We assume that r is large compared to the machine precision, so that roundoff error is negligible compared to the error produced by approximations made in the algorithm. These remarks apply to all the algorithms in this paper.

We note the relations

$$A_m = \frac{x_m + y_m + z_m}{3}, \qquad A_m - x_m = \frac{A_0 - x}{4^m}, \qquad X = 1 - \frac{x_n}{A_n},$$
 (15)

and similar relations obtained by permuting (x, X), (y, Y), (z, Z). Although A_0 can be 0 in the complex case, $A_n \neq 0$ because $Q \geq 0$. (It can be shown that $A_m \neq 0$ if $m \geq 1$.) If x_m, y_m, z_m are real and nonnegative, the inequality of arithmetic and geometric means implies $\lambda_m \leq 3A_m$, whence $A_{m+1} \leq A_m$.

If y = z then R_F reduces to R_C . Because E_2 and E_3 are no longer independent, the series in (14) becomes a series in one variable. By including terms up to degree seven (instead of five), we can usually save one duplication, which seems worthwhile because computation of R_J requires one computation of R_C in each cycle of iteration.

ALGORITHM FOR R_C . Let x and y be nonzero and have phase less in magnitude than π , with the exception that x may be 0. The function $R_C(x,y)$ defined by (3) is to be computed with relative error less in magnitude than r. (We assume $r < 2 \times 10^{-4}$.)

Let $x_0 = x$, $y_0 = y$, and

$$A_0 = \frac{x + 2y}{3}, \qquad Q = (3r)^{-1/8} |A_0 - x|. \tag{16}$$

For $m = 0, 1, 2, \dots$, define

$$\lambda_m = 2\sqrt{x_m}\sqrt{y_m} + y_m, \qquad A_{m+1} = \frac{A_m + \lambda_m}{4}, \qquad (17)$$

$$x_{m+1} = \frac{x_m + \lambda_m}{4}, \qquad y_{m+1} = \frac{y_m + \lambda_m}{4},$$
 (18)

where each square root has nonnegative real part.

Compute A_m for m = 0, 1, ..., n, where $4^{-n}Q < |A_n|$, and define

$$s = \frac{y - A_0}{4^n A_n} \,. \tag{19}$$

Then

$$R_C(x,y) \approx A_n^{-1/2} \left(1 + \frac{3}{10}s^2 + \frac{1}{7}s^3 + \frac{3}{8}s^4 + \frac{9}{22}s^5 + \frac{159}{208}s^6 + \frac{9}{8}s^7 \right)$$
 (20)

with relative error less in magnitude than r.

If the second variable of R_C is real and negative, the Cauchy principal value is

$$R_C(x, -y) = \left(\frac{x}{x+y}\right)^{1/2} R_C(x+y, y), \qquad y > 0,$$
 (21)

by [16, (4.8)] and (73). This vanishes if x = 0.

In the notation of (26) to (28), the duplication theorem for R_J is

$$R_J(x_m, y_m, z_m, p_m) = \frac{1}{4} R_J(x_{m+1}, y_{m+1}, z_{m+1}, p_{m+1}) + \frac{6}{d_m} R_C(1, 1 + e_m).$$
 (22)

The duplication theorem for R_C has been applied to [5, (5.1)] to allow a wider range of complex phase for the variables. Iteration of (22) yields

$$R_J(x_0, y_0, z_0, p_0) = 4^{-n} R_J(x_n, y_n, z_n, p_n) + 6 \sum_{m=0}^{n-1} \frac{4^{-m}}{d_m} R_C(1, 1 + e_m).$$
 (23)

The first term on the right side can be treated as a symmetric function of the five variables x_n , y_n , z_n , p_n , p_n by writing $(t+p)^{-1}$ in (2) as $(t+p)^{-1/2}(t+p)^{-1/2}$. When the fractional differences X, Y, Z, P, P between these five variables and their arithmetic average become small enough, R_J is expanded in powers of the elementary symmetric functions E_2 , E_3 , E_4 , E_5 of X, Y, Z, P, P (since $E_1 = 0$).

ALGORITHM FOR R_J . Let x,y,z have nonnegative real part and at most one of them be 0, while $\operatorname{Re} p > 0$. Alternatively, if $p \neq 0$ and $|\operatorname{ph} p| < \pi$, either let x,y,z be real and nonnegative and at most one of them be 0, or else let two of the variables x,y,z be nonzero and conjugate complex with phase less in magnitude than π and the third variable be real and nonnegative. The function $R_J(x,y,z,p)$ defined by (2) is to be computed with relative error less in magnitude than r. (We assume $r < 10^{-4}$.)

Let $(x_0, y_0, z_0, p_0) = (x, y, z, p)$ and

$$A_0 = \frac{x + y + z + 2p}{5}, \qquad \delta = (p - x)(p - y)(p - z), \tag{24}$$

$$Q = (r/4)^{-1/6} \max\{|A_0 - x|, |A_0 - y|, |A_0 - z|, |A_0 - p|\}.$$
(25)

For $m = 0, 1, 2, \ldots$, define

$$\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}, \qquad A_{m+1} = \frac{A_m + \lambda_m}{4}, \qquad (26)$$

$$x_{m+1} = \frac{x_m + \lambda_m}{4}, \quad y_{m+1} = \frac{y_m + \lambda_m}{4}, \quad z_{m+1} = \frac{z_m + \lambda_m}{4}, \quad p_{m+1} = \frac{p_m + \lambda_m}{4}, \quad (27)$$

$$d_m = (\sqrt{p_m} + \sqrt{x_m})(\sqrt{p_m} + \sqrt{y_m})(\sqrt{p_m} + \sqrt{z_m}), \qquad e_m = \frac{4^{-3m} \delta}{d_m^2}, \qquad (28)$$

where each square root has nonnegative real part.

Compute A_m for m = 0, 1, ..., n, where $4^{-n}Q < |A_n|$. Compute also $R_C(1, 1 + e_m)$ with relative error less in magnitude than r for m = 0, 1, ..., n - 1. Define

$$X = \frac{A_0 - x}{4^n A_n}, \qquad Y = \frac{A_0 - y}{4^n A_n}, \qquad Z = \frac{A_0 - z}{4^n A_n}, \qquad P = (-X - Y - Z)/2, \quad (29)$$

$$E_2 = XY + XZ + YZ - 3P^2, E_3 = XYZ + 2E_2P + 4P^3, (30)$$

$$E_4 = (2XYZ + E_2P + 3P^3)P, E_5 = XYZP^2.$$
 (31)

Then

$$R_J(x, y, z, p) \approx 4^{-n} A_n^{-3/2} \left(1 - \frac{3}{14} E_2 + \frac{1}{6} E_3 + \frac{9}{88} E_2^2 - \frac{3}{22} E_4 - \frac{9}{52} E_2 E_3 + \frac{3}{26} E_5 \right)$$

$$+ 6 \sum_{m=0}^{n-1} \frac{4^{-m}}{d_m} R_C(1, 1 + e_m)$$
(32)

with relative error less in magnitude than r.

When the variables are complex, the bound on relative error is not rigorous because of the possibility of some cancellation between terms that individually have error less than r. In practice, however, the error is usually much smaller than the bound.

If x, y, z are real and nonnegative, at most one of them is 0, and the fourth variable of R_J is negative, the Cauchy principal value is given by [16, (4.6)] and (21):

$$(y+q)R_{J}(x,y,z,-q) = (p-y)R_{J}(x,y,z,p) - 3R_{F}(x,y,z) + 3\left(\frac{xyz}{xz+pq}\right)^{1/2}R_{C}(xz+pq,pq), \quad q > 0,$$
(33)

where p-y=(z-y)(y-x)/(y+q). If we permute the values of x,y,z so that $(z-y)(y-x) \ge 0$, then $p \ge y > 0$ and all terms on the right side can be computed by the preceding algorithms. If x,y,z are complex, conditions of validity have not been established for this method of computing the Cauchy principal value.

It would be desirable to give the algorithm less restrictive conditions on x, y, z, p that do not rule out so many cases in which p = z and R_J reduces to R_D , for in such cases the algorithm gives correct results under the much weaker conditions stated below in the algorithm for R_D . We note that p = z implies $e_m = 0$ and $d_m = 2\sqrt{z_m}(z_m + \lambda_m)$ for all m, whence (23) becomes

$$R_D(x_0, y_0, z_0) = 4^{-n} R_D(x_n, y_n, z_n) + 3 \sum_{m=0}^{n-1} \frac{4^{-m}}{\sqrt{z_m} (z_m + \lambda_m)}.$$
 (34)

The function $R_D(x, y, z)$ is treated as a symmetric function of x, y, z, z, z.

ALGORITHM FOR R_D . Let x, y, z be nonzero and have phase less in magnitude than π , with the exception that at most one of x and y may be 0. The function $R_D(x, y, z)$

defined by (4) is to be computed with relative error less in magnitude than r. (We assume $r < 10^{-4}$.)

Let $x_0 = x$, $y_0 = y$, $z_0 = z$, and

$$A_0 = \frac{x + y + 3z}{5}, \qquad Q = (r/4)^{-1/6} \max\{|A_0 - x|, |A_0 - y|, |A_0 - z|\}.$$
 (35)

For m = 0, 1, 2, ..., define

$$\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}, \qquad A_{m+1} = \frac{A_m + \lambda_m}{4}, \qquad (36)$$

$$x_{m+1} = \frac{x_m + \lambda_m}{4}, \qquad y_{m+1} = \frac{y_m + \lambda_m}{4}, \qquad z_{m+1} = \frac{z_m + \lambda_m}{4},$$
 (37)

where each square root has nonnegative real part.

Compute A_m for m = 0, 1, ..., n, where $4^{-n}Q < |A_n|$. Define

$$X = \frac{A_0 - x}{4^n A_n}, \qquad Y = \frac{A_0 - y}{4^n A_n}, \qquad Z = -(X + Y)/3,$$
 (38)

$$E_2 = XY - 6Z^2$$
, $E_3 = (3XY - 8Z^2)Z$, (39)

$$E_4 = 3(XY - Z^2)Z^2, E_5 = XYZ^3.$$
 (40)

Then

$$R_D(x,y,z) \approx 4^{-n} A_n^{-3/2} \left(1 - \frac{3}{14} E_2 + \frac{1}{6} E_3 + \frac{9}{88} E_2^2 - \frac{3}{22} E_4 - \frac{9}{52} E_2 E_3 + \frac{3}{26} E_5 \right)$$

$$+ 3 \sum_{m=0}^{n-1} \frac{4^{-m}}{\sqrt{z_m} (z_m + \lambda_m)}$$

$$(41)$$

with relative error less in magnitude than r.

The remark about relative error immediately following the algorithm for R_J applies again here. The function R_G can be computed from R_F and R_D by using (7) if at most one of x, y, z is 0. Neither (7) nor the next algorithm can be used to compute $R_G(0,0,z) = \sqrt{z}/2$.

A faster and simpler way of computing the complete case of R_F and R_G with complex variables is useful because Legendre's complete integrals of the first and second kinds are

$$K(k) = R_F(1 - k^2, 1, 0), E(k) = 2R_G(1 - k^2, 1, 0).$$
 (42)

The method of successive arithmetic and geometric means (a special case of successive Landen transformations) serves this purpose for complex $k^2 \notin [1, +\infty)$. (Algorithms using Landen transformations for incomplete R_F and R_G are given in [3] but only for real variables; conditions of validity for complex variables are not obvious.)

ALGORITHM FOR $R_F(x, y, 0)$ AND $R_G(x, y, 0)$. Let x and y be nonzero and have phase less in magnitude than π . The complete elliptic integrals $R_F(x, y, 0)$ and $R_G(x, y, 0)$ are to be computed with relative error less in magnitude than r.

Define
$$x_0 = \sqrt{x}$$
, $y_0 = \sqrt{y}$, and

$$x_{m+1} = \frac{x_m + y_m}{2}, \quad y_{m+1} = \sqrt{x_m y_m}, \quad m = 0, 1, 2, \dots,$$
 (43)

where each square root has positive real part. Compute x_m and y_m for m = 0, 1, ..., n, where

$$|x_n - y_n| < 2.7\sqrt{r}|x_n|. \tag{44}$$

Then

$$R_F(x, y, 0) \approx \frac{\pi}{x_n + y_n} \tag{45}$$

with relative error less in magnitude than r. Also,

$$2R_G(x,y,0) \approx \left(\left(\frac{x_0 + y_0}{2} \right)^2 - \sum_{m=1}^n 2^{m-2} (x_m - y_m)^2 \right) R_F(x,y,0)$$
 (46)

with relative error less in magnitude than r if we neglect terms of order r^2 . The summation is empty if n = 0.

This algorithm can be used to compute also

$$R_D(0, y, z) = \frac{3}{z(y - z)} [2R_G(y, z, 0) - z R_F(y, z, 0)], \qquad 0 \neq y \neq z \neq 0.$$
 (47)

The exceptional case with $y = z \neq 0$ is

$$R_D(0, y, y) = \frac{3\pi}{4} y^{-3/2}.$$
 (48)

3 Numerical checks

Codes based on the algorithms of Section 2 can be checked against the following assortment of numerical values for complete and incomplete integrals with real, conjugate complex, or nonconjugate complex variables.

$$\begin{split} R_F(1,2,0) &= 1.3110\ 28777\ 1461 \\ R_F(i,-i,0) &= R_F(0.5,1,0) = 1.8540\ 74677\ 3014 \\ R_F(i-1,i,0) &= 0.79612\ 58658\ 4234\ -\ i\ 1.2138\ 56669\ 8365 \\ R_F(2,3,4) &= 0.58408\ 28416\ 7715 \\ R_F(i,-i,2) &= 1.0441\ 44565\ 4064 \\ R_F(i-1,i,1-i) &= 0.93912\ 05021\ 8619\ -\ i\ 0.53296\ 25201\ 8635 \\ R_C(0,1/4) &= \pi = 3.1415\ 92653\ 5898 \\ R_C(9/4,2) &= \ln 2 = 0.69314\ 71805\ 5995 \\ R_C(0,i) &= (1-i)\ 1.1107\ 20734\ 5396 \\ R_C(-i,i) &= 1.2260\ 84956\ 9072\ -\ i\ 0.34471\ 13698\ 8768 \\ R_C(1/4,-2) &= \frac{\ln 2}{3} = 0.23104\ 90601\ 8665 \\ R_C(i,-1) &= 0.77778\ 59692\ 0447\ +\ i\ 0.19832\ 48499\ 3429 \\ R_J(0,1,2,3) &= 0.77688\ 62377\ 8582 \\ R_J(2,3,4,5) &= 0.14297\ 57966\ 7157 \\ R_J(2,3,4,-1+i) &= 0.13613\ 94582\ 7771\ -\ i\ 0.38207\ 56162\ 4427 \end{split}$$

$$R_J(-1+i, -1-i, 1, -3+i) = -0.61127\ 97081\ 2028\ -\ i\ 1.0684\ 03839\ 0007$$

$$R_J(-1+i, -2-i, -i, -1+i) = 1.8249\ 02739\ 3704\ -\ i\ 1.2218\ 47578\ 4827$$

 $R_J(i, -i, 0, 1-i) = 1.8260\ 11522\ 9009\ +\ i\ 1.2290\ 66190\ 8643$

 $R_J(i, -i, 0, 2) = 1.6490\ 01166\ 2711$

 $R_J(-1+i,-1-i,1,2) = 0.94148358841220$

The last case does not fit the assumptions, but the algorithm yields a value agreeing with $R_D(-2-i,-i,-1+i)$ below. If x,y,z are strictly positive, the Cauchy principal value of R_J changes sign at a negative value of the fourth variable, as illustrated by

$$R_J(2, 3, 4, -0.5) = 0.24723 81970 3052$$

 $R_J(2, 3, 4, -5) = -0.12711 23004 2964$

In this example R_J vanishes when the fourth variable is approximately -1.2552, and cancellation between terms on the right side of (33) will lead to loss of significant figures.

 $R_D(0,2,1) = 1.7972\ 10352\ 1034$

$$R_D(2,3,4) = 0.16510\ 52729\ 4261$$

 $R_D(i,-i,2) = 0.65933\ 85415\ 4220$
 $R_D(0,i,-i) = 1.2708\ 19627\ 1910\ +\ i\ 2.7811\ 12015\ 9521$
 $R_D(0,i-1,i) = -1.8577\ 23543\ 9239\ -\ i\ 0.96193\ 45088\ 8839$
 $R_D(-2-i,-i,-1+i) = 1.8249\ 02739\ 3704\ -\ i\ 1.2218\ 47578\ 4827$
 $R_G(0,16,16) = 2E(0) = \pi = 3.1415\ 92653\ 5898$
 $R_G(2,3,4) = 1.7255\ 03028\ 0692$
 $R_G(0,i,-i) = 0.42360\ 65423\ 9699$
 $R_G(i-1,i,0) = 0.44660\ 59167\ 7018\ +\ i\ 0.70768\ 35235\ 7515$
 $R_G(-i,i-1,i) = 0.36023\ 39218\ 4473\ +\ i\ 0.40348\ 62340\ 1722$
 $R_G(0,0.0796,4) = E(0.99) = 1.0284\ 75809\ 0288$

Consistency checks that do not use external information are furnished by the following equations, in which x, y, p are positive, $\lambda \mu = xy$, and $|\operatorname{ph} \lambda| < \pi$:

$$R_F(x + \lambda, y + \lambda, \lambda) + R_F(x + \mu, y + \mu, \mu) = R_F(x, y, 0);$$
 (49)

$$R_C(\lambda, x + \lambda) + R_C(\mu, x + \mu) = R_C(0, x); \tag{50}$$

$$R_J(x+\lambda, y+\lambda, \lambda, p+\lambda) + R_J(x+\mu, y+\mu, \mu, p+\mu) = R_J(x, y, 0, p) - 3R_C(a, b),$$
 (51)

where

$$a = p^{2}(\lambda + \mu + x + y), \quad b = p(p + \lambda)(p + \mu), \quad b - a = p(p - x)(p - y);$$
 (52)

$$R_D(\lambda, x + \lambda, y + \lambda) + R_D(\mu, x + \mu, y + \mu) = R_D(0, x, y) - \frac{3}{y\sqrt{x + y + \lambda + \mu}}.$$
 (53)

These equations are special cases of addition theorems. Another consistency check is furnished by

$$R_D(x, y, z) + R_D(y, z, x) + R_D(z, x, y) = \frac{3}{\sqrt{x}\sqrt{y}\sqrt{z}},$$
 (54)

where x, y, z lie in the cut plane and each square root has positive real part.

4 Other integrals

Legendre's complete elliptic integrals K and E are given by

$$K(k) = R_F(0, 1 - k^2, 1), (55)$$

$$E(k) = 2R_G(0, 1 - k^2, 1)$$

$$= \frac{1 - k^2}{3} [R_D(0, 1 - k^2, 1) + R_D(0, 1, 1 - k^2)], \quad (56)$$

$$K(k) - E(k) = \frac{k^2}{3} R_D(0, 1 - k^2, 1), \tag{57}$$

$$E(k) - (1 - k^2)K(k) = \frac{k^2(1 - k^2)}{3}R_D(0, 1, 1 - k^2).$$
 (58)

In Legendre's incomplete integrals we shall use the abbreviation $c = \csc^2 \phi = 1/\sin^2 \phi$:

$$F(\phi, k) = (\sin \phi) R_F(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1) = R_F(c - 1, c - k^2, c), \qquad (59)$$

$$E(\phi, k) = R_F(c - 1, c - k^2, c) - \frac{k^2}{3} R_D(c - 1, c - k^2, c), \qquad (60)$$

$$\Pi(\phi, k, n) = \int_0^{\phi} (1 + n \sin^2 \theta)^{-1} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta$$

$$= R_F(c - 1, c - k^2, c) - \frac{n}{3} R_J(c - 1, c - k^2, c, c + n).$$
(61)

Some related integrals are

$$D(\phi, k) = \int_0^{\phi} \sin^2\theta \, (1 - k^2 \sin^2\theta)^{-1/2} \, d\theta = \frac{1}{3} \, R_D(c - 1, c - k^2, c) \,, \tag{62}$$

$$K(k)Z(\beta,k) = \frac{k^2}{3}\sin\beta\cos\beta\sqrt{1-k^2\sin^2\beta}\,R_J(0,1-k^2,1,1-k^2\sin^2\beta)\,, \quad (63)$$

$$\Lambda_0(\beta, k) = \frac{2}{\pi} \frac{(1 - k^2) \sin \beta \cos \beta}{\Delta}
\cdot \left[R_F(0, 1 - k^2, 1) + \frac{k^2}{3\Delta^2} R_J \left(0, 1 - k^2, 1, 1 - \frac{k^2}{\Delta^2} \right) \right],$$
(64)

where $\Delta = \sqrt{1 - (1 - k^2) \sin^2 \beta}$. The function $Z(\beta, k)$ is Jacobi's zeta function [2, 140.03], and $\Lambda_0(\beta, k)$ is Heuman's lambda function [2, 150.01].

Bulirsch's elliptic integrals [1] are

$$el 1(x, k_c) = x R_F(1, 1 + k_c^2 x^2, 1 + x^2),$$
 (65)

$$el 2(x, k_c, a, b) = ax R_F (1, 1 + k_c^2 x^2, 1 + x^2) + \frac{1}{2} (b - a) x^3 R_D (1, 1 + k_c^2 x^2, 1 + x^2),$$
(66)

$$el 3(x, k_c, p) = x R_F(1, 1 + k_c^2 x^2, 1 + x^2) + \frac{1}{3} (1 - p) x^3 R_J(1, 1 + k_c^2 x^2, 1 + x^2, 1 + px^2),$$
 (67)

$$cel(k_c, p, a, b) = a R_F(0, k_c^2, 1) + \frac{1}{3}(b - pa)R_J(0, k_c^2, 1, p).$$
 (68)

In the real domain, inverse circular and inverse hyperbolic functions are expressed in terms of R_C by

$$\ln\left(\frac{x}{y}\right) = (x-y)R_C\left(\left(\frac{x+y}{2}\right)^2, xy\right), \qquad x > 0, \tag{69}$$

$$\arctan(x/y) = xR_C(y^2, y^2 + x^2), \quad -\infty < x < \infty, \tag{70}$$

$$\operatorname{arctanh}(x/y) = xR_C(y^2, y^2 - x^2), \quad -y < x < y,$$
 (71)

$$\arcsin(x/y) = xR_C(y^2 - x^2, y^2), \quad -y \le x \le y,$$
 (72)

$$\operatorname{arcsinh}(x/y) = xR_C(y^2 + x^2, y^2), \quad -\infty < x < \infty,$$
 (73)

$$\arccos(x/y) = (y^2 - x^2)^{\frac{1}{2}} R_C(x^2, y^2), \qquad 0 \le x \le y, \tag{74}$$

$$\operatorname{arccosh}(x/y) = (x^2 - y^2)^{\frac{1}{2}} R_C(x^2, y^2), \qquad x \ge y, \tag{75}$$

where y > 0 in each case. These equations remain valid in the complex domain provided

that the variables of R_C satisfy the conditions accompanying (3). If y = 1 the function multiplying R_C shows in each case the asymptotic behavior as the left side tends to 0.

Many elliptic integrals of the form

$$\int_{y}^{x} \prod_{i=1}^{n} (a_i + b_i t)^{p_i/2} dt , \qquad (76)$$

where p_1, \ldots, p_n are integers and the integrand is real, are reduced in [7]-[11] to the integrals in Section 2. Use of the algorithm for R_F is illustrated by numerical computation of the integral

$$I = \int_{y}^{x} \frac{dt}{\sqrt{(f_1 + 2g_1t + h_1t^2)(f_2 + 2g_2t + h_2t^2)}},$$
(77)

where all quantities are real, x > y, the two quadratic (or, if $h_i = 0$, linear) polynomials are positive on the open interval of integration, and their product has at most simple zeros on the closed interval. Let

$$q_i(t) = f_i + 2g_i t + h_i t^2, i = 1, 2,$$
 (78)

$$(x-y)U = \sqrt{q_1(x)q_2(y)} + \sqrt{q_1(y)q_2(x)}, \qquad (79)$$

$$T = 2g_1g_2 - f_1h_2 - f_2h_1, (80)$$

$$V = 2\sqrt{(g_1^2 - f_1 h_1)(g_2^2 - f_2 h_2)}. (81)$$

Then

$$I = 2R_F(U^2 + T + V, U^2 + T - V, U^2).$$
(82)

Except for notation this is the same as [4, (34)]. If the interval of integration is infinite, U is obtained by taking a limit; for example, if $x = +\infty$, then $U = \sqrt{h_1 q_2(y)} + \sqrt{q_1(y)h_2}$. If exactly one of q_1 and q_2 has conjugate complex zeros, then V is pure imaginary; otherwise the variables of R_F are real. The algorithm for R_F in this paper can be used in both cases. However, if both polynomials have conjugate complex zeros, the quadrilateral with the zeros as vertices has diagonals that must not intersect at an interior point of the interval of integration. If they do, the integral must be split into two parts at the point of intersection. This restriction is discussed in [4, §4] and removed by a Landen transformation in [11].

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