Bloc-Notes

David Fabre

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1. Noncylindrical Bridges

1.1. *Equilibrium shapes*

The equilibrium shapes are computed through an iterative method, described in the appendix (see also Tong Lei Zhang's master). The program is implemented in Matlab language. The continuation is done with taking as a control parameter either the pressure (=the curvature) inside the bridge (program `Newton_P.m`), or the volume (program `Newton_V.m`). The effect of gravity is also implemented.

Once the ratio $L/a$is fixed, the equilibrium shapes form a family which can be parametrized by either the pressure (or curvature), or the volume.

Figure 1 shows the Pressure/Volume relation for three cases ($L/a = 1.3; 2; 6$), and figure 2 shows a few shapes in the $R - Z$ plane. Note that starting from the cylindrical bridge, the trend is different for short and long bridges : for short ones increasing the pressure (curvature) leads to an increase of the volume, while for long ones increasing the pressure (curvature) leads to an decrease of the volume. In all cases, the curve passes through a state of minimum volume, and terminates at a green point where the shape asymptotes to two touching spheres. We also identify in green the shape which has the same volume as this limit shape. In a coalescence process starting from two touching states, this is the final state, since the volume is conserved during the process and the initial state is unstable.

Note that this final state corresponds corresponds to larger volume than the cylindrical solution for long bridges, and to a smaller volume for short bridges.

The characteristics of these final states are respectively : For $L/a = 1.3$: $[P, V] =$

(a) : $L/a = 1.3$ (b) : $L/a = 2$

(a) : $L/a = 4$ (b) : $L/a = 6$

FIGURE 1. Pressure-volume relations for equilibrium shapes, for $L/a = 1.3; 2; 4$ and 6. Circles identify the cylindrical shape (red), the limit shape of touching spheres (green), the shape with same volume (green), and (for $L/a = 1.3$) the catenoidal shapes (blue).

L/a	shape	$(0, A)$	$(0, S)$	$(1, S)$	$(1, A)$
2	cyl.	3.26637	7.53435	1.49269	4.7160
2	noncyl	2.37693	6.25495	1.40023	4.22306
4	cyl.	7.346183e-01	2.132444e+00	4.996549e-01	1.496437e+00
4	noncyl	7.933437e-01	2.215907e+00	4.697427e-01	1.490868e+00
6	cyl.	1.298953e-01	8.451902e-01	2.907411e-01	7.828465e-01
6	noncyl	3.579927e-01	1.107705e+00	2.218312e-01	7.857503e-01

TABLE 1. Some values of Eigenfrequencies ($m = 1$ to be checked)

$[-0.01257; 2.3296]$; For $L/a = 2$: $[P, V] = [0.7428; 4.1888]$; For $L/a = 4$: $[P, V] = [0.9699; 14.6607]$; For $L/a = 6$: $[P, V] = [0.7926; 37.7]$.

Note also that in the case $L/a = 1.3$, the problem admits two solutions with zero pressure (or curvature), which correspond to classical catenoidal shapes. These cases are identified in blue.

FIGURE 2. A few equilibrium shapes for liquid bridges, for $L/a = 1.3$, 2 and 6. The green shapes correspond to the limit case where the bridge becomes two touching spherical portions, and to the configuration with the same volume.

1.2. *Frequency calculations*

The table gives a few values of the frequencies with cylindrical and non-spherical shape (corresponding to green point in figure 1).

We can see that the departure from the cylindrical shape leads to an increase of frequencies for long bridges and to a decrease for short bridges, mostly for axisymetric modes. The effect is less pronounced on non-axisymetric modes.

Eigenfrequencies for $m = 0$ modes are given in figure 3 as function of pressure, for $L/a = 4$.

TO BE CONTINUED....

FIGURE 3. Frequencies of axisymmetric modes ($L/a = 4$) : ANCIENNE VERSION. Petit problème a régler aux alentour de $P = 0.94$ => ça a l'air réglé avec nouvelle version avec densité suffisante (45)

Appendix A. Stability and oscillation frequencies of cylindrical liquid bridges

A.1. *Problem formulation*

We consider a cylindrical liquid bridge of radius a and length L.

For nondimensionalisation we set $a = 1, \rho = 1, \gamma = 1$.

The unknown are a potential ϕ defined in the volume, and a normal displacement η defined on the surface (extending for $x = -L/2$ to $x = +L/2$):

$$\mathbf{u} = \epsilon \nabla \phi(x,r) e^{i(m\theta + \omega t)}$$

$$r = a + \epsilon \eta(x) e^{i(m\theta + \omega t)}$$

Equations :

$$\begin{aligned}
\Delta\phi &= 0, && \text{(A 1)}\\
\partial_x \phi &= 0 \quad \text{for } x = 0 \text{ and } x = L, && \text{(A 2)}\\
\partial_r \phi &= i\omega\eta \quad \text{for } r = a, && \text{(A 3)}\\
\gamma K_1 = \gamma(-\partial_x^2 + m^2/a^2 - 1/a^2)\eta &= i\omega\phi \quad \text{for } r = a, && \text{(A 4)}\\
\eta &= 0 \quad \text{for } x = 0 \text{ and } x = L. && \text{(A 5)}\\
& && \text{(A 6)}
\end{aligned}$$

A.2. *Analytical solution*

The method is in the line of that of Henderson & Miles (1994) for sloshing modes in a cylindrical container with fixed contact line.

We first expand the potential in the following form (which automatically satisfies the boundary conditions at $x = 0, L$) :

$$\phi = \sum_{n=0}^{\infty} \phi_n \cos(k_n x) \frac{I_m(k_n r)}{I_m(k_n a)} \qquad \text{(A 7)}$$

$$= \sum_{n=0}^{\infty} \phi_{2n} (-1)^n \cos(k_{2n} x') \frac{I_m(k_{2n} r)}{I_m(k_{2n} a)} + \sum_{n=0}^{\infty} \phi_{2n+1} (-1)^n \sin(k_{2n+1} x') \frac{I_m(k_{2n+1} r)}{I_m(k_{2n+1} a)} \qquad \text{(A 8)}$$

where $k_n = n\pi/L$.

The second expression is in terms of $x' = x - L/2$ which spans the bridge from $-L/2$ to $L/2$. In terms of this centered variable, this expression allows to separate the symmetrical and anti symmetrical components, which contain respectively only even and odd terms. However, for the remainder it is simpler to stay with the first expression in terms of x, in order to avoid mixing between sines and cosines.

We first work with the dynamic boundary condition which can be written in the following form :

$$\gamma(-\partial_x^2 + m^2/a^2 - 1/a^2)\eta = \sum_{n=0}^{\infty} (i\omega)\phi_n \cos(k_n x) \qquad \text{(A 9)}$$

Unless if the right-hand side contains resonant terms (see below), the solution of this equation will be generally written as follows :

$$\eta = \sum_{n=0}^{\infty} A_n \cos(k_n x) - (A_s C_s(x) + A_a C_a(x)) \qquad \text{(A 10)}$$

We recognize two parts : the first is the solution proportional to the forcing terms, with amplitudes A_n given as follows :

$$A_n = \frac{-i\omega\phi_n}{\gamma(k_n^2 a^2 + m^2 - 1)}$$

The second part is the solution to the homogeneous equation. This part contains two auxiliary functions, noted $C_s(x)$ and $C_a(x)$. These functions are chosen so that $C_s(x)$ is symmetric and $C_a(x)$ is antisymmetric, and are normalized by $C_s(0) = C_a(0) = 1$. Physically, these two functions represent displacements of the free surface which do not modify the mean curvature. The expressions for these functions depends upon m and will be given below.

We now consider the kinematic boundary condition, which yields :

$$\eta = (i\omega)^{-1} \sum_{n=0}^{\infty} \phi_n \cos(k_n x) T_n, \qquad \text{with } T_n = \frac{I'_m(k_n a)}{I_m(k_n a)} \qquad \text{(A 11)}$$

Using the previous expression for A_n, this expression leads to :

$$\eta = \sum_{n=0}^{\infty} \left(\frac{\omega_n^*}{\omega}\right)^2 A_n \cos(k_n x) \qquad \text{(A 12)}$$

where

$$(\omega_n^*)^2 = \frac{\gamma}{\rho a^3} \left(k_n^2 + m^2 - 1\right) T_n$$

are the "free" frequencies (eigenfrequencies that would be obtained if we relax the fixed-line condition).

We now have two different expressions for $\eta(x)$, namely (A 10) and (A 12), and must equate them. One clearly sees that if the complementary functions are absent, the solution is trivially $\omega = \omega_n^*$: the Fourier components are uncoupled and the frequencies are those of "free problem". These modes do not verify the condition of fixed line, but instead they verify $\partial_x \eta = 0$; physically this represents a bridge located between two parallel planes and allowed to slip along them.

When the fixed constraint is imposed, the complementary functions have to be introduced. To equate the two expressions for η, these functions have to be decomposed along the Fourier basis. The decomposition is as follows :

$$C_s(x) = \sum_{n=0}^{\infty} C_n^s \cos(k_{2n} x)$$

$$C_a(x) = \sum_{n=0}^{\infty} C_n^a \cos(k_{2n+1} x)$$

The Fourier coefficients will be given below. We can now work separately for the symmetric and antisymmetric parts of the expression for η. For symmetric perturbations, equating the Fourier coefficients leads to :

$$\left(\frac{\omega_{2n}^*}{\omega}\right)^2 A_{2n} = A_{2n} + A_s C_n^s \tag{A 13}$$

Which can also be written :

$$A_{2n} = A_c C_n^s \left(\frac{\omega^2}{\omega^2 - \omega_{2n}^*}\right)$$

It remains to sum over all indices n, and recognize that $\sum A_{2n} = A_s$ (which comes from the fixed-line condition $\eta(0) = 0$), to get the following dispersion relation :

$$F_s(\omega^2) = \sum C_n^s \left(\frac{\omega^2}{\omega^2 - \omega_{2n}^*}\right) = 1 \tag{A 14}$$

The case of antisymmetric modes is similar and yields

$$F_a(\omega^2) = \sum C_n^a \left(\frac{\omega^2}{\omega^2 - \omega_{2n+1}^*}\right) = 1. \tag{A 15}$$

We can note that the functions F_s and F_a change sign through a infinite branch at the "free frequencies" ω_n^* and are monotonous between these roots (check ?) ; so we can conclude that the frequencies for the fixed-line problem lie between those for the free-line problem.

A.3. *Details*

- For $m = 0$, the auxiliary functions are given as

$$C_s(x) = \frac{\cos(x - L/2)}{\cos(L/2)}; \quad C_a(x) = \frac{\sin(L/2 - x)}{\sin(L/2)} \tag{A 16}$$

The fourrier coefficients for these functions are as follows :

$$C_n^s = \frac{4L}{L^2 - \pi^2(2n)^2}\tan(L/2) \quad \text{if } n \neq 0$$

$$C_0^s = \frac{2}{L}\tan(L/2)$$

$$C_n^a = -\frac{4L}{L^2 - \pi^2(2n+1)^2}\cot(L/2)$$

Note that if $L = n\pi$, the problem for η becomes non-homogenous and a special treatment is required. However these case are of no particular significance, so we omit the details.

- For $m \geqslant 2$, the auxiliary functions are given as

$$C_s(x) = \frac{\cosh(\mu(x - L/2))}{\cosh(\mu L/2)}; \quad C_a(x) = \frac{\sinh(\mu(L/2 - x))}{\sinh(\mu L/2)} \tag{A 17}$$

The fourrier coefficients for these functions are as follows :

$$C_n^s = \frac{4L\mu}{\mu^2 L^2 + \pi^2(2n)^2}\tanh(\mu L/2) \quad \text{if } n \neq 0$$

$$C_0^s = \frac{2}{L}\tanh(L/2)$$

$$C_n^a = \frac{4L\mu}{\mu^2L^2 + \pi^2(2n+1)^2}\coth(\mu L/2)$$

• The case $m=1$ requires a specific treatment. In that case, the left-hand side of the equation (A 9) is simply $\partial_x^2\eta = 0$, whose homogenous solutions are constants and linear functions, but the right-hand side contains the constant term $i\omega\phi_0$ which proportional to the homogenous solution, hence resonant. In that case, the auxiliary function $C^s(x)$ has to be taken as the response to this resonant term, namely

$$C_s(x) = (2x/L-1)^2$$

The antisymmetric auxiliary function is $C_a(x)$ and is not resonant, and is taken as :

$$C_a(x) = (1-2x/L)$$

The Fourier coefficients for these auxiliary functions are :

$$C_n^s = \frac{16}{\pi^2n^2} \quad \text{if } n \neq 0$$

$$C_0^s = 1/3$$

$$C_n^a = \frac{8}{\pi^2n^2}$$

The solution to the antisymmetric problem is still given by A 15.
The symmetric problem is slightly different. In this case, the expressions for A_n given above (A 13) are still valid for $n \neq 0$, but the case $n=0$ is different, and the expressions has to be replaced by :

$$A_0 = \left(\frac{8}{\pi^2L^2} + C_0^s\right)A_s$$

Hence in this case the dispersion relation reads :

$$F_s(\omega^2) = \left(\frac{8}{\pi^2L^2} + \frac{1}{3}\right) + \sum_{n=1}^{\infty} C_n^s\left(\frac{\omega^2}{\omega^2-\omega_{2n}^*}\right) = 1 \qquad \text{(A 18)}$$

Appendix B. Calcul de la courbure

VERIFIER LES SIGNES

B.1. *Formules de base*

On cherche a exprimer la courbure d'une surface ayant une symétrie de révolution.

La courbure K se décompose en deux termes :

$$K = K^{(a)} + K^{(b)} \quad \text{avec} \quad \left|K^{(a)}\right| = \frac{1}{|MC^{(a)}|} \quad \text{et} \quad \left|K^{(b)}\right| = \frac{1}{|MC^{(b)}|}$$

Le premier terme est la courbure dans le plan méridien ; géométriquement, on l'exprime avec le point $C^{(a)}$ qui est le centre du cercle osculateur à la courbe méridienne. Le second terme est la courbure dans le plan orthogonal ; on l'exprime avec le point $C^{(b)}$ qui est l'intersection entre la normale à la courbe et l'axe de symétrie (voir figure a).

On prend la convention suivante pour le signe de $K^{(a)}$ et $K^{(b)}$: celles-ci sont positives si la surface est convexe et négatives si la surface est concave. Par exemple, dans le cas représenté sur la figure, on a $K^{(a)} > 0$ et $K^{(b)} < 0$.

Le premier terme se calcule a partir des formules de Frénet. On suppose que la courbe méridienne, dans le plan (r,z), admet une représentation paramétrique $M(s)$, où s est

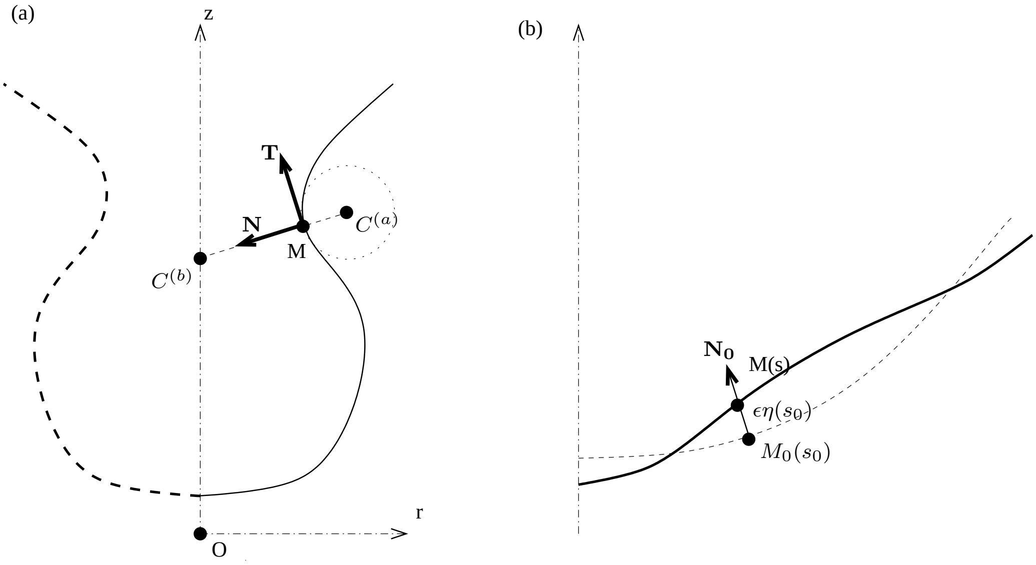

FIGURE 4. Explications de la méthode de calcul de la courbure

l'abscisse curviligne. On note $\mathbf{T}$ le vecteur tangent à la courbe dans le plan méridien, et $\mathbf{N}$ le vecteur normal. On a :

$$\mathbf{T} = \frac{\partial \vec{OM}}{\partial s}$$

$$\frac{\partial \mathbf{T}}{\partial s} = -K^{(a)}\mathbf{N}$$

$$\frac{\partial \mathbf{N}}{\partial s} = K^{(a)}\mathbf{T}$$

En pratique on peut aussi utiliser la formule suivante :

$$K^{(a)} = \mathbf{T} \cdot \frac{\partial \mathbf{N}}{\partial s} = \mathbf{T} \cdot (\nabla \mathbf{N}) \cdot \mathbf{T}$$

Le second terme a l'expression suivante :

$$K^{(b)} = \frac{N_{,r}}{r}$$

où $N_{,r} = \mathbf{N} \cdot \mathbf{e_r}$ est la composante radiale du vecteur normal.

B.2. *Courbure de la forme moyenne*

On suppose que la forme moyenne de l'interface est donnée par un paramétrage de la forme $M_0(s_0)$, où s_0 est l'abscisse curviligne associée. On note T_0, N_0, K_0 les vecteurs tangents, normal, et la courbure associée. Ceux-ci sont donnés par :

$$\mathbf{T}_0 = \frac{\partial \vec{OM}_0}{\partial s_0}$$

$$K_0 = K_0^{(a)} + K_0^{(b)} = \mathbf{T_0} \cdot \frac{\partial \mathbf{N}_0}{\partial s_0} + \frac{\mathbf{N}_{0,r}}{r} \tag{B 1}$$

Appendix C. Perturbation

On suppose maintenant que la surface oscille faiblement autour de la forme moyenne précédemment définie (voir figure b).

On choisit de paramétrer la déformation de la manière suivante :

$$\vec{OM}(s_0) = \vec{OM}_0(s_0) + \epsilon\eta(s_0)\mathbf{N}_0$$

Dans cette expression, ϵ est un petit paramètre, et la fonction η correspond à l'amplitude de la déformation mesurée dans la direction normale à la surface *moyenne*. Notons que l'on garde le paramétrage par la variable s_0 qui est l'abscisse curviligne de la forme moyenne (et qui n'est pas identique à l'abscisse curviligne s de la surface déformée).

On injecte maintenant ce paramétrage dans les formules précédentes, et on linéarise par rapport à ϵ, ce qui aboutit à :

$$\frac{\partial s}{\partial s_0} = \left|\frac{\partial \vec{OM}}{\partial s_0}\right| = 1 - \epsilon\eta K_0^{(a)};$$

$$\mathbf{T} = \left(\frac{\partial s}{\partial s_0}\right)^{-1} \frac{\partial \vec{OM}}{\partial s_0} = \mathbf{T}_0 + \epsilon\mathbf{T}_1; \quad \mathbf{T}_1 = -\frac{\partial \eta}{\partial s_0}\mathbf{N_0}$$

$$\mathbf{N} = \mathbf{N}_0 + \epsilon\mathbf{N}_1; \quad \mathbf{N}_1 = \frac{\partial \eta}{\partial s_0}\mathbf{T_0}$$

$$\begin{aligned} K^{(a)} = \mathbf{T}\cdot\frac{\partial \mathbf{N}}{\partial s} &= (\mathbf{T}_0 + \epsilon\mathbf{T}_1)\left(\frac{\partial s_0}{\partial s}\right)\frac{\partial}{\partial s_0}(\mathbf{N}_0 + \epsilon\mathbf{N}_1) \\ &= K_0^{(a)} + \epsilon K_1^{(a)} \\ K_1^{(a)} &= -\frac{\partial^2 \eta}{\partial s_0^2} - \left(K_0^{(a)}\right)^2 \eta \end{aligned}$$

De même, pour la seconde composante de la courbure :

$$\begin{aligned} K^{(b)} = \frac{N_{,r}}{r} &= \frac{N_{0,r} + \epsilon N_{1,r}}{r + \epsilon\eta N_{0,r}} \\ &= K_0^{(b)} + \epsilon K_1^{(b)} \\ K_1^{(b)} &= -\frac{T_{0,r}}{r}\frac{\partial \eta}{\partial s_0} - \left(K_0^{(b)}\right)^2 \eta \end{aligned}$$

Au final on a donc :

$$K = K_0 + \epsilon K_1$$

$$K_0 = \mathbf{T_0}\frac{\partial \mathbf{N_0}}{\partial s_0} + \frac{N_{0,r}}{r}$$

$$K_1 = -\frac{1}{r}\frac{\partial}{\partial s_0}\left(r\frac{\partial \eta}{\partial s_0}\right) - \left[\left|\frac{\partial \mathbf{N_0}}{\partial s_0}\right|^2 + \frac{N_{0,r}^2}{r^2}\right]\eta$$

Dans cette dernière expression on a utilisé l'identité $T_{0,r} = \partial r/\partial s_0$.

C.1. *Cas particulier : forme moyenne sphérique*

On suppose que la forme moyenne est une sphère de rayon R_0. On utilise les coordonnées sphériques (R,Θ). Dans ce cas, l'abscisse curviligne de la forme moyenne s_0 est donné par $s_0 = R_0\Theta$, et on a :

$$r = R_0 \sin\Theta; \quad z = R_0\cos\Theta; \quad \frac{\partial}{\partial s_0} = \frac{1}{R_0}\frac{\partial}{\partial\Theta}$$

$$\mathbf{N_0} = \mathbf{e_R}; \quad \mathbf{T_0} = \mathbf{e_\Theta}; \quad N_{0,r} = \sin\Theta;$$

En injectant dans les formules précédentes, on aboutit à :

$$K_0 = \frac{2}{R_0}$$

$$K_1 = -\frac{1}{R_0^2\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\eta}{\partial\theta}\right) - \frac{2}{R_0^2}\eta$$

Ce qui correspond bien aux formules obtenues dans ce cas.

C.2. *Paramétrage selon r*

Vérifions que les formules générales trouvée ici est équivalente à celles utilisées dans le cas où la surface est paramétrée par r et non par s_0 C'est-à-dire :

$$z = H(r) = h_0(r) + \epsilon\eta_z(r)$$

Dans ce cas le calcul de la courbure conduit à :

$$K = K_0(r) + \epsilon k(r)$$

avec :

$$K_0(r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\sqrt{1+h_0'^2}}\frac{\partial h_0}{\partial r}\right)$$

$$k(r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\left(1+h_0'^2\right)^3}\frac{\partial \eta_z}{\partial r}\right)$$

(Par rapport aux formules données dans le rapport de Jérôme on a changé les signes afin d'utiliser la même convention sur les normales, et on a rectifié une petite erreur dans le terme k).

La correspondance entre les deux formulations s'établit en utilisant les identités suivantes :

$$\eta_z(r) = \frac{\eta(s_0)}{N_{0,z}}; \quad T_{0,r} = N_{0,z} = \frac{1}{\sqrt{1+h_0'^2}}; \quad T_{0,z} = -N_{0,r} = \frac{h_0'}{\sqrt{1+h_0'^2}}$$

$$\frac{\partial}{\partial r} = N_{0,z}\frac{\partial}{\partial s_0}$$

$$k(r) = K_1(s_0) - \frac{\partial K_0}{\partial s_0}T_{0,z}\eta_z(r)$$

(formules vérifiées avec Maple)

C.3. *Calcul de la forme d'équilibre par méthode de Newton*

Le but est de construire un maillage tel que le long de sa frontière, on ait l'équilibre de Laplace :

$$F = K - \frac{\Delta P}{\sigma} = 0. \tag{C 1}$$

avec

$$\Delta P = \Delta P_b + \Delta \rho g z$$

Ici ΔP_b est la différence de pression à la base de la bulle ($z = 0$), que l'on impose dans le calcul (on pourrait aussi imposer le volume dans la bulle et considérer ΔP_b comme une inconnue, mais cela reste à faire proprement).

La méthode est la suivante :

(a) On part d'un maillage correspondant à une forme approximative de la bulle (par exemple un développement en série de Legendre issu des expériences).

(b) On écrit un développement de Taylor de la fonction F par rapport à des petites variations η de la forme de la surface :

$$F \approx F_0 + F_1 = 0 \quad \text{avec } F_0 = K_0 - (\Delta P + \Delta\rho g z)/\sigma, \quad F_1 = K_1 - \frac{\Delta \rho g}{\sigma} N_{0,z}\eta$$

où K_1 est donné (en fonction de η) par la formule de la section précédente.

(c) On inverse la relation précédente, ce qui donne la fonction η correspondant au déplacement qu'il faut donner à l'interface pour assurer la condition $F = 0$ (sous l'hypothèse de linéarisation).

(d) On construit un champ de vecteurs $\mathbf{U}$, défini à l'intérieur du domaine, correspondant à un déplacement lagrangien vérifiant $\mathbf{U} = \eta \mathbf{N_0}$ sur la frontière du domaine et étant suffisamment régulier à l'intérieur (en pratique on résoud une équation de Poisson).

(e) On déforme le maillage selon le champ de vecteur $\mathbf{U}$, ce qui aboutit à un nouveau maillage en principe plus proche de la solution d'équilibre.

(f) On répète l'opération de manière itérative a partir du point (b), jusqu'à convergence (c'est à dire jusqu'à ce que la quantité F_0 devienne effectivement négligeable.

C.4. *Implémentation avec Freefem*

Pour le calcul de la courbure moyenne, il faut commencer par interpoler les vecteurs normal (et tangent) sous forme de champs P1 définis sur la frontière :

```
(...)

mesh Shempty=emptymesh(Sh);
fespace Wh1(Shempty,P1);
Wh1 N0r,N0z,T0r,T0z,K0a,K0b,test ;

problem CalcN0r(N0r,test)=
 int1d(Shempty,qfe=qf3pE)(N0r*test)-int1d(Shempty,qfe=qf3pE)(N.x*test);
problem CalcN0z(N0z,test)=
 int1d(Shempty,qfe=qf3pE)(N0z*test)-int1d(Shempty,qfe=qf3pE)(N.y*test);

CalcN0r;
CalcN0z;
T0r =  N0z;
T0z = -N0r;
```

```
macro Ds(u1,u2)
[dx(u1)*T0r+dy(u1)*T0z,dx(u2)*T0r+dy(u2)*T0z]
//

problem ComputeK0a(K0a,test)=
 int1d(Shempty,qfe=qf3pE)(K0a*test)
-int1d(Shempty,qfe=qf3pE)(Ds(N0r,N0z)'*[T0r,T0z]*test);
ComputeK0a;

problem ComputeK0b(K0b,test)=
 int1d(Shempty,qfe=qf3pE)(K0b*test*x)
-int1d(Shempty,qfe=qf3pE)(N0r*test);
ComputeK0b;
```

Pour les perturbations, le terme de courbure se traite par intégration par partie :

$$p = \sigma K_1$$

$$\int_{\mathcal{S}} \eta^\dagger p r d\ell = \sigma \int_{\mathcal{S}} \left(\frac{\partial \eta^\dagger}{\partial s_0} \frac{\partial \eta}{\partial s_0} - \left[\left| \frac{\partial \mathbf{N_0}}{\partial s_0} \right|^2 + \frac{N_{0,r}^2}{r^2} \right] \eta^\dagger \eta \right) r d\ell \text{ (+ termes de bord)}$$