Intersection between ellipse and line segment

Kristian Wichmann

March 25, 2017

1 Equation for the ellipse

Let the semimajor and semininor axis of the ellipse be given by a and b. Assuming these to lie along the x and y axis of the coordinate system, the standard equation is:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1.1}$$

2 Equation for the line segment

Let the line segment run bewteen $P(p_1, p_2)$ and $Q(q_1, q_2)$. A directional vector for the segment is:

$$\vec{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} \tag{2.1}$$

Hence, a parametric equation for the line segment is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + t \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} = \begin{pmatrix} p_1 + t(q_1 - p_1) \\ p_2 + t(q_2 - p_2) \end{pmatrix}, \quad t \in [0, 1] \tag{2.2}$$

3 Solving the equations

We now search for values of t which solve both 1.1 and 2.1. According to 2.1 we must have:

$$p_1 + t(q_1 - p_1) \tag{3.1}$$

The square is:

$$x^2 = p_1^2 + t^2(q_1 - p_1)^2 + 2p_1 t(q_1 - p_1) \tag{3.2}$$

Ordering by powers of t this is:

$$x^2 = (q_1 - p_1)^2 t^2 + 2p_1(q_1 - p_1)t + p_1^2 \tag{3.3}$$

By analogy, for y we get:

$$y^2 = (q_2 - p_2)^2 t^2 + 2p_2(q_2 - p_2)t + p_2^2 \tag{3.4}$$

Now insert 3.3 and 3.4 into 1.1:

$$\frac{(q_1 - p_1)^2}{a^2} t^2 + \frac{2p_1(q_1 - p_1)}{a^2} t + \frac{p_1^2}{a^2} + \frac{(q_2 - p_2)^2}{b^2} t^2 + \frac{2p_2(q_2 - p_2)}{b^2} t + \frac{p_2^2}{b^2} = 1 \tag{3.5}$$

Collect terms according to power of t:

$$\underbrace{\left(\frac{(q_1 - p_1)^2}{a^2} + \frac{(q_2 - p_2)^2}{b^2} \right)}_{c_2} t^2 \tag{3.6}$$

$$+ \underbrace{\left(\frac{2p_1(q_1 - p_1)}{a^2} + \frac{2p_2(q_2 - p_2)}{b^2} \right)}_{c_1} t \tag{3.7}$$

$$+ \underbrace{\left(\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} - 1 \right)}_{c_0} = 0 \tag{3.8}$$

This is a quadratic equation in t. The corresponding discriminant is:

$$d = c_1^2 - 4c_0c_2 \tag{3.9}$$

If $d < 0$ there's no solution. If $d > 0$, there's two:

$$t = \frac{-c_1 \pm \sqrt{d}}{2c_2} \tag{3.10}$$

If d is exactly zero, there's one solution $t = -\frac{c_1}{2c_2}$.

4 Checking the solutions

Remember that the line segment only stretches from $t = 0$ to $t = 1$. So it must be checked that the solutions (if any) do indeed lie in the interval $[0, 1]$. Otherwise, there is no intersection (though an extension of the line segment does intersect the ellipse).

5 Special case: Circle

If $a = b = r$ we have a circle with radius r. This means that equation 1.1 becomes:

$$x^2 + y^2 = r^2 \tag{5.1}$$

We now insert equations 3.3 and 3.4 into 5.1:

$$(q_1 - p_1)^2 t^2 + 2p_1(q_1 - p_1)t + p_1^2 + (q_2 - p_2)^2 t^2 + 2p_2(q_2 - p_2)t + p_2^2 = r^2 \tag{5.2}$$

Now collect terms into powers of t:

$$\underbrace{\left((q_1 - p_1)^2 + (q_2 - p_2)^2\right)}_{c_2} t^2 \tag{5.3}$$

$$+ \underbrace{2\left(p_1(q_1 - p_1) + p_2(q_2 - p_2)\right)}_{c_1} t \tag{5.4}$$

$$+ \underbrace{p_1^2 + p_2^2 - r^2}_{c_0} = 0 \tag{5.5}$$

Now proceed as above, finding the discriminant and checking if the solutions (if any) are between 0 and 1.