Transformation Math:

A point P is transformed into P' with the following composite transformation. The 3D composite transformation is defined by a 1st shearing along the Y axis by a shearing factor of (7, 2) about point (90, 10, -20), followed by a uniform scaling by factor 3 again about point (-30, -30, -30), followed by a 30-degree counterclockwise rotation on the X axis about point (2, 10, 12), followed by a reflection in the zy plane and finally followed by a translation of (40, -20, 60).

shx shz

2nd

3rd

4th

5th

→ Always write the order first!!

1st transformation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \text{Translation} \end{bmatrix}}_{T} \times \underbrace{\begin{bmatrix} \text{Reflection} \end{bmatrix}}_{Ref} \times \underbrace{\begin{bmatrix} \text{Rotation about a point} \end{bmatrix}}_{R} \times \underbrace{\begin{bmatrix} \text{Scaling about a point} \end{bmatrix}}_{S} \times \underbrace{\begin{bmatrix} \text{Shearing about a point} \end{bmatrix}}_{Sh} \times \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{P}$$

Last transformation

Shearing along Y-axis [about a point (90,10,-20)] =

$$\begin{bmatrix} 1 & 0 & 0 & 90 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -90 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation (90,10,-20) × Shear along Y-axis × Translation (-90,-10,20)

S (uniform) [about a point (-30,-30,-30)] =

$$\begin{bmatrix} 1 & 0 & 0 & -30 \\ 0 & 1 & 0 & -30 \\ 0 & 0 & 1 & -30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation (-30,-30,-30) × Uniform Scaling × Translation (30,30,30)

Rotation about X-axis [about a point (2,10,12)] =

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 & \\ 0 & \sin 30 & \cos 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation (2,10,12) × Rotation about X-axis × Translation (-2,-10,-12)

Reflection through yz plane $= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Translation (40, -20, 60) $= \begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Full matrix order

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 & 0 \\ 0 & \sin 30 & \cos 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -30 \\ 0 & 1 & 0 & -30 \\ 0 & 0 & 1 & -30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 90 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -90 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{P}$$

Rotation — Scale — Shear — Extreme right

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}_{P'} = \begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Extreme left — Translation — Reflection

To solve now

$$P' = (T \times Ref) \times R \times S \times Sh \times P$$

where

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} \text{Translation} \end{bmatrix}, \quad Ref = \begin{bmatrix} \text{Reflection} \end{bmatrix}, \quad R = \begin{bmatrix} \text{Rotation about a point} \end{bmatrix}, \quad S = \begin{bmatrix} \text{Scaling about a point} \end{bmatrix}, \quad Sh = \begin{bmatrix} \text{Shearing about a point} \end{bmatrix}, \quad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$T \times Ref = M_1$

$\Rightarrow P' = M_1 \times R \times S \times Sh \times P$ [We do matrix multiplication from left into right]

$M_2 \times S \times Sh \times P$

$M_3 \times Sh \times P$

$\therefore P' = M_4 \times P$

NOTE: The ordering of the matrices is very important:

- The first transformation will start from extreme right and the next transformation will follow from left

e.g. TR → First rotate then translate

2D

$$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation matrix

Rotation matrix

$$= \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix}$$

RT → First translate then rotate

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & (\cos\theta dx - \sin\theta dy) \\ \sin\theta & \cos\theta & (\sin\theta dx + \cos\theta dy) \\ 0 & 0 & 1 \end{bmatrix}$$

Not the same!

Inverse Transformation :

Now, for the same question, if we have to go from P' back to P we have to go inverse

- First, the order needs to be reversed

1st transformation ↓

$$\underset{P}{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}} = \underset{Sh}{\begin{bmatrix} \text{Shear} \\ \text{about} \\ \text{a point} \end{bmatrix}} \times \underset{S}{\begin{bmatrix} \text{Scaling} \\ \text{about} \\ \text{a point} \end{bmatrix}} \times \underset{R}{\begin{bmatrix} \text{Rotation} \\ \text{about} \\ \text{a point} \end{bmatrix}} \times \underset{Ref}{\begin{bmatrix} \text{Reflection} \end{bmatrix}} \times \underset{T}{\begin{bmatrix} \text{Translation} \end{bmatrix}} \times \underset{P'}{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}}$$

↑ Last

Previously → 1st Shear
↓ Scale
↓ Rotate
↓ Reflection
↓ Translate

Now → 1st Translate
↓ Reflection
↓ Rotate
↓ Scale
↓ Shear

Also, the parameters to be changed of the inverse transformations.

$T(40,-20,60) \rightarrow T(-40,20,-60)$

$Ref(yz) \rightarrow Ref(yz)$ [Reflection matrix remains the exact same]

$Rotate(30) \rightarrow Rotate(-30)$

$Scale(3,3,3) \rightarrow Scale(1/3, 1/3, 1/3)$

$Shear(7,2) \rightarrow Shear(-7,-2)$

$$\underbrace{\begin{bmatrix}1&0&0&-30\\0&1&0&-30\\0&0&1&-30\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1/3&0&0&0\\0&1/3&0&0\\0&0&1/3&0\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1&0&0&30\\0&1&0&30\\0&0&1&30\\0&0&0&1\end{bmatrix}}_{\text{scale}}\times\underbrace{\begin{bmatrix}1&0&0&2\\0&1&0&10\\0&0&1&12\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1&0&0&0\\0&\cos{-30}&-\sin{-30}&0\\0&\sin{-30}&\cos{-30}&0\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1&0&0&-2\\0&1&0&-10\\0&0&1&-12\\0&0&0&1\end{bmatrix}}_{\text{Rotate}}\times\underbrace{\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\text{Reflect}}\times\underbrace{\begin{bmatrix}1&0&0&-40\\0&1&0&20\\0&0&1&-60\\0&0&0&1\end{bmatrix}}_{\text{Translate}}\times\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}_{P'}$$

$$\begin{bmatrix}x\\y\\z\\1\end{bmatrix}_{P} = \underbrace{\begin{bmatrix}1&0&0&90\\0&1&0&10\\0&0&1&-20\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1&-7&0&0\\0&1&0&0\\0&-2&1&0\\0&0&0&1\end{bmatrix}\times\begin{bmatrix}1&0&0&-90\\0&1&0&-10\\0&0&1&20\\0&0&0&1\end{bmatrix}}_{\text{Shear}}$$