SICP: Exercise 1.13

August 15, 2024

> *Prove that Fib(n) is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers (see 1.2.2) to prove that $Fib(n) = (\varphi^n - \psi^n)/\sqrt{5}$.*

Okay, here I go:

Proof. **Base Cases:**

- Case $n = 1$, which is a simple rearrangement:

$$\frac{\varphi - \psi}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = \mathrm{Fib}(1)$$

- Case $n = 2$ (Here, we use the property of the golden ratios $\varphi^2 = \varphi + 1$, which holds for ψ too):

$$\frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{((\varphi+1) - (\psi+1)}{\sqrt{5}} = \frac{\varphi + 1 - \psi - 1}{\sqrt{5}} = \frac{\varphi - \psi}{\sqrt{5}} = \mathrm{Fib}(1) = 1 = \mathrm{Fib}(2)$$

Induction Step:
We assume that the statement holds for all numbers $\leq n+1$, and will prove it for $n+1$.

$$\begin{aligned}
\frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} &= \frac{\varphi^n \cdot \varphi - \psi^n \cdot \psi}{\sqrt{5}} = \frac{\varphi^{n-1} \cdot \varphi^2 - \psi^{n-1} \cdot \psi^2}{\sqrt{5}} && \text{Rearrange} \\
&= \frac{\varphi^{n-1} \cdot (\varphi+1) - \psi^{n-1} \cdot (\psi+1)}{\sqrt{5}} && \text{Golden ratios} \\
&= \frac{\varphi^n + \varphi^{n-1} - \psi^n + \psi^{n-1}}{\sqrt{5}} && \text{Rearrange} \\
&= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} && \text{Rearrange again} \\
&= \mathrm{Fib}(n) + \mathrm{Fib}(n-1) && \text{Induction requirement} \\
&= \mathrm{Fib}(n+1) && \text{Definition of Fib}
\end{aligned}$$

□

Oh, and I guess I should say that $\lim_{\psi \to \infty} |\psi^n| = 0$. This by itself is not enough, as the statement should hold for all n, not just very large ones. As ψ^n is biggest for $n = 1$ (assuming $n > 0$), the biggest difference will be for Fib(1), where $\left|\frac{\varphi - \psi}{\sqrt{5}} - \frac{\varphi}{\sqrt{5}}\right| \approx 0.28 < 0.5$. Hence, the error will always be below 0.5, so rounding to the nearest number will always lead to the correct Fibonacci number.