

## SICP: Exercise 1.13

Prove that  $\text{Fib}(n)$  is the closest integer to  $\varphi^n/\sqrt{5}$ , where  $\varphi = (1 + \sqrt{5})/2$ . Hint: Let  $\psi = (1 - \sqrt{5})/2$ . Use induction and the definition of the Fibonacci numbers (see 1.2.2) to prove that  $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ .

Okay, here I go:

*Proof.* **Base Cases:**

- Case  $n = 1$ , which is a simple rearrangement:

$$\frac{\varphi - \psi}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = \text{Fib}(1)$$

- Case  $n = 2$  (Here, we use the property of the golden ratios  $\varphi^2 = \varphi + 1$ , which holds for  $\psi$  too):

$$\frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{((\varphi + 1) - (\psi + 1))}{\sqrt{5}} = \frac{\varphi + 1 - \psi - 1}{\sqrt{5}} = \frac{\varphi - \psi}{\sqrt{5}} = \text{Fib}(1) = 1 = \text{Fib}(2)$$

**Induction Step:**

We assume that the statement holds for all numbers  $\leq n + 1$ , and will prove it for  $n + 1$ .

$$\begin{aligned} \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} &= \frac{\varphi^n \cdot \varphi - \psi^n \cdot \psi}{\sqrt{5}} = \frac{\varphi^{n-1} \cdot \varphi^2 - \psi^{n-1} \cdot \psi^2}{\sqrt{5}} && \text{Rearrange} \\ &= \frac{\varphi^{n-1} \cdot (\varphi + 1) - \psi^{n-1} \cdot (\psi + 1)}{\sqrt{5}} && \text{Golden ratios} \\ &= \frac{\varphi^n + \varphi^{n-1} - \psi^n + \psi^{n-1}}{\sqrt{5}} && \text{Rearrange} \\ &= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} && \text{Rearrange again} \\ &= \text{Fib}(n) + \text{Fib}(n-1) && \text{Induction requirement} \\ &= \text{Fib}(n+1) && \text{Definition of Fib} \end{aligned}$$

□

Oh, and I guess I should say that  $\lim_{\psi \rightarrow \infty} |\psi^n| = 0$ . This by itself is not enough, as the statement should hold for all  $n$ , not just very large ones. As  $\psi^n$  is biggest for  $n = 1$  (assuming  $n > 0$ ), the biggest difference will be for  $\text{Fib}(1)$ , where  $\left| \frac{\varphi - \psi}{\sqrt{5}} - \frac{\varphi}{\sqrt{5}} \right| \approx 0.28 < 0.5$ . Hence, the error will always be below 0.5, so rounding to the nearest number will always lead to the correct Fibonacci number.