THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

Solutions to selected exercises for MATH5905, Statistical Inference

Part three: Hypothesis Testing

Question 1: Check the answers and try to explain to yourself why the MLR property indeed holds. Study the examples from lectures first.

Question 2: From 1d), we know that the rejection region of the ump-α test of $H_0 : \sigma \leq \sigma_0$ versus $H_1 : \sigma > \sigma_0$ is in the form $\{\sum_{i=1}^n X_i^2 \geq k\}$. To determine k, we have to "exhaust the level", that is:

$$\alpha = P\Big(\sum_{i=1}^n X_i^2 \geq k|\sigma = \sigma_0\Big) = P\Big(\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} \geq \frac{k}{\sigma_0^2}\Big) = P\Big(\chi_n^2 \geq \frac{k}{\sigma_0^2}\Big).$$

Hence $k/\sigma_0^2 = \chi^2_{n,\alpha} \Rightarrow k = \sigma_0^2\chi^2_{n,\alpha}$ and the UMP-α test is:

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i^2 \geq \sigma_0^2\chi^2_{n,\alpha} \\ 0 \text{ if } \sum_{i=1}^n X_i^2 < \sigma_0^2\chi^2_{n,\alpha} \end{cases}$$

The power function:

$$\text{Power}(t) = P\Big(\sum_{i=1}^n X_i^2 > \sigma_0^2\chi^2_{n,\alpha}|\sigma = t\Big) = P(\chi_n^2 \geq (\frac{\sigma_0}{t})^2\chi^2_{n,\alpha}).$$

Thus Power(t) is increasing function of $t \in (0,\infty)$, and satisfies: Power$(0) = 0$, Power$(\sigma_0) = \alpha$, and $lim_{t\to\infty}Power(t) = 1$.

If $H_0 : \sigma \geq \sigma_0$ versus $H_1 : \sigma < \sigma_0$ was to be tested, then using the **Note** after the Blackwell-Girshick theorem, we know that the ump-α test exists and that now the rejection region is

$$\{\sum_{i=1}^n X_i^2 < \sigma_0^2\chi^2_{n,1-\alpha}\}.$$

The power function is

$$\text{Power}(t) = P\Big(\chi_n^2 \leq (\frac{\sigma_0}{t})^2\chi^2_{n,1-\alpha}\Big).$$

The graph will be "reversed" now since the hypothetical and the alternative region have been changed. We will have: Power$(0) = 1$, Power$(\sigma_0) = \alpha$, and Power(t) is monotonically decreasing when t ranges from 0 to ∞ with $\lim_{t\to\infty}$ Power$(t) = 0$.

Question 3: a) $\mathbb{E}_\theta\varphi = \int_{1/2}^1 \theta x^{\theta-1}dx = 1 - (\frac{1}{2})^\theta, \theta > 0$ is the power function. The size is obtained at $\theta_0 = 1$, so $\mathbb{E}_{\theta_0}\varphi = \frac{1}{2}$.

b) By the Neyman-Pearson lemma, for $H_0 : \theta = 2$ versus $H_1 : \theta = 1$, the best α-test is the one with a rejection region in the form

$$\Big\{\frac{L(\mathbf{X}; 1)}{L(\mathbf{X}; 2)} \geq k\Big\}.$$

Here, the sample size is $n = 1$ and we have

$$\frac{L(\mathbf{X}; 1)}{L(\mathbf{X}; 2)} = \frac{1x^0}{2x} \geq k.$$

Equivalently: $x \leq \frac{1}{2k} = k'$. To make it an α-test, we need

$$\alpha = 0.05 = P(X \leq k'|\theta = 2) = \int_0^{k'} 2xdx = (k')^2.$$

This implies that

$$k' = \sqrt{0.05} \approx .2236.$$

Hence the best 0.05-size test of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$ is

$$\varphi^* = \begin{cases} 1 \text{ if } x \leq .2236 \\ 0 \text{ if } x > .2236 \end{cases}$$

c) $f(x;\theta) = \theta e^{(\theta-1)lnx}$ is a member of one-parameter exponential family with $d(x) = \ln x$ and a monotonically increasing $c(\theta) = \theta - 1$. Hence, according to the **Note** to the Blackwell-Girshick theorem, the UMP-α test exists and has a rejection region

$$S = \{x : \ln x \leq k\}.$$

But $\ln x \leq k \iff x \leq k'$. For $\alpha = 0.05$ we get:

$$\alpha = 0.05 = P(x \leq k' | \theta = 2) = (k')^2$$

and we again get $k' = 0.2236$. This means that the same test as in b) is UMP-α size test of $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$. We could have also argued about this by noticing that in (b) the rejection region **did not** depend on the θ value under the alternative, hence the same test as in (b) will be an **ump-**0.05 test.

d)

$$\lambda = \frac{\max_{\theta \in \Theta_0} L(x;\theta)}{\max_{\theta \in \Theta} L(x;\theta)} = \frac{1}{(-\frac{1}{\ln x})x^{-\frac{1}{\ln x}-1}}$$

on the bottom we have replaced the argument that gives rise to the maximum by the value of the MLE $\hat{\theta}_{mle} = -\frac{1}{\ln x}$. We now observe that $g(x) = \frac{1}{(-\frac{1}{\ln x})x^{-\frac{1}{\ln x}-1}}$ tends to zero when $x \to 0$ or $x \to 1$. Therefore, $\lambda \leq$ constant is equivalent to $\{x \leq k_1 \text{ or } x \geq k_2\}$. The values of k_1 and k_2 must be such that

$$0.1 = \alpha = P(X \leq k_1 \text{ or } x \geq k_2 | \theta = 1) = k_1 + 1 - k_2.$$

Several choices are possible for k_1 and k_2. If we want an **equal-tailed** test then $k_1 = 0.05$ and $k_2 = 1 - 0.05 = 0.95$ should be chosen, that is:

$$\varphi = \begin{cases} 1 \text{ if } x \leq 0.05 \text{ or } x > 0.95 \\ 0 \text{ else} \end{cases}$$

Question 4: Draw a diagram of the first quadrant with axis OX_1 and OX_2 and try to represent the rejection region S as a subset of the unit square. This helps to understand the calculations below.
The joint density, because of the assumed independence, is given by

$$f_{X_1,X_2}(x_1,x_2) = \theta^2 x_1^{\theta-1} x_2^{\theta-1}.$$

Hence

$$\mathbb{E}_\theta \varphi = \int\int_S f_{X_1,X_2}(x_1,x_2)dx_1dx_2 = \int_0^1 (\int_{3x_1/4}^1 \theta^2 (x_1x_2)^{\theta-1} dx_2)dx_1 = \cdots = 1 - \frac{1}{2}(\frac{3}{4})^\theta.$$

The size of this test is $\mathbb{E}_\theta \varphi_{|\theta=1} = \frac{5}{8}$. Note that the test is **not good** since the size is too high and $\mathbb{E}_\theta \varphi_{|\theta=0} = \frac{1}{2}$ which is also very high. Test based on the statistic $T = \ln X_1 + \ln X_2$ should be used instead.

Question 5 The likelihood function is

$$L(\mathbf{x},\theta) = I_{(\theta,\infty)}(x_{(1)})e^{[-\sum_{i=1}^n x_i + n\theta]}$$

The family has a **monotone likelihood ratio in** $X_{(1)}$ (show this by examining the behavior of $\frac{L(\mathbf{X},\theta'')}{L(\mathbf{X},\theta')}, \theta' > \theta'$ as a function of $x_{(1)}$ and convince yourself that the ratio is zero when $x_{(1)} \in (\theta', \theta'')$ but is equal to a positive constant $e^{n(\theta''-\theta')}$ when $x_{(1)} > \theta''$).

This means that a ump-α test exists and has the form

$$\varphi^* = \begin{cases} 1 \text{ if } x_{(1)} > k \\ 0 \text{ if } x_{(1)} < k \end{cases}$$

To find k we need the distribution of $x_{(1)}$. Now:

$$P_\theta(X_{(1)} > k) = (P_\theta(X_1 > k))^n$$

But

$$P_\theta(X_1 > k) = \begin{cases} 1 \text{ if } k \leq \theta \\ 1 - \int_\theta^k e^{-(t-\theta)}dt = e^{-(k-\theta)} \text{ if } \theta < k \end{cases}$$

Hence

$$\mathbb{E}_\theta\varphi^* = P_\theta(X_{(1)} \geq k) = \begin{cases} 1 \text{ if } k \leq \theta \\ e^{-n(k-\theta)}, \text{ if } \theta < k \end{cases}$$

To find k we solve the equation $\mathbb{E}_{\theta_0}\varphi^* = e^{-n(k-\theta)} = \alpha$. This gives $k = \theta_0 - \frac{\ln\alpha}{n}$.
The power function is defined on the positive half axis and is:

- equal to $\alpha e^{-n\theta_0}$ for $\theta = 0$.
- equal to $\alpha e^{-n(\theta_0-\theta)}$ when $\theta \in (0, k)$ (in particular, it is equal to α for $\theta = \theta_0$.)
- is a constant equal to 1 when $\theta > k$.

Question 6: According to 1e) we have a MLR property in $T = \sum_{i=1}^{10} X_i$. Moreover, $T \sim Po(10\lambda)$. For $\lambda_0 = 1$ this is the Poisson distribution with parameter 10. Blackwell-Girshick theorem tells us that a ump-α test ($\alpha = 0.1$) exists and is in the form

$$\varphi^* = \begin{cases} 1 \text{ if } T > 14 \\ \gamma \text{ if } T = 14 \\ 0 \text{ if } T < 14 \end{cases}$$

The value of γ is

$$\gamma = \frac{.1 - .0835}{.9165 - .8644} = \frac{.0165}{.0521} = .317$$

Question 7 We use the density transformation formula:

$$f_Y(y) = f_X(w^{-1}(y))|\frac{dw^{-1}(y)}{dy}|$$

Since $X \sim f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0, \theta > 0$ we get: $Y = \frac{X}{\theta}$ has a standard exponential density $f_Y(y) = e^{-y}, y > 0$ and then, using the properties of Gamma distribution,

$$\sum_{i=1}^n X_i/\theta \sim \text{gamma}(n)$$

with the density

$$f_{\text{gamma}(n)}(x) = \frac{e^{-x}x^{n-1}}{\Gamma(n)}, x > 0.$$

Hence, the UMP-α test is given by

$$\varphi^* = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i \geq k \\ 0 \text{ if } \sum_{i=1}^n X_i < k \end{cases}$$

and

$$P\Big(\frac{1}{\theta_0}\sum_{i=1}^n X_i \geq \frac{k}{\theta_0}|\theta = \theta_0\Big) = \int_{k/\theta_0}^\infty \frac{e^{-x}x^{n-1}}{\Gamma(n)}dx.$$

Hence, the threshold is $k = \theta_0\gamma_{n,\alpha}$ where $\gamma_{n,\alpha}$ is the upper $\alpha * 100\%$ point of the gamma(n) density. This is an **exact** result.

On the other hand, asymptotically (for large n), by using the Central Limit Theorem (CLT) and the fct that $\mathbb{E}X_i = \theta_0$, $\text{Var}X_i = \theta_0^2$ we can get the following approximate value for the threshold:

$$\alpha = P(\sum_{i=1}^{n} X_i \geq k|\theta = \theta_0) = P(\frac{\sqrt{n}(\bar{X} - \theta_0)}{\theta_0} \geq \frac{\sqrt{n}(\frac{k}{n} - \theta_0)}{\theta_0}) \approx 1 - \Phi(\frac{\sqrt{n}(\frac{k}{n} - \theta_0)}{\theta_0})$$

which implies that

$$(\frac{k}{n} - \theta_0)\sqrt{n} = \theta_0 z_\alpha.$$

Hence $k \approx n\theta_0 + \sqrt{n}\theta_0 z_\alpha$ should be chosen in order to have the size asymptotically equal to α.

Question 8: Discussed at lectures.

Question 9:

$$L(\mathbf{X},\mathbf{Y};\mu_1,\mu_2) = (2\pi)^{-\frac{m+n}{2}} e^{-\frac{1}{2}[\sum_{i=1}^{m}(x_i-\mu_1)^2+\sum_{i=1}^{n}(y_i-\mu_2)^2]}.$$

Unrestricted maximisation with respect to both μ_1 and μ_2 leads to $\bar{X}, \bar{Y}$ as solutions. Now, **restricted** maximisation under the restriction $\mu_1 = \mu_2 = \mu$ leads to $\hat{\mu}_{mle(restricted)} = \frac{\sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i}{m+n}$. Therefore,

$$2ln\Lambda_{m,n} = \sum_{i=1}^{m}(x_i - \bar{x})^2 + \sum_{i=1}^{n}(y_i - \bar{y})^2 - \sum_{i=1}^{m}(x_i - \frac{m\bar{x} + n\bar{y}}{m+n})^2 - \sum_{i=1}^{n}(y_i - \frac{m\bar{x} + n\bar{y}}{m+n})^2 =$$

$$-m\bar{x}^2 - n\bar{y}^2 + 2\frac{m\bar{x} + n\bar{y}}{m+n}(\sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i) - (m+n)(\frac{m\bar{x} + n\bar{y}}{m+n})^2 = \cdots = -\frac{mn(\bar{x} - \bar{y})^2}{m+n}$$

Hence

$$-2ln\Lambda_{m,n} = \frac{mn(\bar{x} - \bar{y})^2}{m+n} = T$$

(which can be seen directly to be distributed as chi-square with one degree of freedom under the hypothesis of equal means. Our Generalised LRT test is then:

$$\varphi = \begin{cases} 1 \text{ if } T \geq \chi^2_{1,\alpha} \\ 0 \text{ if } T < \chi^2_{1,\alpha} \end{cases}$$