Exercise 2.9

Fix θ and let $E_\theta[d(X)] = \phi$. Then for d to be an unbiased decision rule, we require that, for all θ';

$$0 \le E_\theta\{L(\theta', d(X))\} - E_\theta\{L(\theta, d(X))\}$$ (Rewriting definition 2.3)

$$= E_\theta[(\theta' - d(X))^2] - E_\theta[(\theta - d(X))^2]$$

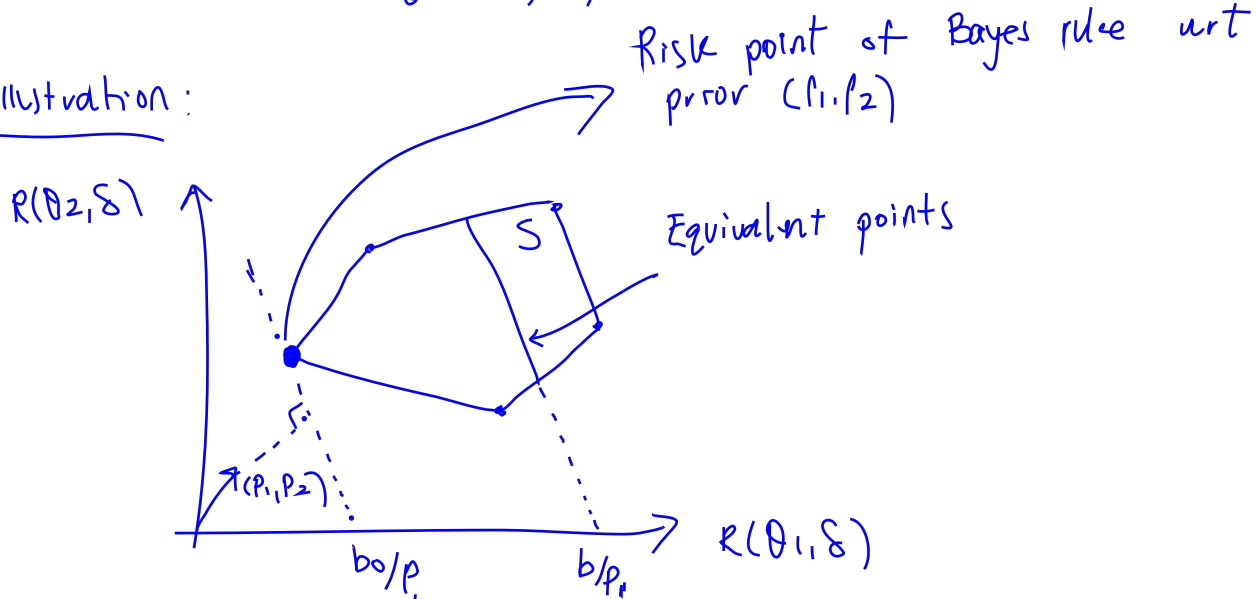
$$= (\theta')^2 - 2\theta'\phi + \overset{+\phi^2}{\cancel{E_\theta[d(X)^2]}} - \left(\theta^2 - 2\theta\phi + \overset{+\phi^2}{\cancel{E_\theta[d(X)^2]}}\right)$$

$$= (\theta' - \phi)^2 - (\theta - \phi)^2$$

$\Leftarrow$ If $\phi = \theta$ then this statement is true.

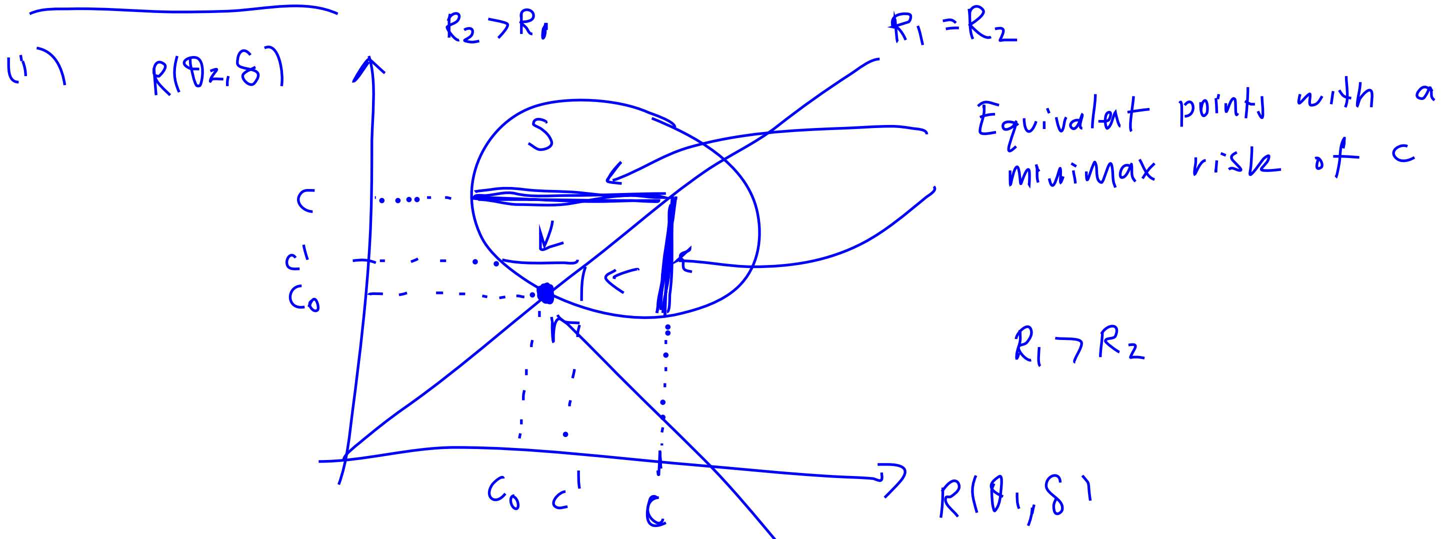
$\Rightarrow$ If $\phi \neq \theta$, then consider setting $\theta' = \phi$. This leads to a contradiction! Hence $E_\theta(d(X)) = \phi = \theta \rightarrow$ unbiased.

Bayes Illustration:



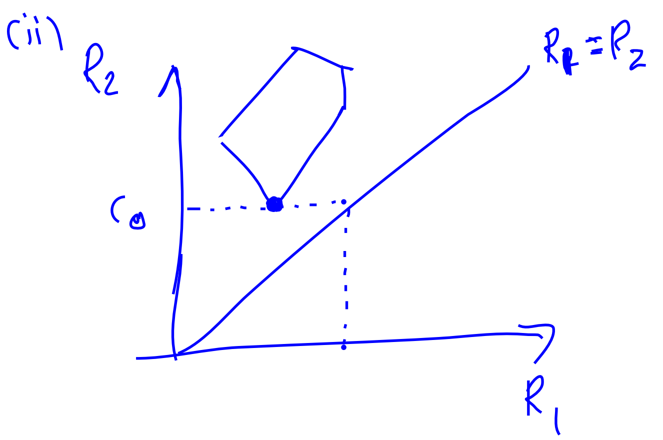
Bayesian risk: $r(\tau, \delta) = p_1 R(\theta_1, \delta) + p_2 R(\theta_2, \delta) = b$

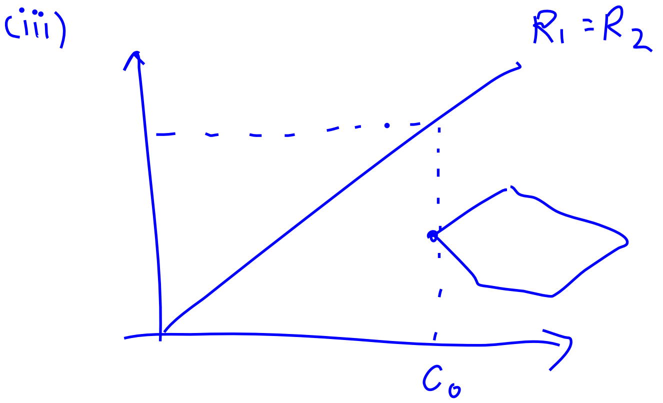
Minimax Illustration



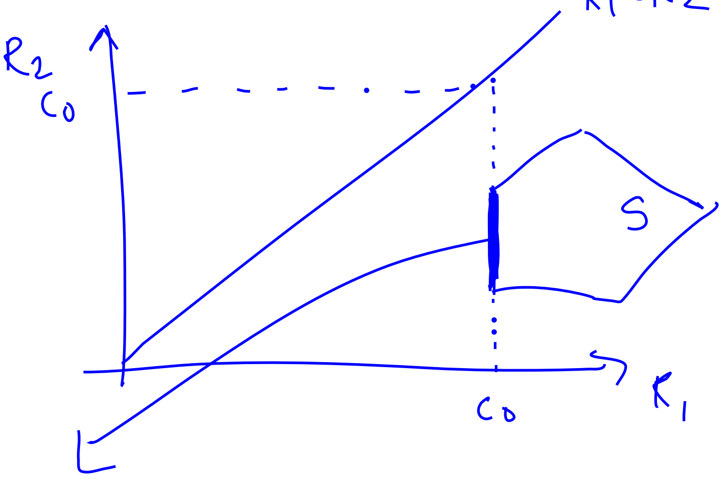
Minimax risk: $\sup_{\theta\in\{\theta_1,\theta_2\}} R(\theta,\delta)$

Risk point of the minimax rule with a minimax risk r_0





Non-uniqueness:



All these risk points represent equally good minimax rules with a minimax risk of c.

Example 2.15

$$h(\theta|X) = \frac{f(X|\theta)\tau(\theta)}{\int_0^1 f(X|\theta)\tau(\theta)\,d\theta} = \frac{\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}\cancel{\frac{1}{B(\alpha,\beta)}}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}\cancel{\frac{1}{B(\alpha,\beta)}}\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta}$$

$$= \frac{\theta^{\Sigma x_i+\alpha-1}(1-\theta)^{n-\Sigma x_i+\beta-1}}{\int_0^1 \theta^{\Sigma x_i+\alpha-1}(1-\theta)^{n-\Sigma x_i+\beta-1}\,d\theta}$$

$$B\left(\sum_{i=1}^n x_i+\alpha,\ n-\sum_{i=1}^n x_i+\beta\right) \longleftrightarrow \int_0^1 \theta^{\Sigma x_i+\alpha-1}(1-\theta)^{n-\Sigma x_i+\beta-1}\,d\theta$$

$$\theta|X \sim \text{Beta}(\quad , \quad)$$

Now Bayes estimator wrt quadratic loss:

$$\hat{\theta}_\tau = E(\theta|X) = \int_0^1 \theta \cdot h(\theta|x)\,d\theta$$

$$= \frac{\int_0^1 \theta^{\Sigma x_i+\alpha\cancel{+1-1}}(1-\theta)^{n-\Sigma x_i+\beta-1}\,d\theta}{B\left(\sum_{i=1}^n x_i+\alpha,\ n-\sum_{i=1}^n x_i+\beta\right)}$$

$$= \frac{B\left(\overset{\tilde{\alpha}}{\sum_{i=1}^n x_i+\alpha+1},\ \overset{\tilde{\beta}}{n-\Sigma x_i+\beta}\right)}{B\left(\sum_{i=1}^n x_i+\alpha,\ n-\Sigma x_i+\beta\right)}$$

$$= \frac{\tilde{\alpha}-1}{\tilde{\alpha}+\tilde{\beta}-1} = \frac{\sum_{i=1}^n x_i+\alpha+\cancel{1}-\cancel{1}}{\cancel{\Sigma x_i}+\alpha+\cancel{1}+n-\cancel{\Sigma x_i}+\beta-\cancel{1}}$$

$$= \frac{\sum_{i=1}^n x_i+\alpha}{\alpha+n+\beta}$$

$X_1, \ldots, X_{100}$

$\Sigma x_i = 77$ $\quad n = 10($

$\alpha = 2$ $\quad \beta = 2$

$79/104$

$$
\begin{aligned}
MSE(\hat\theta_\tau) &= E\left[(\hat\theta_\tau - \theta)^2\right] \\
&= E\left[\left(\underbrace{\hat\theta_\tau - E(\hat\theta_\tau)} + \underbrace{E(\hat\theta_\tau) - \theta}\right)^2\right] \\
&= E\left[(\hat\theta_\tau - E(\hat\theta_\tau))^2\right] + 2\times E\left[(\hat\theta_\tau - E(\hat\theta_\tau))\underbrace{(E(\hat\theta_\tau) - \theta)}_{\text{constant}}\right] \\
&\quad + E\left[\left(\underbrace{E(\hat\theta_\tau) - \theta}\right)^2\right] \\
&= Var(\hat\theta_\tau) + \underbrace{2(E(\hat\theta_\tau) - \theta)}_{\text{constant}} \underbrace{E\left[\hat\theta_\tau - E(\hat\theta_\tau)\right]}_{E(\hat\theta_\tau) - E(\hat\theta_\tau) = 0} + bias(\hat\theta_\tau)^2 \\
&= Var(\hat\theta_\tau) + bias(\hat\theta_\tau)^2
\end{aligned}
$$

Exercise 2.11

For a single observation X we have $f(x|\theta) = \frac{1}{\theta} I_{(x,\infty)}(\theta)$, $\theta > 0$ ($\theta > x$)

which implies that

$$g(x) = \int_0^\infty f(x|\theta)\,\tau(\theta)\,d\theta = \int_x^\infty \frac{1}{\cancel{\theta}}\cdot \cancel{\theta} e^{-\theta}\,d\theta = e^{-x}, \quad x > 0.$$

Hence,

$$h(\theta|x) = \frac{f(x|\theta)\tau(\theta)}{g(x)} = \begin{cases} e^{x-\theta} = \frac{e^{-\theta}}{e^{-x}} & \text{if } \theta > x \\ 0 & \text{if } 0 < \theta < x \end{cases}$$

(i) wrt quadratic loss: the Bayesian estimator is

$$
\begin{aligned}
\delta_\tau(x) = E(\theta|X) &= \int_0^\infty \theta \cdot h(\theta|x)\,d\theta \\
&= \int_x^\infty \theta \cdot e^{x-\theta}\,d\theta \\
&= e^x \int_x^\infty \theta e^{-\theta}\,d\theta \quad \text{integration by parts} \\
&= e^x\left[x e^{-x} + e^{-x}\right] \\
&= x + 1
\end{aligned}
$$

e.g. $X = 1.3$ $\quad X \sim U(0, 2.3)$

$\delta_\tau(1.3) = 2.3$

(ii) Wrt absolute value loss: the Bayesian estimator m is:

$$m = \text{median}(h(\theta|x))$$

$$P(\theta > m | x) \overset{\text{def}}{=} \frac{1}{2} = \int_m^\infty h(\theta|x)\,d\theta = \int_m^\infty e^{x-\theta}\,d\theta$$

$$= -e^{x-\theta}\Big]_m^\infty$$

$$= -0 - (-e^{x-m})$$

$$= e^{x-m}$$

Solve $e^{x-m} = \frac{1}{2}$ or $x - m = \log(1/2)$ or $m = x + \log(2)$

$$= x - \log(1/2)$$

(iii)

Example 2.17

$X_1, X_2, \ldots, X_n$ iid $N(\mu, 1)$

$$f(x|\mu) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2} = (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2}$$

prior: $\tau(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\mu-\mu_0)^2}$ $\qquad \mu \sim N(\mu_0, 1)$

let $x_0 = \mu_0$

Posterior: $h(\mu|x) \propto f(x|\mu)\tau(\mu)$

$$= (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_0-\mu)^2}$$

$$\propto \exp\left(-\frac{1}{2}\sum_{i=0}^n (x_i-\mu)^2\right)$$

$$= \exp\left(-\frac{1}{2}\left[\sum_{i=0}^n x_i^2 - 2\mu\sum_{i=0}^n x_i + (n+1)\mu^2\right]\right)$$

add $\left(\frac{\sum_{i=0}^n x_i}{n+1}\right)^2$

$$\propto \exp\left(-\frac{n+1}{2}\left[\mu - \frac{1}{n+1}\sum_{i=0}^n x_i\right]^2\right)$$

$$\mu | X \sim N\left(\frac{1}{n}\sum_{i=0}^n x_i, \frac{1}{n+1}\right)$$

Exercise 2.12

(i) Posterior: $h(\lambda|N) \propto f(N|\lambda)\,\tau(\lambda)$

$$= \prod_{i=1}^{T} \frac{e^{-\lambda}\lambda^{n_i}}{n_i!} \times \frac{\lambda^{a-1} e^{-\lambda/b}}{\Gamma(a)\, b^a}$$

$$\propto e^{-\lambda T} \lambda^{\sum_{i=1}^{T} n_i} \cdot \lambda^{a-1} e^{-\lambda/b}$$

$$= \lambda^{a-1+\sum_{i=1}^{T} n_i} e^{-\lambda\left(T+\frac{1}{b}\right)}$$

$$\tilde{b} = \frac{1}{T+\frac{1}{b}} = \frac{b}{bT+1}$$

which is proportional to the $\text{Gamma}\left(a+\sum_{i=1}^{T} n_i, \frac{b}{bT+1}\right)$ mean $\tilde{a}\tilde{b}$

$$\hat{\lambda}_{bayes} = E(\lambda|N) = \frac{b}{bt+1}\left(a+\sum_{i=1}^{T} n_i\right)$$

(ii) For the given data $a=2$ and $b=2$, $T=6$, $\sum_{i=1}^{6} n_i = 12$.

Hence

$$h(\lambda|N) \sim \text{Gamma}\left(2+12=14, \frac{2}{2\times 6+1} = \frac{2}{13}\right)$$

The banks claim is tested under $H_0: \lambda \le 2$ vs $H_1: \lambda > 2$.

Then the Bayesian test with 0-1 loss has form:

$$\varphi^* = \begin{cases} 1 & \text{i.e. reject } H_0 \text{ if } P(\lambda<2|N) < 1/2 \\ 0 & \text{i.e. accept } H_0 \text{ if } P(\lambda \ge 2|N) \ge 1/2 . \end{cases}$$

Here $P(\lambda \le 2|N) = \int_0^2 \frac{1}{\Gamma(14)(2/13)^{14}} \lambda^{13} e^{-13\lambda/2}\, d\lambda = 0.427 < 1/2$

"pgamma(2, 14, 2/13)"

Hence we reject H_0 (the banks claim!)