University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2021

Assignment One

Given: Friday 26 February 2021 **Due date: Friday 12 March 2021**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Friday, 12 March 2021. The first page of this PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document. There are four problems each of different weight.

I/We declare that this assessment item is my/our own work, except where acknowledged, and has not been submitted for academic credit elsewhere. I/We acknowledge that the assessor of this item may, for the purpose of assessing this item reproduce this assessment item and provide a copy to another member of the University; and/or communicate a copy of this assessment item to a plagiarism checking service (which may then retain a copy of the assessment item on its database for the purpose of future plagiarism checking). I/We certify that I/We have read and understood the University Rules in respect of Student Academic Misconduct.

Name	Student No.	Signature	Date

Problem One

During the cold winter months, the principle of the Winterville School District must decide whether to call off the next day's school due to the snow conditions. If the principle fails to call off school and there is snow, there are serious concerns for children and teachers not showing up for school and high rates of accidents. However, if the principle does call off school and then it does snow, the students undertake online classes and nothing is lost. On the other hand, if the principle does call of school and it does not snow, the students must make-up the lost day later in the year since it's fine to go outside and students are unlikely to be able to focus in the online class.

The principle decides that the costs of failing to close the school when there is snow is twice the costs of closing the school when there is no snow. Therefore, the principle assigns two units of loss to the first outcome and one to the second. If the principle closes the school when there is snow or keeps it open when there is no snow, then no loss is incurred.

There are two local experts that provide the principle with independent and identically distributed weather forecasts. If there is snow, each expert will independently forecast snow with probability 3/4 and no snow with probability 1/4. However, if there is to be no snow then each expert independently predicts snow with probability 1/2. The principle will listen to both forecasts and then make his decisions based on the data X which is the number of experts forecasting snow.

a) There are two possible actions in the action space $\mathcal{A} = \{a_0, a_1\}$ where action a_0 is to keep the school open and action a_1 is to close the school. There are two states of nature $\Theta = \{\theta_0, \theta_1\}$ where $\theta_0 = 0$ represents "no snow" and $\theta_1 = 1$ represents "snow". Define the appropriate loss function $L(\theta, a)$ for this problem.

b) Compute the probability mass function (pmf) for X under both states of nature.

c) The complete list of all the non-randomized decisions rules D based on x is given by:

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$x = 0$	a_0	a_1	a_0	a_1	a_0	a_1	a_0	a_1
$x = 1$	a_0	a_0	a_1	a_1	a_0	a_0	a_1	a_1
$x = 2$	a_0	a_0	a_0	a_0	a_1	a_1	a_1	a_1

For the set of non-randomized decision rules D compute the corresponding risk points.

d) Find the minimax rule(s) amongst the **non-randomized** rules in D.

e) Sketch the risk set of all **randomized** rules $\mathcal{D}$ generated by the set of rules in D. . You might want to use `R` (or your favorite programming language) to make this sketch more precise.

f) Suppose there are two decisions rules d and d'. The decision d strictly dominates d' if $R(\theta, d) \leq R(\theta, d')$ for all values of θ and $R(\theta, d) < (\theta, d')$ for at least one value θ. Hence, given a choice between d and d' we would always prefer to use d. Any decision rules which is strictly dominated by another decisions rule (as d' is in the above) is said to be inadmissible. Correspondingly, if a decision rule d is not strictly dominated by any other decision rule then it is admissible. Show on the risk plot the set of randomized decisions rules that correspond to the princple's admissible decision rules.

g) Find the risk point of the minimax rule in set of randomized decision rules $\mathcal{D}$ and determine its minimax risk. Compare the minmax risk from the minimax decision rule in D and $\mathcal{D}$.

h) Define the minimax rule in the set $\mathcal{D}$ in terms of rules in D.

i) For which prior on $\{\theta_1, \theta_2\}$ is the minimax rule in the set $\mathcal{D}$ also a Bayes rule?

j) Prior to listening to the forecasts, the principle believes there will be snow with probability 1/2. Find the Bayes rule and the Bayes risk with respect to this prior.

k) For a small positive $\epsilon = 0.1$, illustrate on the risk set the risk points of all rules which are ϵ-minimax. Write down all the vertices that define the region.

Solution:

a) There are two possible actions: a_0 : keep school open and a_1 : close school. There are two state of nature: $\theta = 0$, "no snow" and $\theta = 1$ "snow". Let x be the number of radio stations predicting snow. The loss function is given by

$$L(\theta_1, a_0) = 2 \qquad L(\theta_1, a_1) = 0 \qquad L(\theta_0, a_0) = 0 \qquad L(\theta_0, a_1) = 1$$

b) The pmf for both state of natures:

x	$p(x\|\theta = 0)$
0	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
1	$2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{2}{4}$
2	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

x	$p(x\|\theta = 1)$
0	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$
1	$2 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{6}{16}$
2	$\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

c) Calculation of the risk points $\big\{R(\theta_0, d_i), R(\theta_1, d_i)\big\}$ for $i = 1, 2, \ldots, 8$ are as follows:

– **For** $d_1 = (0, 2)$**:**

$$\begin{aligned} R(\theta_0, d_1) &= L(\theta_0, a_0)P(x = 0|\theta = 0) + L(\theta_0, a_0)P(x = 1|\theta = 0) + L(\theta_0, a_0)P(x = 2|\theta = 0) \\ &= 0 \times \frac{1}{4} + 0 \times \frac{2}{4} + 0 \times \frac{1}{4} \\ &= 0 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_1) &= L(\theta_1, a_0)P(x = 0|\theta = 1) + L(\theta_1, a_0)P(x = 1|\theta = 1) + L(\theta_1, a_0)P(x = 2|\theta = 1) \\ &= 2 \times \frac{1}{16} + 2 \times \frac{6}{16} + 2 \times \frac{9}{16} \\ &= 2 \end{aligned}$$

– **For** $d_2 = (1/4, 30/16)$**:**

$$\begin{aligned} R(\theta_0, d_2) &= L(\theta_0, a_1)P(x=0|\theta=0) + L(\theta_0, a_0)P(x=1|\theta=0) + L(\theta_0, a_0)P(x=2|\theta=0) \\ &= 1 \times \frac{1}{4} + 0 \times \frac{2}{4} + 0 \times \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_2) &= L(\theta_1, a_1)P(x=0|\theta=1) + L(\theta_1, a_0)P(x=1|\theta=1) + L(\theta_1, a_0)P(x=2|\theta=1) \\ &= 0 \times \frac{1}{16} + 2 \times \frac{6}{16} + 2 \times \frac{9}{16} \\ &= \frac{30}{16} \end{aligned}$$

– **For** $d_3 = (2/4, 20/16)$**:**

$$\begin{aligned} R(\theta_0, d_3) &= L(\theta_0, a_0)P(x=0|\theta=0) + L(\theta_0, a_1)P(x=1|\theta=0) + L(\theta_0, a_0)P(x=2|\theta=0) \\ &= 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 0 \times \frac{1}{4} \\ &= \frac{2}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_3) &= L(\theta_1, a_0)P(x=0|\theta=1) + L(\theta_1, a_1)P(x=1|\theta=1) + L(\theta_1, a_0)P(x=2|\theta=1) \\ &= 2 \times \frac{1}{16} + 0 \times \frac{6}{16} + 2 \times \frac{9}{16} \\ &= \frac{20}{16} \end{aligned}$$

– **For** $d_4 = (3/4, 18/16)$**:**

$$\begin{aligned} R(\theta_0, d_4) &= L(\theta_0, a_1)P(x=0|\theta=0) + L(\theta_0, a_1)P(x=1|\theta=0) + L(\theta_0, a_0)P(x=2|\theta=0) \\ &= 1 \times \frac{1}{4} + 1 \times \frac{2}{4} + 0 \times \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_4) &= L(\theta_1, a_1)P(x=0|\theta=1) + L(\theta_1, a_1)P(x=1|\theta=1) + L(\theta_1, a_0)P(x=2|\theta=1) \\ &= 0 \times \frac{1}{16} + 0 \times \frac{6}{16} + 2 \times \frac{9}{16} \\ &= \frac{18}{16} \end{aligned}$$

– **For** $d_5 = (1/4, 14/16)$**:**

$$\begin{aligned} R(\theta_0, d_5) &= L(\theta_0, a_0)P(x=0|\theta=0) + L(\theta_0, a_0)P(x=1|\theta=0) + L(\theta_0, a_1)P(x=2|\theta=0) \\ &= 0 \times \frac{1}{4} + 0 \times \frac{2}{4} + 1 \times \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_5) &= L(\theta_1, a_0)P(x=0|\theta=1) + L(\theta_1, a_0)P(x=1|\theta=1) + L(\theta_1, a_1)P(x=2|\theta=1) \\ &= 2 \times \frac{1}{16} + 2 \times \frac{6}{16} + 0 \times \frac{9}{16} \\ &= \frac{14}{16} \end{aligned}$$

- **For** $d_6 = (2/4, 12/16)$**:**

$$\begin{aligned} R(\theta_0, d_6) &= L(\theta_0, a_1)P(x=0|\theta=0) + L(\theta_0, a_0)P(x=1|\theta=0) + L(\theta_0, a_1)P(x=2|\theta=0) \\ &= 1 \times \frac{1}{4} + 0 \times \frac{2}{4} + 1 \times \frac{1}{4} \\ &= \frac{2}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_6) &= L(\theta_1, a_1)P(x=0|\theta=1) + L(\theta_1, a_0)P(x=1|\theta=1) + L(\theta_1, a_1)P(x=2|\theta=1) \\ &= 0 \times \frac{1}{16} + 2 \times \frac{6}{16} + 0 \times \frac{9}{16} \\ &= \frac{12}{16} \end{aligned}$$

- **For** $d_7 = (3/4, 2/16)$**:**

$$\begin{aligned} R(\theta_0, d_7) &= L(\theta_0, a_0)P(x=0|\theta=0) + L(\theta_0, a_1)P(x=1|\theta=0) + L(\theta_0, a_1)P(x=2|\theta=0) \\ &= 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 1 \times \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_7) &= L(\theta_1, a_0)P(x=0|\theta=1) + L(\theta_1, a_1)P(x=1|\theta=1) + L(\theta_1, a_1)P(x=2|\theta=1) \\ &= 2 \times \frac{1}{16} + 0 \times \frac{6}{16} + 0 \times \frac{9}{16} \\ &= \frac{2}{16} \end{aligned}$$

- **For** $d_8 = (1, 0)$**:**

$$\begin{aligned} R(\theta_0, d_8) &= L(\theta_0, a_1)P(x=0|\theta=0) + L(\theta_0, a_1)P(x=1|\theta=0) + L(\theta_0, a_1)P(x=2|\theta=0) \\ &= 1 \times \frac{1}{4} + 1 \times \frac{2}{4} + 1 \times \frac{1}{4} \\ &= 1 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_8) &= L(\theta_1, a_1)P(x=0|\theta=1) + L(\theta_1, a_1)P(x=1|\theta=1) + L(\theta_1, a_1)P(x=2|\theta=1) \\ &= 0 \times \frac{1}{16} + 0 \times \frac{6}{16} + 0 \times \frac{9}{16} \\ &= 0 \end{aligned}$$

This leads to the following set of risk points in D:

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$R(\theta=0,d)$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
$R(\theta=1,d)$	2	$\frac{30}{16}$	$\frac{20}{16}$	$\frac{18}{16}$	$\frac{14}{16}$	$\frac{12}{16}$	$\frac{2}{16}$	0

d) For each non-randomized decision rule we have

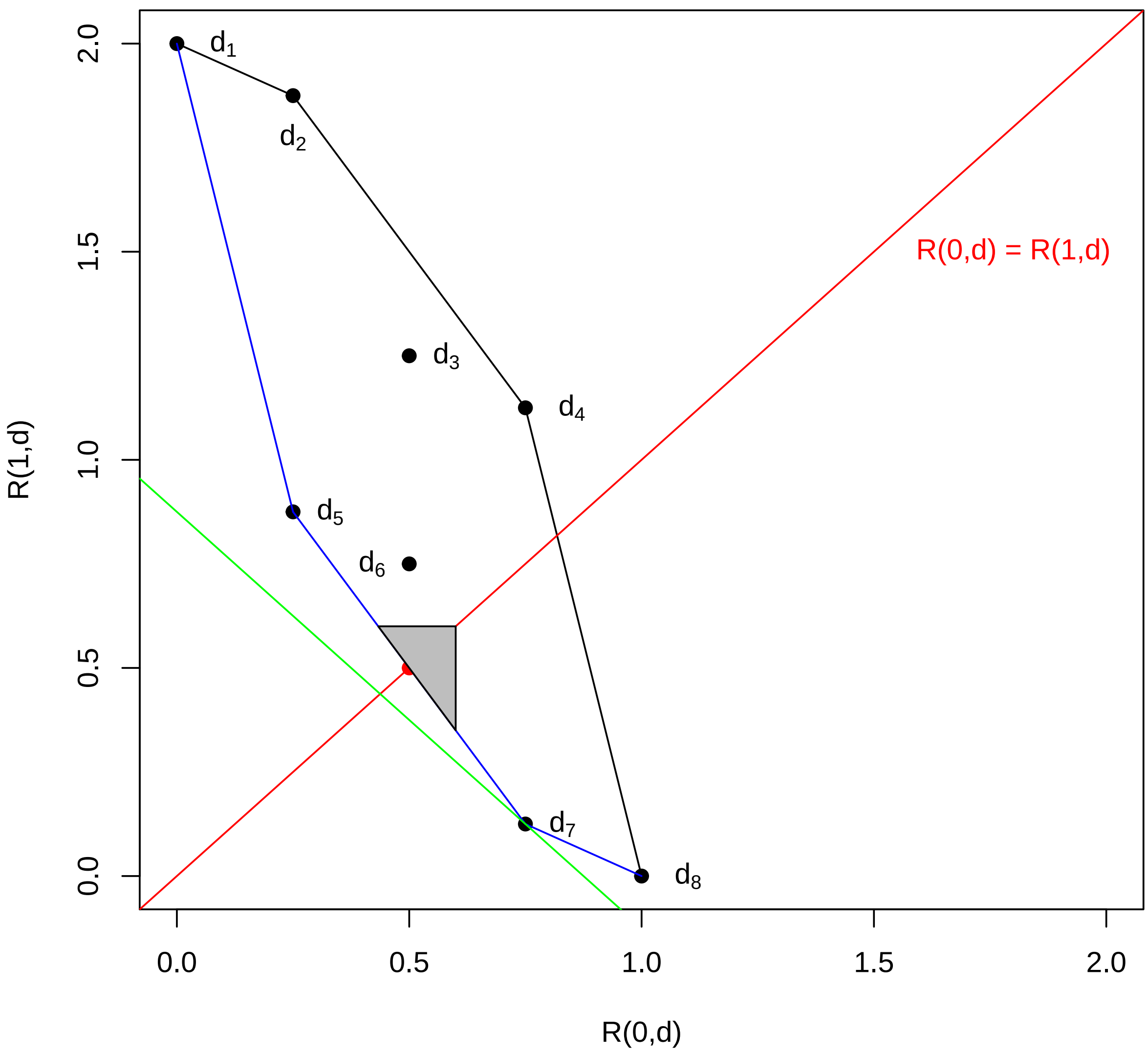
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$\sup\limits_{\theta\in\{\theta_0,\theta_1\}} R(\theta,d_i)$	2	$\frac{30}{16}$	$\frac{20}{16}$	$\frac{18}{16}$	$\frac{14}{16}$	$\frac{12}{16}$	$\frac{3}{4}$	1

Hence,

$$\inf_{d\in D}\sup_{\theta\in\{\theta_0,\theta_1\}} R(\theta,d)=\{d_6,d_7\}$$

which implies that d_6 and d_7 must be the minimax decisions in the set D with a minimax risk of 0.75.

e) Sketch of the randomized rules $\mathcal{D}$ generated by the set of non-randomized decisions rules D:



f) The principles admissible decisions rules are those rules on the "south-west boundary" of the risk set: any convex combination of d_1 and d_5, or d_5 and d_7, or d_7 and d_8. The randomized decisions rules that correspond to admissible rules are colored in blue.

g) The minimax rule is obtained by finding the intersection of the line $y = x$ and the line $\overline{d_5 d_7}$ with points $d_5 = (0.25, 0.875) = (x_0, y_0)$ and $d_7 = (0.75, 0.125) = (x_1, y_1)$. We want to solve the following

$$y = x \qquad \text{and} \qquad y - 0.875 = \frac{0.125 - 0.875}{0.75 - 0.25}(x - 0.25)$$

This is equivalent to

$$x - 0.875 = -1.5(x - 0.25).$$

Hence

$$2.5x = 0.875 + 1.5 \times 0.25$$

and so

$$x = \frac{0.875 + 1.5 \times 0.25}{2.5} = 0.5$$

Therefore the risk point δ_M of the minimax rule in $\mathcal{D}$ is $(0.5, 0.5)$ with a minimax risk of 0.5. This is less than the minimax risk in the set of non-randomised decision rules D with 0.75, as expected, since $D \subset \mathcal{D}$.

h) To express the rule δ_M in terms of d_5 and d_7 we need too find $\alpha \in [0, 1]$ such that

$$0.5 = 0.25\alpha + 0.75(1 - \alpha) \qquad \text{and} \qquad 0.5 = 0.875\alpha + 0.125(1 - \alpha).$$

Looking at the first equation gives (you can solve equation since they lead to the identical solutions):

$$0.5\alpha = 0.25 \qquad \text{therefore} \qquad \alpha = 0.5.$$

Hence the randomized decision rule δ_M is to choose d_5 with probability 0.5 and choose d_7 with probability 0.5 also.

i) Suppose the prior is $(p, 1-p)$ this leads to a line with a normal vector $(p, 1-p)$, that is, a slope with $\frac{-p}{1-p}$ and this slope should coincide with the slop of $\overline{d_5 d_7}$. Hence

$$\frac{-p}{1-p} = \frac{0.125 - 0.875}{0.75 - 0.25} = -\frac{3}{2}$$

should hold. Solving this leads to $p = \frac{3}{5}$ and the least favorable prior with respect to which δ_M is Bayes is $(\frac{3}{5}, \frac{2}{5})$ on (θ_1, θ_2).

j) The line with normal vector $(\frac{1}{2}, \frac{1}{2})$ has slope $\frac{-1/2}{1/2} = -1$. When moving such a line "south-west" as much as possible but retaining intersection with the risk set, we end up with the point d_7 as the decision that corresponds to the Bayes rule.

It's Bayes risk is:

$$\frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times \frac{2}{16} = 0.4375$$

k) See shaded region on risk plot. The risk points are defined by the region with vertices $(0.6, 0.6)$, $(0.6, 0.35)$ and $(0.433, 0.6)$ by using the line $y - 0.875 = -1.5(x - 0.25)$. For instance, set $x = 0.5 + 0.1 = 0.6$ and find y:

$$y - 0.875 = -1.5(0.6 - 0.25)$$

to get $y = 0.35$ and the vertex (0.6, 0.35). Also, set $y = 0.6$ and solve for x:

$$0.6 - 0.875 = -1.5(x - 0.25)$$

to get $x = 0.4333$ and the vertex (0.4333, 0.6).

Problem Two

Suppose that X is a binomial random variable with parameters n and θ where the number of trials n is assumed to be known. Calculate the Bayes rule (based on a single observation of X) for estimating θ when the prior distribution is the uniform distribution on $[0,1]$ and the loss function is given by

$$L(\theta, d) = \frac{1}{\theta(1-\theta)}(\theta - d)^2.$$

Show that the rule you obtained is minimax.

Hint: The Beta function is given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

and the following relation holds:

$$\frac{B(x+1,y)}{B(x,y)} = \frac{x}{x+y}.$$

Solution: The posterior density of θ given $X = x$ is

$$\pi(\theta|x) \propto \theta^x(1-\theta)^{n-x}.$$

The Bayes rule is obtained analytically by minimizing the expected posterior loss (ignoring the normalizing constant):

$$\begin{aligned} f(d) = \mathbb{E}_\theta\big[L(\theta,d)\big] &= \int_\Theta L(\theta,d)\pi(x|\theta)\mathrm{d}\theta \\ &\propto \int_0^1 \frac{1}{\theta(1-\theta)}(\theta-d)^2\theta^x(1-\theta)^{n-x}\mathrm{d}\theta \\ &= \int_0^1 (\theta-d)^2\theta^{x-1}(1-\theta)^{n-x-1}\mathrm{d}\theta \end{aligned}$$

By differentiating with respect to d:

$$\frac{\mathrm{d}}{\mathrm{d}d}f(d) = -2\int_0^1(\theta-d)\theta^{x-1}(1-\theta)^{n-x-1}\mathrm{d}\theta = 0$$

which gives

$$d\int_0^1 \theta^{x-1}(1-\theta)^{n-x-1}\mathrm{d}\theta = \int_0^1 \theta^x(1-\theta)^{n-x-1}\mathrm{d}$$

or

$$d\mathrm{Beta}(x, n-x) = \mathrm{Beta}(x+1, n-x)$$

Hence

$$d = \frac{\mathrm{Beta}(x+1,n-x)}{\mathrm{Beta}(x,n-x)} = \frac{x}{x+n-x} = \frac{x}{n}$$

The Bayes rule is $d(X) = X/n$. By directly calculating the risk:

$$
\begin{aligned}
R(\theta, d) &= \frac{1}{\theta(1-\theta)}\mathbb{E}\left[\left(\theta - \frac{X}{n}\right)^2\right] \\
&= \frac{1}{n^2\theta(1-\theta)}\mathbb{E}\big[(n\theta - X)^2\big] \\
&= \frac{1}{n^2\theta(1-\theta)}\mathbb{E}\big[(\mathbb{E}(X) - X)^2\big] \\
&= \frac{1}{n^2\theta(1-\theta)}\mathbb{V}\mathrm{ar}(X) \\
&= \frac{1}{n^2\theta(1-\theta)}n\theta(1-\theta) \\
&= \frac{1}{n}
\end{aligned}
$$

which shows that the Bayes risk is constant with respect too θ and hence the rule is also a minimax decision rule.

Problem Three
Suppose $X = (X_1, \dots, X_n)$ are i.i.d. Exponential(θ) with density

$$f(x, \theta) = \theta e^{-\theta x}, \qquad x > 0, \quad \theta > 0$$

and let θ have a Gamma(α, β) prior distribution with density

$$\tau(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\alpha-1} e^{-\theta/\beta}, \qquad \alpha, \beta > 0, \quad \theta > 0.$$

a) Find the posterior distribution for $h(\theta|X)$. Use the notation $s = \sum_{i=1}^n X_i$.

b) Hence or otherwise determine the Bayes estimator of θ with respect to the quadratic loss function $L(a, \theta) = (a - \theta)^2$.

c) Suppose the following ten observations were observed:

$$0.12,\ 0.28,\ 0.43,\ 0.34,\ 0.47,\ 0.67,\ 0.82,\ 0.12,\ 0.30,\ 0.45$$

Two actions a_0 (accept H_0) and a_1 (reject H_0) are possible and the losses when using these actions are given by:

$$L(\theta, a_0) = \begin{cases} 0 & \text{if} \quad \theta \in \Theta_0 \\ 2 & \text{if} \quad \theta \in \Theta_1 \end{cases} \qquad \text{and} \qquad L(\theta, a_1) = \begin{cases} 1 & \text{if} \quad \theta \in \Theta_0 \\ 0 & \text{if} \quad \theta \in \Theta_1 \end{cases}$$

Using the loss function above and the parameters $\alpha = 2$ and $\beta = 1$ for the prior, what is your decision when testing $H_0 : \theta \leq 2.5$ versus $H_1 : \theta > 2.5$. You may use the `pgamma` function in `R` or another numerical integration routine from your favourite programming package to answer this question.

Solution:

a) The posterior is proportional to the prior times the likelihood:

$$\begin{aligned} p(\theta|X) &\propto L(X|\theta)\tau(\theta) \\ &= \theta^n \exp(-\theta \sum_{i=1}^n X_i) \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\alpha-1} \exp(-\theta/\beta) \\ &\propto \theta^{n+\alpha-1} \exp\Big(-\theta\big(s + \frac{1}{\beta}\big)\Big) \\ &\propto \theta^{n+\alpha-1} \exp\Big(-\theta\big(\frac{s\beta+1}{\beta}\big)\Big) \end{aligned}$$

This can be identified as a gamma($\tilde{\alpha}, \tilde{\beta}$) density with parameters

$$\tilde{\alpha} = n + \alpha \qquad \text{and} \qquad \tilde{\beta} = \frac{\beta}{s\beta + 1}.$$

b) The Bayes estimator with respect to quadratic loss is given by the posterior mean and hence

$$\hat{\theta}_{\text{bayes}} = \mathbb{E}(\theta|X) = \tilde{\alpha}\tilde{\beta} = \frac{\beta(n+\alpha)}{s\beta+1}$$

c) The structure of the test is as follows:

$$\varphi = \begin{cases} 1 & \text{if} \quad P(\theta < 2.5|X) < 2/3 = 0.667 \\ 0 & \text{if} \quad P(\theta < 2.5|X) \geq 2/3 = 0.667 \end{cases}$$

Here we observe that $n = 10$, $s = \sum_{i=1}^{10} x_i = 4$ the posterior distribution is then:

$$\theta|X \sim \text{Gamma}\left(10+2, \frac{1}{4 \times 1 + 1}\right) = \text{Gamma}(12, 0.2)$$

Then using `R` we can compute the posterior probability under these conditions as :

$$P(\theta < 2.5|X) = \int_0^{2.5} \frac{1}{\Gamma(12)0.2^{12}} \theta^{11} e^{-5\theta} \mathrm{d}\theta \approx 0.5942393$$

Hence, since the posterior probability is less than 0.667 we reject H_0.

Problem Four
Determine the form of the Bayes decision rule in an estimation problem with a one-dimensional parameter $\theta \in \mathbb{R}^1$ and loss function

$$L(\theta, d) = \begin{cases} \alpha(\theta - d) & \text{if} \quad d \leq \theta, \\ \beta(d - \theta) & \text{if} \quad d > \theta, \end{cases}$$

where α and β are known positive constants.

Solution: [5 marks] We need to minimize the following

$$\begin{aligned} Q(X,d) &= \int_\theta L(\theta,d)h(\theta \mid X)\mathrm{d}\theta \\ &= \int_{-\infty}^{d} b(d-\theta)h(\theta \mid X)\mathrm{d}\theta + \int_d^\infty a(\theta - d)h(\theta \mid X)\mathrm{d}\theta \\ &= bd\int_{-\infty}^{d} h(\theta \mid X)\mathrm{d}\theta - b\int_{-\infty}^{d} \theta h(\theta \mid X)\mathrm{d}\theta + a\int_d^\infty \theta h(\theta \mid X)\mathrm{d}\theta - ad\int_d^\infty h(\theta \mid X)\mathrm{d}\theta \end{aligned}$$

Then by taking the derivative with respect to d we obtain

$$\begin{aligned} \frac{\partial}{\partial d}Q(X,d) &= b\int_{-\infty}^{d} h(\theta \mid X)\mathrm{d}\theta + bdh(d \mid X) - adh(d \mid X) - bdh(d \mid X) - a\int_d^\infty h(\theta \mid X)\mathrm{d}\theta + adh(d \mid X) \\ &= b\int_{-\infty}^{d} h(\theta \mid X)\mathrm{d}\theta - a\int_d^\infty h(\theta \mid X)\mathrm{d}\theta \end{aligned}$$

Then set this equal to zero

$$\begin{aligned} 0 &= b\int_{-\infty}^{d^*} h(\theta \mid X)\mathrm{d}\theta - a\int_{d^*}^\infty h(\theta \mid X)\mathrm{d}\theta \\ 0 &= bP(\Theta \leq d^* \mid X = x) - aP(\Theta > d^* \mid X = x) \\ 0 &= bP(\Theta \leq d^* \mid X = x) - a(1 - P(\Theta \leq d^* \mid X = x)) \end{aligned}$$

For which we obtain

$$P(\Theta \leq d^* \mid X = x) = \frac{a}{a+b}$$

which is simply the $\frac{a}{a+b}$ quantile of the posterior distribution. The indeed, delivers a minimum since

$$\frac{\partial^2}{\partial d^2}Q(X,d^*) = (b+a)h(d^* \mid X) > 0$$

since $a, b > 0$ and the posterior evaluated at d^* is also positive.