University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2021

Assignment One

Given: Friday 26 February 2021 **Due date: Friday 12 March 2021**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Friday, 12 March 2021. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document.

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Problem 1

a)

$$\mathcal{A} = \{a_0, a_1\}$$

$$a_0: keep\ school\ open$$

$$a_1: close\ the\ school$$

The loss function $L(\theta, \alpha)$

$$L(\theta, a_0) = \begin{cases} 0 & if \quad \theta = \theta_0 \\ 2 & if \quad \theta = \theta_1 \end{cases}, \qquad L(\theta, a_1) = \begin{cases} 1 & if \quad \theta = \theta_0 \\ 0 & if \quad \theta = \theta_1 \end{cases}$$

b)

X: the number of experts forecasting snow.

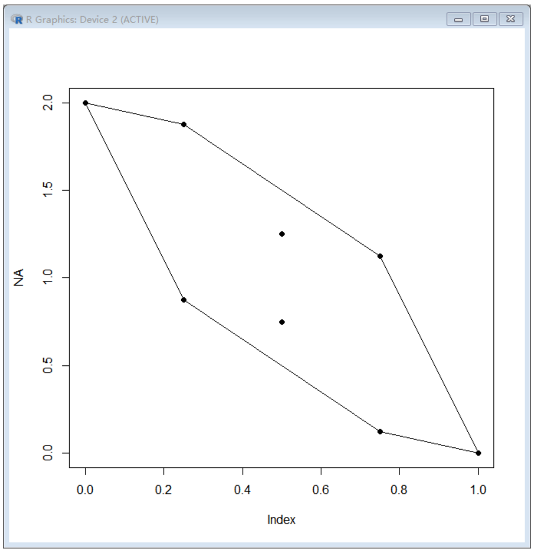
the probability mass function (pmf) for X is :

X	0	1	2
$P(X\|\theta_0)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

X	0	1	2
$P(X\|\theta_1)$	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$

c)

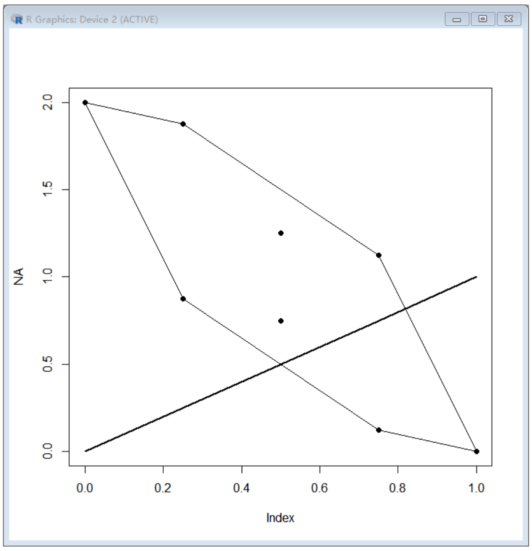
Rule	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$R(\theta_0, d_i)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$R(\theta_1, d_i)$	2	$\frac{15}{8}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{8}$	0
Max	2	$\frac{15}{8}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	1



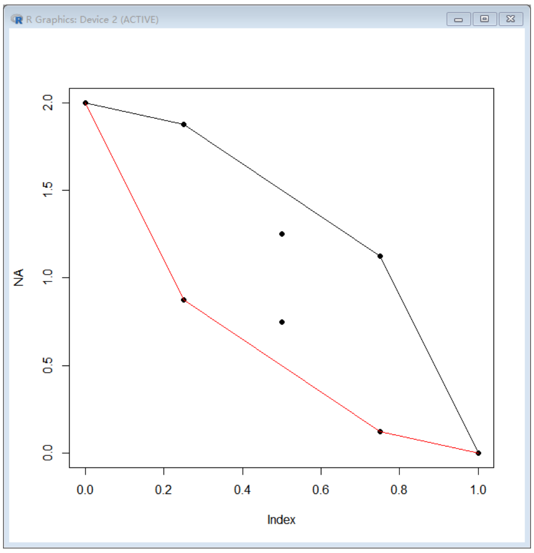
d) As shown in the above chart, the minimum of these maxima is 0.75.

Hence, d_6 and d_7 are the minimax decision rules with minimax risk.

e)



f) the risk plot the set of randomized decisions rules that correspond to the principle's admissible decision rules.



g) The minimax rule in the set $\mathcal{D}$ of randomize decision rules is obtained by examining the intersection of the line y = x with the "most south-west" part of the convex risk set. And the solution x = y = 0.5

The value of the minimax risk is given by sup{0.5, 0.5} = 0.5

The minimax decision rule in D(0.75) is greater than that in $\mathcal{D}(0.5)$.

h) $\alpha \cdot \frac{1}{4} + (1-\alpha) \cdot \frac{3}{4} = 0.5 \qquad \alpha = 0.5$

choose d_5 with probability 0.5

choose d_7 with probability 0.5

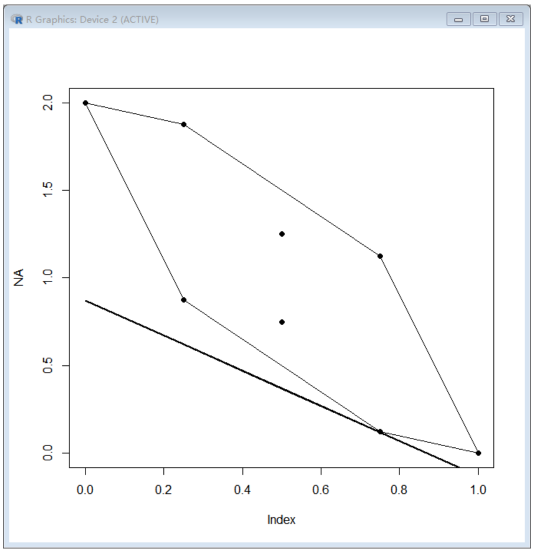
i) $\theta_1 x + \theta_2 y = c, y = -\frac{\theta_1}{\theta_2} x + \frac{c}{\theta_2}$

Given the points $\left(\frac{1}{4}, \frac{7}{8}\right)$ and $\left(\frac{3}{4}, \frac{1}{8}\right)$, solving this give

$$\theta_1 = 0.6, \qquad \theta_2 = 0.4$$

j) The line 0.5x + 0.5y = c. The slope of the lines is -1.

Hence, to find the Bayes rule with respect to this prior, we need to move lines with a slope (-1) "most south-west" while still maintaining

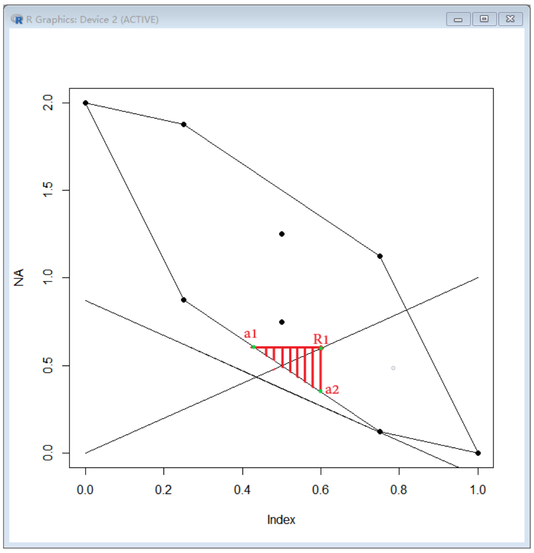


an intersection with the risk set.

$$y = -x + \frac{7}{8}$$

And the Bayes risk is equal to $\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{8} = \frac{7}{16}$

k) The intersection region is (Graph below)



all the vertices that define the region:

$$a_1(\frac{13}{30}, \frac{3}{5})$$

$$a_2(\frac{3}{5}, \frac{7}{20})$$

$$R_1(\frac{3}{5}, \frac{3}{5})$$

Problem 2

Prior Distribution is the Uniform Distribution on $[0,1]$, *thus*

$\tau(\theta) = 1$

Recall that X *is a binomial random variable with* n *and* θ,

for a single observation x *of* X, *we have*

$f(x|\theta) = \theta^x (1-\theta)^{n-x}$

Loss funcion $L(\theta, d) = \dfrac{1}{\theta(1-\theta)}(\theta - d)^2$

According to Theorem 2.7, *we have* :

$$Q(x,d) = \int_\Theta L(\theta,d)\, h(\theta|x)\, d\theta = \int_0^1 \frac{1}{\theta(1-\theta)}(\theta-d)^2 h(\theta|x)\, d\theta$$

$$= \int_0^1 \frac{1}{\theta(1-\theta)}\left(\theta^2 - 2\theta d + d^2\right) h(\theta|x)\, d\theta$$

$$= \int_0^1 (1-\theta)^{-1}\theta h(\theta|x)\, d\theta - \int_0^1 (1-\theta)^{-1} 2dh(\theta|x)\, d\theta + \int_0^1 \theta^{-1}(1-\theta)^{-1} d^2 h(\theta|x)\, d\theta$$

$$h(\theta|x) = \frac{f(X|\theta)\,\tau(\theta)}{g(x)}, \textit{where } g(x) = \int_\Theta f(X|\theta)\,\tau(\theta)\, d\theta \textit{ is a constant}$$

$$i.e.\ \ h(\theta|x) \propto f(X|\theta)\,\tau(\theta) = \theta^x(1-\theta)^{n-x} \sim Beta(x+1, n-x+1)$$

Recall that

$$Q(x,d) = \int_0^1 (1-\theta)^{-1}\theta h(\theta|x)\, d\theta - \int_0^1 (1-\theta)^{-1} 2dh(\theta|x)\, d\theta + \int_0^1 \theta^{-1}(1-\theta)^{-1} d^2 h(\theta|x)\, d\theta$$

$$Q(x,d) = \frac{1}{g(x)}\left[\int_0^1 (1-\theta)^{-1}\theta\theta^x(1-\theta)^{n-x}\, d\theta - \int_0^1 (1-\theta)^{-1} 2d\theta^x(1-\theta)^{n-x}\, d\theta + \int_0^1 \theta^{-1}(1-\theta)^{-1} d^2\theta^x(1-\theta)^{n-x}\, d\theta\right]$$

$$= \frac{1}{g(x)}\Big[\ \underline{\int_0^1 \theta^{x+1}(1-\theta)^{n-x-1}\, d\theta} \qquad (1)$$

$$\underline{-\,2d\int_0^1 \theta^{x}(1-\theta)^{n-x-1}\, d\theta} \qquad (2)$$

$$\underline{+d^2\int_0^1 \theta^{x-1}(1-\theta)^{n-x-1}\, d\theta} \qquad (3)\ \ \Big]$$

$$\frac{\partial\left(Q\left(x,d\right)\right)}{\partial d}=\frac{1}{g(x)}\left[\frac{\partial(1)}{\partial d}-\frac{\partial(2)}{\partial d}+\frac{\partial(3)}{\partial d}\right],$$

$(1)=B\left(x+2,n-x\right)$
$(2)=2dB(x+1,n-x)$

$(3)=d^2B(x,n-x)$
$Thus\ \frac{\partial(1)}{\partial d}=0,\ \ \frac{\partial(2)}{\partial d}=2B(x+1,n-x),\ \ \frac{\partial(3)}{\partial d}=2dB(x,n-x)$

$$Set\ \frac{\partial\left(Q\left(x,d\right)\right)}{\partial d}=0,\ \ then\ 2dB(x,\ n-x)-2B(x+1,\ n-x)=0,$$

$$d=\frac{B(x+1,\ n-x)}{B(x,\ n-x)}=\frac{x}{x+n-x}=\frac{x}{n}$$

$\frac{x}{n}$ *is the Bayes rule for estimating* θ.

Then show d is minimax.

*According to the lemma in page*128, *Week*2 *Lecture, this can be done by prove*

that the risk of the Bayes estimator $\left(\frac{x}{n}\right)$ *does not depend on* θ.

Problem 3

a)

$$h(\theta|x) \propto L(x|\theta)\tau(\theta)$$

We only consider the part which is relevant to θ, so remove the part which is not relevant to θ from $L(x|\theta)\ and\ \tau(\theta)$.

$$L(x|\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

$$= \prod_{i=1}^{n} \theta e^{-\theta X_i} = \theta e^{-\theta X_1} \cdot \theta e^{-\theta X_2} \cdot \theta e^{-\theta X_2} \cdot \ldots$$

$$L(x|\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} X_i}$$

$$L(x|\theta) \cdot \tau(\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} X_i} \cdot \theta^{\alpha-1} \cdot e^{\frac{-\theta}{\beta}}$$

$$= \theta^{n+\alpha-1} \cdot e^{-\theta(\sum_{i=1}^{n} X_i + \frac{1}{\beta})}$$

since

$$s = \sum_{i=1}^{n} X_i$$

then

$$L(x|\theta) \cdot \tau(\theta) = \theta^{n+\alpha-1} \cdot e^{-\theta(s+\frac{1}{\beta})}$$

$$L(x|\theta) \cdot \tau(\theta) \sim Gamma(n+\alpha, s+\frac{1}{\beta})$$

so

$$h(\theta|x) \sim Gamma(n+\alpha, s+\frac{1}{\beta})$$

b)

For squared error loss,

$$\delta_\tau(x) = E(\theta|x)$$

$$shape = n + \alpha, rate = s + \frac{1}{\beta}$$

$$E(\theta|x) = \frac{shape}{rate} = \frac{n+\alpha}{s + \frac{1}{\beta}} = \frac{n+\alpha}{\sum_{i=1}^{n} X_i + \frac{1}{\beta}}$$

$$\hat{\theta}_{Bayes} = E(\theta|x) = \frac{n+\alpha}{\sum_{i=1}^{n} X_i + \frac{1}{\beta}}$$

c)

$$L(\theta, a_0) = \begin{cases} 0 \ \ if \ \ \theta \in \Theta_0 \\ 2 = C_2 \ \ if \ \ \theta \in \Theta_1 \end{cases}, \qquad L(\theta, a_1) = \begin{cases} 1 = C_1 \ \ if \ \ \theta \in \Theta_0 \\ 0 \ \ if \ \ \theta \in \Theta_1 \end{cases}$$

$$\alpha = 2, \beta = 1$$

the number of observations(X_i): $n = 10$

$$\sum_{i=1}^{n} X_i = 4, so\ s = 4$$

$$Gamma\left(n + \alpha, s + \frac{1}{\beta}\right) = Gamma(12{,}5)$$

with generalized 0-1 loss function, then the test:

$$\varphi^* = \begin{cases} Reject\ \mathrm{H}_0 \ \ if \ \ P(\theta \in \Theta_0|X) < \frac{2}{3} = (\frac{C_2}{C_1 + C_2}) \\ Accept\ \mathrm{H}_0 \ \ if \ \ P(\theta \in \Theta_0|X) > \frac{2}{3} = (\frac{C_2}{C_1 + C_2}) \end{cases}$$

since

$$\mathrm{H}_0 \colon \theta \leq 2.5$$

By using ***pgamma*** function in R,

$$\hat{\theta}_{Bayes} = \boldsymbol{pgamma}(2.5,12,5) \approx 0.594$$

$$\hat{\theta}_{Bayes} < \frac{2}{3}$$

So the conclusion is: $Reject$ H_0.

Problem 4

General Theorem regarding calculation of Bayesian decision rules:

$$Q(x,d) = \int_{\Theta} L(\theta|d)\, h(\theta|x) d\theta$$

$$L(\theta,d) = \begin{cases} \alpha(\theta - d) & if \quad d \le \theta \\ \beta(d-\theta) & if \quad d > \theta \end{cases}$$

and

$$h(\theta|x) \; is \; the \; posterior \; density \; of \; \theta \; given \; X \; = \; (X_1, X_2, \ldots, X_n);$$

then

$$Q(x,d) = \int_{-\infty}^{d} \beta(d-\theta)\, h(\theta|x) d\theta + \int_{\mathrm{d}}^{+\infty} \alpha(\theta - d)\, h(\theta|x) d\theta$$

$$= \beta d \int_{-\infty}^{d} h(\theta|x) d\theta - \beta \int_{-\infty}^{d} \theta h(\theta|x) d\theta$$

$$+\alpha \int_{d}^{+\infty} \theta h(\theta|x) d\theta - \alpha d \int_{d}^{+\infty} h(\theta|x) d\theta$$

Let

$$\mathcal{A} = \beta d \int_{-\infty}^{d} h(\theta|x) d\theta$$

$$\mathcal{B} = \beta \int_{-\infty}^{d} \theta h(\theta|x) d\theta$$

$$\mathcal{C} = \alpha \int_{d}^{+\infty} \theta h(\theta|x)d\theta$$

$$\mathcal{D} = \alpha d \int_{d}^{+\infty} h(\theta|x)d\theta$$

$$Q(x, d) = \mathcal{A} - \mathcal{B} + \mathcal{C} - \mathcal{D}$$

Hence, in order to looking for d that gives an infimum to

$$inf Q(x, d)$$

so we need to calculate

$$\frac{\partial(Q(x,d))}{\partial d} = \frac{\partial(\mathcal{A})}{\partial d} - \frac{\partial(\mathcal{B})}{\partial d} + \frac{\partial(\mathcal{C})}{\partial d} - \frac{\partial(\mathcal{D})}{\partial d}$$

$$Let\ \mathcal{E} = \frac{\partial(\mathcal{A})}{\partial d} = \beta \int_{-\infty}^{d} h(\theta|x)d\theta + \beta d h(d|x)$$

$$Let\ \mathcal{F} = \frac{\partial(\mathcal{B})}{\partial d} = \beta d h(d|x)$$

$$Let\ \mathcal{G} = \frac{\partial(\mathcal{C})}{\partial d} = -\alpha d h(d|x)$$

$$Let\ \mathcal{H} = \frac{\partial(\mathcal{D})}{\partial d} = \alpha \int_{d}^{+\infty} h(\theta|x)d\theta - \alpha d h(d|x)$$

since

$$P(\theta \in \Theta) = \int_{\Theta} h(\theta|x)d\theta$$

then

$$\mathcal{E} - \mathcal{F} = \beta \int_{-\infty}^{d} h(\theta|x)d\theta = \beta P(\theta < d)$$

$$\mathcal{G} - \mathcal{H} = -\alpha \int_{d}^{+\infty} h(\theta|x)d\theta = -\alpha(1 - P(\theta < d))$$

$$\frac{\partial(Q(x,d))}{\partial d} = \mathcal{E} - \mathcal{F} + \mathcal{G} - \mathcal{H}$$

$$= \beta P(\theta < d) - \alpha(1 - P(\theta < d))$$

$$= (\alpha + \beta)\big(P(\theta < d)\big) - \alpha$$

$$= (\alpha + \beta)\left(\int_{-\infty}^{d} h(\theta|x)d\theta\right) - \alpha$$

Setting the derivative with respect to d to zero we get

$$\frac{\partial(Q(x,d))}{\partial d} == (\alpha + \beta)\big(P(\theta < d)\big) - \alpha = 0$$

$$P(\theta < d) = \frac{\alpha}{\alpha + \beta}$$

$$P(\theta > d) = \frac{\beta}{\alpha + \beta}$$

should check that the second derivative is positive:

$$\frac{\partial^2(Q(x,d))}{\partial d^2} = (\alpha + \beta)\frac{\partial(\int_{-\infty}^{d} h(\theta|x)d\theta)}{\partial d}$$

$$= (\alpha + \beta)h(d|x)$$

$$\alpha, \beta\text{: } known\ positive\ constants.$$

so

$$\frac{\partial^2(Q(x,d))}{\partial d^2} > 0$$