THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

PRACTICE Mid-Session Test - 2021 - Wednesday 24th March (Week 6)
Solutions

MATH5905

Time allowed: 135 minutes

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be i.i.d. Poisson(θ) random variables with density function

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \ldots, \quad \text{and} \quad \theta > 0.$$

   a) The statistic $T(X) = \sum_{i=1}^n X_i$ is complete and sufficient for θ. Provide justification for why this statement is true.

   Using the property of the one-parameter exponential family, we observe that

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!} = e^{-\theta}\frac{1}{x!}\exp\big(x\log\theta\big)$$

   Thus, the Poisson distribution belongs to the one-parameter exponential family. Then this implies that $T(X) = \sum_{i=1}^n X_i$ is complete and (minimal) sufficient for θ.

   b) Derive the UMVUE of $h(\theta) = e^{-k\theta}$ where $k = 1, 2, \ldots, n$ is a known integer. You must justify each step in your answer. Hint: Use the interpretation that $P(X_1 = 0) = e^{-\theta}$ and therefore $P(X_1 = 0, \ldots, X_k = 0) = P(X_1 = 0)^k = e^{-k\theta}$.

   Since T is sufficient and complete for θ, we first need to find an unbiased estimator of h. Let $W = I_{\{X_1=0,\ldots,X_k=0\}}(X)$ and then using the fact that $P(X_i = 0) = e^{-\theta}$ we see that

$$E(W) = P(X_1 = 0, \ldots, X_k = 0) = P(X_1 = 0)^k = \left[e^{-\theta}\right]^k = e^{-k\theta},$$

   which is unbiased for $h(\theta)$. Now, we apply the Theorem of Lehmann-Scheffe, and obtain the following

$$\begin{aligned}
\hat{\tau}(T) = E(W|T = t) &= E\Bigg(I_{\{X_1=0,\ldots,X_k=0\}}|\sum_{i=1}^n X_i = t\Bigg) \\
&= P\Bigg(X_1 = 0, \ldots, X_k = 0|\sum_{i=1}^n X_i = t\Bigg) \\
&= \frac{P\Big(X_1 = 0, \ldots, X_k = 0, \sum_{i=1}^n X_i = t\Big)}{P\Big(\sum_{i=1}^n X_i = t\Big)}
\end{aligned}$$

Now we observe that the events in the numerator must be satisfied simultaneously to have a non-zero probability and hence this reduces to

$$\begin{aligned}
\hat{\tau}(T) &= \frac{P\Big(X_1 = 0, \ldots, X_k = 0, \sum_{i=k+1}^{n} X_i = t\Big)}{P\Big(\sum_{i=1}^{n} X_i = t\Big)} \\
&= e^{-k\theta} \frac{P\Big(\sum_{i=k=1}^{n} X_i = t\Big)}{P\Big(\sum_{i=1}^{n} X_i = t\Big)} \\
&= e^{-k\theta} \frac{e^{-\theta(n-k)}(\theta(n-k))^t}{t!} \cdot \frac{t!}{e^{-n\theta}(n\theta)^t} \\
&= \left(\frac{n-k}{n}\right)^t \\
&= \left(1 - \frac{k}{n}\right)^{n\bar{X}}
\end{aligned}$$

which is the UMVUE for $h(\theta) = e^{-k\theta}$.

c) Calculate the Cramer-Rao lower bound for the minimal variance of an unbiased estimator of $h(\theta) = e^{-k\theta}$.

First we calculate the fisher information as follows. We have that

$$\begin{aligned}
\log f(x) &= -\theta + x \log \theta - \log x! \\
\frac{\partial}{\partial \theta} \log f(x) &= -1 + \frac{x}{\theta} \\
\frac{\partial^2}{\partial \theta^2} \log f(x) &= -\frac{x}{\theta^2}
\end{aligned}$$

and so the Fisher information in a single sample is

$$I_{X_1}(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \log f(x)\right] = \frac{E(X)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

Hence the Fisher information for the whole sample is

$$I_{\mathbf{X}} = n I_{X_1}(\theta) = \frac{n}{\theta}.$$

Then notice that

$$\frac{\partial}{\partial \theta} h(\theta) = -k e^{-k\theta}.$$

Hence the Cramer-Rao lower bound is

$$\frac{\left(\frac{\partial}{\partial \theta} h(\theta)\right)^2}{I_{\mathbf{X}}(\theta)} = \frac{\theta}{n} k^2 e^{-2k\theta}$$

d) Show that there does not exist an integer k for which the variance of the UMVUE of $h(\theta)$ attains this bound.

To show that the Cramer-Rao bound is attainable we will look at the score function:

$$\begin{aligned}
V(\boldsymbol{X},\theta) &= -n + \frac{1}{\theta}\sum_{i=1}^{n} X_i \\
&= -n + \frac{n\bar{X}}{\theta} \\
&= \frac{n}{e^{-k\theta}}\left(\frac{\bar{X}e^{-k\theta}}{\theta} - e^{-k\theta}\right)
\end{aligned}$$

Therefore, no matter what integer k we choose the term $\frac{\bar{X}e^{-k\theta}}{\theta}$ cannot be a statistic as it will still depend on θ.

e) Determine the MLE $\hat{h}$ of $h(\theta)$.

The MLE for θ is simply $\hat{\theta} = \bar{X}$ and hence the

$$\widehat{h(\theta)}_{\mathbf{mle}} = h(\hat{\theta}_{\mathbf{mle}}) = e^{-k\bar{X}}.$$

f) Suppose that $n = 5$, $T = 10$ and $k = 1$ compute the numerical values of the UMVUE in part (b) and the MLE in part (e). Comment on these values.

The UMVUE is

$$\widehat{h(\theta)}_{\mathbf{umvue}} = \left(1 - \frac{1}{5}\right)^{5\times 2} = 0.107$$

The MLE is

$$\widehat{h(\theta)}_{\mathbf{mle}} = e^{-2} = 0.135$$

As $n \to \infty$ the UMVUE would converge to the MLE while for finite sample size these values are slightly different.

g) Consider testing $H_0 : \theta \leq 2$ versus $H_1 : \theta > 2$ with a 0-1 loss in Bayesian setting with the prior $\tau(\theta) = 4\theta^2 e^{-2\theta}$. What is your decision when $n = 5$ and $T = 10$. You may use:

$$\int_0^2 x^{12} e^{-7x} \mathrm{d}x = 0.00317$$

Note: The continuous random variable X has a gamma density f with parameters $\alpha > 0$ and $\beta > 0$ if

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

and

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha!$$

First we need to compute the posterior by observing that it proportional to the likelihood times the prior. The likelihood for this same is

$$L(X,\theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} X_i}}{\prod_{i=2}^{n} X_i!} \propto e^{-5}\theta^{10}$$

therefore

$$h(\theta|x) \propto 4\theta^2 e^{-2\theta} \times e^{-5\theta}\theta^{10} = \theta^{12}e^{-7\theta}$$

which implies that

$$\theta|X \sim \textbf{gamma}(13, \frac{1}{7}).$$

Hence we are interested in computing the posterior probability

$$\begin{aligned} P(\theta < 2|X) &= \int_0^2 \frac{1}{\Gamma(13)\left(\frac{1}{7}\right)^{13}}\theta^{12}e^{-7\theta}\mathrm{d}\theta \\ &= 202.27 \times 0.00317 \\ &= 0.64 \end{aligned}$$

We compare this posterior probability with 0.5 **since we are dealing with a 0-1 loss. Since this probability is greater than 0.5 we must NOT reject** H_0**.**

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with a density

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, x > \theta, \\ 0 \text{ else} \end{cases}$$

where $\theta \in \mathbb{R}^1$ is an unknown parameter. Let $T = \min\{X_1, \ldots, X_n\} = X_{(1)}$ be the minimal of the n observations.

a) Show that T is a sufficient statistic for the parameter θ.
First calculate the likelihood as follows

$$L(\boldsymbol{X}, \theta) = \prod_{i=1}^{n} e^{-(X_i-\theta)} I_{(-\infty, X_i)}(\theta) = \exp\Big(- \sum_{i=1}^{n} X_i + n\theta \Big) I_{(-\infty, X_{(1)})}(\theta)$$

Therefore the likelihood can be written as $L(X,\theta) = g(T,\theta)h(X)$ **where**

$$g(T,\theta) = \exp(n\theta) I_{(-\infty,T)}(\theta) \qquad \textbf{and} \qquad h(X) = \exp\Big(- \sum_{i=1}^{n} X_i \Big)$$

Hence, $T = X_{(1)}$ **is sufficient by the Neyman Fisher Factorization Criterion.**

b) Show that the density of T is

$$f_T(t) = \begin{cases} ne^{-n(x-\theta)}, t > \theta, \\ 0 \text{ else} \end{cases}$$

Hint: You may find the CDF first by using

$$P(X_{(1)} < x) = 1 - P(X_1 > x \cap X_2 > x \cdots \cap X_n > x).$$

First note that for $x < \theta$ **we have** $P(X_1 \geq x) = 1$ **and for** $x \geq \theta$ **we have,**

$$P(X_1 \geq x) = \int_x^{\infty} e^{-(y-\theta)} \mathrm{d}y = \Big[- e^{-(y-\theta)} \Big]_{y=x}^{y=\infty} = e^{-(x-\theta)}$$

Hence,

$$\begin{aligned} F_T(x,\theta) &= P(T \leq t) \\ &= 1 - P(X_1 \geq t, X_2 \geq t, \ldots, X_n \geq t) \\ &= 1 - P(X_1 \geq t)^n \\ &= \begin{cases} 1 - e^{-n(t-\theta)} & \textbf{if} \quad t \geq \theta \\ 0 & \textbf{if} \quad t < \theta. \end{cases} \end{aligned}$$

Then by differentiation

$$f_T(t,\theta) = ne^{-n(x-\theta)}, \quad t \geq 0,$$

otherwise zero.

c) Find the maximum likelihood estimator of θ and provide justification.
The MLE for θ is calculated by maximizing over all θ values

$$L(\boldsymbol{X}, \theta) = \exp\Big(- \sum_{i=1}^{n} X_i + n\theta \Big) I_{(-\infty, X_{(1)})}(\theta)$$

The graph of the Likelihood function is at zero at minus infinity and then continues to increase until the value $X_{(1)}$ and then from here stays at zero. Hence $X_{(1)}$ must be the MLE.

d) Show that the MLE is a biased estimator. Hint: You might want to consider the transformation $Y = T - \theta$ when performing the integral and then utilize the density of an exponential distribution.
By calculating

$$E(X_{(1)}) = \int_{\theta}^{\infty} t \cdot n e^{-n(t-\theta)} \mathrm{d}t$$

Now let $y = t - \theta$ then we get

$$\begin{aligned} E(X_{(1)}) &= n \int_{0}^{\infty} (y + \theta) e^{-ny} \mathrm{d}y \\ &= \int_{0}^{\infty} y \cdot n e^{-ny} \mathrm{d}y + \theta \int_{0}^{\infty} n e^{-ny} \mathrm{d}y \\ &= \frac{1}{n} + \theta \\ &\neq \theta \end{aligned}$$

and hence the MLE is a biased estimator.

e) Show that $T = X_{(1)}$ is complete for θ.
Take any function g such that $E_\theta g(T) = 0 \quad \forall \theta$. Then

$$\begin{aligned} 0 = E_\theta g(T) &= \int_{-\infty}^{\infty} g(t) f_T(t) \mathrm{d}t \\ &= \int_{\theta}^{\infty} g(t) \cdot n e^{-n(t-\theta)} \mathrm{d}t \\ &= n e^{n\theta} \int_{\theta}^{\infty} g(t) e^{-nt} \mathrm{d}t \end{aligned}$$

This implies that

$$\int_{\theta}^{\infty} g(t) e^{-nt} \mathrm{d}t = 0 \qquad \forall \theta$$

By taking the derivative with respect to θ to both sides we get

$$-g(\theta) e^{-n\theta} = 0 \quad \forall\, \theta$$

but since $e^{-n\theta} \neq 0$ we must have $g(\theta) = 0$ for all θ. Since θ is just acting as a variable we have $g(t) = 0$ for all t. Hence $P(g(T) = 0) = 1$ for all θ and hence $T = X_{(1)}$ is complete for θ.

f) Hence determine the UMVUE of θ.

From part d) we know that $W = X_{(1)} - \frac{1}{n}$ is an unbiased estimator for θ. Thus by Lehmann-Scheffe theorem,

$$\hat{\theta} = E(W|X_{(1)}) = X_{(1)} - \frac{1}{n}$$

is the unique UMVUE of θ.