Example 8.65

Here $f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

The CGF $K_X(t) = \log M_X(t) = \log e^{t^2/2} = t^2/2$

$K_X'(t) = t$ and $K_X''(t) = 1$

Saddlepoint equation: $K_X'(\hat{t}) = \hat{t} = \bar{x}$.

Hence, the first order saddlepoint approximation: for the density:

$$\hat{f}(\bar{x}) = \sqrt{\frac{n}{2\pi K''(\hat{t})}} \exp\{n K_X(\hat{t}) - n\hat{t}\bar{x}\}$$

$$= \sqrt{\frac{n}{2\pi}} \exp\{n \cdot \tfrac{1}{2}\bar{x}^2 - n\bar{x}\bar{x}\}$$

$$= \sqrt{\frac{n}{2\pi}} e^{-\frac{1}{2}n\bar{x}^2}$$

which is precisely the density of a $N(0, \frac{1}{n})$ - and we know $\bar{X} \sim N(0, \frac{1}{n})$ in this case i.e. no error!

For the CDF we can apply the Lugannani-Rice formula and we get

$$\hat{w}_n = \text{sign}(\hat{t}) \sqrt{2n(\hat{t}\bar{x} - K_X(\hat{t}))}$$

$$= \sqrt{2n(\bar{x} \cdot \bar{x} - \tfrac{1}{2}\bar{x}^2)} = \sqrt{n\bar{x}^2} = \bar{x}\sqrt{n}$$ and

$$\hat{u}_n = \hat{t}\sqrt{nK_X''(\hat{t})}$$

$$= \bar{x}\sqrt{n}$$

Hence, $\phi(\hat{w}_n)\left(\frac{1}{\hat{w}_n} - \frac{1}{\hat{u}_n}\right) = 0$ and so

$$\frac{\bar{X} - 0}{1/\sqrt{n}} = \sqrt{n}\,\bar{X}$$

$$F_{\bar{X}}(\bar{x}) = \Phi(\hat{w}_n) + \phi(\hat{w}_n)\left(\frac{1}{\hat{w}_n} - \frac{1}{\hat{u}_n}\right) = \Phi(\bar{x}\sqrt{n})$$



which is the CDF of a $N(0, \frac{1}{n})$ - exact.

Exercise 8.40

(i) $K_X(t) = \log e^{\mu t + \frac{1}{2}\sigma^2 t^2} = \mu t + \frac{1}{2}\sigma^2 t^2$

(ii) $K_X'(t) = \mu + \frac{1}{2}\sigma^2(2t) = \mu + \sigma^2 t$

$K_X''(t) = \sigma^2$

Then the saddlepoint is $K_X'(\hat{t}) = \mu + \sigma^2 \hat{t} = \bar{x}$

$$\iff \hat{t} = \frac{\bar{x}-\mu}{\sigma^2}$$

Clearly $K_X''(\hat{t}) = \sigma^2$

(iii) $$\hat{f}(\bar{x}) = \sqrt{\frac{n}{2\pi K_X''(\hat{t})}} \exp\left\{ n K_X(\hat{t}) - n\hat{t}\bar{x} \right\}$$

$$= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{ n\left[ \left(\frac{\bar{x}-\mu}{\sigma^2}\right)\mu + \frac{1}{2}\sigma^2\left(\frac{\bar{x}-\mu}{\sigma^2}\right)^2 - \left(\frac{\bar{x}-\mu}{\sigma^2}\right)\bar{x} \right] \right\}$$

$$= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{ \frac{n}{2\sigma^2}\left[ 2\mu(\bar{x}-\mu) + (\bar{x}-\mu)^2 - 2(\bar{x}-\mu)\bar{x} \right] \right\}$$

$$\vdots$$

$$= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{ -\frac{n}{2\sigma^2}\left(\bar{x}^2 - 2\mu\bar{x} + \mu^2\right) \right\}$$

$$= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{ -\frac{n}{2\sigma^2}(\bar{x}-\mu)^2 \right\}$$

$X_1 \dots X_n$ iid $N(\mu, \sigma^2)$

$\Rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

which is the <u>exact</u> density of a $N(\mu, \sigma^2/n)$

## Exercise 8.41

(i) $M_X(t) = \dfrac{\lambda}{\lambda - t} \qquad t < \lambda$

Hence $K_X(t) = \log\lambda - \log(\lambda - t) \qquad t < \lambda$

(ii) $K_X'(t) = -\dfrac{(-1)}{\lambda - t} = \dfrac{1}{\lambda - t} = (\lambda - t)^{-1}$

$K_X''(t) = (-1)(\lambda - t)^{-2}(-1) = \dfrac{1}{(\lambda - t)^2}$

Saddlepoint: $K_X'(\hat{t}) = \dfrac{1}{\lambda - \hat{t}} \overset{\text{set}}{=} \bar{x} \iff \lambda - \hat{t} = \dfrac{1}{\bar{x}}$

$\iff \hat{t} = \lambda - \dfrac{1}{\bar{x}}$

$K_X(\hat{t}) = \log\lambda - \log(\cancel{\lambda} - (\cancel{\lambda} - \frac{1}{\bar{x}}))$

$= \log\lambda - \log\frac{1}{\bar{x}}$

$= \log\lambda + \log\bar{x}$

$= \log\lambda\bar{x}$

$K_X''(\hat{t}) = \dfrac{1}{(\cancel{\lambda} - (\cancel{\lambda} - \frac{1}{\bar{x}}))^2}$

$= \dfrac{1}{(1/\bar{x})^2} = \bar{x}^2$

(iii) $\hat{f}(\bar{x}) = \sqrt{\dfrac{n}{2\pi\bar{x}^2}} \exp\left\{n \cdot \log\lambda\bar{x} - n \cdot (\lambda - \frac{1}{\bar{x}})\bar{x}\right\}$

$= \sqrt{\dfrac{n}{2\pi}}\, \bar{x}^{-1} \exp\left\{\log(\lambda\bar{x})^n - n(\lambda\bar{x} - 1)\right\}$

$= \sqrt{\dfrac{n}{2\pi}}\, \bar{x}^{-1} \lambda^n \bar{x}^n e^{-n\lambda\bar{x}} e^n$

$= \sqrt{\dfrac{n}{2\pi}}\, e^n \lambda^n \bar{x}^{n-1} e^{-n\lambda\bar{x}}$

(iv) The above is NOT the true density of $\bar{x}$. The true density is given by

$$f(\bar{x}) = \frac{(n\lambda)^n}{(n-1)!} \bar{x}^{n-1} e^{-n\lambda\bar{x}}$$

or $\bar{X} \sim \text{Gamma}(n, n\lambda)$

Exact: $\frac{n^n}{(n-1)!}$ $\approx$ $\sqrt{\frac{n}{2\pi}} e^n$ Approximation

are quite close for large $n$!

$\boxed{\text{CLT}}$ $E(X) = \lambda$

$\text{Var}(X) = \lambda^2$

$\bar{X} \overset{a}{\sim} N(\lambda, \frac{\lambda^2}{n})$

$\lambda = 1$

$n = 3$

Exercise 8.42

(i) To calculate the mgf of a Poisson($\lambda$) (but you can simply look at the table of common distributions):

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

Now for $Y \sim \text{Poisson}(e^t\lambda)$ we have

$$\sum_{y=0}^{\infty} \frac{e^{-\lambda e^t}(\lambda e^t)^y}{y!} = 1 \iff \sum_{y=0}^{\infty} \frac{(e^t\lambda)^y}{y!} = e^{\lambda e^t}$$

since if we some each possible value of the pmf we get one. Hence,

$$M_X(t) = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

The cgf for a single observation is

$$K_X(t) = \log M_X(t) = \log e^{\lambda(e^t-1)} = \lambda(e^t-1)$$

with $K_X'(t) = K_X''(t) = \lambda e^t$

(ii) The saddlepoint equation $K_X'(\hat{t}) = \bar{x}$ gives the saddlepoint as

$$\lambda e^{\hat{t}} = \bar{x} \iff \hat{t} = \log\left(\frac{\bar{x}}{\lambda}\right)$$

First order saddlepoint approximation is

$$\hat{f}(\bar{x}) = \sqrt{\frac{n}{2\pi K_X''(\hat{t})}} \exp\{nK_X(\hat{t}) - n\hat{t}\bar{x}\}$$

$$= \sqrt{\frac{n}{2\pi\lambda e^{\log(\bar{x}/\lambda)}}} \exp\left\{n\lambda\left(e^{\log(\bar{x}/\lambda)}-1\right) - n\log\left(\frac{\bar{x}}{\lambda}\right)\bar{x}\right\}$$

$$= \sqrt{\frac{n}{2\pi\lambda\cdot\bar{x}/\lambda}} \exp\left\{n\lambda\left(\frac{\bar{x}}{\lambda}-1\right) - n\bar{x}\log\bar{x} + n\bar{x}\log\lambda\right\}$$

$$= \sqrt{\frac{n}{2\pi\bar{x}}}\, e^{n\bar{x}} e^{-n\lambda} \bar{x}^{-n\bar{x}} \lambda^{n\bar{x}}$$

$$= \sqrt{\frac{n}{2\pi}} \; \frac{\lambda^{n\bar{x}} e^{n\bar{x}} e^{-n\lambda}}{\bar{x}^{n\bar{x}+1/2}}$$

Now let $Y = \sum_{i=1}^{n} X_i = n\bar{X}$ then the density transformation formula gives

since $\bar{X} = \frac{Y}{n}$ and $\frac{d\bar{x}}{dy} = \frac{1}{n}$

$$f_Y(y) = f_{\bar{X}}\left(\frac{y}{n}\right) \cdot \frac{1}{n}$$

$$= \sqrt{\frac{n}{2\pi}} \; \frac{e^{-n\lambda} \lambda^{n \cdot \frac{y}{n}} e^{n \cdot \frac{y}{n}}}{\left(\frac{y}{n}\right)^{\frac{ny}{n}+1/2}}$$

$$= \sqrt{\frac{n}{2\pi}} \; \frac{e^{-n\lambda} (\lambda)^y e^y}{y^{y+1/2} \left(\frac{1}{n}\right)^{y+1/2}} \cdot \frac{1}{n}$$

$$= \sqrt{\frac{1}{2\pi}} \; \frac{e^{-n\lambda} \lambda^y e^y \, n^{y+1/2} \cdot \sqrt{n} \cdot \frac{1}{n}}{y^{y+1/2}}$$

$$= \sqrt{\frac{1}{2\pi}} \; \frac{e^{-n\lambda} (n\lambda)^y e^y}{y^{y+1/2}}$$

Then Stirlings formula

$$n! \approx \sqrt{2\pi n}\, e^{-n} n^n$$

makes the following approximation

$$\hat{f}(y) = \frac{e^{-n\lambda} (n\lambda)^y}{y!}$$

which is the density of a Poisson$(n\lambda)$.