THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

PRACTICE MID SESSION TEST - 2021 - Wedneday, 24th March (Week 6)

MATH5905

Time allowed: 135 minutes

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be i.i.d. Poisson(θ) random variables with density function

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \ldots, \quad \text{and} \quad \theta > 0.$$

   a) The statistic $T(X) = \sum_{i=1}^n X_i$ is complete and sufficient for θ. Provide justification for why this statement is true.

   b) Derive the UMVUE of $h(\theta) = e^{-k\theta}$ where $k = 1, 2, \ldots, n$ is a known integer. You must justify each step in your answer. Hint: Use the interpretation that $P(X_1 = 0) = e^{-\theta}$ and therefore $P(X_1 = 0, \ldots, X_k = 0) = P(X_1 = 0)^k = e^{-k\theta}$.

   c) Calculate the Cramer-Rao lower bound for the minimal variance of an unbiased estimator of $h(\theta) = e^{-k\theta}$.

   d) Show that there does not exist an integer k for which the variance of the UMVUE of $h(\theta)$ attains this bound.

   e) Determine the MLE $\hat{h}$ of $h(\theta)$.

   f) Suppose that $n = 5$, $T = 10$ and $k = 1$ compute the numerical values of the UMVUE in part (b) and the MLE in part (e). Comment on these values.

   g) Consider testing $H_0 : \theta \leq 2$ versus $H_1 : \theta > 2$ with a 0-1 loss in Bayesian setting with the prior $\tau(\theta) = 4\theta^2 e^{-2\theta}$. What is your decision when $n = 5$ and $T = 10$. You may use:

$$\int_0^2 x^{12} e^{-7x} \mathrm{d}x = 0.00317$$

   Note: The continuous random variable X has a gamma density f with parameters $\alpha > 0$ and $\beta > 0$ if

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

   and

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) = \alpha!$$

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with a density

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, x > \theta, \\ 0 \text{ else} \end{cases}$$

where $\theta \in \mathbb{R}^1$ is an unknown parameter. Let $T = \min\{X_1, \ldots, X_n\} = X_{(1)}$ be the minimal of the n observations.

a) Show that T is a sufficient statistic for the parameter θ.

b) Show that the density of T is

$$f_T(t) = \begin{cases} ne^{-n(x-\theta)}, t > \theta, \\ 0 \text{ else} \end{cases}$$

Hint: You may find the CDF first by using

$$P(X_{(1)} < x) = 1 - P(X_1 > x \cap X_2 > x \cdots \cap X_n > x).$$

c) Find the maximum likelihood estimator of θ and provide justification.

d) Show that the MLE is a biased estimator. Hint: You might want to consider using a substitution and then utilize the density of an exponential distribution when computing the integral.

e) Show that $T = X_{(1)}$ is complete for θ.

f) Hence determine the UMVUE of θ.