Some Notation

First, let's introduce some commonly encountered notation in asymptotics about he behavior of functions of a real variable x as $x \to \infty$. Given two such functions $f(x)$ and $g(x)$, defined for all sufficiently large real numbers x, we write

$$f(x) = O(g(x))$$

as short-hand for the following statement: there exists constants x_0 and M such that

$$\frac{|f(x)|}{|g(x)|} \leq M \qquad \text{for all } x > x_0.$$

If this holds, we say that $f(x)$ is of order $g(x)$ and we call the above estimate a "big Oh estimate".

The notation $f(x) = o(g(x))$ means that $g(x) \neq 0$ for sufficiently large x and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

If this holds, we say $f(x)$ is of smaller order than $g(x)$.

For our purposes, we need to define the relation

$$f(n) = O(g(n)) \qquad \text{and} \qquad f(n) = o(g(n))$$

for functions f and g of integer variable n - the sample size.

Normal Distribution Illustration

Suppose that Z is $N(0,1)$ then

$-\infty < z < \infty$

$$
\begin{aligned}
M_Z(t) = \mathbb{E}\big[e^{tZ}\big] &= \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2 + tz} \mathrm{d}z
\end{aligned}
$$

Now note that:

$$
-\frac{z^2}{2} + tz = -\frac{1}{2}(\underbrace{z^2 - 2tz}_{+t^2-t^2}) = -\frac{1}{2}\Big[(z-t)^2 - t^2\Big] = -\frac{1}{2}(z-t)^2 + \frac{t^2}{2}
$$

Therefore

$$
\begin{aligned}
M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}(z-t)^2 + \frac{t^2}{2}\Big) \mathrm{d}z \\
&= e^{t^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(z-t)^2\Big) \mathrm{d}z}_{\text{density of } N(t,1)} \\
&= e^{t^2/2} \cdot 1 = e^{t^2/2}
\end{aligned}
$$

$EX = \mu + \sigma \cdot 0 = \mu$

$\operatorname{Var} X = \sigma^2 \cdot 1 = \sigma^2$

$X \sim N(\mu, \sigma^2)$

Now let $X \sim N(\mu, \sigma^2)$ then $X = \mu + \sigma Z$ and hence

$$
\begin{aligned}
M_X(t) = \mathbb{E}\big[e^{tX}\big] &= \mathbb{E}\big[e^{t(\mu+\sigma Z)}\big] \\
&= \mathbb{E}\big[e^{\mu t} e^{t\sigma Z}\big] \\
&= e^{\mu t}\mathbb{E}\big[e^{(t\sigma)Z}\big] \\
&= e^{\mu t} M_Z(t\sigma) \\
&= e^{\mu t} e^{t^2\sigma^2/2} \\
&= \exp\Big(\mu t + \frac{1}{2}t^2\sigma^2\Big)
\end{aligned}
$$

Using $M_X(t)$ we can recover the mean and variance:

$$
\begin{aligned}
\mu_1' = \quad \mathbb{E}(X^1) = \frac{\mathrm{d}}{\mathrm{d}t} M_X(t)\Big|_{t=0} &= (\mu + t\sigma^2)\exp\Big(\mu t + \frac{1}{2}t^2\sigma^2\Big)\Big|_{t=0} \\
&= (\mu + 0)e^0 \\
&= \mu
\end{aligned}
$$

$$
\begin{aligned}
\mu_2' = \quad \mathbb{E}(X^2) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} M_X(t)\Big|_{t=0} &= \sigma^2 \exp\Big(\mu t + \frac{1}{2}t^2\sigma^2\Big) + (\mu + t\sigma^2)^2 \exp\Big(\mu t + \frac{1}{2}t^2\sigma^2\Big)\Big|_{t=0} \\
&= \sigma^2 e^0 + (\mu + 0)^2 e^0 \\
&= \sigma^2 + \mu^2
\end{aligned}
$$

Therefore,

$$
\operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2} = \sigma^2
$$

$= \mu_2' - (\mu_1')^2$

The cumulant generating function for $N(\mu, \sigma^2)$ is

$$K_X(t) = \log M_X(t) = \log \exp\left(\mu t + \frac{1}{2}t^2\sigma^2\right) = \mu t + \frac{1}{2}t^2\sigma^2$$

There,

$$\begin{aligned}
\kappa_1 &= \frac{\mathrm{d}}{\mathrm{d}t}K_X(t)\Big|_{t=0} = \mu + \sigma^2 t\Big|_{t=0} = \mu \to \text{mean} \; // \\
\kappa_2 &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}K_X(t)\Big|_{t=0} = \sigma^2\Big|_{t=0} = \sigma^2 \to \text{variance} \; // \\
\kappa_3 &= \frac{\mathrm{d}^3}{\mathrm{d}t^3}K_X(t)\Big|_{t=0} = 0 \to \text{skewness} \; //
\end{aligned}$$

Relationship between moments and cumulants

The Taylor expansion of $K_X(t)$ is:

$$K_X(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2!} + \cdots + \kappa_r \frac{t^r}{r!} + O(t^{r+1})$$

Then using the relation $e^{K_X(t)} = M_X(t)$ we get

$$e^{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \cdots + \kappa_r \frac{t^r}{r!} \cdots} = e^{\kappa_1 t} e^{\kappa_2 \frac{t^2}{2!}} e^{\kappa_3 \frac{t^3}{3!}} \cdots = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots \qquad (1)$$

Now expand the exponents on the LHS to give

$$\left(1 + \kappa_1 t + \frac{(\kappa_1 t)^2}{2!} + \cdots\right)\left(1 + \kappa_2 \frac{t^2}{2!} + \frac{1}{2!}\left(\frac{(\kappa_2 t^2)}{2!}\right)^2 + \cdots\right) \cdots$$

Then equating the coefficients in front of the powers of t in the LHS and RHS of (1), we get from the coefficient of t:

$$\kappa_1 = \mu_1' = \mathbb{E}(X)$$

and the coefficient of t^2:

$$\frac{\kappa_1^2}{\cancel{2!}} + \frac{\kappa_2}{\cancel{2!}} = \frac{\mu_2'}{\cancel{2!}} \qquad \Longrightarrow \qquad \kappa_2 = \mu_2' - (\mu_1')^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathrm{Var}(X)$$

Exercise 8.38:

$$e^{t(X_1+\cdots+X_n)} = \left[e^{tX_1}\right]^n$$

iid

$$\begin{aligned}
K_{\sum_{i=1}^n X_i}(t) &= \log\left\{\mathbb{E}e^{t\sum_{i=1}^n X_i}\right\} \\
&= \log\left\{\left(\mathbb{E}e^{tX_1}\right)^n\right\} \qquad \text{using independence} \\
&= n\log\mathbb{E}e^{tX_1} \\
&= nK_{X_1}(t) \quad //
\end{aligned}$$

$$\begin{aligned}
K_{aX_1+b}(t) &= \log\left\{\mathbb{E}e^{aX_1t+bt}\right\} \\
&= \log\left\{e^{bt}\mathbb{E}e^{(at)X_1}\right\} = \log e^{bt}M_X(at) \\
&= bt + K_{X_1}(at)
\end{aligned}$$

$$= \log e^{bt} + \log M_X(at)$$

$$= bt + K_{X_1}(at)$$

Theorem 8.32: Cornish-Fisher expansions

We will give a heuristic justification for the Cornish-Fisher expansion formula given in Theorem 8.32. Since

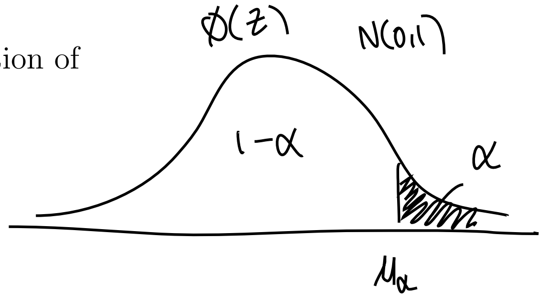
$$Z_n = \sqrt{n}\frac{\bar{X} - \mu}{\sigma}$$

is approximately standard normal, the quantile z_α which is theoretically defined as the solution of:

$$F_{Z_n}(z_\alpha) = 1 - \alpha$$

should be in the vicinity of the u_α quantile defined a the solution of

$$\Phi(u_\alpha) = 1 - \alpha$$



↗ Edgeworth Expansion

Then by Theorem 8.31:

$$1 - \alpha = F_{Z_n}(z_\alpha) \approx \Phi(z_\alpha) - \frac{C_1(F)p_1(z_\alpha)\phi(z_\alpha)}{\sqrt{n}} - \frac{C_2(F)p_2(z_\alpha) + C_3(F)p_3(z_\alpha)}{n}\phi(z_\alpha)$$

and by applying a Taylor series expansion everywhere around u_α we get

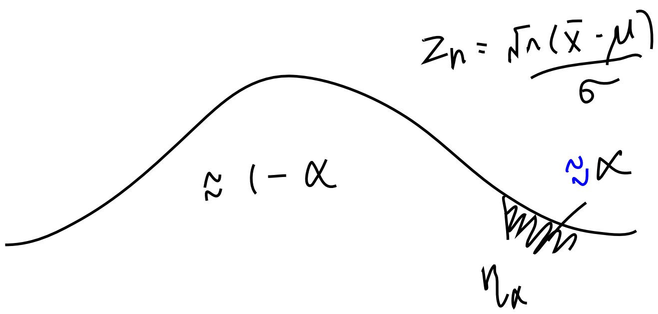
$$\cancel{1 - \alpha} = \cancel{\Phi(u_\alpha)} + \phi(u_\alpha)(z_\alpha - u_\alpha) + \phi(u_\alpha)\Big[\text{polynomials in } z_\alpha, u_\alpha\Big]$$

Since $\Phi(u_\alpha) = 1 - \alpha$ we can cancel with the $(1 - \alpha)$ on the LHS and we get:

$$\underbrace{\cancel{\phi(u_\alpha)}}_{\neq 0}\Big[\text{some polynomials in } z_\alpha, u_\alpha\Big] = 0$$

Since $\phi(u_\alpha)$ is non-zero the terms in the bracket must be set to zero and we can express z_α by using u_α from the resulting relation. In this was we finally obtain the expression in Theorem 8.32:

$$\eta_\alpha = u_\alpha + \frac{(u_\alpha^2 - 1)\rho_3}{6\sqrt{n}} + \frac{(u_\alpha^3 - 3u_\alpha)\rho_4}{24n} - \frac{(2u_\alpha^3 - 5u_\alpha)\rho_3^2}{36n} + o(n^{-1})$$



Exercise 8.39

The MGF for a χ_1^2 random variable (denoted generically as X here) is known to be

$$M_X(t) = (1-2t)^{\overset{-1/2}{\cancel{1/2}}}, \qquad t < \frac{1}{2}$$

which leads to the following CGF:

$$K_X(t) = -\frac{1}{2}\log(1-2t), \qquad t < \frac{1}{2}$$

Hence we get:

$$\begin{aligned}
K_X'(t) &= \frac{1}{1-2t} \\
K_X''(t) &= \frac{2}{(1-2t)^2} \\
K_X'''(t) &= \frac{8}{(1-2t)^3} \\
K_X^{(4)}(t) &= \frac{48}{(1-2t)^4}
\end{aligned}$$

which gives

$$\begin{aligned}
\kappa_1 &= K_X'(0) = 1 \\
\kappa_2 &= K_X''(0) = 2 \\
\kappa_3 &= K_X'''(0) = 8 \\
\kappa_4 &= K_X^{(4)}(0) = 48
\end{aligned}$$

Finally, the standardized cumulants are:

$$\rho_3 = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{8}{2^{3/2}} = 2\sqrt{2}$$

$$\rho_4 = \frac{\kappa_4}{\kappa_2^{4/2}} = \frac{48}{2^2} = 12$$

Show that $\mathbb{E}(X_t) = K'_X(t)$

$X_t : f(x;t) = e^{tX - K_X(t)} f(x)$

Start with

$$\log f(x;t) = tx - K_X(t) + \log f(x)$$

and take the first with respect to t:

$$\frac{\partial \log f(x;t)}{\partial t} = x - K'_X(t)$$

$$\Updownarrow$$

and this is equivalent to:

$$\frac{\frac{\partial}{\partial t} f(x;t)}{f(x;t)} = x - K'_X(t)$$

and hence

$$\frac{\partial}{\partial t} f(x;t) = (x - K'_X(t)) f(x;t)$$

Then integrate with respect to x to arrive at the following:

$$\frac{\partial}{\partial t} \underbrace{\int f(x;t)\mathrm{d}x}_{1} = \underbrace{\int x f(x;t)\mathrm{d}x}_{E X_t} - K'_X(t) \underbrace{\int f(x;t)\mathrm{d}x}_{1}$$

$$\underbrace{\frac{\partial}{\partial t} 1}_{0 =} = \mathbb{E}(X_t) - K'_X(t)$$

which implies that

$$\mathbb{E}(X_t) = K'_X(t)$$

Going from Eq. ~~24~~ 21 to Eq. ~~25~~ 22

The formula

$$
\begin{aligned}
f(x) &= \exp\{-\hat{t}x + K_X(\hat{t})\}f(x;\hat{t}) \\
&\approx \frac{1}{\sqrt{2\pi K_X''(\hat{t})}} \exp\{K_X(\hat{t}) - \hat{t}x\}\left[1 + \left(\frac{1}{8}\rho_4(\hat{t}) - \frac{5}{24}\rho_3^2(\hat{t})\right)\right]
\end{aligned}
$$

is essentially true for any random variable X. Suppose we want to apply it to a specific random variable say to the arithmetic mean of n *i.i.d.* random variables $X_1, X_2, \ldots, X_n$. $\bar{X}$ By utilizing the relationship between cumulant generation functions:

$$K_{\sum_{i=1}^n X_t}(t) = nK_{X_1}(t)$$

we can get the saddlepoint approximation formula for the density of $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ given by $f_{\bar{X}}(\bar{x})$ as

$$\hat{f}(\bar{x}) = \sqrt{\frac{n}{2\pi K_X''(\hat{t})}} \exp\{nK_X(\hat{t}) - n\hat{t}\bar{x}\}\left[1 + \left(\frac{1}{8n}\hat{\rho}_4 - \frac{5}{24n}\hat{\rho}_3^2\right)\right]$$

where $K_X(t)$ is the CGF for a SINGLE observation X, $\hat{t}$ is the saddlepoint (i.e. it has to be recalculated for every $\bar{x}$ value) and the solution to $K_X'(\hat{t}) = \bar{x}$ and $\hat{\rho}_i = n^{1-i/2}\rho_i(\hat{t})$ where

$$\rho_i(t) = \frac{K_X^{(i)}(t)}{\left[K_X''(t)\right]^{i/2}}, \qquad i \geq 3$$

Even the simpler version of the above **second order** saddlepoint approximation, namely the **first order** saddlepoint approximation

$$\hat{f}(\bar{x}) = \sqrt{\frac{n}{2\pi K_X''(\hat{t})}} \exp\{nK_X(\hat{t}) - n\hat{t}\bar{x}\}$$

is very precise for sample sizes such as $n = 5, 6, 10$.