University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2021

Assignment Two Solutions

Given: Wednesday 7 April 2021 **Due date: Wednesday 21 April 2021**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Wednesday, 21 April 2021. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document.

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Name	Student No.	Signature	Date

Problem 1

Let $X = (X_1, X_2, \ldots, X_n)$ be sample of n i.i.d. random variables, each with a density

$$f(x, \theta) = \frac{\sqrt{\theta}}{x\sqrt{2\pi}} \exp\Big(-\frac{\theta}{2}\log^2(x)\Big)$$

when $x > 0$ otherwise zero and where $\theta > 0$ is a parameter.

a) Find the distribution of $Y_i = \log X_i$ and hence or otherwise compute $\mathbb{E}(\log^2 X_i)$.

b) Find the Fisher information about θ in one observation and in the sample of n observations.

c) Find the Maximum Likelihood Estimator (MLE) of $h(x) = \frac{1}{\theta}$ and show that it is unbiased.

d) Does the variance of the MLE for $h(\theta)$ attain the Cramer Rao bound? **Note:** a χ^2_k distribution has mean k and variance $2k$.

e) Determine the asymptotic distribution of the MLE of θ and also the asymptotic distribution of $\tau(\theta) = e^{-\theta}$.

Solution:

a) By applying the density transformation formula to $Y = g(X) = \log X$ which is a one-to-one transformation from $\mathcal{X} = \{x|x > 0\}$ into $\mathcal{Y} = \{y|y \in \mathbb{R}\}$ with inverse $X = g^{-1}(Y) = e^Y$ and derivative

$$\frac{\mathrm{d}X}{\mathrm{d}Y} = e^Y.$$

Therefore, by applying the transformation density formula, the density function of Y becomes

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{\mathrm{d}X}{\mathrm{d}Y}\right| \\ &= \frac{\sqrt{\theta}}{e^y\sqrt{2\pi}} \exp\Big(-\frac{\theta}{2}\log^2 e^y\Big)|e^y| \\ &= \frac{\sqrt{\theta}}{\sqrt{2\pi}} \exp\Big(-\frac{\theta}{2}y^2\Big) \end{aligned}$$

Hence, this is the density of a normal distribution with mean 0 and variance $\frac{1}{\theta}$. That is $Y_i = \log X_i \sim N(0, \frac{1}{\theta})$ with $\mathbb{E}(Y_i) = 0$ and $\mathrm{Var}(Y_i) = \frac{1}{\theta}$ and so

$$\mathbb{E}(\log^2 X) = \mathbb{E}(Y^2) = \mathrm{Var}(Y) + (\mathbb{E}(Y))^2 = \frac{1}{\theta} + 0 = \frac{1}{\theta}.$$

b) First, compute the likelihood for one observation, for example X_1 as follows

$$f(x, \theta) = \frac{\sqrt{\theta}}{x\sqrt{2\pi}} \exp\Big(-\frac{\theta}{2}\log^2(x)\Big),$$

and the log-likelihood function is

$$\log f(x, \theta) = -\frac{1}{2}\log 2\pi - \log x + \frac{1}{2}\log\theta - \frac{\theta}{2}\log^2 x.$$

Take the first derivative of the log-likelihood to compute the score as

$$V(X,\theta) = \frac{\partial \log f(X,\theta)}{\partial \theta} = \frac{1}{2\theta} - \frac{1}{2}\log^2 x$$

and the second derivative

$$\frac{\partial^2 \log f(X,\theta)}{\partial \theta^2} = -\frac{1}{2\theta^2}$$

The the information in a single observation is

$$\begin{aligned} I_{X_1}(\theta) &= -E\bigg(\frac{\partial^2 \log f(X,\theta)}{\partial \theta^2}\bigg) \\ &= -E\bigg(-\frac{1}{2\theta^2}\bigg) \\ &= \frac{1}{2\theta^2} \end{aligned}$$

For a sample of n i.i.d. observations,

$$I_X(\theta) = nI_{X_1}(\theta) = \frac{n}{2\theta^2}$$

c) The likelihood of the sample

$$\begin{aligned} L(X,\theta) &= \prod_{i=1}^{n} \frac{\sqrt{\theta}}{x_i\sqrt{2\pi}} \exp\bigg(-\frac{\theta}{2}\log^2(x_i)\bigg) \\ &= \theta^{n/2}(2\pi)^{-n/2}\prod_{i=1}^{n}\frac{1}{x_i}\exp\bigg(-\frac{\theta}{2}\sum_{i=1}^{n}\log^2 x_i\bigg) \end{aligned}$$

with log-likelihood

$$\log L(X,\theta) = \frac{n}{2}\log\theta - \frac{n}{2}\log 2\pi - \sum_{i=1}^{n}\log x_i - \frac{\theta}{2}\sum_{i=1}^{n}\log^2 x_i.$$

Then the score function becomes

$$V(X,\theta) = \frac{\partial \log L(X,\theta)}{\partial\theta} = \frac{n}{2\theta} - \frac{1}{2}\sum_{i=1}^{n}\log^2 x_i$$

Then by setting the score to zero we can obtain the MLE

$$\hat{\theta}_{\text{mle}} = \frac{n}{\sum_{i=1}^{n}\log^2 x_i}.$$

Then the invariance transformation property gives the mle for $h(\theta) = \frac{1}{\theta}$ as

$$\widehat{h(\theta)}_{\text{mle}} = h(\hat{\theta}_{\text{mle}}) = \frac{1}{n}\sum_{i=1}^{n}\log^2(X_i).$$

Now consider

$$\mathbb{E}(\hat{\theta}_{\text{mle}}) = \mathbb{E}\bigg(\frac{1}{n}\sum_{i=1}^{n}\log^2 X_i\bigg) = \frac{1}{n}n\mathbb{E}(\log^2 X_1) = \frac{1}{n}n\frac{1}{\theta} = \frac{1}{\theta} = h(\theta)$$

Hence the MLE for $\widehat{h(\theta)}_{\text{mle}}$ is unbiased for $h(\theta)$.

d) Now we can compute the variance as follows

$$\text{Var}(\widehat{h(\theta)}_{\text{mle}}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n}\log^2 X_i\right) = \frac{1}{n^2}n\text{Var}(\log^2 X_1) = \frac{1}{n}\text{Var}(\log^2 X_i)$$

Now recall that $Y_i = \log X_i \sim N(0, \frac{1}{\theta})$ and so $Y_i\sqrt{\theta} \sim N(0,1)$ and therefore

$$Y_i^2\theta \sim \chi_1^2$$

Hence

$$2 = \text{Var}(\chi_1^2) = \text{Var}(\theta Y_i^2) = \theta^2\text{Var}(Y_i^2)$$

which implies that

$$\text{Var}(Y_i^2) = \text{Var}(\log^2 X_i) = \frac{2}{\theta^2}.$$

Hence, the variance of the MLE of $h(\theta)$ is

$$\text{Var}(\widehat{h(\theta)}_{\text{mle}}) = \frac{1}{n}\text{Var}(\log^2 X_i) = \frac{2}{n\theta^2}$$

which is the same as the CRLB since

$$\frac{(h'(\theta))^2}{nI_{X_1}(\theta)} = \frac{\left(\frac{1}{\theta^2}\right)^2}{\frac{n}{2\theta^2}} = \frac{2}{n\theta^2}$$

and hence the CRLB is attained by the variance of the MLE which makes it also the UMVUE.

e) The delta method sates that the asymptotic distribution of a smooth function $\tau(\theta)$ is given by

$$\sqrt{n}(\widehat{\tau(\theta)} - \tau(\theta)) \xrightarrow{d} N\left(0, \frac{(\tau'(\theta)^2)}{I_{X_1}(\theta)}\right).$$

Then the asymptotic distribution for θ is

$$\sqrt{n}(h(\hat{\theta}) - h(\theta)) \xrightarrow{d} N\left(0, \frac{2}{\theta^2}\right).$$

Then the asymptotic distribution for $h(\theta) = e^{\theta}$ can be computed by noting that $h'(\theta) = e^{\theta}$ then

$$\sqrt{n}(e^{-\hat{\theta}} - e^{-\theta}) \xrightarrow{d} N\left(0, 2\theta^2 e^{-2\theta}\right).$$

Problem 2

Suppose $X = X_1, X_2, \ldots, X_n$ is a sample of n i.i.d. random variables from a population with a density

$$f(x;\theta) = \begin{cases} \dfrac{\tau x^{\tau-1}}{\theta^\tau} & \text{if} \quad 0 < x < \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

where $\tau > 0$ is a known constant and $\theta > 0$ is an unknown parameter.

a) Show that the density of $T = X_{(n)}$ is

$$f_T(t) = \begin{cases} \dfrac{n\tau t^{n\tau-1}}{\theta^{n\tau}} & \text{if} \quad 0 < t < \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

b) Show that the family $\{L(X,\theta\}, \theta > 0$ has a monotone likelihood ratio in the statistic $T = X_{(n)}$.

c) Find the uniformly most powerful α-size test φ^* of

$$H_0 : \theta \leq \tau \qquad \text{versus} \qquad H_1 : \theta > \tau.$$

d) Calculate the power function of φ^*.

e) Calculate the value of the power function at the threshold constant, τ and as $\theta \to \infty$. Then, sketch a graph of the power function as precisely as possible.

Solution:

a) First, the CDF is calculated as follows

$$F(x) = \frac{\tau}{\theta^\tau}\int_0^x u^{\tau-1}\mathrm{d}u = \frac{\tau}{\theta^\tau}\left[\frac{u^\tau}{\tau}\right]_{u=0}^{u=x} = \left(\frac{x}{\theta}\right)^\tau.$$

Hence,

$$\begin{aligned} F_T(t) = P(X_{(1)} \leq t) &= P(X_{(n)} < t) \\ &= P(X_1 < t, X_2 < t, \ldots, X_n < t) \\ &= P(X_1 < t)^n \\ &= \left(\frac{t}{\theta}\right)^{\tau n} \end{aligned}$$

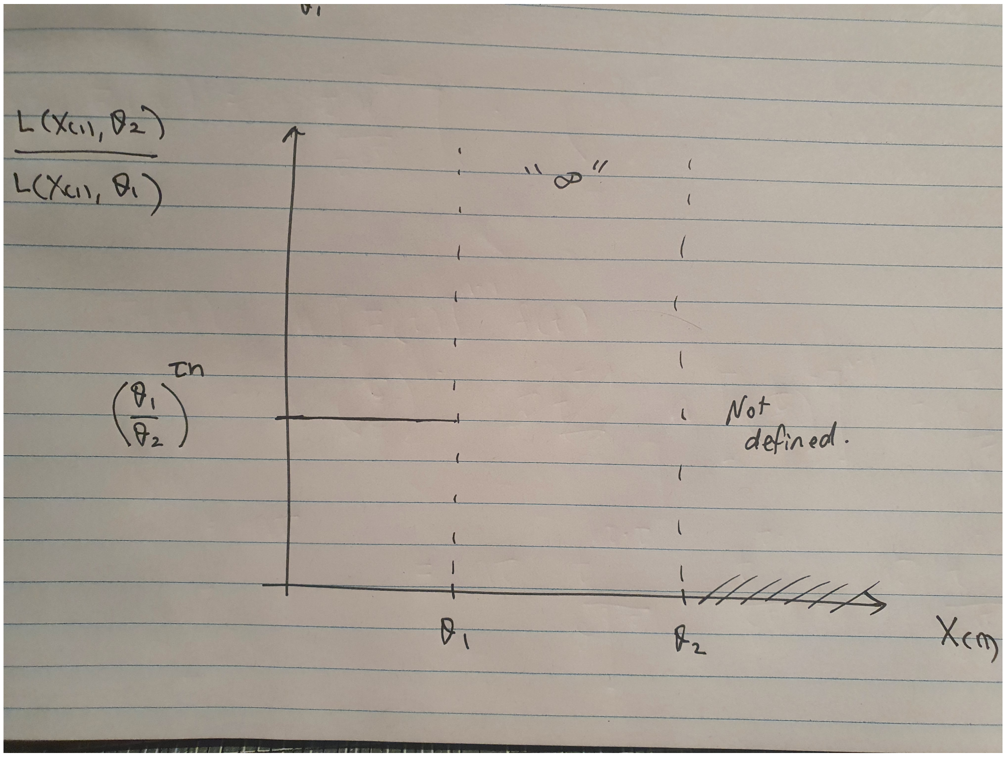
for $0 < x < \theta$. Then by differentiating the CDF we obtain the density

$$f_T(t) = \begin{cases} \dfrac{\tau n t^{\tau n-1}}{\theta^{\tau n}} & \text{if} \quad 0 < t < \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

b) Let $\theta_1 < \theta_2$ and consider the ratio

$$\frac{L(X,\theta_2)}{L(X,\theta_1)} = \left(\frac{\theta_1}{\theta_2}\right)^{\tau n}\frac{I_{(X_{(n)},\infty)}(\theta_2)}{I_{(X_{(n)},\infty)}(\theta_1)} = \begin{cases} \left(\dfrac{\theta_1}{\theta_2}\right)^{\tau n} & \text{if} \quad X_{(n)} < \theta_1 \\ \infty & \text{if} \quad \theta_1 < X_{(n)} < \theta_2 \\ \text{undefined} & \text{if} \quad X_{(n)} > \theta_2 \end{cases}.$$

A plot of this ratio as a function of $T = X_{(n)}$ can be seen below. It can be seen to be non-decreasing as $X_{(n)}$ increases (as we go along the x axis) when the ratio is defined, i.e. when $X_{(n)} < \theta_2$.



c) Since the family $\{L(X,\theta\}, \theta > 0$ has a monotone likelihood ratio in the statistic $T = X_{(n)}$, we know the structure of the test is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X_{(n)} > k \\ 0 & \text{if} \quad X_{(n)} \leq k \end{cases}.$$

To determine k we must "exhaust the α-level" as follows:

$$\alpha = P(X_{(n)} > k|\theta = \tau) = 1 - P(X_{(n)} < k|\theta = \tau) = 1 - \left(\frac{k}{\tau}\right)$$

or equivalently

$$k = \tau(1-\alpha)^{\frac{1}{\tau n}}.$$

Hence, the UMP α-test is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X_{(n)} > \tau(1-\alpha)^{\frac{1}{\tau n}} \\ 0 & \text{if} \quad X_{(n)} \leq \tau(1-\alpha)^{\frac{1}{\tau n}} \end{cases}.$$

d) The power can be calculated as follows,

$$\begin{aligned} \text{Power}(\theta) = E_\theta(\varphi^*(X)) &= P(X_{(n)} > \tau(1-\alpha)^{\frac{1}{\tau n}}) \\ &= 1 - P(X_{(n)} < \tau(1-\alpha)^{\frac{1}{\tau n}}) \\ &= 1 - \left(\frac{\tau(1-\alpha)^{\frac{1}{\tau n}}}{\theta}\right)^{\tau n} \\ &= 1 - (1-\alpha)\left(\frac{\tau}{\theta}\right)^{\tau n} \end{aligned}$$

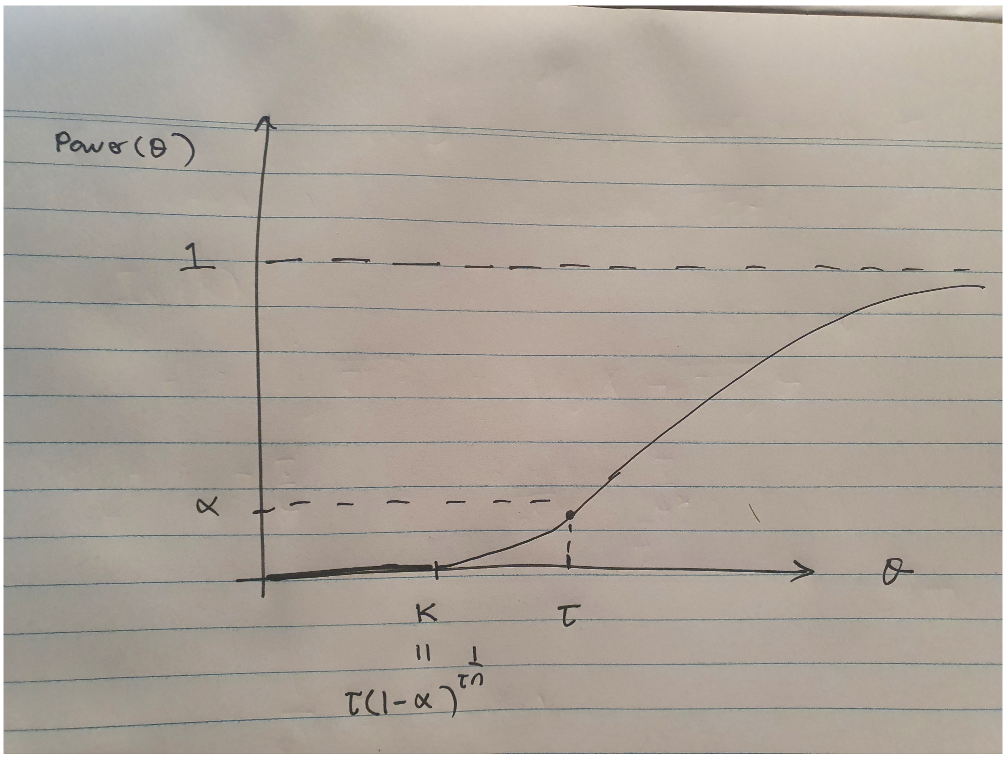
for $k = \tau(1-\alpha)^{\frac{1}{\tau n}} < \theta < \infty$.

e) Now we can compute the power function at the following to get some idea how to plot it:

$$\text{Power}(\tau(1-\alpha)^{\frac{1}{\tau n}}) = 1 - (1-\alpha)\left(\frac{\tau}{\tau(1-\alpha)^{\frac{1}{\tau n}}}\right)^{\tau n}(1-\alpha) = 1 - 1 = 0$$

$$\text{Power}(\tau) = 1 - (1-\alpha)\left(\frac{\tau}{\tau}\right)^{\tau n} = 1 - (1-\alpha) = \alpha$$

$$\text{Power}(\infty) = 1 - \left(\frac{\tau}{\infty}\right)^{\tau n}(1-\alpha) = 1 - 0 = 1$$



Problem 3
Suppose that X is a random variable with density function

$$f(x,\theta) = e^{-(x-\theta)}, \qquad \theta < x < \infty,$$

and zero elsewhere.

a) Let $X = (X_1, \dots, X_n)$ be a sample of n i.i.d. observations from this distribution.

   i) Compute the distribution and density function for $T = X_{(1)}$.
   ii) Find a statistic that has the MLR property.
   iii) Determine the uniformly most powerful α-size test of

$$H_0 : \theta \geq \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0.$$

   iv) Suppose the following data was collect $\mathbf{x} = (1, 2, 1.01, 3, 1.45)$. Test the hypothesis that $H_0 : \theta \geq 1$ versus $\theta < 1$ with a significance level $\alpha = 0.10$.
   v) Let $Z_n = n\big(X_{(1)} - \theta\big)$. Find the distribution Z_n converges to as $n \to \infty$.
   vi) Hence or otherwise justify that $X_{(1)}$ is a consistent estimator of θ.

b) Now suppose that $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ are the order statistics of a random sample of size five from this distribution. Let the observed value of $X_{(1)}$ be $x_{(1)}$. The test rejects $H_0 : \theta = 2$ and accepts $H_1 : \theta \neq 1$ when either $x_{(1)} \geq 2$ or $x_{(1)} < 1$.

   i) Find the power function $\gamma(\theta)$ for all values θ for this particular test.

   ii) Plot the power function $\gamma(\theta)$ for all values θ.

Solution:

a) i) The density can be directly computed from the formula in the lecture slides by noting that the distribution function for each X_i is

$$F_X(x) = \int_\theta^x e^{-(y-\theta)} \mathrm{d}y = -e^{-(y-\theta)}\Big|_\theta^x = 1 - e^{-(x-\theta)}, \qquad \theta < x < \infty.$$

which implies that

$$f_T(t) = n[1 - F_X(t)]^{n-1} f_X(t) = n\Big[1 - (1 - e^{-(t-\theta)})\Big]^{n-1} e^{-(t-\theta)} = ne^{-n(t-\theta)}$$

for $\theta < t < \infty$ and zero elsewhere. Then the distribution function can be computed as follows

$$F_T(t) = \int_\theta^t f_T(y)\mathrm{d}y = \int_\theta^t ne^{-n(y-\theta)}\mathrm{d}y = -e^{-n(y-\theta)}\Big|_\theta^t = 1 - e^{-n(t-\theta)}$$

for $\theta < t < \infty$ and when $t < \theta$ we have $F_T(t) = 0$.

ii) Suppose that $\theta'' > \theta'$ and consider the ratio

$$\frac{L(X,\theta'')}{L(X,\theta')} = \frac{\exp\Big(-\Big(\sum_{i=1}^n X_i - \theta''\Big)\Big) I_{(-\infty, X_{(1)})}(\theta'')}{\exp\Big(-\Big(\sum_{i=1}^n X_i - \theta'\Big)\Big) I_{(-\infty, X_{(1)})}(\theta')} = \exp\big(n(\theta''-\theta')\big)\frac{I_{(-\infty, X_{(1)})}(\theta'')}{I_{(-\infty, X_{(1)})}(\theta')}$$

which is monotonically increasing in the statistic $T = X_{(1)}$.

iii) From part ii) we know that the structure of the UMP level α test for testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ takes the form

$$\varphi^* = \begin{cases} 1 & \text{if} \quad X_{(1)} < c \\ 0 & \text{if} \quad X_{(1)} \geq c \end{cases}.$$

Then using part i) we are able to "exhaust the α level" as follows

$$\mathbb{E}_{\theta_0}(\varphi^*) = P_{\theta_0}(X_{(1)} < k) = 1 - e^{-n(c-\theta_0)} = \alpha$$

This implies that

$$e^{-n(c-\theta_0)} = 1-\alpha \implies -n(c-\theta_0) = \ln(1-\alpha)$$

Therefore,

$$c = -\frac{1}{n}\ln(1-\alpha) + \theta_0$$

Hence, the UMP α size test is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X_{(1)} < -\frac{1}{n}\ln(1-\alpha)+\theta_0 \\ 0 & \text{if} \quad X_{(1)} \geq -\frac{1}{n}\ln(1-\alpha)+\theta_0 \end{cases}.$$

iv) The test becomes

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X_{(1)} < 1.02 \\ 0 & \text{if} \quad X_{(1)} \geq 1.02 \end{cases}.$$

and since $x_{(1)} = 1.01$ we reject H_0.

v) First, from part (i) we have

$$f_T(t) = ne^{-n(t-\theta)} \qquad \text{and} \qquad F_T(t) = 1 - e^{-n(t-\theta)}$$

Then define the random variable $Z_n = n(X_{(1)} - \theta)$ which then has distribution function:

$$\begin{aligned} F_{Z_n}(z) = P(Z_n \leq z) &= P\Big(n(X_{(1)}-\theta)) \leq z\Big) \\ &= P\bigg(X_{(n)} \leq \frac{z}{n} + \theta\bigg) \\ &= 1 - \exp\Big(-n\Big(\frac{z}{n}+\theta-\theta\Big)\Big) \\ &= 1 - \exp(-z) \end{aligned}$$

for $0 < z < \infty$. Here, $F_Z(z)$ is the distribution function of an exponential random variable with mean one. Therefore, Z_n converges in distribution to an exponential random variable with mean one as $n \to \infty$.

vi) We consider evaluating $P(|X_{(1)} - \theta| < \epsilon)$ directly by noting that $X_{(1)}$ cannot possibly be smaller than θ. Hence

$$P(|X_{(1)} - \theta| < \epsilon) = P(X_{(1)} - \theta < \epsilon) = P(X_{(1)} \leq \epsilon + \theta) = 1 - e^{-nn(\epsilon+\theta-\theta)} = 1 - e^{-n\epsilon}$$

we conclude no matter what positive value ϵ takes,

$$P(X_{(1)} \leq \epsilon + \theta) \to 0$$

as desired.

b) i) First, from part (i) we have

$$F_{X_{(1)}}(t) = 1 - e^{-n(t-\theta)}, \qquad \theta < t < \infty$$

When $\theta < 1$ the power function $\gamma(\theta)$ is

$$\begin{aligned}
\gamma(\theta) &= P(X_{(1)} < 1 \quad \text{or} \quad X_{(1)} > 2) \\
&= \int_\theta^1 5e^{-5(t-\theta)}\mathrm{d}t + \int_2^\infty 5e^{-5(t-\theta)}\mathrm{d}t \\
&= -e^{-5(t-\theta)}\Big|_\theta^1 - e^{-5(t-\theta)}\Big|_2^\infty \\
&= \left(1 - e^{-5(1-\theta)}\right) - \left(0 - e^{-5(t-2)}\right) \\
&= 1 - e^{-5(1-\theta)} + e^{-5(2-\theta)}
\end{aligned}$$

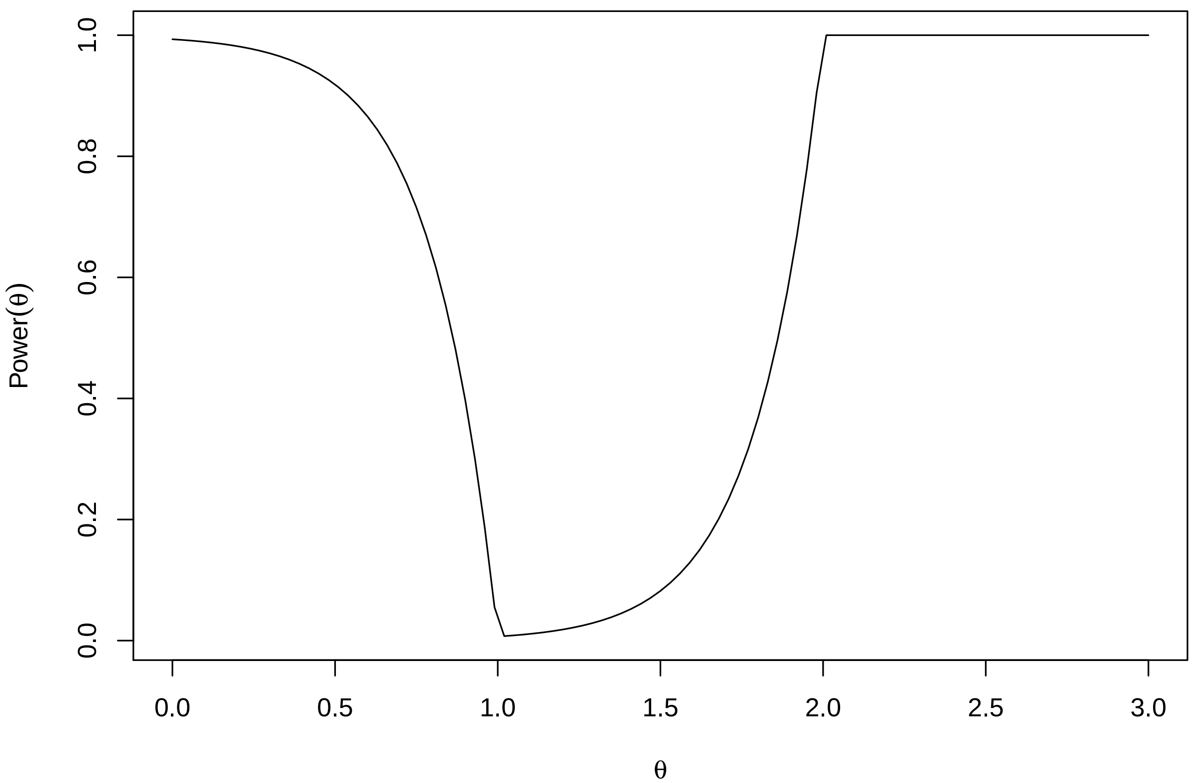
When $1 < \theta < 2$ the power function $\gamma(\theta)$ is

$$\begin{aligned}
\gamma(\theta) &= P(X_{(1)} < 1 \quad \text{or} \quad X_{(1)} > 2) \\
&= P(X_{(1)} > 2) \\
&= \int_2^\infty 5e^{-5(t-\theta)}\mathrm{d}t \\
&= -e^{-5(t-\theta)}\Big|_2^\infty \\
&= -\left(0 - e^{-5(t-2)}\right) \\
&= e^{-5(2-\theta)}
\end{aligned}$$

When $\theta > 2$ the power function $\gamma(\theta)$ is

$$\begin{aligned}
\gamma(\theta) &= P(X_{(1)} > 2) \\
&= P(X_{(1)} > 2) \\
&= \int_\theta^\infty 5e^{-5(t-\theta)}\mathrm{d}t \\
&= -e^{-5(t-\theta)}\Big|_\theta^\infty \\
&= 1
\end{aligned}$$

ii)


Power(θ)
θ
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Problem Four

Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is a random sample from the density

$$f(x; \alpha, \beta) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} I_{[\beta,\infty)}(x), \quad \alpha > 0, \quad \beta > 0.$$

a) Find the Maximum Likelihood Estimator (MLE) for both α and β. Write the MLE for α in terms of T where

$$T = \log\left(\frac{\prod_{i=1}^n X_i}{X_{(1)}^n}\right)$$

b) Consider testing

$$H_0 : \alpha = 1, \beta > 0 \qquad \text{versus} \qquad H_1 : \alpha \neq 1, \beta > 0.$$

Show that the likelihood ratio is given by the following

$$\lambda(X) = \left(\frac{T}{n}\right)^n e^{n-T}$$

c) Show that the Likelihood Ratio test (LRT) has rejection region of the form

$$\{X : T(X) \leq k_1 \quad \text{or} \quad T(X) \geq k_2\}$$

where $0 < k_1 < k_2$.

Solution:

a) The likelihood is given by

$$L(X, \alpha, \beta) = \prod_{i=1}^n \frac{\alpha\beta^\alpha}{x^{\alpha+1}} I_{[\beta,\infty)}(x_i) = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^n \frac{1}{x_i}\right)^{\alpha+1} I_{(-\infty, X_{(1)}]}(\beta)$$

with log-likelihood

$$\log L(X, \alpha, \beta) = n\log\alpha + n\alpha\log\beta - (\alpha+1)\log\prod_{i=1}^n X_i, \qquad \beta < X_{(1)}$$

Now, suppose that α is fixed (at any value) then the log-likelihood is an increasing function of β for $\beta < X_{(1)}$. Therefore, both the restricted and unrestricted MLEs of β are $\hat{\beta} = X_{(1)}$. For the MLE of α, we take the derivative and set it zero

$$\frac{\partial}{\partial\alpha}\log L(X, \alpha, X_{(1)}) = \frac{n}{\alpha} + n\log X_{(1)} - \log\prod_{i=1}^n X_i = 0$$

which gives the MLE:

$$\frac{n}{\hat{\alpha}} = \log\prod_{i=1}^n X_i - \log X_{(1)}^n$$

or

$$\hat{\alpha} = \frac{n}{\log\left(\prod_{i=1}^n X_i / X_{(1)}^n\right)} = \frac{n}{T}$$

Need to confirm that this delivers a maximum by checking the second derivative evaluated at the MLE:

$$\frac{\partial^2}{\partial \alpha^2} \log L(X, \alpha, \beta) = -\frac{n}{\alpha^2}$$

which is less than zero since $\alpha > 0$.

b) Under the null hypothesis, the MLE of α is $\hat{\alpha}_0 = 1$ and the MLE for β is still $\hat{\beta} = X_{(1)}$. The likelihood ratio statistic is given by

$$\begin{aligned}
\lambda(X) &= \frac{\hat{\alpha}_0^n \hat{\beta}^{n\hat{\alpha}_0} \left(\prod_{i=1}^n \frac{1}{x_i} \right)^{\hat{\alpha}_0+1} I_{(-\infty, X_{(1)}]}(\hat{\beta})}{\hat{\alpha}^n \beta^{n\hat{\alpha}} \left(\prod_{i=1}^n \frac{1}{x_i} \right)^{\hat{\alpha}+1} I_{(-\infty, X_{(1)}]}(\hat{\beta})} \\
&= \frac{X_{(1)}^n \Big/ \left(\prod_{i=1}^n X_i \right)^2}{(n/T)^n X_{(1)}^{n^2/T} \Big/ \left(\prod_{i=1}^n X_i \right)^{n/T+1}} \\
&= \left(\frac{T}{n}\right)^n \frac{X_{(1)}^n \Big/ \left(\prod_{i=1}^n X_i \right)}{X_{(1)}^{n^2/T} \Big/ \left(\prod_{i=1}^n X_i \right)^{n/T}} \\
&= \left(\frac{T}{n}\right)^n \frac{X_{(1)}^n \Big/ \left(\prod_{i=1}^n X_i \right)}{\left(X_{(1)}^n \Big/ \left(\prod_{i=1}^n X_i \right) \right)^{n/T}} \\
&= \left(\frac{T}{n}\right)^n \frac{e^{-T}}{(e^{-T})^{n/T}} \\
&= \left(\frac{T}{n}\right)^n e^{n-T}
\end{aligned}$$

c) Notice that the log-ratio $\log \lambda(X)$ has derivative with respect to T since:

$$\frac{\partial}{\partial T} \log \lambda(X) = \frac{\partial}{\partial T} n \log(T) - n\log(n) + n - T = \frac{n}{T} - 1$$

which implies that $\lambda(X)$ is increasing when $T \leq n$ and decreasing when $T \geq n$. Therefore, $T \leq k$ is the same as $T \leq k_1$ or $T \geq k_2$ for appropriately chosen constant k_1 and k_2 with $k_1 < k_2$.

Problem 5
Suppose $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$ are the order statistics based on a random sample of size four from the density $f(x) = 2e^{-2x}$, $x > 0$.

a) Find $\mathbb{E}(X_{(3)})$. You will ned to use a computer package to approximate the integral. E.g. the `integrate` function in `R`.

b) Find the density of the median $M = \frac{1}{2}(X_{(2)} + X_{(3)})$.

c) Using this result (or otherwise), find $P(M > \mathbb{E}(X))$. You will ned to use a computer package to approximate the integral. E.g. the `integrate` function in `R`.

Solution:

a) Using the formula in the lecture notes with $f_X(x) = 2e^{-2x}$, $x > 0$ and $F_X(x) = 1 - e^{-2x}$ with $n = 4$ and $r = 3$:

$$f_{X_{(r)}}(y_r) = \frac{n!}{(r-1)!(n-r)!}\big[F_X(y_r)\big]^{r-1}\big[1 - F_X(y_r)\big]^{n-r} f_X(y_r)$$

we obtain

$$\begin{aligned} f_{X_{(3)}}(y_3) &= \frac{4!}{(3-1)!(4-3)!}\Big[1 - e^{-2y_3}\Big]^{3-1}\Big[1 - (1 - e^{-2y_3})\Big]^{4-3} 2e^{-2y_3} \\ &= 24(1 - e^{-2y_3})^2 e^{-4y_3} \end{aligned}$$

Therefore,

$$\mathbb{E}(X_{(3)}) = \int_0^\infty 24y_3(1 - e^{-2y_3})^2 e^{-4y_3}\mathrm{d}y_3 = 0.542$$

This requires numerical integration using a computer program such as `R Studio`.

b) The joint density of $X_{(i)}$ and $X_{(i)}$ is

$$f_{X_{(i)},X_{(j)}}(x,y) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(x) f_X(y)\Big[F_X(x)\Big]^{i-1}\Big[F_X(y) - F_X(x)\Big]^{j-1-i}\Big[1 - F_X(y)\Big]^{n-j}$$

where here we have $i = 2$, $j = 3$, $n = 4$ which gives

$$\begin{aligned} f_{X_{(2)},X_{(3)}}(x,y) &= \frac{4!}{1!0!1!}2e^{-2x}2e^{-2y}\Big[1 - e^{-2x}\Big]^{2-1}\Big[1 - e^{-2y} - (1 - e^{-2x})\Big]^{3-1-2}\Big[1 - (1 - e^{-2y})\Big]^{4-3} \\ &= 96e^{-2x}e^{-2y}(1 - e^{-2x})e^{-2y} \\ &= 96e^{-2x-4y}(1 - e^{-2x}) \end{aligned}$$

for $0 < x < y < \infty$. Now by applying the density transformation formula with

$$U = \frac{1}{2}(X_{(2)} + X_{(3)}) \qquad \text{and} \qquad V = X_{(2)},$$

then we can equivalently write this as

$$X_{(2)} = V \qquad \text{and} \qquad X_{(3)} = 2U - V.$$

The value of the Jacobian of this transformation is equal to -2 since:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = 0 \times -1 - 2 \times 1 = -2$$

Therefore the absolute value of the Jacobian is equal to two. Using the density transformation formula we have:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_{(2)},X_{(3)}}(x_{(2)}(u,v), x_{(3)}(u,v))|J(u,v)| \\ &= f_{X_{(2)},X_{(3)}}(v, 2u-v) \cdot 2 \\ &= 96e^{-2v-4(2u-v)}(1-e^{-2v}) \cdot 2 \\ &= 192e^{-2v-8u+4v}(1-e^{-2v}) \\ &= 192e^{2v-8u}(1-e^{-2v}) \end{aligned}$$

for $0 < v < u < \infty$ since $0 < X_{(2)} < \frac{1}{2}(X_{(2)} + X_{(3)}) < \infty$ which implies $2X_{(2)} < X_{(2)} + X_{(3)}$ or $X_{(2)} < X_{(3)}$. Then if $M = \frac{1}{2}(X_{(2)} + X_{(3)})$, then the marginal density can be computed using integration as follows

$$\begin{aligned} f_M(u) &= 192 \int_0^u e^{2v-8u} - e^{-8u} \mathrm{d}v \\ &= 192 \Big[\frac{1}{2}e^{2v-8u} - ve^{-8u}\Big]_{v=0}^{v=u} \\ &= 192 \Big[\frac{1}{2}e^{-6u} - ue^{-8u} - \Big(\frac{1}{2}e^{-8u} - 0\Big)\Big] \\ &= 192 \Big[\frac{1}{2}e^{-6u} - ue^{-8u} - \frac{1}{2}e^{-8u}\Big] \end{aligned}$$

for $0 < u < \infty$.

c)

$$P(M > 0.5) = \int_{0.5}^{\infty} 192 \Big[\frac{1}{2}e^{-6u} - ue^{-8u} - \frac{1}{2}e^{-8u}\Big] \mathrm{d}u \approx 0.3020708.$$