Start: 6:05 pm

Faculty of Science

School of Mathematics and Statistics

MATH5905
Statistical Inference

Lecture Notes

written by Dr. Tom Stindl

Term 1, 2021

- 3.1 Data reduction in statistical inference
- 3.2 Sufficent partition example
- 3.3 Sufficiency principle
- 3.4 Neyman Fisher factorization criterion
- 3.5 Sufficiency examples
- 3.6 Lehmann and Scheffe's method for constructing a minimal sufficient partition
- 3.7 Minimal sufficient examples
- 3.8 One parameter exponential family densities
- 3.9 Generalization to a k- parameter exponential family
- 3.10 Ancillary statistic and ancillarity principle

6 Hypothesis testing

1 Elements of probability

Who am I?

Dr. Tom Stindl

Statistics lecturer at the School of Mathematics and Statistics.

About me:

- PhD in Statistics at UNSW
- Degree in Actuarial Science
- Rugby League Referee

Research interest is in developing statistical inferential methods for self-exciting point processes. Application domains include finance, seismology, bushfires and crime.

A face to the voice!

Who are you?

What students do we have here today?

Contact

Dr. Tom Stindl

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My office is on the first floor of the Red Centre building in RC-1037. Consultation will be avaliable with appointments to be made via email **and conducted using Zoom.**

For administrative problems, contact the **Student Services Office** (Mrs Markie Lugton, RC-3072, m.lugton@unsw.edu.au).

Lectures

I will give four hours of lectures per week except for week 6.

Wednesday	**18:00 - 20:00**	**Online BBC**
Friday	**18:00 - 20:00**	**Online BBC**

Note: the Friday lecture of week seven will be replaced with a recorded lecture.

Tutorials

Tutorials and computer labs (using RStudio) for this course are flexible and will be held during the lectures. More precise information will be given during lectures.

Online materials: Further information, skeleton lecture notes, and other materials will all be provided on Moodle.

A set of tutorial exercises will be available on Moodle. These problems are for you to enhance your mastery of the course. Some of the problems will be done in lectures, but you will learn a lot more if you try to do them before class.

Software used

We will use the R [R Core Team, 2017] software during the term. It is one of the most widely used software for Statistical computation and Graphics.

- Install it on your laptop: `https://cran.r-project.org`
- Next install `RStudio`, a nice Graphical User Interface to **R**: `http://www.rstudio.com/products/rstudio/download`

Computer laboratories

Computer laboratories (RC-G012 and RC-M020) are open 9-5 Monday-Friday on teaching days. RC-M020 has extended teaching hours (usually 8:30-9pm Monday-Friday, and 9-5 Monday-Friday on non-teaching weeks).

Course aims

- The aim of the course is to introduce the main ideas and principles behind parametric and non-parametric inference procedures.

频率观点和贝叶斯观点

- Both frequentist and Bayesian perspectives will be discussed.
- Estimation, confidence set construction and hypothesis testing are discussed within a decision-theoretic framework.
- Both finite sample optimality and asymptotic optimality will be defined and discussed.
- Computationally intensive methods such as bootstrap are discussed and are compared to asymptotic approximations such as Edgeworth expansions and saddlepoint method.
- Students will learn how to determine appropriate inference procedure and to draw inferences using the chosen procedure.

Assessment Details

Task	**Due**	**Weight**	**Duration**
Assignment 1	End of week 4	10%	2 weeks
Assignment 2	End of week 9	10%	2 weeks
Mid-session test	Week 7	20%	2 hours*
Final examination	May	60%	3 hours*

(*) Mid-session test and Final examination will be time-released assignments with additional time provided to upload the solutions already included.

In all assessments marks will be awarded for correct working and appropriate explanations and not just the final answer.

Consult the course outline on Moodle for more details.

Late assignments will *not* be accepted.

Skills to be developed

- Learn how statistical inference arises from the first principles of probability theory;
- Learn the fundamental principles of inference: sufficiency, likelihood, ancillarity, and equivariance;
- Learn the concepts of finite-sample and asymptotic efficiency of an inference procedure;
- Master the parametric and non-parametric delta method, asymptotic normality, Edgeworth expansions and saddlepoint method;
- Be able to estimate key population parameters of interest, to test hypotheses about them and to construct confidence regions;
- Be able to use in practice the parametric, nonparametric, Bayes and robust inference;
- Learn how to use the computer package **R** to generate output for the most common inference procedures and for computer-intensive calculations such as bootstrapping and robust estimation.

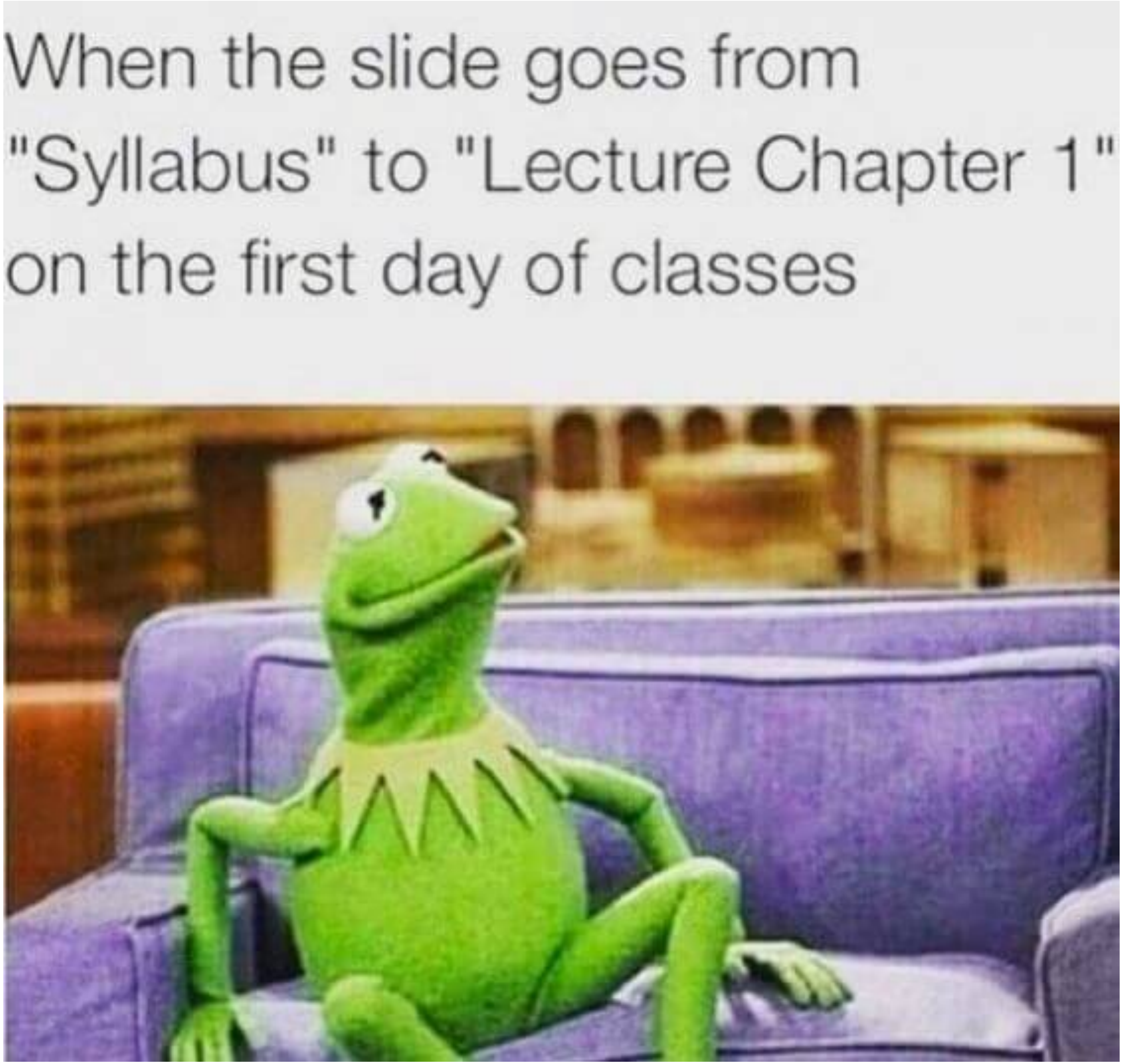
My philosophy for MATH5905

- Lecture notes provide a brief reference source for this course.
- At this stage, these are skeleton lecture notes only. Throughout the course other materials and textbooks will be used for deeper understanding.
- New ideas and skills are first introduced and demonstrated in lectures, then students develop these skills by applying them to specific tasks in tutorials and assessments.
- Computing skills will be used to some extent but this is not a course in computing; the computing part is mainly used to illustrate the theory/methodology.

Any questions?

Please feel free to interrupt me at any time!

Really!


When the slide goes from
"Syllabus" to "Lecture Chapter 1"
on the first day of classes

1.2 Probability

These lecture notes were originally written and developed by Professor Spiridon Penev.

二项分布，泊松分布，正态分布，指数分布

Standard univariate distributions such as the binomial, Poisson, normal, exponential are assumed to be known and are summarised in the Table of Common Distributions on pages 621–626 of **CB**. A copy can be found on Moodle.

These distributions will be used a lot throughout the course so don't worry if you don't know these off by heart just yet.

The revision mainly the follows the sections of the CB reference.

1.2.1 Events and probabilities

An experiment that includes randomness can be modelled with probabilities. An event A is assigned a probability, $P(A)$, which is a number between 0 and 1. The certain event has probability 1, while the impossible event has probability 0.

The simplest probabilistic model has a finite number m of possibilities (often called outcomes) and each of them has the same probability $1/m$.

A collection of k outcomes, where k is less than or equal to m, is called an event A and its probability is calculated as

$$P(A) = \frac{\text{the number of outcomes in A}}{\text{the total number of outcomes}} = \frac{k}{m}.$$

Example 1.1

Suppose there are n people in a Zoom meeting.

i) Find the probability that at least two people have the same birthday.

ii) Calculate the probability for $n = 22$.

iii) Calculate the probability for $n = 23$.

Solution:

i) Let A_n be the event that at least two people have the same birthday.
The number of outcomes **not** in the event A_n is

A_n^c "not A"

$$k = 365 \times 364 \times \cdots \times (365 - n + 1).$$

① ② ⓝ

# of outcomes in A_n^c

The total number of possible outcomes in the sample space of all birthday combinations is

$$m = 365 \times 365 \times \ldots 365 = 365^n.$$

1 2 n

The probability that all birthdays are distinct is

$$P(A_n^c) = \frac{k}{m} = \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}.$$

A_n

Hence the probability that two or more people have the same birthday in the Zoom meeting is

$$P(A_n) = 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}.$$

$= 1 - P(A_n^c)$

ii) For $n = 22$ this probability is

```
n <- 22
1 - prod(365:(365 - n + 1))/365^n
```

```
#> [1] 0.4756953
```

that is $P(A_{22}) = 0.48$.

iii) For $n = 23$ this probability is

```
n <- 23
1 - prod(365:(365 - n + 1))/365^n
```

```
#> [1] 0.5072972
```

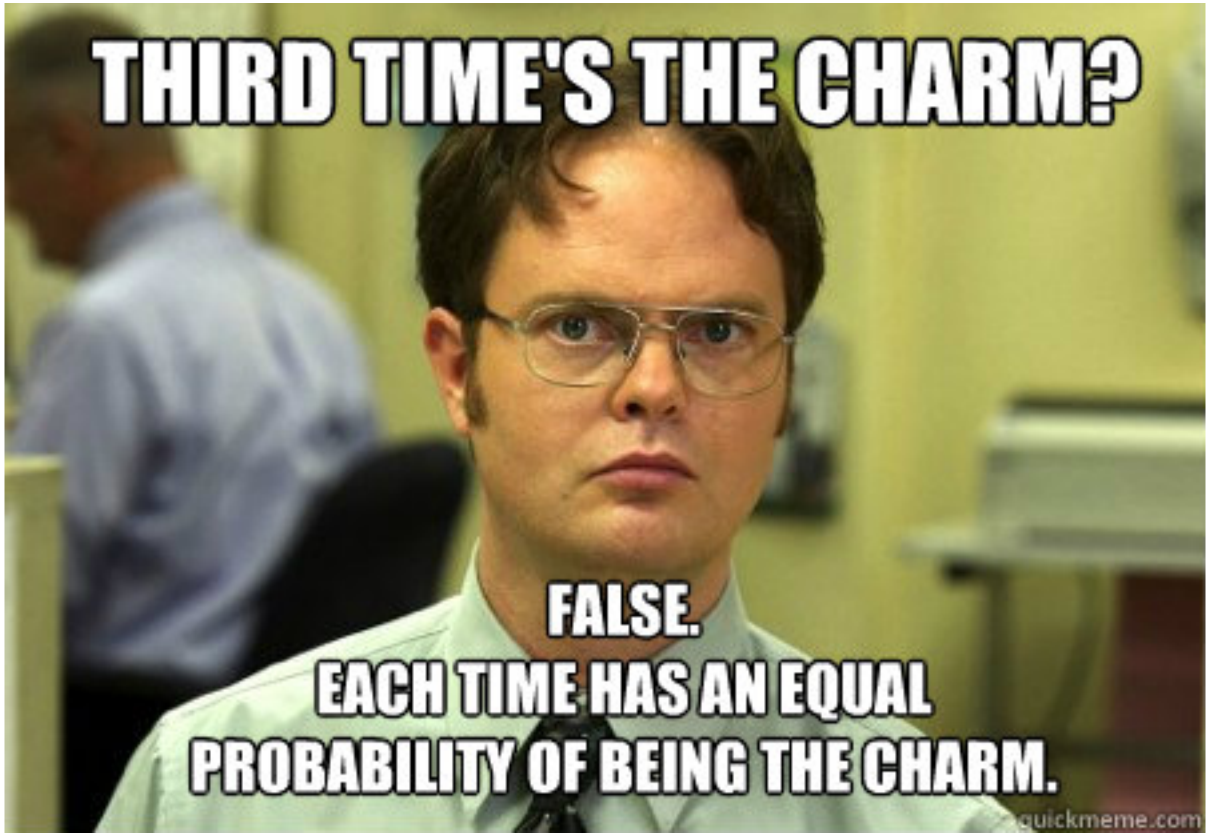
that is $P(A_{22}) = 0.51$.

Exercise 1.1 (revision)

Suppose there are n people in a Zoom meeting.

i) Find the probability that at least one person has the same birthday as you.

ii) Find the value of n, that is the number of people needed in the Zoom meeting, so that the probability that at least one person has the same birthday as you approaches $\frac{1}{2}$.

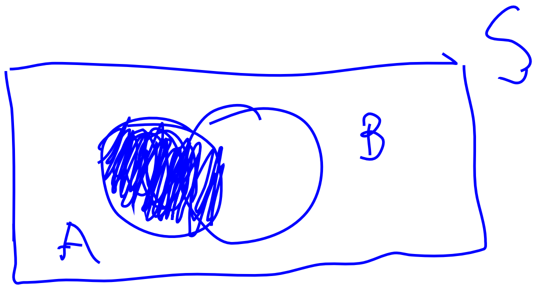

THIRD TIME'S THE CHARM?
FALSE.
EACH TIME HAS AN EQUAL
PROBABILITY OF BEING THE CHARM.
quickmeme.com

1.2.2 Conditional probability and independence

Conditional probabilities $P(A|B)$ are calculated by updating the probability $P(A)$ of a particular event under the additional information that a second event B has occurred.

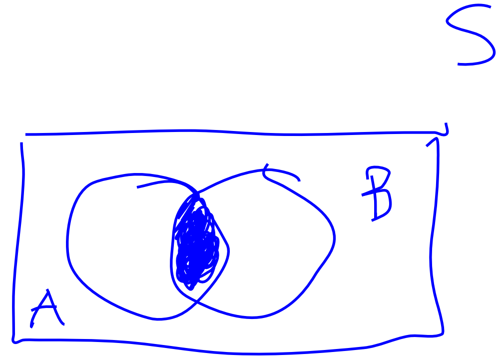
Conditional probabilities can be calculated as follows

$\frac{P(A)}{P(S)}$



$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

$P(A|B)$



where $P(A \cap B)$ is the probability that both A and B occur.

If the events A and B are independent then

$P(B|A) = P(B)$

$$P(A \cap B) = P(A)P(B) \quad \text{and} \quad P(A|B) = P(A).$$

Example 1.2

Suppose four cards are dealt from the top of a well-shuffled deck of 52 playing cards. Find the probability that all four cards are aces.

Solution:

The number of distinct groups of four cards is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{52}{4} = 270725$$

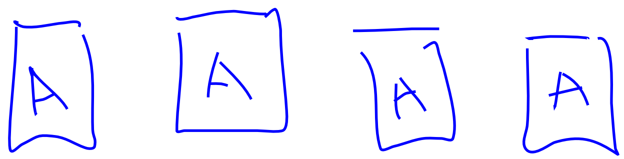
$$n! = n \times (n-1)(n-2)\cdots 3 \times 2 \times 1$$

Only one of these groups consists of all four aces and every group is equally likely. Therefore, the probability of being dealt all four aces is

$$\frac{k}{m} = \frac{1}{270725}.$$

Another way to calculate this probability is to first consider the probability that the first card is an ace, which is $4/52$. Given that the first card is an ace, the probability that the second card is an ace is $3/51$. Continuing this argument, we obtain

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{1}{270725}.$$



Example 1.3

Let us now see how the conditional probabilities change given that some aces have already been drawn from the deck. Four cards will again be dealt from a well-shuffled deck, and we now want to calculate

$$P(4 \text{ aces in } 4 \text{ cards} \,|\, i \text{ aces in } i \text{ cards}), \quad i = 1, 2, 3$$

The event "4 aces in 4 cards" is a subset of the event "i aces in i cards". Hence, from the definition of conditional probability we have

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} & P(4 \text{ aces in } 4 \text{ cards} \,|\, i \text{ aces in } i \text{ cards}) \\ = & \frac{P(\{4 \text{ aces in } 4 \text{ cards }\} \cap \{i \text{ aces in } i \text{ cards}\})}{P(i \text{ aces in } i \text{ cards})} \\ = & \frac{P(4 \text{ aces in } 4 \text{ cards })}{P(i \text{ aces in } i \text{ cards})} \end{aligned}$$

The numerator has already been calculated and the denominator can be calculated using a similar argument

$$\binom{n}{k} \overset{\text{def}}{=} \frac{n!}{k!(n-k)!}$$

$$P(i \text{ aces in } i \text{ cards}) = \frac{k}{m} = \frac{\binom{4}{i}}{\binom{52}{i}}.$$

Hence, the conditional probability is

$$\begin{aligned} P(4 \text{ aces in } 4 \text{ cards } | i \text{ aces in } i \text{ cards}) &= \frac{\binom{52}{i}}{\binom{52}{4}\binom{4}{i}} \\ &= \frac{(4-i)!48!}{(52-i)!} \\ &= \frac{1}{\binom{52-i}{4-i}} \end{aligned}$$

Try to show this.

$i = 1, 2, 3$

//

For $i = 1, 2$ and 3 the conditional probabilities are 0.00005, 0.00082 and 0.02041.

$$\frac{1}{\binom{52-i}{4-i}}$$

```r
i <- 1
1/choose(52 - i, 4 - i)
```

```
#> [1] 4.801921e-05
```

```r
i <- 2
1/choose(52 - i, 4 - i)
```

```
#> [1] 0.0008163265
```

```r
i <- 3
1/choose(52 - i, 4 - i)
```

```
#> [1] 0.02040816
```

1.2.3 Bayes' Theorem

Consider the following two equations that arise from the definition of the conditional probability

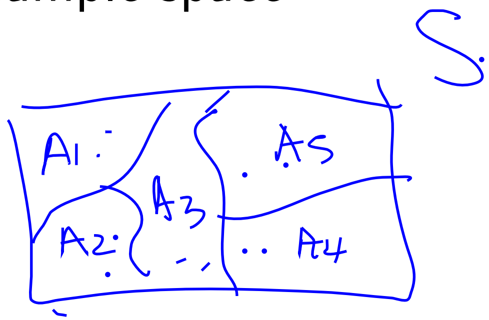
$$P(A \cap B) = P(A|B)P(B) \quad \text{and} \quad P(A \cap B) = P(B|A)P(A).$$

By equating and rearranging these two formulas gives **Bayes' theorem**:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

→ Reversing the condition

Additionally, for a general partition $A_1, A_2, \ldots, A_n$ of the sample space S with $P(A_i) > 0$ for all $i = 1, \ldots, n$, we have



$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

P(B)

for each $j = 1, \ldots, n$ which follows from the law of total probability

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Example 1.4

When a coded telegraph message is sent, there are sometimes errors in the transmission. In particular, Morse code uses "dots" and "dashes", which are known to occur in the proportion of $3:4$.

This means that for any given symbol

$$P(\text{dot sent}) = \frac{3}{7} \quad \text{and} \quad P(\text{dash sent}) = \frac{4}{7}.$$

Suppose there is interference on the transmission line, with probability $\frac{1}{8}$ a dot is mistakenly received as a dash, and vice 反之亦然 versa. Suppose we receive a dot, find the probability that a dot was actually transmitted.

p(dot sent | dot recieved)

Solution:

From the information given, we know that

$$P(\text{dot sent}) = \frac{3}{7} \quad \text{and} \quad P(\text{dot received} \mid \text{dot sent}) = \frac{7}{8}. = 1 - \frac{1}{8}$$

Using Bayes' theorem, we can write:

$$P(\text{dot sent} \mid \text{dot received}) = P(\text{dot received} \mid \text{dot sent}) \frac{P(\text{dot sent})}{P(\text{dot received})}$$

Additionally, we can write by the law of total probability

$$
\begin{aligned}
P(\text{dot received}) &= P(\text{dot received} \cap \text{dot sent}) + P(\text{dot received} \cap \text{dash sent}) \\
&= P(\text{dot received} \mid \text{dot sent})P(\text{dot sent}) + \\
&\qquad P(\text{dot received} \mid \text{dash sent})P(\text{dash sent}) \\
&= \frac{7}{8} \times \frac{3}{7} + \frac{1}{8} \times \frac{4}{7} \\
&= \frac{25}{56}
\end{aligned}
$$

$P(A \cap B) = P(A \mid B)P(B)$

3:7

Combining these results, we have that the probability of correctly receiving a dot

$$P(\text{dot sent} \mid \text{dot received}) = \frac{(7/8) \times (3/7)}{25/56} = \frac{21}{25}.$$

Exercise 1.2 (revision)

Suppose that 5% of men and 0.25% of women are colour-blind. A person is chosen at random and that person is colour-blind. Find the probability that the randomly selected person is male. Assume males and females to be in equal numbers.

Exercise 1.3 (at lecture)

66% 2/3 Two litters of a particular rodent species have been born, one with two brown-haired and one grey-haired (litter 1), and the other with three brown-haired and two grey-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

60% ≈ 3/5

i) Find the probability that the animal chosen is brown-haired.

ii) Given that a brown-haired offspring was selected, find the probability that the sampling was from litter 1.

$P(L1 \mid B)$?

1.3.1 Random variables

In many experiments, it is easier to deal with a summary variable, called a random variable, than with the original probability structure.

A random variable X is defined as a function from a sample space S into the set of real numbers

$$X : S \to \mathbb{R}$$

For example, consider an experiment where two dice are thrown, we can define the random variable X as a sum of the numbers rolled.

$S = \{(1,1), (1,2), (1,3), \ldots \ldots, (5,6), (6,6)\}$

$X : S \longrightarrow \{2, 3, \ldots, 12\}$ $P(X=2) = 1/36$

1.3.2 Probability mass function (pmf) for discrete random variables

Let us now consider real-valued realisations x of a discrete random variable X. The probability mass function (pmf) of a discrete random variable X is

概率质量函数

$$f(x) = P(X = x),$$

which describes the distribution of X by assigning probabilities for the events $\{X = x\}$.

1.3.3 Cumulative distribution function (cdf)

累积分布函数

A cumulative distribution function (cdf) of a random variable X is defined by

$$F(x) = P(X \leq x), \text{ for all } x$$

$F(4) = P(X \leq 4)$
$= P(X=4) + P(X=3)$
$+ P(X=2)$

Theorem 1.1

The function $F(x)$ is a cdf if and only if the following three conditions hold

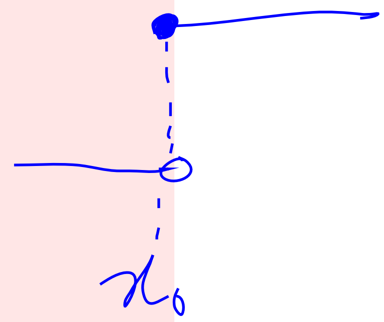
i) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

ii) $F(x)$ is a nondecreasing function of x

iii) $F(x)$ is right-continuous, that is, for every number x_0,

右连续

$$\lim_{x \downarrow x_0} F(x) = F(x_0).$$



This theorem is useful to determine whether a function is a valid cdf.

Example 1.5

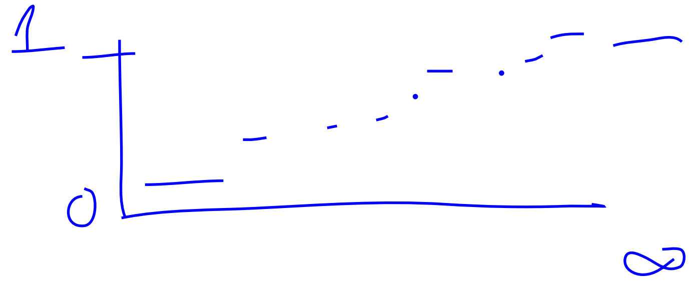
Consider the function $F(x) = 1 - (1-p)^x$ where $x = 1, 2, \ldots$ and $0 < p < 1$. Show that the conditions in the above theorem are satisfied.

$X \sim \text{Geometric}(p)$

$f(x) = (1-p)^{x-1} p$

Solution:

Since $F(x) = 0$ for all $x < 0$,



$$\lim_{x \to -\infty} F(x) = 0$$

and

$1 - 0$

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 - (\underbrace{1-p}_{<1})^x = 1$$

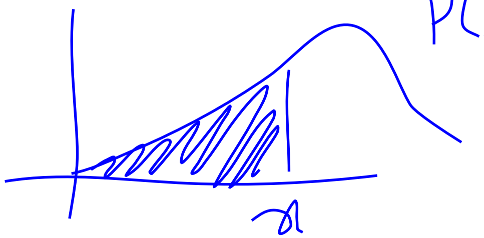
where x goes through only integer values when this limit is taken. To verify property (ii), we note that the sum: $F(x) = \sum_{i=1}^{x} (1-p)^{i-1} p$ contains more positive terms as x increases.

Geometric series

1.3.4 Probability density function (pdf) theorem

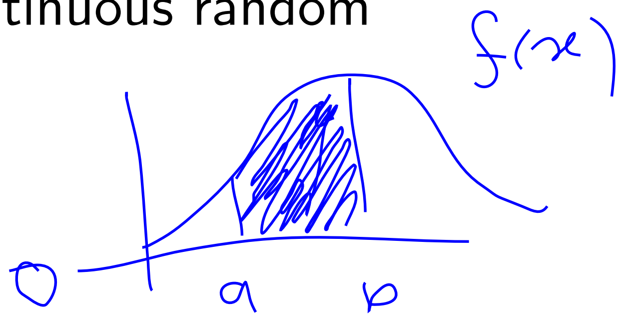
概率密度函数

The probability density function (pdf) $f(x)$ of a continuous random variable X is a function that satisfies



$P(X \le x)$

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t \quad \text{for all } x.$$



Theorem 1.2

A function $f(x)$ is a pdf (or pmf) of a random variable X if and only if the following two conditions hold

i) $f(x) \geq 0$ for all x

ii) $\sum_x f(x) = 1$ (pmf) or $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ (pdf)

1.3.5 Transformations

A transformation of random variable is a function of a random variable X with a known cdf $F(x)$. We will often be able to gain complete knowledge about the distribution of the transformed variable, or in other cases, will be able to acquire some understanding of the average behaviour of this transformed random variable.

If X is a random variable with a cdf $F(x)$, then any function of X, such that $Y = g(X)$, is also a random variable. $F_Y(y)$

We will introduce a subscript in the notation of a cdf and pdf to distinguish between two different random variables X and Y.

F_X f_X F_Y f_Y

6:05 pm start

Theorem 1.3

Let X be a random variable with cdf function $F_X(.)$ and density $f_X(.)$. Let $Y = g(X)$ and $F_Y(.)$ be the cdf of Y. Denote

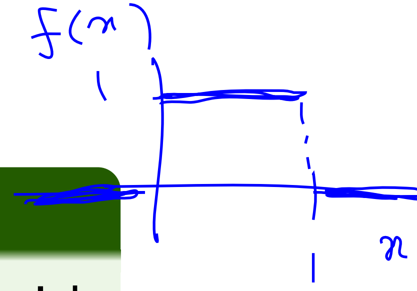
$$S_X = \{x : f_X(x) > 0\} \quad \text{and} \quad S_Y = \{y : y = g(x) \text{ for some } x \in S_X\}.$$

i) If g is increasing on S_X then $F_Y(y) = F_X(g^{-1}(y))$ for $y \in S_y$.

ii) If g is decreasing on S_X and X is continuous random variable then $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in S_y$.

Proof: at lecture.

$S_X = [0, 1]$

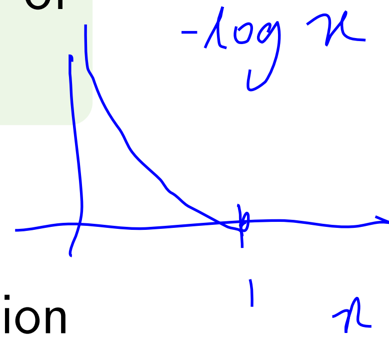


Example 1.6

均匀分布

Let X be a uniformly distributed random variable $X \sim U(0, 1)$ with density $f_X(x) = 1$ if $0 < x < 1$ and 0 otherwise. Find the density of the transformed variable $Y = g(X) = -\log(X)$.

g



Solution:

The cdf is $F_X(x) = x$ when $0 \leq x \leq 1$. We now make a transformation

$$Y = g(X) = -\log(X)$$

We can verify that $g(x)$ is a decreasing function of x and since X ranges from 0 to 1, $Y = -\log(X)$ ranges from 0 to ∞, that is $S_Y = (0, \infty)$.

$F_X(x) = \int_0^x 1 \, dx = x \quad 0 < x < 1$

For $y > 0$, $y = -\log(x)$ implies that $x = e^{-y}$ and hence $g^{-1}(y) = e^{-y}$.

Hence, using the above theorem, we have for $y > 0$

$$F_Y(y) = 1 - F_X(g^{-1}(y)) = 1 - F_X(e^{-y}) = 1 - e^{-y}. \qquad y \geq 0$$

Additionally, $F_Y(y) = 0$ for $y \leq 0$. We can recognise $F_Y(y)$ as the cdf of the standard exponential distribution.

In summary, the $-\log$ transformed uniform $[0, 1]$ random variable is standard exponentially distributed.

$X \sim U[0,1]$ then $-\log X = Y \sim \exp(1)$

1.3.6 Density transformation formula

Theorem 1.4

单调的

Let X have a pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let S_X and S_Y be as defined previously and suppose that $f_X(x)$ is continuous on S_X and that $g^{-1}(y)$ has a continuous derivative on S_Y. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & \text{for } \; y \in S_Y \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Follows from Theorem 1.3, by taking the derivative of the cdf $F_y(y)$ to obtain the pdf $f_Y(y)$.

$X \sim \text{Gamma}(n, \beta)$

Example 1.7

Let X be gamma distributed with density

$$f_X(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, \quad 0 < x < \infty$$

where β is a positive constant and n is a positive integer. Find the pdf of transformed variable $g(X) = 1/X$.

$g(x)$

$S_x = [0, \infty]$

$S_y = [0, \infty]$

Solution:

First note that $S_Y = S_X = (0, \infty)$. Now since $y = g(x) = \frac{1}{x}$, then $g^{-1}(y) = 1/y$ and $\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = -1/y^2$. Then by applying the density transformation formula for $y \in (0, \infty)$ we obtain

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y} \right)^{n-1} e^{-1/\beta y} \frac{1}{y^2} \\
&= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y} \right)^{n+1} e^{-1/\beta y}
\end{aligned}
$$

$x \to \frac{1}{y}$

which is a special case of an inverse gamma pdf.

$Y \sim$ Inv-Gamma(n, β)

1.3.7 Probability integral transform

概率积分变换

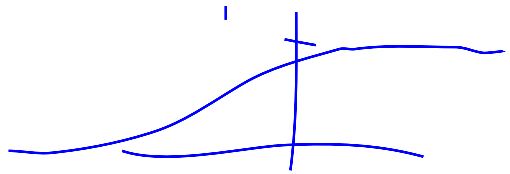
$g = F_X$

Theorem 1.5

Let X be a continuous random variable with a cdf $F_X(.)$. Then the random variable $Y = F_X(X)$ is uniformly distributed on $[0, 1]$.

Proof.

Using Theorem 1.3 and noting that F_X is a continuous and monotone increasing transformation, we have that



$$F_Y(y) = F_X(F_X^{-1}(y)) = y$$

$S_X \in \mathbb{R}$

and $S_Y = \{0 \le y \le 1\}$. Hence Y is uniformly distributed on $[0, 1]$.

$S_Y \in [0,1]$

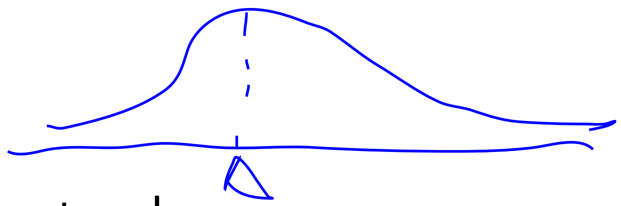
□

This fact is very useful in random number generation from a given distribution. If it is required to generate an observation X from a population with cdf $F_X(x)$, we need only to generate a uniform random number U, between 0 and 1 and solve for x in the equation $F_X(x) = u$.

$x = F^{-1}(u)$ where $U \sim \text{Unif}[0,1]$

There are often more computationally efficient methods for random number generation. However, this method is still useful because of its general applicability.

1.4.1 Expected values



The expected value of a distribution can be understood as a measure of the centre of a distribution, which is obtained by weighting the values of the random variable according to the probability distribution.

The formal definition states, that the expected value or mean of a random variable $g(X)$, denoted by $\mathbb{E}(g(X))$ is

$$\mathbb{E}(g(X)) = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{if } X \text{ is continuous} \\ \sum_{x \in S_x} g(x) f_X(x) = \sum_{x \in S_x} g(x) P(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

given that the interval or sum exists.

Example 1.8

Suppose X has an exponential λ distribution, with density

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad 0 \le x < \infty, \quad \lambda > 0.$$

Calculate the expected value of X.

Solution:

$$E(X) = \int_0^\infty x \cdot \frac{1}{\lambda} e^{-x/\lambda} dx$$

The $\mathbb{E}(X)$ is given by

$$\mathbb{E}(X) = \int_0^\infty \frac{1}{\lambda} x e^{-x/\lambda} dx$$

? ? ? ? ?

$$= \cancel{-xe^{-x/\lambda}\Big|_0^\infty} + \int_0^\infty e^{-x/\lambda} dx \qquad \text{(integration by parts)}$$

$$= \int_0^\infty e^{-x/\lambda} dx = \lambda.$$

1.4.2 Moments 矩

n阶矩被定义为一变量的n次方与其概率密度函数（Probability Density Function, PDF）之积的积分

Moments of a distribution are an important class of expectations. For each integer n, the nth moment of X,

$$\mu_n' = \mathbb{E}(X^n).$$

$\mu_n' = E(X^n)$

The nth central moment of X, μ_n, is defined by

$$\mu_n = \mathbb{E}((X - \mu)^n),$$

where $\mu = \mu_1' = \mathbb{E}(X)$.

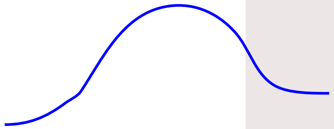
In particular, the variance of a random variable X is its second central moment,

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

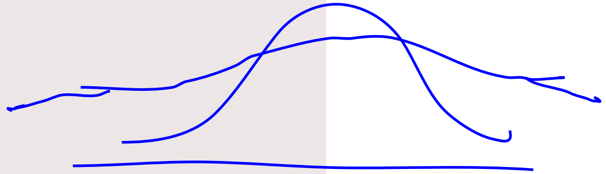
Hence, the variance is a measure of the degree of spread of a distribution around its mean.

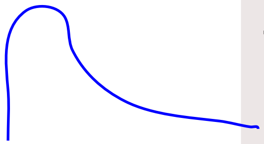
Exercise 1.4 (revision)

Let μ_n denote the nth central moment of a random variable X. Two quantities of interest, in addition to mean and variance, are



$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}.$$





偏度 峰度

The value of α_3 is called the skewness and α_4 the kurtosis. The skewness measures the lack of symmetry in the pdf. The kurtosis measures the peakedness or flatness of the pdf.

偏度（skewness），是统计数据分布偏斜方向和程度的度量，是统计数据分布非对称程度的数字特征。定义上偏度是样本的三阶标准化矩。

峰度（peakedness；kurtosis）又称峰态系数。表征概率密度分布曲线在平均值处峰值高低的特征数。直观看来，峰度反映了峰部的尖度。随机变量的峰度计算方法为：随机变量的四阶中心矩与方差平方的比值。

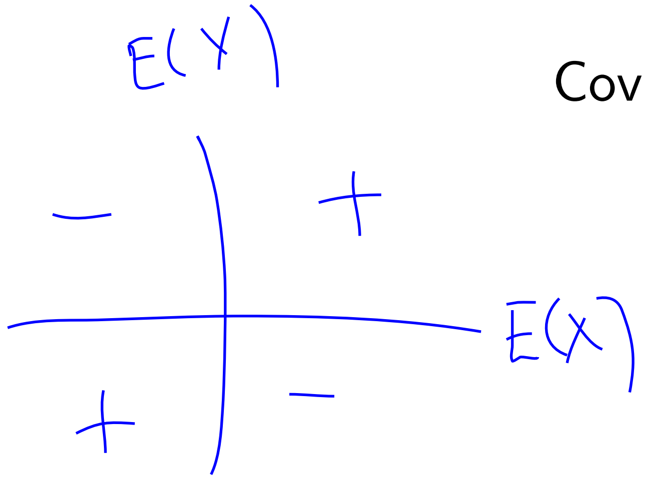
i) Show that if a pdf is symmetric about a point a, then $\alpha_3 = 0$.
Hint: Show that $\mu_3 = 0$ for a general density function $f(x)$.

ii) Calculate α_4 for the following:

$$f(x) = \frac{1}{2}, \quad -1 < x < 1.$$

1.4.3 Covariance and correlation

Covariance and correlation are given by the following formulas



$$\mathsf{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))]$$
$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\rho = \mathsf{Cor}(X,Y)$$
$$= \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}}$$

$-1 \le \rho \le 1$

1.5.1 Random vector

Now we will generalise the concepts of cumulative distribution function, probability mass function and density function for univariate random variables to allow multivariate modelling. We also explain how the independence of random variables relates to the construction of appropriate models.

Let consider a random vector

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \in R^p$$

where $p \geq 2$ and there are p different components, each of which is a random variable with a cumulative distribution function $F_{X_i}(x_i), i = 1, 2, \ldots, p$. Each of the functions $F_{X_i}(x_i)$ is called a marginal distribution.

边缘分布

1.5.2 Joint cumulative distribution function

联合累计分布函数

The joint cdf of the random vector $\boldsymbol{X}$ is:

$$\begin{aligned} F_{\boldsymbol{X}}(\boldsymbol{x}) &= P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_p \leq x_p) \\ &= F_{\boldsymbol{X}}(x_1, x_2, \ldots, x_p) \end{aligned}$$

1.5.3 Joint probability mass/density function

In case of a discrete vector $\boldsymbol{X}$ the probability mass function is defined as

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 = x_1, X_2 = x_2, \ldots, X_p = x_p)$$

If a density $f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{X}}(x_1, x_2, \ldots, x_p)$ exists such that

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} f_{\boldsymbol{X}}(\boldsymbol{t}) \mathrm{d}t_1 \ldots \mathrm{d}t_p$$

then $\boldsymbol{X}$ is a continuous random vector with a joint density function of p arguments $f_{\boldsymbol{X}}(\boldsymbol{x})$. In this case the following holds

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\partial^p F_{\boldsymbol{X}}(\boldsymbol{x})}{\partial x_1 \partial x_2 \ldots \partial x_p}$$

If $\boldsymbol{X}$ has p independent components then

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{X_1}(x_1)F_{X_2}(x_2)\dots F_{X_p}(x_p)$$

cdf

holds and, equivalently, also

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = p_{X_1}(x_1)p_{X_2}(x_2)\dots p_{X_p}(x_p)$$

pmf

and

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1}(x_1)f_{X_2}(x_2)f_{X_p}(x_p)$$

pdf

holds.

Exercise 1.5 (revision)

A pdf is defined by $\begin{pmatrix} X \\ Y \end{pmatrix}$

$$f(x,y) = \begin{cases} C(x+2y) & \text{if} \quad 0<y<1 \quad \text{and} \quad 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of C.

Solve: $\int_0^2 \int_0^1 c(x+2y)\, dy\, dx = 1$

$C = 0.25$

1.5.4 Marginal distributions

The previous slides defined what a joint distribution of a random vector $\mathbf{X}$ is. This section will explain how to obtain the marginal distribution or conditional distribution for some components of the random vector $\mathbf{X}$when the joint distribution of $\mathbf{X}$ is known.

$\underbrace{X_1, X_2, \ldots}_{k}, X_p$

The marginal cdf of the first $k < p$ components of the vector $\mathbf{X}$ is defined in a natural way as follows

$F(x_1, x_2, \ldots, x_k)$

$$
\begin{aligned}
= \; & P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k) \\
&= P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k, \underbrace{X_{k+1} \leq \infty, \ldots, X_p \leq \infty}) \\
&= F_{\boldsymbol{X}}(x_1, x_2, \ldots, x_k, \infty, \infty, \ldots, \infty)
\end{aligned}
$$

The marginal density of the first k components can be obtained by partial differentiation

$$
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\boldsymbol{X}}(x_1, x_2, \ldots, x_p) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_p
$$

still a function $x_1, \ldots, x_k$

Exercise 1.6 (revision)

Consider the pdf:

$$f(x,y) = \begin{cases} C(x+2y) & \text{if} \quad 0<y<1 \quad \text{and} \quad 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal distribution of X.

Solve:

$f_X(x) = \int_0^1 C(x+2y)\,dy \qquad 0<x<2$

1.5.5 Conditional distributions

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

The conditional density X when $X_{r+1} = x_{r+1}, \dots, X_p = x_p$ is defined by

$$f_{(X_1,\dots,X_r|X_{r+1},\dots,X_p)}(x_1,\dots,x_r|x_{r+1},\dots,x_p) = \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{f_{X_{r+1},\dots,X_p}(x_{r+1},\dots,x_p)}$$

The above conditional density is interpreted as the joint density of $X_1,\dots,X_r$ when $X_{r+1} = x_{r+1}, \dots, X_p = x_p$ and is only defined when $f_{X_{r+1},\dots,X_p}(x_{r+1},\dots,x_p) \neq 0$.

In the case of mutual independence the p components, all conditional distributions do not depend on the conditions and it holds that

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{p} F_{X_i}(x_i) \quad \text{and} \quad f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{p} f_{X_i}(x_i).$$

1.5.6 Moments

$\in \mathbb{R}^p$

Given the density $f_{\boldsymbol{X}}(\boldsymbol{x})$ of the random vector $\boldsymbol{X}$ the joint moments of order $s_1, s_2 \ldots, s_p$ is given by

$$\mathbb{E}(X_1^{s_1} \ldots X_p^{s_p}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{s_1} \ldots x_p^{s_p} f_{\boldsymbol{X}}(x_1, \ldots, x_p) \mathrm{d}x_1 \ldots \mathrm{d}x_p$$

1.5.7 Density transformation formula

Assume that the p existing random variables $X_1, X_2, \ldots, X_p$ with given density $f_{\boldsymbol{X}}(\boldsymbol{x})$ has been transformed by a smooth (i.e. differentiable) one-to-one transformation into p new random variables $Y_1, Y_2, \ldots, Y_p$, i.e. a new random vector $\boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{p}}$ has been created by calculating

$$Y_i = y_i(X_1, X_2, \ldots, X_p), \quad i = 1, 2, \ldots, p$$

The question is how to calculate the density $g_{\boldsymbol{Y}}(\boldsymbol{y})$ of $\boldsymbol{Y}$ by knowing the transformation functions $y_i(X_1, X_2, \ldots, X_p)$, $i = 1, 2, \ldots, p$ and the density $f_{\mathbf{X}}(\mathbf{x})$ of the original random vector.

Since the transformation of the X into Y is assumed to be one-to-one, its inverse transformation $X_i = x_i(Y_1, Y_2, \ldots, Y_p)$, $i = 1, 2, \ldots, p$ also exists and then the following density transformation formula applies

$$g_{\mathbf{Y}}(y_1, \ldots, y_p) = f_{\mathbf{X}}\Big(x_1(y_1, \ldots, y_p), \ldots, x_p(y_1, \ldots, y_p)\Big)\Big|J(y_1, \ldots, y_p)\Big|$$

where $J(y_1, \ldots, y_p)$ is the Jacobian of the transformation

$$J(y_1, \ldots, y_p) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \ldots & \frac{\partial x_1}{\partial y_p} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \ldots & \frac{\partial x_2}{\partial y_p} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x_p}{\partial y_1} & \frac{\partial x_p}{\partial y_2} & \ldots & \frac{\partial x_p}{\partial y_p} \end{vmatrix}$$

Exercise 1.7 (revision)

Given the pdf

$$f(x,y) = \begin{cases} C(x+2y) & \text{if} \quad 0<y<1 \quad \text{and} \quad 0<x<2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the pdf of the random variable

$$Z = \frac{9}{(X+1)^2}.$$

$$f_Z(z) = f_X(x(z))\left|\frac{dx(z)}{dz}\right|$$

Notice that this is a one-dimensional density transformation.

Exercise 1.8 (at lecture)

Let X and Y be independent, standard normal random variables. Consider the transformation $U = X + Y$ and $V = X - Y$. Find the joint density of U and V.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} X+Y \\ X-Y \end{pmatrix}$$

1.5.8 Multivariate normal distribution

多元正态分布

We will only need the **non-degenerated** multivariate normal.

Consider the term

$$\left(\frac{x-\mu}{\sigma}\right)^2 = (x-\mu)(\sigma^2)^{-1}(x-\mu)$$

in the univariate normal density of

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty.$$

We will generalize this to

$$(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}).$$

Here $\boldsymbol{\mu} = E(\boldsymbol{X}) \in \boldsymbol{R}^p$ is the expected value of the random vector $\boldsymbol{X} \in \boldsymbol{R}^p$ and the (assumed to be positive definite) matrix

Cov(X2,Xp)

$$\Sigma = E\left[(X-\mu)(X-\mu)'\right] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{pmatrix} \in \mathcal{M}_{p,p}$$

is the covariance matrix.

Cov(Xp,X2)

- The diagonals elements are the variances of the p random variables which we simply denote σ_{ii} by σ_i^2;
- σ_{ij}, $i \neq j$ are the covariances.

The final result

$$f(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} e^{-(x-\mu)'\Sigma^{-1}(x-\mu)/2}, \quad -\infty < x_i < \infty, \;\; i = 1, 2 \dots, p$$