Theorem 7.27 (Probability-integral transform)

$Y = F_X(X)$

For $Y = F_Y(y)$ we have for $0 < y < 1$:

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P(F_X(X) \le y) \\ &= P\Big(F_X^{-1}(F_X(X)) \le F_X^{-1}(y)\Big) \\ &= P(X \le F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y \end{aligned}$$

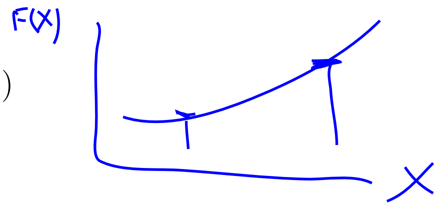
$P(Y \le y) = 0$

At the end points we have $P(Y \le y) = 1$ for $y > 1$ and $\cancel{P(Y \le y) = 0}$ for $y \le 0$, showing that Y has a uniform distribution on the unit interval. Now let

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)}$$

F

be the order statistics for a random sample from a continuous distribution. Since F is an increasing function, this implies that



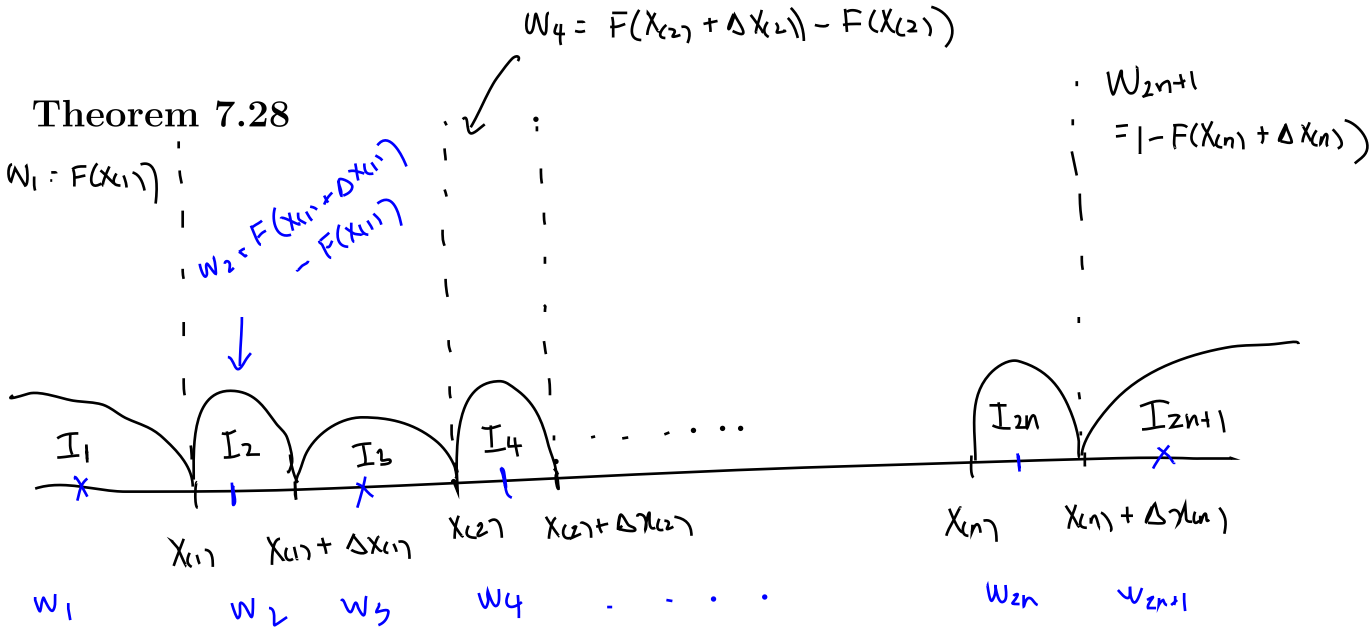
$$F_X(X_{(1)}) < F_X(X_{(2)}) < \ldots < F_X(X_{(n)})$$

or

$$U_{(1)} < U_{(2)} < \ldots < U_{(n)}$$

are distributed as the order statistics from a uniform distribution on $(0, 1)$.

Theorem 7.28



Important: the real line $(-\infty, \infty)$ is subdivided into $2n+1$ disjoint intervals $I_1, I_2, \ldots, I_{2n+1}$. For these intervals we calculate the probabilities $w_1, w_2, \ldots, w_{2n+1}$ for having a realization in each of them using the CDF $F(\cdot)$ only.

Therefore, by looking at a particular realization of the Multinomial$(n, w_1, w_2, \ldots, w_{2n+1})$ distribution where precisely one realization falls in each of the intervals $I_2, I_4, \ldots, I_{2n}$ and from the properties of the multinomial distribution we get this probability as

$$n!, \quad \frac{n!}{0!1!0!1!\ldots0!1!0!}\underbrace{[F_X(x_{(1)} + \Delta x_{(1)}) - F_X(x_{(1)})]}_{\Delta x_{(1)}} \ldots [F_X(x_{(n)} + \Delta x_{(n)}) - F_X(x_{(n)})]$$

By dividing this product by the product of the increments

$$\Delta x_{(1)} \Delta x_{(2)} \ldots \Delta x_{(n)}$$

and letting $\max_{i=1,2,\ldots,n} \Delta x_{(i)} \to 0$ we get on one hand:

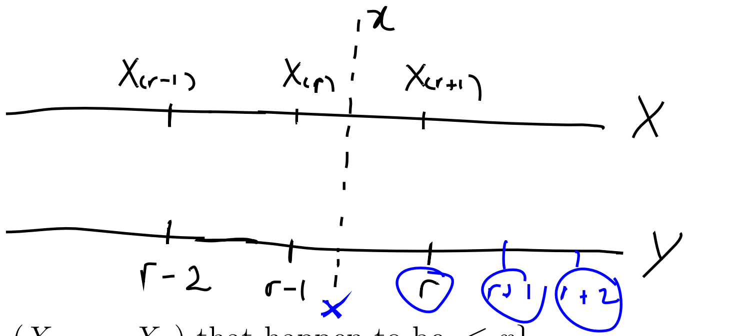
$$n! \prod_{i=1}^{n} f_X(x_{(i)})$$

in the limit. On the other hand: this limit per definition delivers the mixed partial derivatives of the CDF, that is, gives us the density

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)}, \ldots, x_{(n)})$$

□

$$F(x,y) \longrightarrow \frac{\partial}{\partial x}\frac{\partial}{\partial y} F(x,y) = f(x,y)$$



$Y \in \{0, 1, \ldots, n\}$

Theorem 7.29

Let's introduce a discrete random variable

$$Y = \{\text{number of observations from the sample } (X_1, \ldots, X_n) \text{ that happen to be } \le x\}$$

Therefore, $Y \sim \text{Bin}(n, F_X(x))$ and we make the critical observation that

$$F_{X_{(r)}}(x) = P(X_{(r)} \le x) = P(Y \ge r)$$

$Y \sim Bin(n, F(x))$
$P(Y=k) = \binom{n}{k} F(x)^k (1-F(x))^{n-k}$

Hence, we need to evaluate the RHS:

$$F_{X_{(r)}}(x) = P(Y \ge r) = \sum_{k=r}^{n} \binom{n}{k} F_X(x)^k (1 - F_X(x))^{n-k}$$

To obtain the density we need to differentiate each of the terms in the summands in $\sum_{k=r}^{n}$ for $k = r, \ldots, n$ by applying the product rule $(uv)' = u'v + uv'$ formula each time. For $k = r$ we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\binom{n}{r} F_X(x)^r (1 - F_X(x))^{n-r}\right] \\
&= \binom{n}{r} r F_X(x)^{r-1} f_X(x) (1 - F_X(x))^{n-r} - \binom{n}{r} F_X(x)^r (1 - F_X(x))^{n-r-1} (n-r) f_X(x) \\
&= \frac{n!}{(r-1)!(n-r)!} F_X(x)^{r-1} (1 - F_X(x))^{n-r} f_X(x) - A_1
\end{aligned}$$

$\frac{n!}{r!(n-r)!} r$

Now when we apply similar rules to each of the summands we get:

$$\begin{aligned}
f_{X_{(r)}}(x) = &\frac{n!}{(r-1)!(n-r)!} F_X(x)^{r-1} (1 - F_X(x))^{n-r} f_X(x) \cancel{- A_1} \\
&+ \cancel{A_1} - \cancel{A_2} \\
&+ \cancel{A_2} - \cancel{A_3} \\
&+ \cancel{\ldots} - \cancel{\ldots} \\
&\ldots \\
&+ (n-n) \times (\ldots)
\end{aligned}$$

$0 \times 0 = 0$

Large cancellations occur since $\binom{n}{r}(n-r) = \binom{n}{r+1}(r+1)$ and each of the summands after the first disappear! Therefore, we are left with:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) F_X(x)^{r-1} (1 - F_X(x))^{n-r}$$

□