**THE UNIVERSITY OF NEW SOUTH WALES**

**DEPARTMENT OF STATISTICS**

**Additional exercises for MATH5905, Statistical Inference**

**Part four: Multinomial distribution. Order statistics**

1. a) $X_1, X_2, X_3$ has a multinomial $(8; 0.2, 0.3, 0.5)$ distribution. Find $P(X_1 = 2, X_2 = 2, X_3 = 4)$, $E(X_2), Var(X_2)$ and $Cov(X_1, X_3)$.

   b) $X_1, X_2, X_3$ has a multinomial $(6; 0.5, 0.2, 0.3)$ distribution. Find $P(X_1 = 3, X_2 = 1, X_3 = 2)$ and $P(X_1 + X_2 = 2)$.

2. Find the probability density function of the second order statistic $X_{(2)}$ in a random sample of size four from a population with the density function

$$f(x) = \begin{cases} e^{1-x}, 1 < x < \infty, \\ 0 \text{ elsewhere} \end{cases}$$

3. Find the probability density function of $X_{(4)}$ in a random sample of size five from a population with the density function

$$f(x) = \begin{cases} \frac{1}{x^2}, 1 \leq x < \infty, \\ 0 \text{ elsewhere} \end{cases}$$

4. The opening prices per share of two similar stocks $Y_1$ and $Y_2$ are independent random variables, each with density function

$$f(y) = \begin{cases} \frac{1}{2}e^{-\frac{y-4}{2}}, y \geq 4, \\ 0 \text{ elsewhere} \end{cases}$$

   On a given morning Mr. A is going to buy shares of whichever stock is less expensive. Find the probability density function and the expected value for the price per share that Mr. A will have to pay.

5. Find the expected value of the largest order statistic in a random sample of size 3 from:

   a) the exponential distribution with density $f(x) = e^{-x}, x \geq 0$.

   b) the standard normal distribution

6. Electric components of a certain type have lifetime $Y$ with probability density given by

$$f(y) = \begin{cases} \frac{1}{100}e^{-\frac{y}{100}}, y > 0, \\ 0 \text{ elsewhere} \end{cases}$$

   a) Suppose that two such components operate independently and in series in a certain system (that is, the system fails when either component fails). Find the density function for $X$, the lifetime of the system.

   b) Now suppose that the components operate in parallel (that is, the system does not fail until both components fail). Find the density function for $X$, the lifetime of the system.

7. A continuous random variable $X$ has a standard exponential distribution

$$f(x) = \begin{cases} e^{-x}, x > 0, \\ 0 \text{ elsewhere} \end{cases}$$

   For a random sample of size 3, let $X_{(1)}, X_{(2)}, X_{(3)}$ denote the ordered sample.

a) Write down the joint distribution of $X_{(1)}$ and $X_{(3)}$.

b) Obtain the distributions of $X_{(1)}$ and $X_{(3)}$.

c) Evaluate $EX_{(1)}$ and $EX_{(3)}$.

d) Find the sampling distribution of the range $R = X_{(3)} - X_{(1)}$.

8. A random sample of size 3 is taken from a population with density

$$f(x) = \begin{cases} 2x, 0 \leq x < 1, \\ 0 \text{ elsewhere} \end{cases}$$

Find the sampling distribution of the range $R$.

9. For a random sample of size 2 from a standard normal distribution, find the distribution of the range.

**Answers**

1 a) 0.0945, 2.4, 1.68, -0.8 b) 0.135 0.059535

2) $12exp(3(1 - x_{(2)}))(1 - exp(1 - x_{(2)})), 1 \leq x_{(2)} < \infty$

3) $20(x_{(4)} - 1)^3/x_{(4)}^6, 1 \leq x_{(4)} < \infty$

4) $exp(-y_{(1)} - 4), y_{(1)} \geq 4$. The expected value is 5.

5) a) 11/6, b) $\frac{3}{2\sqrt{\pi}}$.

6) a) $\frac{1}{50}exp(-\frac{x}{50})$; b) $\frac{1}{50}(1 - exp(-\frac{x}{100}))exp(-\frac{x}{100})$

7) a) $6(e^{-x_{(1)}} - e^{-x_{(3)}})e^{-(x_{(1)}+x_{(3)})}, 0 < x_{(1)} < x_{(3)} < \infty$.

b) $3exp(-3x_{(1)}), x_{(1)} > 0; 3(1 - exp(-x_{(3)}))^2 exp(-x_{(3)}), x_{(3)} > 0$.

c) $1/3, 11/6$; d) $2exp(-u)(1 - exp(-u)), u > 0$.

8) $12u(1 - u)^2, 0 < u < 1$.

9) $\frac{1}{\sqrt{\pi}}exp(-\frac{u^2}{4}), u > 0$.