Exercise 1.3

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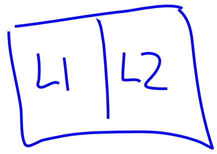


Let L1 and L2 denote the first and second litter.

Let B denote a brown haired rodent.

$$P(B|L1) = \frac{2}{3} \quad \text{and} \quad P(B|L_2) = \frac{3}{5}$$

Random sample from litters $\Rightarrow$ $P(L_1) = P(L_2) = \frac{1}{2}$.

i) Law of total probability:

$$P(B) = P(L1)P(B|L1) + P(L2)P(B|L2)$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{3}{5}$$

$$= \frac{19}{30}$$

ii) Use Bayes theorem:

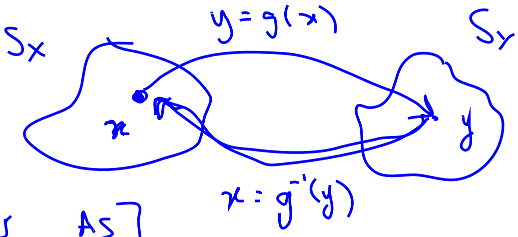
$$P(L1|B) = \frac{P(B|L1)\,P(L1)}{P(B)}$$

$$= \frac{\left(\frac{2}{3}\right) \times \left(\frac{1}{2}\right)}{\left(\frac{19}{30}\right)} = \frac{10}{19} > 0.50$$

Theorem 1.3

Proof: The transformation $x \mapsto g(x)$ is monotone and each x goes to only one y and each y comes from at most one x (one-to-one from $S_X \to S_Y$)

Also, for each $y \in S_Y$ there is an $x \in S_X$ such that $g(x) = y$ (onto from $S_X \to S_Y$)



[$\to$ The transformation g uniquely pairs x's and y's. As g is monotone then g^{-1} is singled valued that is $g^{-1}(y) = x$ iff $y = g(x)$.]

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

(i) If g is increasing

$$\{x \in S_X : g(x) \leq y\} = \{x \in S_X : g^{-1}(g(x)) \leq g^{-1}(y)\}$$

$$= \{x \in S_X \quad x \leq g^{-1}(y)\}$$

Hence $F_Y(y) = \int_{\{x \in S_X : x \leq g^{-1}(y)\}} f_X(x)\,dx$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx$$

Since "$F_X(-\infty)$" $= 0$

$$= F_X(g^{-1}(y)) - \lim_{x \to -\infty} F_X(x)$$

$$\lim_{x \to -\infty} F(x) = 0$$

$$= F_X(g^{-1}(y))$$

Case (ii) Now if g is monotonically decreasing then

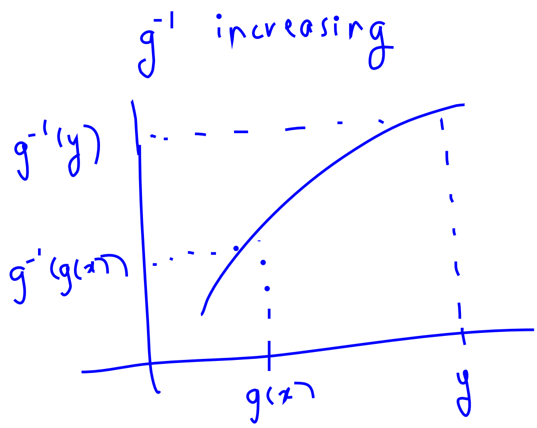
$F_Y(y) = P(g(X) \le y)$

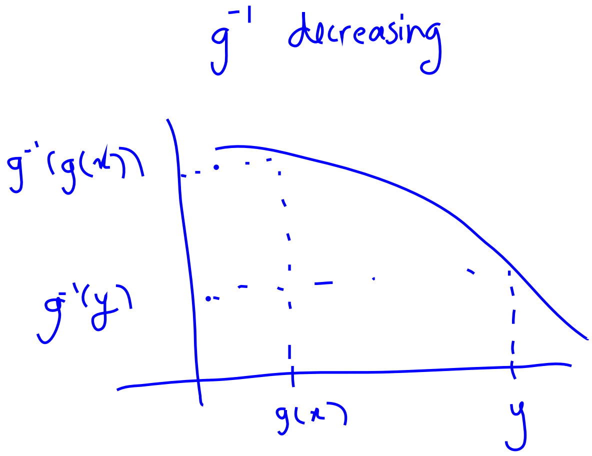
$Y = g(X)$

$$\{x \in S_X : g(x) \le y\} = \{x \in S_X : g^{-1}(g(x)) \ge g^{-1}(y)\}$$
$$= \{x \in S_X : x \ge g^{-1}(y)\}$$

Notice that the inequality sign $\le$ has been flipped to $\ge$.

Reason:





Hence when g is montonically decreasing and $g(x) \le y$ by applying g^{-1} we see that $g^{-1}(g(x)) \ge g^{-1}(y)$.

Hence,
$$F_Y(y) = \int_{\{x \in S_X : x \ge g^{-1}(y)\}} f_X(x)\,dx$$
$$= \int_{g^{-1}(y)}^{\infty} f_X(x)\,dx$$

$\int f(x)\,dx = F(x)$

$$= \lim_{x \to \infty} F_X(x) - F_X(g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

$\square$

Exercise 1.8

The joint density of X and Y

$$f_{XY}(x,y) \underset{\text{independence}}{=} f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = (2\pi)^{-1} e^{-x^2/2} e^{-y^2/2}$$

$-\infty < x < \infty$
$-\infty < y < \infty$

By solving $u = x+y$ and $v = x-y$ for x and y in terms of u and v:

$$u+v = x+y+(x-y) = 2x \Rightarrow x = h_1(u,v) = \frac{u+v}{2}$$

$$u-v = (x+y)-(x-y) = 2y \Rightarrow y = h_2(u,v) = \frac{u-v}{2}$$

The Jacobian is given

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

The joint density of U and V

$$f_{UV}(u,v) = f_{XY}(h_1(u,v), h_2(u,v)) \, |J(u,v)|$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2\right\} \exp\left\{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2\right\} \left|-\frac{1}{2}\right| \quad \begin{matrix} -\infty < u < \infty \\ -\infty < v < \infty \end{matrix}$$

$\vdots$ (Try this)

$$= \left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-u^2/4}\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-v^2/4}\right)$$

$$= f_U(u) f_V(v)$$

U and V are independent

$U \sim N(0,2)$
$V \sim N(0,2)$