Example 4.29

$$E_\theta(T(X)) = \cancel{\theta} = \sum_{x=1}^{\infty} T(x)\, \cancel{\theta}\, (\underbrace{1-\theta}_{\eta})^{x-1}$$

Let $\eta = 1-\theta$

$$\text{LHS}: 1 = T(1)\cdot 1 + T(2)\eta + T(3)\eta^2 + \cdots\cdots = \text{RHS}$$

Let $T(1) = 1$ and $T(2) = T(3) = T(4) = \cdots\cdots = 0$

$$T(X) = \begin{cases} 1 & \text{if } X = 1 \\ 0 & \text{if } X = 2, 3, \ldots\ldots \end{cases} \qquad \theta$$

Note: the estimator $\tilde{T}(X) = \frac{1}{X}$ which is the MLE for this example.

Exercise 4.18

(i) Exponential(θ) $\quad f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$

$E(X) = \theta$

$$\log f(x,\theta) = -\log\theta - \frac{x}{\theta}$$

$$\frac{\partial \log f(X,\theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{X}{\theta^2} \qquad \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2X}{\theta^3}$$

Therefore,

$$I_{X_1}(\theta) = -E\left[\frac{1}{\theta^2} - \frac{2X}{\theta^3}\right] = -\frac{1}{\theta^2} + \frac{2E(X)}{\theta^3} = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$$

Hence the CRLB is

$$\frac{(\tau'(\theta))^2}{nI_{X_1}(\theta)} = \frac{1^2}{n \times \frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

Consider the statistic $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{n\theta}{n} = \theta \rightarrow \text{unbiased estimator}$$

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{n^2} n\theta^2 = \frac{\theta^2}{n} = \text{CRLB attained.}$$

Hence $\bar{X}$ is the UMVUE for θ.

(ii) Bernoulli(θ)

$E(X) = \theta$ $\quad f(x,\theta) = \theta^x (1-\theta)^{1-x}$ $\quad x \in \{0,1\}$ $\quad \theta \in (0,1)$

$$\log f(x,\theta) = x\log\theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f(x,\theta)}{\partial\theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta} \qquad \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = -\frac{x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

Hence, $I_{X_1}(\theta) = -E\left[-\frac{X}{\theta^2} - \frac{(1-X)}{(1-\theta)^2}\right] = \frac{\theta}{\theta^2} + \frac{(1-\theta)}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{1-\theta}$

$$= \frac{1}{\theta(1-\theta)}$$

Hence, the CRLB

$$\frac{1^2}{n \times \frac{1}{\theta(1-\theta)}} = \frac{\theta(1-\theta)}{n} .$$

Consider the statistic $T = \bar{X}$:

$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{1}{n} n\theta = \theta \rightarrow$ Unbiased

$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i) = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}$

which attains the CRLB and hence $\bar{X}$ must be the UMVUE for θ.

(iii) Normal$(\theta,1)$

$$f(x,\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$$

$$\log f(x,\theta) = -\frac{1}{2}\log 2\pi - \frac{1}{2}(x-\theta)^2$$

$$\frac{\partial \log f(x,\theta)}{\partial\theta} = -\frac{1}{2} 2(x-\theta)(-1) = x-\theta$$

$$\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = -1$$

Hence, $I_{X_1}(\theta) = -E(-1) = 1$ and the CRLB is $\frac{(\tau'(\theta))^2}{n I_{X_1}(\theta)} = \frac{1^2}{n} = \frac{1}{n}$

Now consider $T = \bar{X}$ then $E(T) = \frac{1}{n} n\theta = \theta$ $\rightarrow$ unbiased

and $Var(T) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\cdot n = \frac{1}{n}$ $\rightarrow$ attains the CRLB

Hence, $\bar{X}$ must be the UMVUE.

(iv) Normal$(0,\theta)$: $$f(x,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}x^2}$$

$$\log f(x,\theta) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta - \frac{1}{2\theta}x^2$$

$$\frac{\partial \log f(x,\theta)}{\partial\theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} \quad \text{and} \quad \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

Note: $Var(X) = E(X^2) - E(X)^2$ or $E(X^2) = Var(X) + E(X)^2$
$$= \theta + 0^2$$
$$= \theta$$

then $$I_{X_1}(\theta) = -E\left[\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}\right] = -\frac{1}{2\theta^2} + \frac{E(x^2)}{\theta^3} = -\frac{1}{2\theta^2} + \frac{1}{\theta^2} = \frac{1}{2\theta^2}$$

Then the CRLB is $$\frac{(\tau'(\theta))^2}{n I_{X_1}(\theta)} = \frac{1^2}{n\cdot\frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

Consider $T = \frac{1}{n}\sum_{i=1}^{n}(X_i - 0)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ then

$$E(T) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{n} n\theta = \theta \longrightarrow \text{unbiased.}$$

Note: $X_i \sim N(0,\theta)$ then $\frac{X_i}{\sqrt{\theta}} \sim N(0,1)$ then $\frac{X_i^2}{\theta} \sim N(0,1)^2 \sim \chi_1^2$

where χ_1^2 is chi-squared with one degree of freedom and has mean 1 and variance 2.

$$Var\left(\frac{X_i^2}{\theta}\right) = Var(X_1^2) = 2 \quad \Rightarrow \quad \frac{1}{\theta^2} Var(X_i^2) = 2$$

$$\Rightarrow \quad Var(X_i^2) = 2\theta^2$$

Therefore

$$Var\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i^2) = \frac{1}{n^2} \cdot n \cdot 2\theta^2 = \frac{2\theta^2}{n} = CRLB$$

Hence $T = \frac{1}{n}\sum_{i=1}^n X_i^2$ is the UMVUE ~~for~~ θ.

Example 4.31

(i) We can also directly check the attainability in this case:

$$\frac{\partial^2 \log L(x,\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(-n + \frac{1}{\theta}\sum_{i=1}^n x_i\right) = -\frac{1}{\theta^2}\sum_{i=1}^n X_i$$

Therefore,

$$I_X(\theta) = \frac{1}{\theta^2}\sum_{i=1}^n E(X_i) = \frac{n\theta}{\theta^2} = \frac{n}{\theta}$$

and the CR bound for the variance of an unbiased estimator for $\tau(\theta) = \theta$ is

$$\frac{(\tau'(\theta))^2}{I_X(\theta)} = \frac{1}{n/\theta} = \frac{\theta}{n}$$

A direct calculation gives:

$$Var(\bar{X}) = \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \qquad \text{since } Var(X_i) = \theta$$

$$= \frac{1}{n^2} n\theta$$

$$= \frac{\theta}{n}$$

Note: We did not have to do this. We know from the score factorization

$$V(x,\theta) = \frac{n}{\theta}(\bar{x} - \theta)$$

that the bound is attainable for $\bar{X}$.

(ii) We know that $I_X(\theta) = \frac{n}{\theta}$ and $\tau'(\theta) = -e^{-\theta}$.

This gives the CR bound as:

$$\frac{(\tau'(\theta))^2}{I_X(\theta)} = \frac{(-e^{-\theta})^2}{n/\theta} = \frac{\theta e^{-2\theta}}{n}.$$

Exercise 4.19

(i) We showed in Example 4.33 that the statistic $T(X) = \sum_{i=1}^{n} X_i$ is complete for $\theta \in (0,1)$ – could also argue this since the Bernoulli belongs to the one-parameter exponential family. (also is sufficient)

Therefore, if we start with an unbiased estimator W of $\tau(\theta) = \theta(1-\theta)$ and then calculated $\hat{\tau}(T) = E(W|T)$ in a second step, we will get the UMVUE of τ.

Suggestion for W:

$$W = X_1(1-X_2) \qquad \text{or} \qquad \tilde{W} = I_{\{X_1=1\}}(X) \cdot I_{\{X_2=0\}}(X)$$

We see that

$$E_\theta(W) = E(X_1) - E(X_1X_2) = E(X_1) - E(X_1)E(X_2) = \theta - \theta \cdot \theta = \theta(1-\theta)$$

and

$$E_\theta(\tilde{W}) = E\left[I_{\{X_1=1\}}(X)\right] \cdot E\left[I_{\{X_2=0\}}(X)\right] = P(X_1=1)\,P(X_2=0) = \theta(1-\theta)$$

Now we get

$$\hat{\tau}(T) = E(W|T) = 1 \times P(X_1=1, X_2=0 \mid T=t) + 0\, P(W=0 \mid T=t)$$

$$= \frac{P(X_1=1, X_2=0, \sum_{i=1}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)}$$

$$= \frac{P(X_1=1, X_2=0, \sum_{i=3}^{n} X_i = t-1)}{P(\sum_{i=1}^{n} X_i = t)}$$

Now $\sum_{i=1}^{n} X_i \sim \text{Binomial}(n, \theta)$

$$\hat{\tau}(T) = \frac{P(X_1=1)\,P(X_2=0)\,\underbrace{P\left(\sum_{i=3}^{n} X_i = t-1\right)}_{Bin(n-2,\theta)}}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$$= \frac{\cancel{\theta}(\cancel{1-\theta})\binom{n-2}{t-1}\cancel{\theta^{t-1}}\cancel{(1-\theta)^{n-2-(t-1)}}}{\binom{n}{t}\cancel{\theta^{t}}\cancel{(1-\theta)^{n-t}}}$$

$$= \binom{n-2}{t-1}\Big/\binom{n}{t} \qquad \binom{n}{t} = \frac{n!}{t!\,(n-t)!}$$

Further,

$$= \frac{(n-2)!}{\cancel{(t-1)!}(n-t-1)!} \times \frac{t!\,(n-t)!}{n!} \qquad t! = t.(t-1)!$$

$$n! = n(n-1).(n-2)!$$

$$= \frac{t(n-t)}{n(n-1)}$$

$$T = \sum X_i \qquad \frac{T}{n} = \bar{X}$$

$$= \frac{\cancel{n}\bar{X}(n-n\bar{X})}{\cancel{n}(n-1)} \qquad T = n\bar{X}$$

$$= \frac{n}{n-1}\,\bar{X}(1-\bar{X})$$

being the UMVUE of $\tau(\theta) = \theta(1-\theta)$

Note: the MLE in the case is $\bar{X}(1-\bar{X})$ which would be biased

(ii) We have already seen in Example 4.30 that $W = \frac{n+1}{n} X_{(n)}$ is unbiased for θ and in Exercise 3.15 (ii) that $T = X_{(n)}$ is minimal sufficient for θ. Now we show that $X_{(n)}$ is complete for θ

Recall: $$f_T(t) = \frac{n t^{n-1}}{\theta^n} \qquad 0 < t < \theta$$

Then $E g(T) = 0$ for all $\theta \in (0, \infty)$ then

$$\int_0^\theta g(t) \cdot \frac{n t^{n-1}}{\theta^n} dt = 0 \qquad \text{here } \frac{n}{\theta^n} \neq 0 > 0$$

$$\Rightarrow \int_0^\theta g(t) t^{n-1} dt = 0$$

Now take the derivative wrt θ:

$$\frac{d}{d\theta} 0 = 0 = \frac{d}{d\theta} \int_0^\theta g(t) t^{n-1} dt = g(\theta) \theta^{n-1}$$

E.g $\frac{d}{dx} \int_0^x t^2 dt = \frac{d}{dx} \left. \frac{t^3}{3} \right]_0^x = \frac{d}{dx} \frac{x^3}{3} = \frac{3x^2}{3} = x^2$

But $\theta > 0$ implies $g(\theta) = 0$ for all $\theta > 0$. That is,

$P(g(T) = 0) = 1$ and $T = X_{(n)}$ is complete

Since $W = \frac{n+1}{n} X_{(n)} = \frac{n+1}{n} T$ is unbiased for θ and a function of a complete and sufficient statistic T then

$$\hat{\tau}(T) = E\left[\frac{n+1}{n} T \,\middle|\, T \right] = \frac{n+1}{n} T = \frac{n+1}{n} X_{(n)}$$

which is the UMVUE for θ

Here $X_{(n)}$ is the MLE which is again biased. (Why!)

(iii) $X_1, X_2, \ldots, X_n$ iid sample Poisson(θ). First note that $T = \sum_{i=1}^n X_i \sim \text{Poisson}(n\theta)$

(1) is sufficient for θ as f belongs to the one parameter exponential family. This also implies that T is complete. However, we will show that it is also complete directly.

Consider $E[g(T)] = 0$ for all $\theta > 0$. Then

$$\sum_{t=0}^{\infty} g(t) \cdot \frac{e^{-n\theta}(n\theta)^t}{t!} = 0 \quad \text{for all } \theta > 0.$$

But since $e^{-n\theta} \neq 0$ we have

$$\sum_{t=0}^{\infty} g(t) \frac{n^t}{t!} \theta^t = g(0)\frac{n^0}{0!}\theta^0 + g(1)\frac{n^1}{1!}\theta^1 + g(2)\frac{n^2}{2!}\theta^2 + \cdots = 0$$

and this is only zero when the coefficients $g(t)\frac{n^t}{t!}$ are all zero, which implies $g(t) = 0$ $t = 0, 1, 2, \ldots$ and hence

$$P(g(T) = 0) = 1 \quad \text{for all } \theta.$$

(2) To start with an unbiased estimator for $\tau(\theta) = e^{-\theta}$ we use the interpretation that $e^{-\theta} = P(X_1 = 0)$ of $\tau(\theta)$. Hence,

$W = I_{\{X_1 = 0\}}(X)$ would be unbiased for $\tau(\theta)$:

$$E_\theta W = E\, I_{\{X_1=0\}}(X) = P(X_1 = 0) = e^{-\theta}$$

(3) Now by conditioning on the complete and sufficient statistic $T = \sum_{i=1}^n X_i$ we obtain the UMVUE:

$$\hat{\tau}(T) = E[W \mid T] = E\left[I_{\{X_1=0\}}(X) \,\middle|\, \sum_{i=1}^n X_i = t\right]$$

$$= P\left(X_1 = 0 \,\middle|\, \sum_{i=1}^n X_i = t\right)$$

$$= \frac{P\left(X_1 = 0, \sum_{i=1}^n X_i = t\right)}{P\left(\sum_{i=1}^n X_i = t\right)}$$

$$= \frac{P\left(X_1 = 0, \sum_{i=2}^n X_i = t\right)}{P\left(\sum_{i=1}^n X_i = t\right)}$$

$$\hat{\tau}(T) = \frac{P(X_1 = 0)\, P\left(\sum_{i=2}^{n} X_i = t\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$\sum_{i=2}^{n} X_i \sim \text{Poisson}((n-1)\theta)$

$\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$

$$= \cancel{e^{-\theta}} \cdot \frac{\cancel{e^{-(n-1)\theta}}\, [(n-1)\cancel{\theta}]^t}{\cancel{t!}} \cdot \frac{\cancel{t!}}{\cancel{e^{-n\theta}}\, [n\cancel{\theta}]^t}$$

$$= \left(\frac{n-1}{n}\right)^t$$

$$= \left(1 - \frac{1}{n}\right)^{n\bar{X}}$$

$t = \sum X_i = n\bar{X}$

is the UMVUE for $\tau(\theta) = e^{-\theta}$.