Exercise 1.1

Part (i)

Let A be the event that none of the n people have the same birthday as you. The number of outcomes in the event A is

$$k = \underbrace{364 \times 364 \times \ldots \times 364}_{n \text{ times}} = 364^n.$$

The total number of birthday combinations for n people is:

$$m = \underbrace{365 \times 365 \times \ldots \times 365}_{n \text{ times}} = 365^n.$$

Hence,

$$P(A) = \frac{k}{m} = \left(\frac{364}{365}\right)^n$$

Finally, the probability that at least one person has the same birthday as you is is obtained by applying the compliment rule

$$P(A^c) = 1 - P(A) = 1 - \left(\frac{364}{365}\right)^n.$$

Part (ii)

We want it to be near $\frac{1}{2}$:

$$1 - \left(\frac{364}{365}\right)^n \approx \frac{1}{2}$$

or

$$\left(\frac{364}{365}\right)^n \approx \frac{1}{2}$$

By taking the natural log

$$n \log\left(\frac{364}{365}\right) = \log(0.5)$$

and solving for n we get:

$$n \approx \frac{\log(0.5)}{\log(364/365)} \approx 252.65.$$

Exercise 1.2

Let M and F denote the event that the randomly selected person is male or female, respectively. Let C denotes the event that the randomly selected person is colour-blind. From the question, we know the following probabilities:

$$P(C|M) = 0.05, \qquad P(C|F) = 0.0025 \qquad \text{and} \qquad P(M) = P(F) = 0.5.$$

We are interested in the conditional probability that the randomly selected person is a male given that the randomly selected person is colour-blind. In probability notation that is $P(M|C)$. Now the law of total probability gives

$$P(C) = P(C|M)P(M) + P(C|F)P(F) = 0.05 \times 0.5 + 0.0025 \times 0.5 = 0.02625.$$

Then by applying Bayes' Theorem, we obtain:

$$P(M|C) = \frac{P(C|M)P(M)}{P(C)} = \frac{0.05 \times 0.5}{0.02625} = 0.9524.$$

Think: would you expect the conditional probability of the randomly selected person being a male given that we know that they are colour-blind to be so high, given that males and females have the same 50% chance of being selected?

Exercise 1.4

Part (i)

Since f is symmetric around a we have $\mu = a$. Then

$$\begin{aligned}
\mu_3 &= \int_{-\infty}^{\infty} (x-a)^3 f(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} (x-a)^3 f(x)\mathrm{d}x + \int_{a}^{\infty} (x-a)^3 f(x)\mathrm{d}x \quad \text{(splitting the integral)} \\
&= \int_{-\infty}^{0} y^3 f(y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \quad \text{(change of variable } y = x-a\text{)} \\
&= \int_{0}^{\infty} -y^3 f(-y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \\
&= 0 \quad \text{(since } f(-y+a) = f(y+a)\text{)}
\end{aligned}$$

Part (ii)

The mean.:

$$\mu = \mathbb{E}(X) = \int_{-1}^{1} x\frac{1}{2}\mathrm{d}x = \left.\frac{x^2}{4}\right|_{-1}^{1} = \frac{1^2}{4} - \frac{(-1)^2}{4} = \frac{1}{4} - \frac{1}{4} = 0.$$

The variance:

$$\mu_2 = \mathbb{E}[(X-\mu)^2] = \int_{-1}^{1} (x-0)^2\frac{1}{2}\mathrm{d}x = \left.\frac{x^3}{6}\right|_{-1}^{1} = \frac{1^3}{6} - \frac{(-1)^3}{6} = \frac{2}{6} = \frac{1}{3}.$$

Third and fourth central moments:

$$\mu_3 = \mathbb{E}[(X-\mu)^3] = \int_{-1}^{1} (x-0)^3\frac{1}{2}\mathrm{d}x = \left.\frac{x^4}{8}\right|_{-1}^{1} = \frac{1}{8} - \frac{(-1)^4}{4} = \frac{1}{8} - \frac{1}{8} = 0.$$

$$\mu_4 = \mathbb{E}[(X-\mu)^4] = \int_{-1}^{1} (x-0)^4 \frac{1}{2} \mathrm{d}x = \frac{x^5}{10}\bigg|_{-1}^{1} = \frac{1^5}{10} - \frac{(-1)^5}{10} = \frac{2}{10} = \frac{1}{5}.$$

Therefore, the skewness is

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{0}{(1/3)^{(3/2)}} = 0$$

and kurtosis

$$\alpha_4 = \frac{\mu_4}{(\mu_2)^2} = \frac{(1/5)}{(1/3)^2} = 1.8.$$

Exercise 1.5

The density function must integrate to one over the range of x and y values. Therefore

$$\iint f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^2 \int_0^1 c(x+2y)\mathrm{d}y\mathrm{d}x = 1$$

Now

$$\begin{aligned}
c\int_0^2 \int_0^1 x + 2y\mathrm{d}y\mathrm{d}x &= c\int_0^2 \Big[xy + y^2\Big]_0^1 \mathrm{d}x \\
&= c\int_0^2 (x+1) - (0+0)\mathrm{d}x \\
&= c\int_0^2 x + 1\mathrm{d}x \\
&= c\left[\frac{x^2}{2} + x\right]_0^2 \\
&= c\left(\frac{2^2}{2} + 2 - (0+0)\right) \\
&= 4c
\end{aligned}$$

Therefore $4c = 1$ and hence $c = 0.25$.

Exercise 1.6

The marginal distribution of X is:

$$\begin{aligned}
f_X(x) = \int_0^1 f(x,y)\mathrm{d}y &= \frac{1}{4}\int_0^1 x + 2y\mathrm{d}y \\
&= \frac{1}{4}\Big[xy + y^2\Big]_0^1 \\
&= \frac{x+1}{4} \qquad 0 < x < 2.
\end{aligned}$$

Exercise 1.7

The marginal density of X was

$$f(x,y) = \frac{1}{4}(x+1) \qquad \text{for} \quad 0 < x < 2.$$

Notice that the transformation

$$Z = \frac{9}{(X+1)^2}$$

is monotonically decreasing on S_X with inverse

$$X(Z) = \frac{3}{\sqrt{Z}} - 1.$$

Furthermore, before applying the density transformation formula, notice that

$$\frac{\mathrm{d}X}{\mathrm{d}Z} = \frac{\mathrm{d}}{\mathrm{d}Z} 3Z^{-1/2} = -\frac{3}{2} Z^{-3/2}.$$

Therefore,

$$\begin{aligned} g(z) &= f(x(z)) \left| \frac{\mathrm{d}X}{\mathrm{d}Z} \right| \\ &= f\Big(\frac{3}{\sqrt{z}} - 1\Big) \Big| -\frac{3}{2} z^{-3/2} \Big| \\ &= \frac{\frac{3}{\sqrt{z}} - 1 + 1}{4} \cdot \frac{3}{2} z^{-3/2} \\ &= \frac{9}{8z^2} \end{aligned}$$

Since $0 < x < 2$ it implies that

$$\frac{9}{(0+1)^2} < z < \frac{9}{(2+1)^2}$$

or $S_Z = \{z\,;\, 1 < z < 9\}$.