Exercise 6.26

By the Neyman-Pearson lemma for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$, the best α-test is the one with rejection region of the form:

$$\left\{ \frac{L(X,1)}{L(X,0)} \geq k \right\}$$

Here,

$$L(X,\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}$$

and thus

$$\frac{L(X,1)}{L(X,0)} = \frac{e^{(x-1)}}{(1+e^{(x-1)})^2} \cdot \frac{(1+e^{(x-0)})^2}{e^{(x-0)}} = \frac{(e^x+1)^2}{e(e^{x-1}+1)^2} \geq k$$

Now by ignoring the constant $\frac{1}{e}$ and square operation we have that (using the quotient rule):

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{e^x+1}{e^{x-1}+1} = \frac{e^x - e^{x-1}}{(1+e^{x-1})^2}$$

and since $e^x > e^{x-1}$ or $e^x - e^{x-1} > 0$, this ratio is increasing in x and the inequality $\frac{L(X,1)}{L(X,0)} \geq k$ must be the same as rejection when $X \geq k'$. Therefore, to make this an α-test we need to "exhaust the level":

$$\mathbb{E}_{\theta_0}\varphi^* = \alpha = 0.1 = P(X \geq k' | \theta = 0)$$

The CDF here is

$$F(x,\theta) = \int_{-\infty}^{x} \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2} \mathrm{d}x = \ldots = \frac{e^{x-\theta}}{1+e^{x-\theta}}$$

and thus

$$0.10 = P(X > k') = 1 - F(k',0) = 1 - \frac{e^{k'}}{1+e^{k'}} = \frac{1}{e^{k'}+1}$$

which leads to

$$k' = \log(\frac{1}{0.1} - 1) = \log(9) = 2.19722$$

The best 0.10-test of $H_0 : \theta = 0$ vs $H_1 : \theta = 1$ is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X \geq 2.19722 \\ 0 & \text{if} \quad X < 2.19722 \end{cases}$$

Example 6.51

Assume, a sample of one observation ($n = 1$) from an exponential family with density

$$f(x,\theta) = \frac{1}{\theta}\exp\Big(-\frac{x}{\theta}\Big), \quad x > 0$$

is available. The parameter $\theta > 0$ is to be tested. Construct an UMP unbiased α-test of:

(i)

$$H_0 : \theta \in [1,2] \quad \text{versus} \quad H_1 : \theta \notin [1,2].$$

This case is covered by Theorem 6.24 and here we have

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < C_1 \quad \text{or} \quad T > C_2 \\ 0 & \text{if} \quad C_1 \le T \le C_2 \end{cases}$$

where $T(x) = x$ in this case (one-parameter exponential family with $d(x) = x, n = 1$). We only need to find C_1 and C_2 in order to uniquely specify the above test.

Since $X \sim \text{Exp}(\theta)$ we have $F(x) = 1 - e^{-x/\theta}$ and therefore

$$\begin{aligned} \mathbb{E}_{\theta_0}(\varphi^*) &= P(T < C_1 \quad \text{or} \quad T > C_2) \\ &= P(T < C_1) + P(T > C_2) \\ &= 1 - e^{-C_1/\theta} + (1 - (1 - e^{-C_2/\theta})) \\ &= 1 - e^{-C_1/\theta} + e^{-C_2/\theta} \end{aligned}$$

Therefore to determine C_1 and C_2 we need to satisfy the two equations

$$\begin{cases} \mathbb{E}_{\theta=1}(\varphi^*) = 1 - e^{-C_1} + e^{-C_2} = \alpha \\ \mathbb{E}_{\theta=2}(\varphi^*) = 1 - e^{-C_1/2} + e^{-C_2/2} = \alpha \end{cases}$$

This system can be solved by substitution. Set $k_1 = e^{-C_1/2}$ and $k_2 = e^{-C_2/2}$ then we have

$$\begin{cases} k_1^2 - k_2^2 = 1 - \alpha \\ k_1 - k_2 = 1 - \alpha \end{cases}$$

which the first equation gives

$$(k_1 + k_2)\underbrace{(k_1 - k_2)}_{1-\alpha} = 1 - \alpha$$

and hence $k_1 + k_2 = 1$. This implies

$$\begin{cases} k_1 + k_2 = 1 \\ k_1 - k_2 = 1 - \alpha \end{cases}$$

Then adding these two equations yields

$$2k_1 = 2 - \alpha \qquad \text{or} \qquad k_1 = 1 - \alpha/2.$$

and then substituting this back into the first equation gives

$$k_1 + k_2 = 1 \implies k_2 = 1 - k_1 = 1 - (1 - \alpha/2) = \alpha/2$$

Hence,

$$C_1 = -2\log(1 - \alpha/2) \quad \text{and} \quad C_2 = -2\log(\alpha/2).$$

since

$$k_1 = e^{-C_1/2} \quad \text{and} \quad k_2 = e^{-C_2/2}.$$

Therefore, the test has structure

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < -2\log(1-\alpha/2) \quad \text{or} \quad T > -2\log(\alpha/2) \\ 0 & \text{if} \quad -2\log(1-\alpha/2) \leq T \leq -2\log(\alpha/2) \end{cases}$$

with power function

$$\text{Power}(\theta) = \mathbb{E}_\theta(\varphi^*) = 1 - e^{-C_2/\theta} + e^{-C_2/\theta} = 1 - (1 - \alpha/2)^{2/\theta} + (\alpha/2)^{2/\theta}$$

(ii)

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1.$$

According to the Theorem 6.25, the UMPU α-test φ^* for (ii) has the same structure as part (i):

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < C_1 \quad \text{or} \quad T > C_2 \\ 0 & \text{if} \quad C_1 \leq T \leq C_2 \end{cases}$$

where $T(x) = x$ in this case (one-parameter exponential family with $d(x) = x, n = 1$).We only need to find C_1 and C_2 in order to uniquely specify the above test.

Note that

$$\mathbb{E}_\theta(\varphi^*) = P_\theta(X \notin (C_1, C_2)) = 1 - \exp(-C_1/\theta) + \exp(-C_2/\theta)$$

since the cdf is $F(x, \theta) = 1 - \exp(-x/\theta), x > 0)$. The two conditions on $\mathbb{E}_\theta(\varphi^*)$ are:

- $$\mathbb{E}_\theta(\varphi^*)\Big|_{\theta=1} = \alpha = 1 - \exp(-C_1) + \exp(-C_2)$$

- $$\begin{aligned} \frac{\partial}{\partial\theta}\mathbb{E}_\theta(\varphi^*)\Big|_{\theta=1} &= -\frac{C_1}{\theta^2}\exp(-\frac{C_1}{\theta}) + \frac{C_2}{\theta^2}\exp(-\frac{C_2}{\theta})\Big|_{\theta=1} \\ &= -C_1\exp(-C_1) + C_2\exp(-C_2) = 0 \end{aligned}$$

We get a system of two equations with respect to C_1 and C_2. It can be solved numerically (iteratively) given the level α and hence the UMPU α-test will be completely specified. We will skip over these details but the key is a solution can be found from here. We need to find C_1 and C_2 that satisfy

$$\begin{cases} \alpha = 1 - e^{-C_1} + e^{-C_2} \\ 0 = -C_1 e^{-C_1} + C_2 e^{-C_2} \end{cases}.$$

Exercise 6.33

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are *i.i.d.* Cauchy$(\theta, 1)$ and we are testing

$$H_0 : \theta \leq 0 \quad \text{vs} \quad H_1 : \theta > 0$$

The likelihood is:

$$L(X, \theta) = \prod_{i=1}^{n} \frac{1}{\pi} \frac{1}{1 + (x_i - \theta)^2} = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2}$$

By taking the logarithm we arrive at the log-likelihood:

$$\log L(X, \theta) = -n \log \pi - \sum_{i=1}^{n} \log(1 + (x_i - \theta)^2)$$

which gives the score:

$$V(X, \theta) = \frac{\partial}{\partial \theta} \log L(X, \theta) = -\sum_{i=1}^{n} \frac{-2(x_i - \theta)}{1 + (x_i - \theta)^2} = \sum_{i=1}^{n} \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2}$$

Then

$$V(X, \theta)\Big|_{\theta=0} = \sum_{i=1}^{n} \frac{2x_i}{1 + x_i^2}$$

which leads to the LMP α-test

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad T = \displaystyle\sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2} > k \\ 0 & \text{if} \quad T = \displaystyle\sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2} < k \end{cases}$$

Now we need to find k (but we can only do this asymptotically). Let

$$U = \frac{2X}{1 + X^2}$$

where $X \sim \text{Cauchy}(0, \theta)$ then

$$\mathbb{E}(U) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(1 + x^2)} \cdot \frac{1}{1 + x^2} \mathrm{d}x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(1 + x^2)^2} \mathrm{d}x = 0$$

since the integral of an odd function $g(x) = -g(-x)$ is zero i.e.

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{-\infty}^{0} g(x)\mathrm{d}x + \int_{0}^{\infty} g(x)\mathrm{d}x \\ &= -\int_{-\infty}^{0} g(-x)\mathrm{d}x + \int_{0}^{\infty} g(x)\mathrm{d}x \\ &= \int_{0}^{\infty} g(x)\mathrm{d}x - \int_{0}^{\infty} g(x)\mathrm{d}x \\ &= 0 \end{aligned}$$

Furthermore,

$$\mathrm{Var}(U) = \mathbb{E}(U^2) = \frac{4}{\pi^2}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\mathrm{d}x = \ldots = \frac{1}{2}.$$

Therefore,

$$U_i = \frac{2X_i}{1+X_i^2}$$

has zero mean and variance $1/2 < \infty$. Now by applying the CLT we obtain

$$\frac{T-0}{\sqrt{\frac{n}{2}}} = \sqrt{\frac{2}{n}}T \xrightarrow{d} N(0,1)$$

Hence,

$$\begin{aligned}
\alpha &= P(T > k|\theta = 0) \\
&= P\Big(\sqrt{\frac{2}{n}}T > \sqrt{\frac{2}{n}}k|\theta = 0\Big) \\
&= P(Z > \sqrt{\frac{2}{n}}k\Big)
\end{aligned}$$

which implies that

$$\sqrt{\frac{2}{n}}k = z_\alpha \qquad \text{or} \qquad k = z_\alpha\sqrt{\frac{n}{2}}$$

and now the test is completely determined.