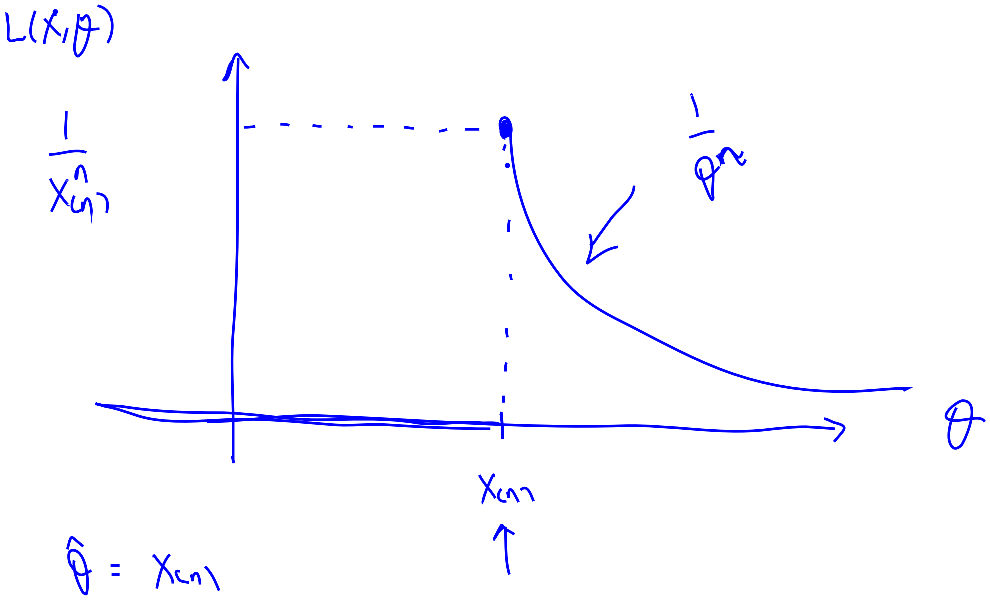
<u>Example 5-37</u>

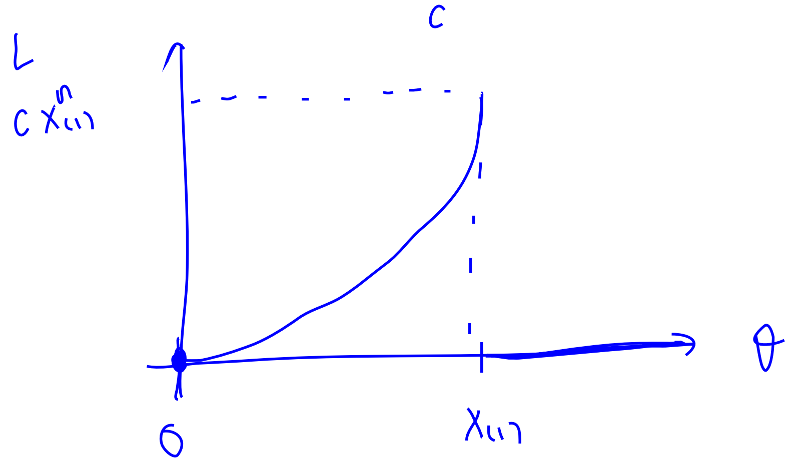
$$L(X,\theta) = \frac{1}{\theta^n} I_{(X_{(n)},\infty)}(\theta)$$



Hence $\hat{\theta} = X_{(n)}$

<u>Example 5.21</u>

$$L(X,\theta) = \theta^n \underbrace{\left(\prod_{i=1}^n x_i^{-2}\right)}_{C} I_{(-\infty, X_{(1)})}(\theta) = C\theta^n \qquad \theta > 0$$



$\hat{\theta} = X_{(1)}$

Example 5.42 (Variance stabilization)

$X_1, \ldots, X_n$ iid Poisson(λ) $\Rightarrow$ $L(x,\lambda) = \dfrac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}$

$$\log L(x,\lambda) = -n\lambda + \log\lambda \sum_{i=1}^{n} x_i - \sum \log x_i!$$

$$V(x,\lambda) = \frac{\partial}{\partial\lambda}\log L(x,\lambda) = -n + \frac{1}{\lambda}\sum x_i \overset{\text{set } 0}{\Rightarrow} \hat{\lambda}_{mle} = \bar{X}$$

Furthermore,

$$E\left(-\frac{\partial^2 \log L(x,\lambda)}{\partial\lambda^2}\right) = \frac{1}{\lambda^2}E\left(\sum X_i\right) = \frac{1}{\lambda^2}n\lambda = \frac{n}{\lambda} = I_X(\lambda)$$

and $I_{X_1}(\lambda) = \dfrac{1}{\lambda}$

Hence, the asymptotic distribution of $\hat{\lambda}_{mle} = \bar{X}$ is

$$\sqrt{n}(\bar{X} - \lambda) \xrightarrow{d} N(0, \lambda)$$

Note that the asymptotic variance is λ and it depends on the unknown parameter and in that sense it is <u>not</u> stable.

The transformation $h(\lambda) = \sqrt{\lambda}$ is a variance stabilizing transformation since the asymptotic variance of

$$\sqrt{n}\left(h(\hat{\lambda}) - h(\lambda)\right)$$

turns out to be constant (in fact 1/4)!

We have $\widehat{h(\lambda)}_{mle} = h(\hat{\lambda}_{mle}) = h(\bar{X}) = \sqrt{\bar{X}}$ and $h'(\lambda) = \dfrac{1}{2\sqrt{\lambda}}$

then the delta method yeilds:

$$\sqrt{n}\left(\sqrt{\bar{x}} - \sqrt{\lambda}\right) \xrightarrow{d} N\left(0, (h'(\lambda))^2 I_{X_1}^{-1}(\lambda)\right)$$

$$\xrightarrow{d} N\left(0, \frac{1}{4\lambda}\cdot\lambda\right)$$

$$\xrightarrow{d} N\left(0, \frac{1}{4}\right) \qquad \sqrt{\bar{X}} \overset{a}{\sim} N\left(0, \frac{1}{4n}\right)$$

We can construct asymptotic variance stabilizing CI's:

- First for $\sqrt{\lambda}$: $\quad \sqrt{\bar{X}} \pm z_{\alpha/2} \frac{1}{2\sqrt{n}}$

- Then for λ: $\quad \left(\left(\sqrt{\bar{X}} - z_{\alpha/2} \frac{1}{2\sqrt{n}} \right)^2 , \left(\sqrt{\bar{X}} + z_{\alpha/2} \frac{1}{2\sqrt{n}} \right)^2 \right)$

Exercise 5.22

(i) The Likelihood function

$$L(X,\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod X_i \right)^{\theta-1}$$

with log-likelihood function:

$$\log L(X,\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log X_i$$

Setting the score to zero we obtain:

$$V(X,\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i \overset{set}{=} 0 \implies \hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log X_i}$$

Then applying the transformation invariance property:

$$\widehat{\tau(\theta)}_{mle} = \tau(\hat{\theta}_{mle}) = \frac{\hat{\theta}}{\hat{\theta} + 1}$$

(ii) We have

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \underset{\theta^2}{\overset{\shortparallel}{I_{X_1}^{-1}(\theta)}})$$

where $I_{X_1}(\theta) = \frac{1}{\theta^2}$.

Now $\tau'(\theta) = \frac{d}{d\theta} \theta \cdot (1+\theta)^{-1} = 1 \cdot (1+\theta)^{-1} + \theta(-1)(1+\theta)^{-2}$

$$= \frac{1}{1+\theta} - \frac{\theta}{(1+\theta)^2} = \frac{1+\theta}{(1+\theta)^2} - \frac{\theta}{(1+\theta)^2}$$

$$= \frac{1}{(1+\theta)^2}$$

Hence the delta method gives:

$$\sqrt{n}(\hat{\tau}-\tau) \xrightarrow{d} N\left(0, (\tau'(\theta))^2 I_{X_1}^{-1}(\theta)\right)$$
$$\xrightarrow{d} N\left(0, \frac{1}{(1+\theta)^4} \cdot \theta^2\right)$$
$$\xrightarrow{d} N\left(0, \frac{\theta^2}{(1+\theta)^4}\right)$$

Exercise 5.23

(i) The MLE of λ is $\bar{X}$ hence the MLE of $\tau(\lambda)=\frac{1}{\lambda}$ is $\frac{1}{\bar{X}}$.

(ii) $$E\left[\frac{1}{\bar{X}}\right] = \sum_{\{\bar{x}:\bar{x}\geq 0\}} \frac{1}{\bar{x}} \cdot P(\bar{X}=\bar{x}) = \underbrace{\frac{1}{0}P(\bar{X}=0)}_{\to\infty} + \sum_{\{\bar{x}:\bar{x}>0\}} \frac{1}{\bar{x}} P(\bar{X}=\bar{x})$$

$$P(\bar{X}=0) = P(X_1=0,\dots,X_n=0) = P(X_1=0)^n = (e^{-\lambda})^n = e^{-n\lambda} \neq 0 \xrightarrow{n\to\infty} 0$$

Since $P(\bar{X}=0)>0$ we get the first moment of the MLE is infinite (not to mention the second) and hence there is no finite variance.

(iii) Here we have $h(\lambda)=\frac{1}{\lambda}$ and so $h'(\lambda) = -\frac{1}{\lambda^2}$. We also know that $I_{X_1}(\lambda)=\frac{1}{\lambda}$. Hence the delta method gives

$$\sqrt{n}\left(\frac{1}{\bar{X}}-\frac{1}{\lambda}\right) \xrightarrow{d} N\left(0, (h'(\lambda))^2 \cdot I_{X_1}^{-1}(\lambda)\right)$$
$$\xrightarrow{d} N\left(0, \frac{1}{\lambda^4}\cdot\lambda\right)$$
$$\xrightarrow{d} N\left(0, \frac{1}{\lambda^3}\right)$$

By comparing (ii) and (iii) we see that although the finite variance does not exist, the asymptotic variance is well defined.

Exercise 5.24

(i) The density of a single observation $\qquad$ μ - known.//

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$L(X,\sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\right)$$

$$\log L(X,\sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$V(X,\sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{\partial \log L(X,\sigma^2)}{\partial\sigma^2}$$

Setting this to zero:

$$\hat{\sigma}^2_{mle} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

Information:

$$I_X(\sigma^2) = -E\left[\frac{n}{2\sigma^4} - \frac{2}{2\sigma^6}\sum_{i=1}^{n}(X_i-\mu)^2\right]$$

$X_i \sim N(\mu,\sigma^2)$

$X_i-\mu \sim N(0,\sigma^2)$

$$= -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{i=1}^{n}\underbrace{E[(X_i-\mu)^2]}_{\text{var}X_i}$$

$$= -\frac{n}{2\sigma^4} + \frac{n\sigma^2}{\sigma^6} = -\frac{n}{2\sigma^4} + \frac{n}{\sigma^4} = \frac{n}{2\sigma^4}$$

$$I_{X_1}(\sigma^2) = \frac{1}{2\sigma^4}$$

Hence,

$$\sqrt{n}\,(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$$

(ii) First notice $\hat{\sigma}_{mle} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2} = h(\hat{\sigma}^2)$

and since $h(\sigma^2) = \sqrt{\sigma^2}$ we have $h'(\sigma^2) = \frac{1}{2\sqrt{\sigma^2}} = \frac{1}{2\sigma}$

Then the delta method:

$$\sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N\left(0, \frac{1}{4\sigma^2} \cdot 2\sigma^4\right)$$

$$\xrightarrow{d} N\left(0, \frac{\sigma^2}{2}\right)$$

Example 5.44

First, $\log L(X,\mu) = \frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$

and the MLE is $\hat{\mu} = \bar{X}$ //

The deviance is

$$D(\mu) = -2\log\frac{L(X,\mu)}{L(X,\hat{\mu})}$$

$$= -2\left[\log L(X,\mu) - \log L(X,\bar{X})\right]$$

$$= -2\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$$

$$= \frac{2}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i-\mu)^2 - \sum_{i=1}^{n}(X_i-\bar{X})^2\right]$$

$$= \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2$$

'' $-$ '' $=$ ''

Since $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \sqrt{n}\frac{(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

$$\Rightarrow \frac{n(\bar{X}-\mu)^2}{\sigma^2} = D(\mu) \sim N(0,1)^2 = \chi^2_1$$

and this distribution is <u>exact</u>.

Example 5.45

$$D(\theta) = 2n\left[\frac{\bar{X}}{\theta} - \log\frac{\bar{X}}{\theta} - 1\right]$$

Now $\log\frac{\bar{X}}{\theta} = \log\left(1 + \frac{\bar{X}-\theta}{\theta}\right) \approx \frac{\bar{X}-\theta}{\theta} - \frac{1}{2}\left(\frac{\bar{X}-\theta}{\theta}\right)^2$

$$D(\theta) = 2n\left[\frac{\bar{X}}{\theta} - \left(\left(\frac{\bar{X}-\theta}{\theta}\right) - \frac{1}{2}\left(\frac{\bar{X}-\theta}{\theta}\right)^2\right) - 1\right]$$

$$= 2n\left[\cancel{\frac{\bar{X}}{\theta}} - \cancel{\frac{\bar{X}}{\theta}} + \cancel{1} + \frac{1}{2}\left(\frac{\bar{X}-\theta}{\theta}\right)^2 \cancel{-1}\right]$$

$$= \frac{n(\bar{X}-\theta)^2}{\theta^2}$$

and this very close asymptotically to $\frac{n(\theta-\bar{X})^2}{\bar{X}^2}$ since $\bar{X} \approx \theta$ for large n.