University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2021

Assignment Two

Given: Wednesday 7 April 2021 **Due date: Wednesday 21 April 2021**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Wednesday, 21 April 2021. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document.

I/We declare that this assessment item is my/our own work, except where acknowledged, and has not been submitted for academic credit elsewhere. I/We acknowledge that the assessor of this item may, for the purpose of assessing this item reproduce this assessment item and provide a copy to another member of the University; and/or communicate a copy of this assessment item to a plagiarism checking service (which may then retain a copy of the assessment item on its database for the purpose of future plagiarism checking). I/We certify that I/We have read and understood the University Rules in respect of Student Academic Misconduct.

Name	Student No.	Signature	Date
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Problem 1

Let $X = (X_1, X_2, \ldots, X_n)$ be sample of n i.i.d. random variables, each with a density

$$f(x,\theta) = \frac{\sqrt{\theta}}{x\sqrt{2\pi}} \exp\left(-\frac{\theta}{2} \log^2(x) \right)$$

when $x > 0$ otherwise zero and where $\theta > 0$ is a parameter.

a) Find the distribution of $Y_i = \log X_i$ and hence or otherwise compute $\mathbb{E}(\log^2 X_i)$.

b) Find the Fisher information about θ in one observation and in the sample of n observations.

c) Find the Maximum Likelihood Estimator (MLE) of $h(\theta) = \frac{1}{\theta}$ and show that it is unbiased.

d) Does the variance of the MLE for $h(\theta)$ attain the Cramer Rao bound? **Note:** a χ^2_k distribution has mean k and variance $2k$.

e) Determine the asymptotic distribution of the MLE of $h(\theta) = \frac{1}{\theta}$ and also the asymptotic distribution of $\tau(\theta) = e^{-\theta}$.

Problem 2

Suppose $X = X_1, X_2, \ldots, X_n$ is a sample of n i.i.d. random variables from a population with a density

$$f(x;\theta) = \begin{cases} \dfrac{\tau x^{\tau-1}}{\theta^{\tau}} & \text{if} \quad 0 < x < \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

where $\tau > 0$ is a known constant and $\theta > 0$ is an unknown parameter.

a) Show that the density of $T = X_{(n)}$ is

$$f_T(t) = \begin{cases} \dfrac{n\tau t^{n\tau-1}}{\theta^{n\tau}} & \text{if} \quad 0 < t < \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

b) Show that the family $\{L(X,\theta\}, \theta > 0$ has a monotone likelihood ratio in the statistic $T = X_{(n)}$.

c) Find the uniformly most powerful α-size test φ^* of

$$H_0 : \theta \leq \tau \qquad \text{versus} \qquad H_1 : \theta > \tau.$$

d) Calculate the power function of φ^*.

e) Calculate the value of the power function at the threshold constant, τ and as $\theta \to \infty$. Then, sketch a graph of the power function as precisely as possible.

Problem 3
Suppose that X is a random variable with density function

$$f(x, \theta) = e^{-(x-\theta)}, \qquad \theta < x < \infty,$$

and zero elsewhere.

a) Let $X = (X_1, \dots, X_n)$ be a sample of n i.i.d. observations from this distribution.

   i) Compute the distribution and density function for $T = X_{(1)}$.
   ii) Find a statistic that has the MLR property.
   iii) Determine the uniformly most powerful α-size test of

$$H_0 : \theta \geq \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0.$$

   iv) Suppose the following data was collect $\mathbf{x} = (1, 2, 1.01, 3, 1.45)$. Test the hypothesis that $H_0 : \theta \geq 1$ versus $\theta < 1$ with a significance level $\alpha = 0.10$.
   v) Let $Z_n = n\big(X_{(1)} - \theta\big)$. Find the distribution Z_n converges to as $n \to \infty$.
   vi) Hence or otherwise justify that $X_{(1)}$ is a consistent estimator of θ.

b) Now suppose that $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ are the order statistics of a random sample of size five from this distribution. Let the observed value of $X_{(1)}$ be $x_{(1)}$. The test rejects $H_0 : \theta = 2$ and accepts $H_1 : \theta \neq 1$ when either $x_{(1)} \geq 2$ or $x_{(1)} < 1$.

   i) Find the power function $\gamma(\theta)$ for all values θ for this particular test.

   ii) Plot the power function $\gamma(\theta)$ for all values θ.

Problem 4
Suppose $\mathbf{X} = (X_1, \dots, X_n)$ is a random sample from the density

$$f(x; \alpha, \beta) = \frac{\alpha \beta^{\alpha}}{x^{\alpha+1}} I_{[\beta, \infty)}(x), \quad \alpha > 0, \quad \beta > 0.$$

a) Find the Maximum Likelihood Estimator (MLE) for both α and β. Write the MLE for α in terms of T where

$$T = \log\left(\frac{\prod_{i=1}^{n} X_i}{X_{(1)}^{n}}\right)$$

b) Consider testing

$$H_0 : \alpha = 1, \beta > 0 \qquad \text{versus} \qquad H_1 : \alpha \neq 1, \beta > 0.$$

Show that the likelihood ratio is given by the following

$$\lambda(X) = \left(\frac{T}{n}\right)^{n} e^{n-T}$$

c) Show that the Likelihood Ratio test (LRT) has rejection region of the form

$$\{X : T(X) \leq k_1 \quad \text{or} \quad T(X) \geq k_2\}$$

where $0 < k_1 < k_2$.

Problem 5

Suppose $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$ are the order statistics based on a random sample of size four from the density $f(x) = 2e^{-2x}$, $x > 0$.

a) Find $\mathbb{E}(X_{(3)})$. You will need to use a computer package to approximate the integral. E.g. the `integrate` function in `R`.

b) Find the density of the median $M = \frac{1}{2}(X_{(2)} + X_{(3)})$.

c) Using this result (or otherwise), find $P(M > \mathbb{E}(X))$. You will need to use a computer package to approximate the integral. E.g. the `integrate` function in `R`.