Exercise 7.38:

First note that the CDF is $F_Y(y) = \int_0^y \frac{1}{100} e^{-t/100}\,dt = 1 - e^{-y/100}, \; y > 0.$

(i) We are interested in $X = \min(Y_1, Y_2) = Y_{(1)}$ and need to calculate:

$$f_{Y_{(1)}}(x) = f_X(x) = n\left[1 - F_Y(x)\right]^{n-1} f_Y(x)$$

$$= 2\left[1 - (1 - e^{-x/100})\right]^{2-1} \frac{1}{100} e^{-x/100}$$

$$= \frac{1}{50} e^{-x/100} \cdot e^{-x/100}$$

$$= \frac{1}{50} e^{-x/50}, \quad x > 0$$

$Y_{(1)} = X \sim Exp(50)$

(ii) Now we want $X = \max(Y_1, Y_2) = Y_{(2)}$ given by

$$f_X(x) = n\left[F_Y(x)\right]^{n-1} f_Y(x)$$

$$= 2(1 - e^{-x/100})^{2-1} \frac{1}{100} e^{-x/100}$$

$$= \frac{1}{50}(1 - e^{-x/100})\, e^{-x/100}. \qquad x > 0.$$

Exercise 7.37

(i) $F(x) = \int_0^x e^{-t}\,dt = 1 - e^{-x}, \quad x > 0.$

Now $i=1, j=3, n=3$ for Theorem 7.30:

$$f_{X_{(1)},X_{(3)}}(x_{(1)}, x_{(3)}) = \frac{3!}{(1-1)!\,(3-1-1)!\,(3-3)!} f(x_{(1)}) f(x_{(3)}) [F(x_{(1)})]^{1-1} [F(x_{(3)}) - F(x_{(1)})]^{3-1-1} [1 - F(x_{(3)})]^{3-3}$$

$$= 6 e^{-x_{(1)}} e^{-x_{(3)}} [1 - e^{-x_{(3)}} - (1 - e^{-x_{(1)}})]^1$$

$$= 6 e^{-x_{(1)}} e^{-x_{(3)}} (e^{-x_{(1)}} - e^{-x_{(3)}}), \quad 0 < x_{(1)} < x_{(3)} < \infty$$

(ii)
$$f_{X_{(1)}}(x_{(1)}) = n[1 - F(x_{(1)})]^{n-1} f(x_{(1)})$$

$$= 3(1 - (1 - e^{-x_{(1)}}))^{3-1} e^{-x_{(1)}}$$

$$= 3 e^{-2x_{(1)}} e^{-x_{(1)}}$$

$$= 3 e^{-3x_{(1)}} \qquad x_{(1)} > 0$$

$X_{(1)} \sim Exp(3)$

$$f_{X_{(3)}}(x_{(3)}) = 3 e^{-x_{(3)}} (1 - e^{-x_{(3)}})^2 \qquad 0 < x_{(3)} < \infty$$

(iii) $E(X_{(1)}) = \frac{1}{3}$

(iv)
$$E(X_{(3)}) = \int_0^\infty x \cdot 3e^{-x} (1 - e^{-x})^2\,dx$$

$$= 3\int_0^\infty x e^{-x}(1 - 2e^{-x} + e^{-2x})\,dx$$

$$= 3\left[\int_0^\infty x e^{-x}\,dx - 2\int_0^\infty x e^{-2x}\,dx + \int_0^\infty x e^{-3x}\,dx\right]$$

$\alpha=2 \quad \beta=1 \qquad \alpha=2 \quad \beta=2 \qquad \alpha=2 \quad \beta=3$

If X is Gamma(α, β) then

$$\boxed{\begin{aligned} &\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx = 1 \\ &\int_0^\infty x^{\alpha-1} e^{-\beta x}\, dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \end{aligned}}$$

$$EX_{(3)} = 3\left[\frac{\Gamma(2)}{1^2} - 2\frac{\Gamma(2)}{2^2} + \frac{\Gamma(2)}{3^2}\right]$$

$\Gamma(n) = (n-1)!$

$\Gamma(2) = 1! = 1$

$$= 3\left(1 - \frac{1}{2} + \frac{1}{9}\right)$$

$$= \frac{11}{6}$$

(V) Let consider the transformation

$$U = X_{(3)} - X_{(1)} \quad \text{and} \quad V = X_{(1)}$$

This implies that

$$X_{(1)} = V \quad \text{and} \quad X_{(3)} = U + X_{(1)} = U + V$$

Then the Jacobian:

$$J(u,v) = \begin{vmatrix} \frac{\partial X_{(1)}}{\partial u} & \frac{\partial X_{(1)}}{\partial v} \\ \frac{\partial X_{(3)}}{\partial u} & \frac{\partial X_{(3)}}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 \times 1 - 1 \times 1 = -1$$

Hence the joint density is

$$f_{UV}(u,v) = f_{X_{(1)},X_{(3)}}\left(x_{(1)}(u,v),\, x_{(3)}(u,v)\right) |J(u,v)|$$

$$= f_{X_{(1)},X_{(3)}}(v,\, u+v) \cdot |-1|$$

$$= 6e^{-v}e^{-(u+v)}\left(e^{-v} - e^{-(u+v)}\right)$$

and since $0 < X_{(1)} < X_{(3)} < \infty$

$$0 < v < u+v < \infty$$

$\Rightarrow \quad 0 < u < \infty$ and $0 < v < \infty$

$$f_U(u) = \int_0^\infty 6e^{-v}e^{-(u+v)}\left(e^{-v} - e^{-(u+v)}\right) dv$$

$$= 6\int_0^\infty e^{-u-3v} - e^{-2u-3v}\, dv$$

$$= 6\left[-\frac{1}{3}e^{-u-3v} + \frac{1}{3}e^{-2u-3v}\right]_0^\infty$$

$$= 2\left[0+0-\left(-e^{-u} + e^{-2u}\right)\right]$$

$$= 2e^{-u}\left(1-e^{-u}\right) \qquad 0 < u < \infty$$