Example 5.39

Suppose that $X \sim \text{Uniform}(0,1)$ and let $\bar{X}_n = \frac{s_1}{n}$ where $s_1 = \sum_{i=1}^{n} X_i$. Then by the property of expected value:

$$
\begin{aligned}
\mathbb{E}(s_1) &= \mathbb{E}\Big(\sum_{i=1}^{n} X_i\Big) = n\mu_1' \\
\mathbb{E}(s_1^2) &= \mathbb{E}\left(\Big(\sum_{i=1}^{n} X_i\Big)^2\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{n} X_i^2 + \sum\sum_{i\neq j} X_i X_j\right) \\
&= \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum\sum_{i\neq j} \mathbb{E}(X_i)\mathbb{E}(X_j) \\
&= n\mu_2' + n(n-1)(\mu_1')^2
\end{aligned}
$$

where $\mu_1' = \mathbb{E}(X)$ and $\mu_2' = \mathbb{E}(X^2)$.

Therefore, the mean square error is:

$$
\begin{aligned}
\text{MSE}_\theta(\bar{X}_n) &= \mathbb{E}\Big[\big(\frac{s_1}{n} - \theta\big)^2\Big] \\
&= \mathbb{E}\Big[\frac{s_1^2}{n^2} - 2\frac{s_1}{n}\theta + \theta^2\Big] \\
&= \frac{n\mu_2' + n(n-1)(\mu_1')^2}{n^2} - \frac{2\theta}{n\mu_1'}n + \theta^2 \\
&= \theta^2 - 2\theta\mu_1' + \frac{n-1}{n}(\mu_1')^2 + \frac{\mu_2'}{n}
\end{aligned}
$$

For the specific case when $X \sim \text{Uniform}(0,1)$ we have $\mu_1' = \frac{1}{2}$ and

$$
\mu_2' = \mathbb{E}(X^2) = \int_0^1 x^2 \cdot 1\mathrm{d}x = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.
$$

Therefore,

$$
\begin{aligned}
\text{MSE}_\theta(\bar{X}_n) &= \theta^2 - \theta + \Big(1 - \frac{1}{n}\Big)\frac{1}{4} + \frac{1}{3n} \\
&= \theta^2 - \theta + \frac{1}{12n} + \frac{1}{4}
\end{aligned}
$$

Example 5.43

Here $h(\mu_X, \mu_Y) = \frac{\mu_X}{\mu_Y}$ with

$$\frac{\mathrm{d}}{\mathrm{d}\mu_X} h(\mu_X, \mu_Y) = \frac{1}{\mu_Y} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_Y} h(\mu_x, \mu_Y) = -\frac{\mu_X}{\mu_Y^2}$$

Recall the fist order Taylor approximation:

(1)

$$\mathbb{E}_\theta h(T) \approx h(\theta) + \sum h_i'(\theta)\mathbb{E}_\theta(T_i - \theta_i) \approx h(\theta)$$

since T_i has mean θ_i.

(2)

$$\begin{aligned} \mathrm{Var}_\theta h(T) &\approx \mathbb{E}_\theta\Big(\big[h(T) - h(\theta)\big]^2\Big) && \text{using above approximation} \\ &\approx \mathbb{E}_\theta\Bigg(\Big(\sum_{i=1}^{k} h_i'(\theta)(T_i - \theta_i)\Big)^2\Bigg) && \text{taylor series expansion} \\ &= \sum_{i=1}^{k}[h_i'(\theta)]^2\mathrm{Var}(T_i) + 2\sum_{i>j} h_i'(\theta)h_j'(\theta)\mathrm{Cov}(T_i, T_j) \end{aligned}$$

Approximation (2) is very useful since it provide us with a variance formula for a general function. By (1) and (2) we get

$$\mathbb{E}\Big(\frac{X}{Y}\Big) = \frac{\mu_X}{\mu_Y}$$

and

$$\begin{aligned} \mathrm{Var}\Big(\frac{X}{Y}\Big) &\approx \Big(\frac{1}{\mu_Y}\Big)^2\mathrm{Var}(X) + \Big(-\frac{\mu_X}{\mu_Y^2}\Big)^2\mathrm{Var}(Y) + 2\Big(\frac{1}{\mu_Y}\Big)\Big(-\frac{\mu_X}{\mu_Y^2}\Big)\mathrm{Cov}(X, Y) \\ &= \Big(\frac{\mu_X}{\mu_Y}\Big)^2\Big(\frac{\mathrm{Var}(X)}{\mu_X^2} + \frac{\mathrm{Var}(X)}{\mu_Y^2} - 2\frac{\mathrm{Cov}(X, Y)}{\mu_X\mu_Y}\Big) \end{aligned}$$

We now have an approximation for the mean and variance of the ratio estimator and this approximation involved the means, variances and covariance of X and Y.

Note: Exact calculation would be quite hopeless with closed form expression being unattainable.

Now we will find the asymptotic distribution. The inverse of the information matrix is:

$$I_n^{-1}(\mu_X, \mu_Y) = \frac{1}{n}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Applying the delta method gives:

$$\sqrt{n}\Big(\frac{\bar{X}}{\bar{Y}} - \frac{\mu_X}{\mu_Y}\Big) \xrightarrow{d} N\left(0, \Big(\frac{1}{\mu_Y}, -\frac{\mu_X}{\mu_Y^2}\Big)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} \frac{1}{\mu_Y} \\ -\frac{\mu_X}{\mu_Y^2} \end{pmatrix}\right)$$

Now just focusing on the asymptotic variance:

$$
\begin{aligned}
\Big(\frac{1}{\mu_Y}\sigma_{11} - \frac{\mu_X}{\mu_Y}\sigma_{21}, \frac{\sigma_{12}}{\mu_Y} - \frac{\mu_X}{\mu_Y}\sigma_{22}\Big)\begin{pmatrix} \frac{1}{\mu_Y} \\ -\frac{\mu_X}{\mu_Y^2} \end{pmatrix} &= \frac{\sigma_{11}}{\mu_Y^2} - \frac{\mu_X}{\mu_Y^3}\sigma_{21} + \frac{\sigma_{12}}{\mu_Y^2} - \frac{\mu_X^2}{\mu_Y^4}\sigma_{22} \\
&= \frac{\mu_X^2}{\mu_Y^2}\big[\frac{\sigma_{11}}{\mu_X^2} + \frac{\sigma_{22}}{\mu_Y^2} - 2\frac{\sigma_{12}}{\mu_X\mu_Y}\big]
\end{aligned}
$$

In other words,

$$
\frac{\bar{X}}{\bar{Y}} \approx N\left(\frac{\mu_X}{\mu_Y}, \frac{1}{n}\frac{\mu_X^2}{\mu_Y^2}\left[\frac{\sigma_{11}}{\mu_X^2} + \frac{\sigma_{22}}{\mu_Y^2} - 2\frac{\sigma_{12}}{\mu_X\mu_Y}\right]\right)
$$