THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

MATH5905 STATISTICAL INFERENCE

Revision exercises for week 1

1) How many children should a family be planning for if they want to have with probability 0.95 at least one boy and at least one girl? (Assuming that the probability of having a boy is the same as the probability of having a girl).

2) An insurance company has three types of customers-high, medium and low risk. Twenty percent of its customers are high risk, 30 percent are medium risk and 50 percent are low risk. The probability that a customer has at least one accident in the current year is 0.25 for high risk, 0.16 for medium risk, and 0.10 for low risk.

a) Find the probability that a customer chose at random will have at least one accident in the current year.
b) Find the probability that a customer is high risk, given that the person has had at least one accident during the current year.

3) Let X be a continuous random variable with density $f(x)$ and a cumulative distribution function $F(x)$. Fix certain x_0 and assume that $F(x_0) < 1$. Define the function

$$g(x) = \begin{cases} \dfrac{f(x)}{1 - F(x_0)}, & x \geq x_0 \\ 0 & \text{else.} \end{cases}$$

Prove that $g(x)$ is a density function.

4) If $Z \sim N(0, 1)$ find the density of Z^2, (i.e., the density of the chi-square with one degree of freedom.

5) Recall the definition of the Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$. It generalizes the factorial function since for positive integers $\Gamma(n) = (n-1)!$ holds. Using it, a two-parameter family of distributions Gamma (α, β) of non-negative random variables can be defined with densities given by

$$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp(-x/\beta) x^{\alpha-1}, x > 0.$$

Here $\alpha > 0$ is called a shape parameter and $\beta > 0$ is called the scale parameter. Show that if X is Gamma(α, β) distributed then

$$\mathbb{E}(X) = \alpha\beta, \quad \mathbb{E}(X^2) = \beta^2\alpha(\alpha + 1), \quad \text{and} \quad \text{Var}(X) = \alpha\beta^2.$$

Solutions:

1) Drawing a tree diagram for the different stages of the "birth experiment", we see that at the second attempt, the probability to have a girl and a boy is 1/2, further, after the third attempt it becomes $\frac{1}{2}+\frac{1}{4}$, after the fourth it is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, then $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=0.9375$ which is still less than 0.95. After the sixth experiment it jumps above 0.95 already. Hence the answer is: at least 6 children should be planned for.

2) Denote by H,M,L the three types of customers. Denote by A the event that a customer has at least one accident in the current year. It is given that: $P(H)=0.2, P(M)=0.3, P(L)=0.5, P(A|H)=0.25, P(A|M)=0.16, P(A|L)=0.10$.

a) Find the probability that a customer chose at random will have at least one accident in the current year: $P(A)=P(A|H)P(H)+P(A|M)P(M)+P(A|L)P(L)$ (total probability formula). This gives $P(A)=0.148$.

b) Find the probability that a customer is high risk, given that the person has had at least one accident during the current year. We need

$$P(H|A)=P(H\bigcap A)/P(A)=P(A|H)P(H)/[P(A|H)P(H)+P(A|M)P(M)+P(A|L)P(L)]$$

(Bayes formula). Hence we get $0.25\cdot 0.2/0.148=0.3378$.

3) We need to show that $g(x)$ is non-negative and integrates to 1 over the range where it is non-zero. The non-negativity is obvious since $f(x)$ is a density and $1-F(x_0)>0$ is given. Using the fact that $F(x)$ is a cdf, the integral is:

$$\int_{x_0}^{\infty}\frac{f(x)}{1-F(x_0)}\mathrm{d}x=\frac{F(x)}{1-F(x_0)}\Big|_{x_0}^{\infty}=\frac{1-F(x_0)}{1-F(x_0)}=1.$$

4) We find the cdf first and then differentiate it to find the density. We also not that $f_{Z^2}(x)=0$ when $x<0$. When $x>0$ on the other hand, we have for the cdf

$$F_{Z^2}(x)=P(Z^2\le x)=P(Z^2\le x)=P(-\sqrt{x}\le Z\le\sqrt{x})=\Phi(\sqrt{x})-\Phi(-\sqrt{x}),$$

where $\Phi(x)$ denotes the cdf of the standard normal distribution (with the property $\Phi(x)=1-\Phi(-x)$.) Hence $F_{Z^2}(x)=2\Phi(\sqrt{x})-1$ when $x>0$. Differentiating, we get

$$f_{Z^2}(x)=\phi(\sqrt{x})\frac{1}{\sqrt{x}}=\frac{1}{\sqrt{2\pi x}}\exp(-x/2),\qquad x>0.$$

5) We have

$$EX=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}\exp(-x/\beta)x^{\alpha}dx.$$

We now change the variables by setting: $\frac{x}{\beta}=y\to\mathrm{d}x=\beta\mathrm{d}y$ and getting

$$\mathbb{E}(X)=\frac{\beta^{\alpha+1}}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}\exp(-y)y^{\alpha}\mathrm{d}y=\beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

Now we utilise the property $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ of the gamma function (this property can be checked through integration by parts). Hence we get $\mathbb{E}(X)=\alpha\beta$. The remaining equalities in this question can be shown by following the same pattern.