Exercise 6.26

By the Neyman-Pearson lemma for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$, the best α-test is the one with rejection region of the form:

$$\left\{ \frac{L(X,1)}{L(X,0)} \geq k \right\}$$

$$\frac{e^x+1}{e^{x-1}+1} \geq \sqrt{ke} = k'$$

Here, $n=1$

$$L(X,\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}$$

$$\iff x > k''$$

and thus

$$\frac{L(X,1)}{L(X,0)} = \frac{e^{(x-1)}}{(1+e^{(x-1)})^2} \cdot \frac{(1+e^{(x-0)})^2}{e^{(x-0)}} = \frac{(e^x+1)^2}{e(e^{x-1}+1)^2} \geq k$$

Now by ignoring the constant $\frac{1}{e}$ and square operation we have that (using the quotient rule):

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x+1}{e^{x-1}+1} = \frac{e^x - e^{x-1}}{(1+e^{x-1})^2}$$

$$\left\{ \frac{L(X,1)}{L(X,0)} \geq k \right\} \iff \{ X \geq k' \}$$

and since $e^x > e^{x-1}$ or $e^x - e^{x-1} > 0$, this ratio is increasing in x and the inequality $\frac{L(X,1)}{L(X,0)} \geq k$ must be the same as rejection when $X \geq k'$. Therefore, to make this an α-test we need to "exhaust the level":

$$\mathbb{E}_{\theta_0}\varphi^* = \alpha = 0.1 = P(X \geq k' | \theta = 0)$$

$$\varphi^* = \begin{cases} 1 & X \geq k' \\ 0 & X < k' \end{cases}$$

The CDF of X here is $P(X \leq x)$

$$F(x,\theta) = \int_{-\infty}^{x} \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}\mathrm{d}x = \ldots = \frac{e^{x-\theta}}{1+e^{x-\theta}}$$

and thus

$$0.10 = P(X > k') \underset{= 1 - P(X \leq k')}{=} 1 - F(k',0) = 1 - \frac{e^{k'}}{1+e^{k'}} = \frac{1}{e^{k'}+1} = 0.1$$

which leads to

$$k' = \log(\frac{1}{0.1} - 1) = \log(9) = 2.19722$$

The best 0.10-test of $H_0 : \theta = 0$ vs $H_1 : \theta = 1$ is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad X \geq 2.19722 \\ 0 & \text{if} \quad X < 2.19722 \end{cases}$$

reject
do not reject

Example 6.51

Assume, a sample of one observation ($n = 1$) from an exponential family with density

$$f(x, \theta) = \frac{1}{\theta} \exp\Big(-\frac{x}{\theta}\Big), \quad x > 0$$

is available. The parameter $\theta > 0$ is to be tested. Construct an UMP unbiased α-test of:

(i)

$$H_0 : \theta \in [1, 2] \quad \text{versus} \quad H_1 : \theta \notin [1, 2].$$

$H_1 \leftarrow \quad H_0 \quad \rightarrow H_1$

$1 \qquad 2$

reject H_0 $\leftarrow$ do not reject $\rightarrow$ reject H_0

$C_1 \qquad C_2$

This case is covered by Theorem 6.24 and here we have

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < C_1 \quad \text{or} \quad T > C_2 \\ 0 & \text{if} \quad C_1 \leq T \leq C_2 \end{cases}$$

where $T(x) = x$ in this case (one-parameter exponential family with $d(x) = x, n = 1$). We only need to find C_1 and C_2 in order to uniquely specify the above test. → No randomization

Since $X \sim \text{Exp}(\theta)$ we have $F(x) = 1 - e^{-x/\theta}$ and therefore

$$\begin{aligned}
\mathbb{E}_{\theta}(\varphi^*) &= P(T < C_1 \quad \text{or} \quad T > C_2) \\
&= P(T < C_1) + P(T > C_2) \\
&= 1 - e^{-C_1/\theta} + (\cancel{1} - (\cancel{1} - e^{-C_2/\theta})) \\
&= 1 - e^{-C_1/\theta} + e^{-C_2/\theta}
\end{aligned}$$

Therefore to determine C_1 and C_2 we need to satisfy the two equations

$$\begin{cases} \mathbb{E}_{\theta=1}(\varphi^*) = 1 - e^{-C_1} + e^{-C_2} = \alpha \\ \mathbb{E}_{\theta=2}(\varphi^*) = 1 - e^{-C_1/2} + e^{-C_2/2} = \alpha \end{cases}$$

This system can be solved by substitution. Set $k_1 = e^{-C_1/2}$ and $k_2 = e^{-C_2/2}$ then we have

$$\begin{cases} k_1^2 - k_2^2 = 1 - \alpha \\ k_1 - k_2 = 1 - \alpha \end{cases}$$

$K_1^2 = e^{-C_1}$

$K_2^2 = e^{-C_2}$

$a^2 - b^2 = (a+b)(a-b)$

which the first equation gives

$$(k_1 + k_2) \underbrace{(k_1 - k_2)}_{1-\alpha} = 1 - \alpha$$

and hence $k_1 + k_2 = 1$. This implies

$$\begin{cases} k_1 + k_2 = 1 \\ k_1 - k_2 = 1 - \alpha \end{cases}$$

Then adding these two equations yields

$$2k_1 = 2 - \alpha \qquad \text{or} \qquad k_1 = 1 - \alpha/2.$$

and then substituting this back into the first equation gives

$$k_1 + k_2 = 1 \implies k_2 = 1 - k_1 = 1 - (1 - \alpha/2) = \alpha/2$$

Hence,

$$C_1 = \overset{-2\log(k_1)}{-2\log(1 - \alpha/2)} \quad \text{and} \quad C_2 = \overset{-2\log k_2}{-2\log(\alpha/2)}.$$

since

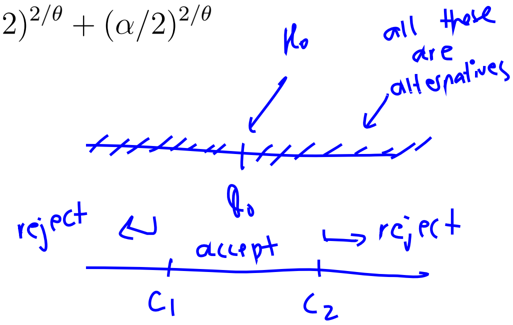
$$k_1 = e^{-C_1/2} \quad \text{and} \quad k_2 = e^{-C_2/2}.$$

Therefore, the test has structure

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < -2\log(1-\alpha/2) \quad \text{or} \quad T > -2\log(\alpha/2) \\ 0 & \text{if} \quad -2\log(1-\alpha/2) \le T \le -2\log(\alpha/2) \end{cases}$$

with power function

$$\text{Power}(\theta) = \mathbb{E}_\theta(\varphi^*) = 1 - e^{-C_2/\theta} + e^{-C_2/\theta} = 1 - (1-\alpha/2)^{2/\theta} + (\alpha/2)^{2/\theta}$$



(ii)

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1.$$

According to the Theorem 6.25, the UMPU α-test φ^* for (ii) has the same structure as part (i):

$$\varphi^*(x) = \begin{cases} 1 & \text{if} \quad T < C_1 \quad \text{or} \quad T > C_2 \\ 0 & \text{if} \quad C_1 \le T \le C_2 \end{cases}$$

where $T(x) = x$ in this case (one-parameter exponential family with $d(x) = x, n = 1$).We only need to find C_1 and C_2 in order to uniquely specify the above test.

Note that

$$\mathbb{E}_\theta(\varphi^*) = P_\theta(X \notin (C_1, C_2)) = 1 - \exp(-C_1/\theta) + \exp(-C_2/\theta)$$

Same as before!

since the cdf is $F(x, \theta) = 1 - \exp(-x/\theta), x > 0)$. The two conditions on $\mathbb{E}_\theta(\varphi^*)$ are:

- $$\mathbb{E}_\theta(\varphi^*)\Big|_{\theta=1} = \alpha = 1 - \exp(-C_1) + \exp(-C_2)$$

- $$\begin{aligned} \frac{\partial}{\partial\theta}\mathbb{E}_\theta(\varphi^*)\Big|_{\theta=1} &= -\frac{C_1}{\theta^2}\exp(-\frac{C_1}{\theta}) + \frac{C_2}{\theta^2}\exp(-\frac{C_2}{\theta})\Big|_{\theta=1} \\ &= -C_1\exp(-C_1) + C_2\exp(-C_2) = 0 \end{aligned}$$

We get a system of two equations with respect to C_1 and C_2. It can be solved numerically (iteratively) given the level α and hence the UMPU α-test will be completely specified. We will skip over these details but the key is a solution can be found from here. We need to find C_1 and C_2 that satisfy

$$\begin{cases} \alpha = 1 - e^{-C_1} + e^{-C_2} \\ 0 = -C_1 e^{-C_1} + C_2 e^{-C_2} \end{cases}.$$

Exercise 6.33

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are *i.i.d.* Cauchy$(\theta, 1)$ and we are testing

$$H_0 : \theta \leq 0 \quad \text{vs} \quad H_1 : \theta > 0$$

The likelihood is:

$$L(X, \theta) = \prod_{i=1}^{n} \frac{1}{\pi} \frac{1}{1 + (x_i - \theta)^2} = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2}$$

By taking the logarithm we arrive at the log-likelihood:

$$\log L(X, \theta) = -n \log \pi - \sum_{i=1}^{n} \log(1 + (x_i - \theta)^2)$$

which gives the score:

$$V(X, \theta) = \frac{\partial}{\partial \theta} \log L(X, \theta) = -\sum_{i=1}^{n} \frac{-2(x_i - \theta)}{1 + (x_i - \theta)^2} = \sum_{i=1}^{n} \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2}$$

Then

$$V(X, \theta)\Big|_{\theta=0} = \sum_{i=1}^{n} \frac{2x_i}{1 + x_i^2}$$

which leads to the LMP α-test

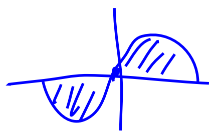
$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad \text{V} = \sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2} > k \\ 0 & \text{if} \quad \text{V} = \sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2} < k \end{cases}$$

$E_{\theta=0} \varphi^* = \alpha$

Now we need to find k (but we can only do this asymptotically). Let

$$U = \frac{2X}{1 + X^2}$$

where $X \sim$ Cauchy$(0, \theta)$ then



$$\mathbb{E}(U) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(1 + x^2)} \cdot \frac{1}{1 + x^2} \mathrm{d}x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(1 + x^2)^2} \mathrm{d}x = 0$$

$= 0$

since the integral of an odd function $g(x) = -g(-x)$ is zero i.e.

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{-\infty}^{0} g(x)\mathrm{d}x + \int_{0}^{\infty} g(x)\mathrm{d}x \\
&= -\int_{-\infty}^{0} g(-x)\mathrm{d}x + \int_{0}^{\infty} g(x)\mathrm{d}x \\
&= \int_{0}^{\infty} g(x)\mathrm{d}x - \int_{0}^{\infty} g(x)\mathrm{d}x \\
&= 0 \;//
\end{aligned}$$

$E(U^2) - (\cancel{EU})^2$

Furthermore,

$$\mathrm{Var}(U) = \mathbb{E}(U^2) = \frac{4}{\pi^2}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\mathrm{d}x = \ldots = \frac{1}{2}.$$

Therefore, $T = r(X, \theta)$

$$U_i = \frac{2X_i}{1+X_i^2}$$

has zero mean and variance $1/2 < \infty$. Now by applying the CLT we obtain

$$\frac{T-0}{\sqrt{\frac{n}{2}}} = \sqrt{\frac{2}{n}}T \xrightarrow{d} N(0,1)$$

Hence,

$$\begin{aligned}
\alpha &= P(T > k|\theta = 0) \\
&= P\Big(\sqrt{\frac{2}{n}}T > \sqrt{\frac{2}{n}}k|\theta = 0\Big) \\
&= P(Z > \sqrt{\frac{2}{n}}k)
\end{aligned}$$

$P(Z > z_\alpha) = \alpha$

which implies that

$$\sqrt{\frac{2}{n}}k = z_\alpha \qquad \text{or} \qquad k = z_\alpha\sqrt{\frac{n}{2}}$$

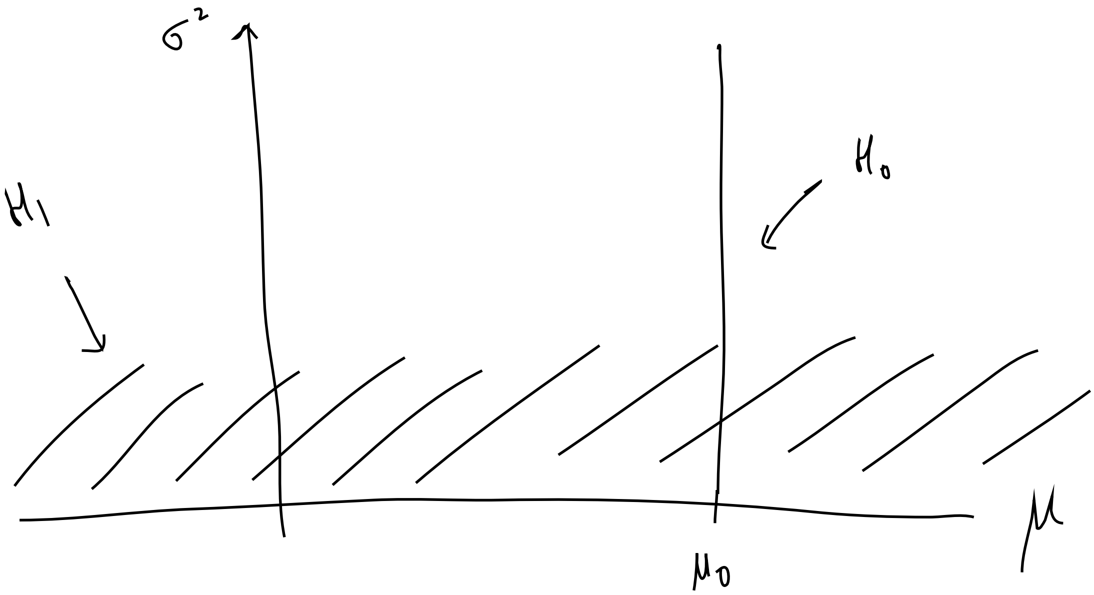
and now the test is completely determined.

Example 6.54

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a normal sample with mean μ and variance σ^2. Let's test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ but now both μ and σ^2 are unknown. Hence, in effect, we are testing

$$H_0 : \begin{cases} \mu = \mu_0 \\ \sigma^2 > 0 \end{cases} \qquad \text{versus} \qquad H_1 : \begin{cases} \mu \neq \mu_0 \\ \sigma^2 > 0 \end{cases}$$

In term of the notation of the GLRT we have a parameter vector $\theta = (\mu, \sigma^2)^\top$, H_0 is a one-dimensional subspace as sketched below whereas H_1 is "anything" above the x-axis:



To perform the GLRT we need to maximize $L(X, \theta)$ under the null hypothesis and under the alternative.

(1) Under the null hypothesis: $\mu = \mu_0$ and so we need to optimize with respect to σ only:

$$\begin{aligned} L(X, \theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(X_i - \mu_0)^2 \Big) \\ &= (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu_0)^2 \Big) \end{aligned}$$

and

$$\log L(X, \theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2$$

$\theta = \sigma^2$

By setting the score to zero we obtain:

$$V(X,\theta) = \frac{\partial}{\partial\sigma^2}\log L(X,\theta) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i-\mu_0)^2 = 0$$

which implies that

$\sigma^4 \neq 0$

$$-n\sigma^2 + \sum_{i=1}^{n}(X_i-\mu_0)^2 = 0$$

$$\hat\sigma^2_{H_0} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2 \quad //$$

Also, can check second derivative to see if we arrive at a maximum:

$$\begin{aligned}
\frac{\partial^2}{\partial\sigma^2\partial\sigma^2}\log L(X,\theta)\Big|_{\sigma^2=\hat\sigma^2_{H_0}} &= \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(X_i-\mu_0)^2\Big|_{\sigma^2=\hat\sigma^2_{H_0}} \\
&= \frac{n}{2\hat\sigma^4_{H_0}} - \frac{n}{\hat\sigma^4_{H_0}} \\
&= -\frac{n}{2\hat\sigma^2_{H_0}} < 0
\end{aligned}$$

$\hat\sigma_{H_0} > 0$.

which means that $\hat\sigma^2_{H_0}$ is a maximum and the MLE for σ^2 under H_0 and

$$\begin{aligned}
\sup L(X,\theta)\Big|_{\sigma^2=\hat\sigma^2_{H_0}} &= (2\pi)^{-n/2}(\hat\sigma^2_{H_0})^{-n/2}\exp\Big(-\frac{1}{\cancel{2\hat\sigma^2_{H_0}}}\cancel{\sum_{i=1}^{n}(X_i-\mu_0)^2}\Big) \\
&= (2\pi\hat\sigma^2_{H_0})^{-n/2}\exp(-\frac{n}{2})
\end{aligned}$$

(2) Without the restriction of H_0: we must maximize $L(X,\theta)$ with respect to both μ and σ^2. This mean that we need to solve the system:

$$\begin{cases} \dfrac{\partial}{\partial\mu}\log L(X,\theta) = 0 \\ \dfrac{\partial}{\partial\sigma^2}\log L(X,\theta) = 0 \end{cases}$$

This leads to

$$\hat\mu = \bar X \qquad \text{and} \qquad \hat\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar X)^2.$$

When plugging in, we get the sup $L(X,\theta)$ without any restrictions and it is given by:

$\hat\mu \quad \hat\sigma^2$

$$\begin{aligned}
\sup L(X,\theta)\Big|_{\theta=\hat\theta} &= (2\pi)^{-n/2}(\hat\sigma^2)^{-n/2}\exp\Big(-\frac{1}{2\hat\sigma^2}\sum_{i=1}^{n}(X_i-\bar X)^2\Big) \\
&= (2\pi\hat\sigma^2)^{-n/2}\exp(-\frac{n}{2})
\end{aligned}$$

$$-2\cdot\left(-\frac{n}{2}\right) = n$$

Hence

$$-2\log\lambda(X) = -2\log\frac{(\cancel{2\pi}\hat{\sigma}^2_{H_0})^{-n/2}\cancel{\exp(-\frac{n}{2})}}{(\cancel{2\pi}\hat{\sigma}^2)^{-n/2}\cancel{\exp(-\frac{n}{2})}} = -2\log\left(\frac{\hat{\sigma}^2_{H_0}}{\hat{\sigma}^2}\right)^{-n/2} = n\log\left(\frac{\hat{\sigma}^2_{H_0}}{\hat{\sigma}^2}\right) \xrightarrow{d} \chi^2_1$$

Since the rejection region is

$$\left\{X\middle|\lambda(X) \le C\right\}$$

$2 > 1$

$-2 < -1$

this leads to the rejection region

$$\left\{X\middle| -2\log\lambda(X) \ge -2\log C = K\right\}$$

$P(\chi^2_1 > K) = \alpha$

$K = \chi^2_{1,\alpha}$

and the GLRT is

$$\varphi^*(X) = \begin{cases} 1 & \text{if} \quad n\log\left(\frac{\hat{\sigma}^2_{H_0}}{\hat{\sigma}^2}\right) \ge \chi^2_{1,\alpha} \\ 0 & \text{if} \quad n\log\left(\frac{\hat{\sigma}^2_{H_0}}{\hat{\sigma}^2}\right) < \chi^2_{1,\alpha} \end{cases}$$

Note: that the degree's of freedom are equal to one since in this case $r = 1$ and $k - s = 2 - 1 = 1$ where $k = 2$ and $s = 1$.

Note: Now the convergence of $-2\log\lambda(X)$ to the limiting χ^2_1 is only asymptotic (not precise as in Example 6.53 where σ was known). However,

$$-2\log\lambda(X) = n\log\left(1 + \frac{(\bar{X} - \mu_0)^2}{\hat{\sigma}^2}\right) \approx \frac{n(\bar{X} - \mu_0)^2}{s^2}$$

$\log(1+y) \approx y$

where S^2 is the sample variance given by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

So it is almost equivalent to the standard t-test for testing $\mu = \mu_0$ when σ^2 is unknown.