Sila nabijene šipke na naboj

Vektorski zapis diferencijala sile iznosi:

$$
\begin{aligned}
d\vec{F}_q &= k\frac{q \cdot dQ}{r_{12}^2}\hat{r}_{21} && (1)\\
&= k\frac{q \cdot \lambda dL}{r_{12}^2}\hat{r}_{21} && (2)\\
&= k\frac{q \cdot \lambda dL}{x^2 + L^2}\hat{r}_{21} && (3)
\end{aligned}
$$

Diferencijal po vektorskim komponentama iznosi:

$$
\begin{aligned}
dF_x &= dF\frac{x}{\sqrt{x^2 + L^2}} && (4)\\
dF_y &= dF\frac{L}{\sqrt{x^2 + L^2}} && (5)
\end{aligned}
$$

Iznos komponente u odnosu na $|d\vec{F}_q|$ (3) proizlazi iz geometrije zadatka:

$$
\begin{aligned}
dF_x &= dF\frac{x}{\sqrt{x^2 + L^2}} && (6)\\
&= k\frac{q\lambda dL}{x^2 + L^2}\frac{x}{\sqrt{x^2 + L^2}} && (7)\\
&= kq\lambda x\frac{dL}{\sqrt{(x^2 + L^2)^3}} && (8)
\end{aligned}
$$

Ukupna sila F_x iznosi, gdje (12) proizlazi iz integrala (15):

$$
\begin{aligned}
F_x &= \int dF_x && (9)\\
&= \int_0^a kq\lambda x\frac{dL}{\sqrt{(x^2 + L^2)^3}} && (10)\\
&= kq\lambda x\int_0^a \frac{dL}{\sqrt{(x^2 + L^2)^3}} && (11)\\
&= kq\lambda x\left[\frac{1}{x^2}\frac{L}{\sqrt{L^2 + x^2}}\right]_0^a && (12)\\
&= kq\lambda x\frac{1}{x^2}\left(\frac{a}{\sqrt{a^2 + x^2}} - 0\right) && (13)\\
&= kq\frac{\lambda a}{x\sqrt{a^2 + x^2}} = \frac{kQq}{x\sqrt{a^2 + x^2}} && (14)
\end{aligned}
$$

Prikazani integral možemo riješiti tan substitucijom:

$$\int \frac{dL}{\sqrt{(x^2+L^2)^3}} = \frac{1}{x^3}\int \frac{dL}{\left(1+\frac{L^2}{x^2}\right)^{\frac{3}{2}}} \tag{15}$$

$$= \left| \begin{matrix} \tan\theta = \frac{L}{x} \\ \frac{1}{\cos^2\theta}d\theta = \frac{dL}{x} \end{matrix} \right| \tag{16}$$

$$= \frac{1}{x^3}\int \frac{x}{\cos^2\theta}\frac{d\theta}{\left(1+\tan^2\theta\right)^{\frac{3}{2}}} \tag{17}$$

$$= \frac{x}{x^3}\int \frac{1}{\cos^2\theta}\frac{d\theta}{\left(\frac{1}{\cos^2\theta}\right)^{\frac{3}{2}}} \tag{18}$$

$$= \frac{1}{x^2}\int \cos\theta d\theta \tag{19}$$

$$= \frac{1}{x^2}\sin\theta + C \tag{20}$$

$$= \frac{1}{x^2}\sin\tan^{-1}\left(\frac{L}{x}\right) + C \tag{21}$$

$$= \frac{1}{x^2}\frac{L}{\sqrt{L^2+x^2}} + C \tag{22}$$

Ukupna sila F_y iznosi, gdje (26) proizlazi iz integrala (28):

$$F_y = \int dF \frac{L}{\sqrt{x^2+L^2}} \tag{23}$$

$$= \int_0^a k\frac{q\lambda dL}{x^2+L^2}\frac{L}{\sqrt{x^2+L^2}} \tag{24}$$

$$= kq\lambda\int_0^a \frac{LdL}{(x^2+L^2)^{\frac{3}{2}}} \tag{25}$$

$$= kq\lambda\left[-\frac{1}{\sqrt{x^2+L^2}}\right]_0^a \tag{26}$$

$$= \frac{kqQ}{a}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right) \tag{27}$$

Prikazani integral možemo riješiti u-substitucijom:

$$\int \frac{LdL}{(x^2+L^2)^{\frac{3}{2}}} = \left| \begin{matrix} u = x^2+L^2 \\ du = 2LdL \end{matrix} \right| \tag{28}$$

$$= \frac{1}{2}\int \frac{du}{u^{\frac{3}{2}}} \tag{29}$$

$$= -\frac{1}{2\cdot\frac{1}{2}}u^{-\frac{1}{2}} + C \tag{30}$$

$$= -\frac{1}{\sqrt{x^2+L^2}} + C \tag{31}$$

Dodatno

Bez dokaza navodimo da električno polje u točki (x, y) prostora iz zadatka iznosi:

$$\vec{E} = \frac{kQ}{ax}\left(\frac{y}{\sqrt{x^2+y^2}} - \frac{y-a}{\sqrt{x^2+(y-a)^2}}\right)\hat{i} + \frac{kQ}{a}\left(\frac{1}{\sqrt{x^2+(y-a)^2}} - \frac{1}{\sqrt{x^2+y^2}}\right)\hat{j}$$