ME2 Computing- Session 6: Numerical solution of differential equations: initial value problems

Learning outcomes:

- Being able to solve first order ODEs with explicit methods
- Being able to solve first order ODEs with implicit methods
- Being able to solve a system of first order ODEs and Higher order ODEs

Before you start:

In your H drive create a folder *H:\ME2MCP\Session6* and work within it.

We will be testing Tasks A and B with the ODE: $\frac{dy}{dt} = -2yt - 2t^3$, whose analytical solution is: y$= 1 - t^2 + ce^{-t^2}$

Task A: Explicit methods: Forward Euler and RK4

1. Write a function, *FwEuler*, to solve a general ODE $\frac{dy}{dt} = F(t, y)$ with a forward Euler scheme. The function receives the initial condition, t_0 and y_0, the time step *h* and the desired final computational time t_{end} (all these input arguments are scalars). The function outputs two arrays, *t* and *y*, describing the specific solution *y(t)*. Within *FwEuler*, *F(t,y),* can be evaluated by invoking a separate function *func*.
   Explicit methods are subject to instabilities: consider this when choosing the value of *h*.

2. Write a function RK4 to perform as the function at point 1, but implementing a RK4 method instead.

Task B: Implicit methods: Backward Euler

1. Write a function, *BwEuler*, to solve the specific ODE under study, with a backward Euler scheme. The function receives the initial condition, t_0 and y_0, the time step $\mathrm{h}$and the desired final computational time t_{end}. The function outputs two arrays, *t* and *y*, describing the specific solution *y(t)*.

Task C: System of ODEs, with explicit methods

1. Modify the function *FwEuler*, into a new function *FwEulerTwo*, to solve the set of two ODEs:

$$\begin{cases} \frac{dy_1}{dt} = F_1(t, y_1, y_2) \\ \frac{dy_2}{dt} = F_2(t, y_1, y_2) \end{cases} \quad \text{with initial conditions} \quad \begin{cases} y_1(t = t_0) = y_1^0 \\ y_2(t = t_0) = y_2^0 \end{cases}$$

FwEulerTwo receives the vector Y_0 of initial values, the initial and final time t_0 and t_{end}, and the time step *h*. *FwEulerTwo* should output an array *t* and a two rows array *Y* with the solutions, $Y = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.
On the side, you also need to define two separate functions, *func1*, *func2*, to evaluate F_1 and F_2, respectively.

2. Test the function to solve the *predator-prey* problem applied to the London property rental market. The cycles of property rental prices vs population are described by the Lokta-Volterra's set of first order ODEs:

$$\begin{cases} \frac{d£}{dt} = 0.3£N - 0.8£ \\ \frac{dN}{dt} = 1.1N - N£ \end{cases} \quad \text{with initial conditions} \quad \begin{cases} £(t=0) = 0.8 \\ N(t=0) = 7 \end{cases}$$

   where £ is the average price of house rentals (in thousands pounds) and N is the number of inhabitants (in millions). Compute the cycles in the period [0:40] years, in steps of 0.019 (i.e. weekly).

3. Plot, on the same graph, the results vs time. Plot also, in a different figure, the number of inhabitants vs the rental price.

Task D: Higher order ODEs: damped non-linear motion of a pendulum

The oscillation of a pendulum of mass *m*, attached to a weightless string, is described by the second order ODE:

$$\frac{d^2\theta}{dt^2} + \frac{c}{m}\frac{d\theta}{dt} + \frac{g}{L}sin\theta = 0$$

where *c* is the damping coefficient, *g* the gravitational acceleration and *L* the length of the string.
The pendulum initially is at rest, displaced at an angle θ_0.

The second order ODE can be reduced to a set of two first order ODEs, introducing a new dependent (artificial) variable *w*:

$$\begin{cases} w = \frac{d\theta}{dt} \\ \frac{dw}{dt} = \left(\frac{d^2\theta}{dt^2}\right) = -\frac{c}{m}\frac{d\theta}{dt} - \frac{g}{L}sin\theta = -\frac{c}{m}w - \frac{g}{L}sin\theta \end{cases}$$

1. Determine the motion of the pendulum, $\theta(t)$, for the first initial 15 seconds, with initial condition $\theta(t=0) = \pi/4$. (Use *FwEulertwo* with $\Delta t = 0.005s$). Use a mass of 0.5Kg and a string L = 1m. Observe the difference between the swinging in a dry place (c = 0.05Ns/m) and within a humid viscous environment (c = 0.18Ns/m).