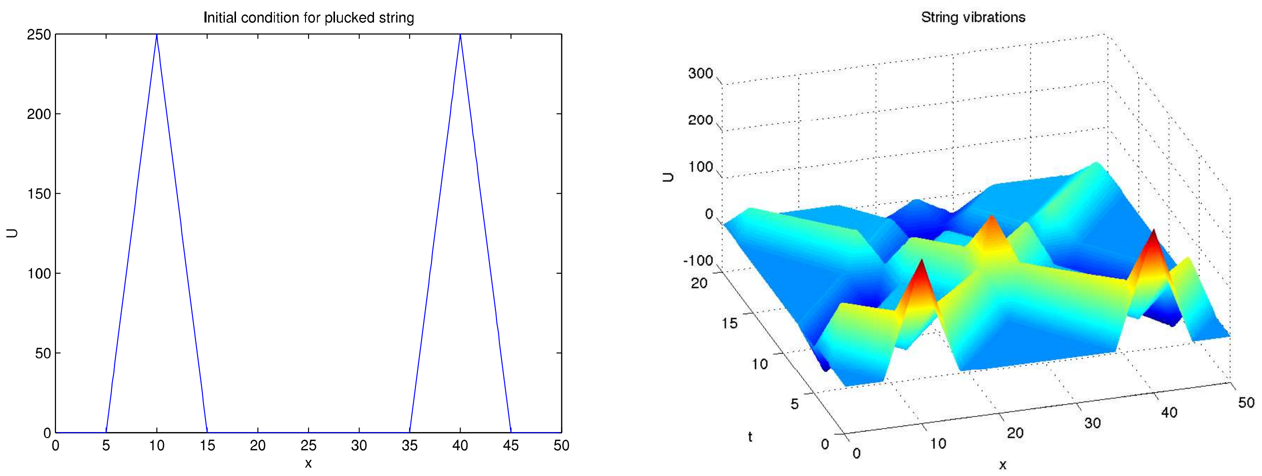


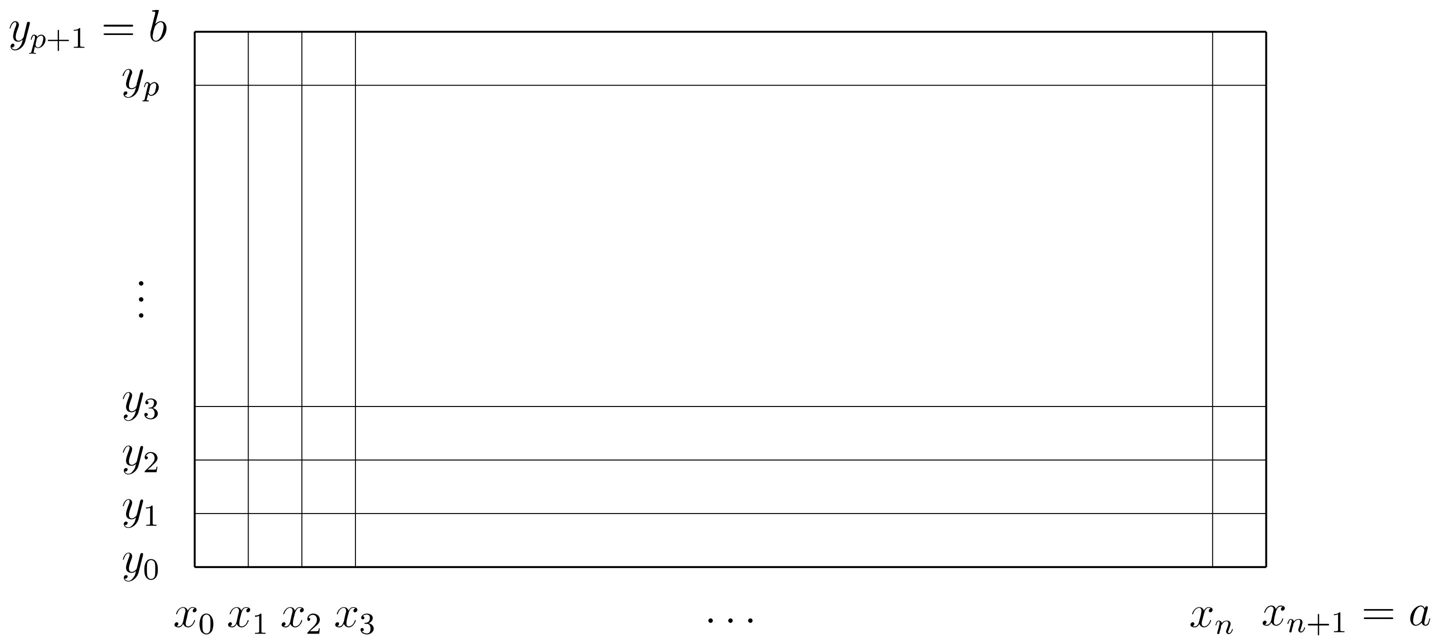
Figure 8.3: Initial conditions in (a) and matlab solution using explicit central difference method for 1D wave equation with friction in (b)

8.2 2-D Wave Equation

$$U_{tt} = \beta(U_{xx} + U_{yy}), \quad 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq t \leq T$$

8.2.1 Example: vibrations of a thin elastic membrane fixed at its walls

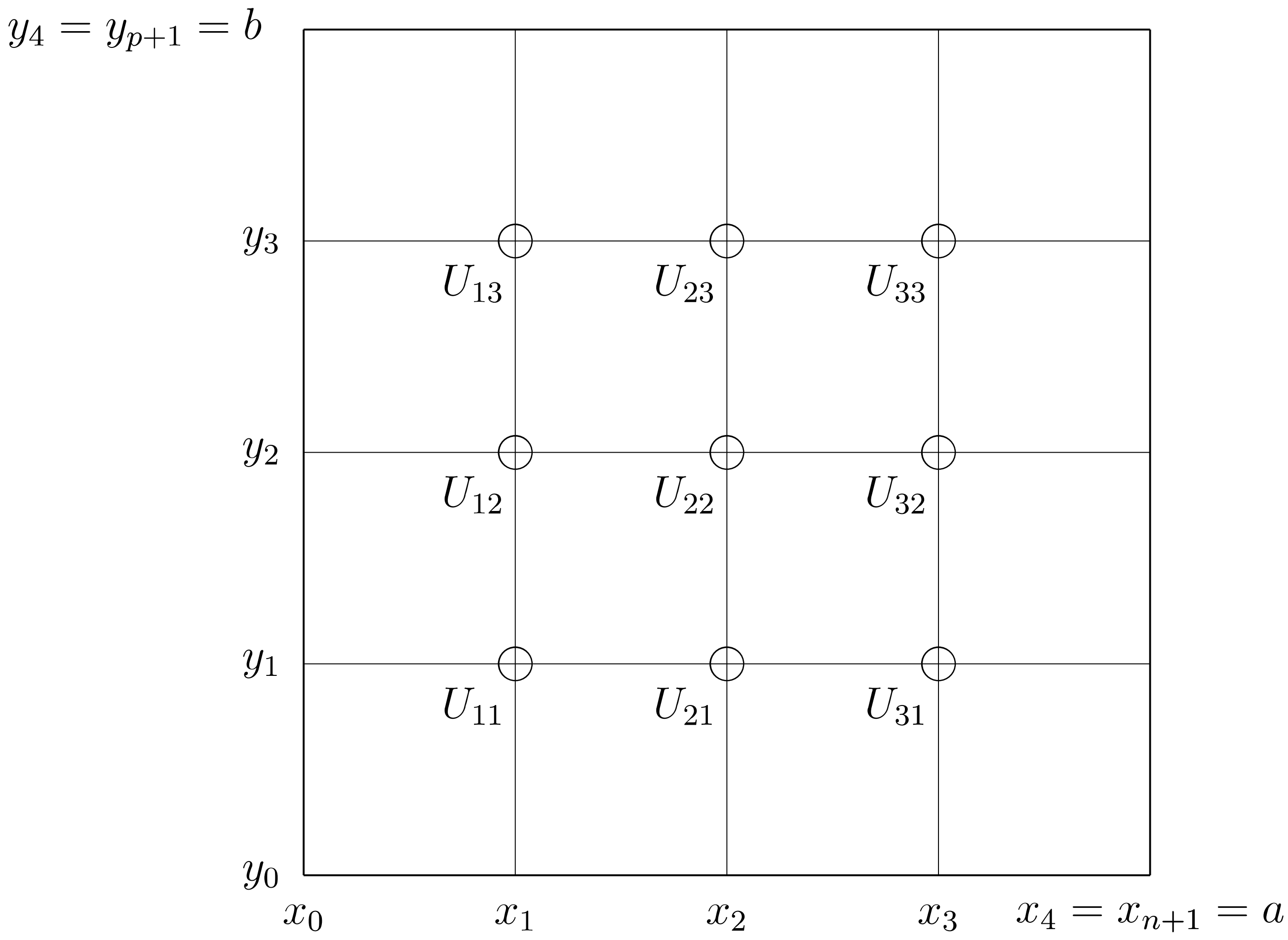
We discretise in x and y-directions:



We discretise: $\Delta t = \frac{T}{m}$, $\Delta x = \frac{a}{n+1}$, $\Delta y = \frac{b}{p+1}$, $t_k = k\Delta t$, $x_i = i\Delta x$, $y_j = j\Delta y 0 \leq k \leq m$, $0 \leq i \leq n+1$, $0 \leq j \leq p+1$, and let $U_{ij}^k = U(t_k, x_i, y_j)$

Suppose we solve for $n = 3$ and $p = 3$ and have Dirichlet boundary conditions:
$U(0, y, t) = 0 = U^k_{oj}, \quad U(a, y, t) = 0 = U^k_{n+1,j} = U^k_{4j}, \quad U(x, 0, t) = 0 = U^k_{i0}, \quad U(x, b, t) = 0 = U^k_{i,p+1} = U^k_{i4}$
and initial conditions:
$U(x, y, 0) = f(x, y) = f_{ij} \quad U_t(x, y, 0) = g(x, y) = g_{ij}$.

Since we have Dirichlet boundary conditions: the outer boundaries of the region we are solving for are known: $U^k_{0,j}, U^k_{n+1,j}, U^k_{i,0}, U^k_{i,p+1}$, and we need to find the interior values: $U^k_{i,j}$ for $1 \le i \le n$ and $1 \le j \le p$.



We will use the *2-D Central Difference Method*

$$
\begin{aligned}
U_{tt} &= \frac{U^{k+1}_{ij} - 2U^k_{ij} + U^{k-1}_{ij}}{\Delta t^2}, \\
U_{xx} &= \frac{U^k_{i+1,j} - 2U^k_{ij} + U^k_{i-1,j}}{\Delta x^2}, \\
U_{yy} &= \frac{U^k_{i,j+1} - 2U^k_{ij} + U^k_{i,j+1}}{\Delta y^2}
\end{aligned}
$$

We let $s_x = \frac{\beta \Delta t^2}{\Delta x^2}, \ s_y = \frac{\beta \Delta t^2}{\Delta y^2}$ and substitute the central difference approximations into our PDE, $U_{tt} = \beta(U_{xx} + U_{yy})$ we solve for U_{ij}^{k+1}:

$$U_{ij}^{k+1} = 2U_{ij}^{k}(1 - s_x - s_y) - U_{ij}^{k-1} + s_x(U_{i+1,j}^{k} + U_{i-1,j}^{k}) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})$$

computing $\vec{U}^{k+1}$ uses the solution at $\vec{U}^{k}$ and $\vec{U}^{k-1}$.
For first time step U_{ij}^{1} needs U_{ij}^{0} and U_{ij}^{-1}. Again we need to use the initial conditions to find the ghost point, U_{ij}^{-1}:

$$\frac{\partial U_{ij}^{0}}{\partial t} = U_t(x, y, 0) = \frac{U_{ij}^{1} - U_{ij}^{-1}}{2\Delta t} = g(x_i, y_j) = g_{ij} \Rightarrow U_{ij}^{-1} = U_{ij}^{1} - 2\Delta t g_{ij}$$

Solution at first time step $k = 1$:

$$U_{ij}^{1} = U_{ij}^{0}(1 - s_x - s_y) + \Delta t g_{ij} + \frac{s_x}{2}(U_{i+1,j}^{0} + U_{i-1,j}^{0}) + \frac{s_y}{2}(U_{i,j+1}^{0} + U_{i,j-1}^{0})$$

If we let $\vec{U}^{k} = \begin{pmatrix} U_{11}^{k} \\ U_{12}^{k} \\ U_{13}^{k} \\ U_{21}^{k} \\ U_{22}^{k} \\ U_{23}^{k} \\ U_{31}^{k} \\ U_{32}^{k} \\ U_{33}^{k} \end{pmatrix}$
then for time steps, $k > 1$, the solution is:

$$U_{ij}^{k+1} = 2U_{ij}^{k}(1 - s_x - s_y) - U_{ij}^{k-1} + s_x(U_{i+1,j}^{k} + U_{i-1,j}^{k}) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})$$

and we can write this in vector form:

$$\vec{U}^{k+1} = A\vec{U}^{k} + \vec{b} - \vec{U}^{k-1}$$

where:

$$\begin{aligned} A &= \begin{pmatrix} \lambda & s_y & 0 & s_x & 0 & 0 & 0 & 0 & 0 \\ s_y & \lambda & s_y & 0 & s_x & 0 & 0 & 0 & 0 \\ 0 & s_y & \lambda & 0 & 0 & s_x & 0 & 0 & 0 \\ s_x & 0 & 0 & \lambda & s_y & 0 & s_x & 0 & 0 \\ 0 & s_x & 0 & s_y & \lambda & s_y & 0 & s_x & 0 \\ 0 & 0 & s_x & 0 & s_y & \lambda & 0 & 0 & s_x \\ 0 & 0 & 0 & s_x & 0 & 0 & \lambda & s_y & 0 \\ 0 & 0 & 0 & 0 & s_x & 0 & s_y & \lambda & s_y \\ 0 & 0 & 0 & 0 & 0 & s_x & 0 & s_y & \lambda \end{pmatrix} \\ \text{and } \lambda &= 2(1 - s_x - s_y) \end{aligned}$$

$$
b = \begin{pmatrix} s_x U_{01}^k + s_y U_{10}^k \\ s_x U_{02}^k \\ s_x U_{03}^k + s_y U_{14}^k \\ s_y U_{20}^k \\ 0 \\ s_y U_{24}^k \\ s_x U_{41}^k + s_y U_{30}^k \\ s_x U_{42}^k \\ s_x U_{43}^k + s_y s_{34}^k \end{pmatrix}
$$

8.2.2 Examples of wave equation

1. **Elastic wave propagation through rocks** in *1-D*

$$\sigma_{xx,x} = \rho U_{tt} \tag{8.1}$$

where

$$
\begin{aligned}
\sigma_{xx} &= E\varepsilon_{xx},\ \sigma_{xx} = \text{stress},\ \varepsilon_{xx} = \text{strain} \\
&= E\frac{\partial U}{\partial x}
\end{aligned}
$$

$$8.1 \Rightarrow EU_{xx} = \rho U_{tt} \quad \text{or } U_{tt} = \frac{E}{\rho}U_{xx}$$

elastic waves propagate with speed $\sqrt{\frac{E}{\rho}}$

2. **Electromagnetic Wave Equation**

$$c^2\nabla^2 E = \ddot{E} \text{ and } c^2\nabla^2 B = \ddot{B} \tag{8.2}$$

From Maxwell's equations where E is electric field, B is magnetic field. Derived using:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times E = -\frac{\partial B}{\partial t} \tag{8.3}$$

$$\nabla \cdot B = 0, \quad \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \tag{8.4}$$

taking curl of 8.3 and 8.4 and using $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$ and $\nabla(\nabla \cdot E) = \nabla\left(\frac{\rho}{\epsilon_0}\right) = 0, \quad \nabla(\nabla \cdot B) = 0$ gives Equation 8.2 where $c = \sqrt{\frac{1}{\mu_0\varepsilon_0}} = 3 \times 10^8$m/s.