2D Wave eqn Discretization

- Hyperbolic PDE

~~Examples~~

allows non-smooth solutions

- $U_{tt} = \beta(U_{xx} + U_{yy})$

Use (β or c^2)

$$\Rightarrow \frac{\delta^2 \varphi}{\delta t^2} = \beta\left(\frac{\delta^2 \varphi}{\delta x^2} + \frac{\delta^2 \varphi}{\delta y^2}\right)$$

Example: 2D thin elastic membrane fixed at its walls

Discretise:

$U_{i,j}^{k} = U(t_k, x_i, y_j)$

$0 \le x \le a$
$0 \le y \le b$
~~$0 \le t \le m$~~

Boundary Conditions:

Dirichlet $\rightarrow U(0,y,t) = U(a,y,t) = U(x,0,t) = U(x,b,t)$

Initial conditions:

$U(x,y,0) = f(x,y) = f_{ij}$

$U_t(x,y,0) = g(x,y) = g_{ij}$

2D Central Difference

$$U_{tt} = \frac{U_{i,j}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1}}{\Delta t^2}$$

$$U_{xx} = \frac{U_{i+1,j}^{k} - 2U_{i,j}^{k} + U_{i-1,j}^{k}}{\Delta x^2}$$

$$U_{yy} = \frac{U_{i,j+1}^{k} - 2U_{i,j}^{k} + U_{i,j-1}^{k}}{\Delta y^2}$$

Let $s_x = \beta \frac{\Delta t^2}{\Delta x^2}$, $s_y = \frac{\beta \Delta t^2}{\Delta y^2}$ and substitute

into $U_{tt} = B(U_{xx} + U_{yy})$ to solve for $U_{i,j}^{k+1}$

$$\boxed{U_{i,j}^{k+1} = 2U_{i,j}^{k}(1 - s_x - s_y) - U_{i,j}^{k-1} + s_x(U_{i+1,j}^{k} + U_{i-1,j}^{k}) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})}$$

Hence computing U^{k+1} requires U^{k} and U^{k-1}

For the first time step we need U^{0}_{k} and U^{-1}

Use initial conditions to find U^{-1}

$$U_t(x,y,0) = \frac{\delta U_{i,j}^{0}}{\delta t} = \frac{U_{i,j}^{1} - U_{i,j}^{-1}}{2\Delta t} = g_{i,j} \Rightarrow U_{i,j}^{-1} = U_{i,j}^{1} - 2\Delta t\, g_{i,j}$$

Hence for first time step

$$\boxed{U_{i,j}^{1} = U_{i,j}^{0}(1 - s_x - s_y) + \Delta t\, g_{i,j} + \frac{s_x}{2}(U_{i+1,j}^{0} + U_{i-1,j}^{0}) + \frac{s_y}{2}(U_{i,j+1}^{0} + U_{i,j-1}^{0})}$$

Convergence

$s_x < \frac{1}{2}$, $s_y < \frac{1}{2}$