ME2 Computing- Coursework summary

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A) What physics are we trying to model and analyse?

We are trying to model and analyse the propagation of waves in a two dimensional medium over time. We wish to analyse cases where the initial application of a force on a 2D membrane causes a deformation in the third (z) dimension and propagates with time.

In this case we will study the vibration of a drumskin, which can be modelled as a uniform membrane stretched over the solid drum body, which creates fixed boundary conditions.

For the initial conditions we can simulate a sharp point impact in a particular spot and observe the subsequent vibration behaviour.

By changing the boundary and initial conditions this method could be used to solve the propagation of waves in any 2D membranes, e.g. a trampoline.

B) What PDE are we trying to solve?

We wish to solve the 2D wave equation.

$$\nabla^2\varphi - c^2\frac{\partial^2\varphi}{\partial t^2} = 0$$

$$In\ 2D:\ c^2\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

Hence φ = $\varphi(x, y, t)$. φ is the displacement in the z direction.

C) Boundary value and/or initial values for my specific problem: (be consistent with what you wrote in A)

Boundary Conditions:

All edges of the drumskin will be fixed ends. Thus, there will be no displacement at the edges and the boundaries will have a Dirichlet condition $\varphi = 0$ at all the edges.

I.e. $\varphi(0, y, t) = \varphi(x_{max}, y, t) = \varphi(x, 0, t) = \varphi(0, y_{max}, t) = 0$

Initial conditions:

We want an initial random disturbance in the drumskin due to it being hit with a drumstick.

The initial condition was defined as $\varphi(x, y, 0) = f(x, y) = -e^{(-\left(\frac{(x-2)^2}{W}+\frac{(y-2)^2}{W}\right))}$

This models a narrow gaussian peak pointing downwards at (x, y) = (2, 2) with width W, which was defined as 0.1.

And we chose the initial derivative with respect to time to be zero:

$\frac{\partial \varphi(x,y,0)}{\partial t} = g(x,y) = 0$

D) What numerical method are we going to deploy and why?

We are going to use the explicit central difference method. It has non-stringent convergence conditions, so it doesn't require absurdly small timesteps, allowing for relatively fast calculations.

We already have experience implementing it for other PDE's from our maths course. Hyperbolic equations like the wave equations allow non smooth solutions, so any imperfections should be fine.

If we wanted to calculate numerical results for vibration frequencies and amplitudes, an estimate for the actual wave propagation speed would be needed. Since we are only interested in the qualitative behaviour, we chose values that produce a clear plot.

We could also introduce some damping, which would make these results even more realistic, this is however outside the scope of this brief exploration.

E) We are going to discretise the PDE as the following:

The trampoline will be rectangular, 5 by 5 decimetres. The end time will be 5 seconds. We will employ uniform timesteps and spatial steps as such:

$$\Delta x = \Delta y = 0.1 \quad \Delta t = 0.05 \quad 0 \leq x, y, t \leq 5$$

Using the central difference method for the second derivative and rearranging the wave equation we get the following expression we will use to implement the method:

$$U_{i,j}^{k+1} = 2U_{i,j}^{k}\left(1 - s_x - s_y\right) - U_{i,j}^{k-1} + s_x\left(U_{i+1,j}^{k} + U_{i-1,j}^{k}\right) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})$$

$$where\ s_x = c^2 \frac{\Delta t^2}{\Delta x^2}\ \ and\ s_y = c^2 \frac{\Delta t^2}{\Delta y^2}$$

Computing U requires the function values at the two previous timesteps. So, for the first timestep after the initial condition we need a different approach:

$$From\ the\ initial\ derivative: U_t(x,y,0) = \frac{\partial U_{i,j}^0}{\partial t} = \frac{\left(U_{i,j}^1 - U_{i,j}^{-1}\right)}{2\Delta t} = g(x,y)\ or\ g_{ij}$$

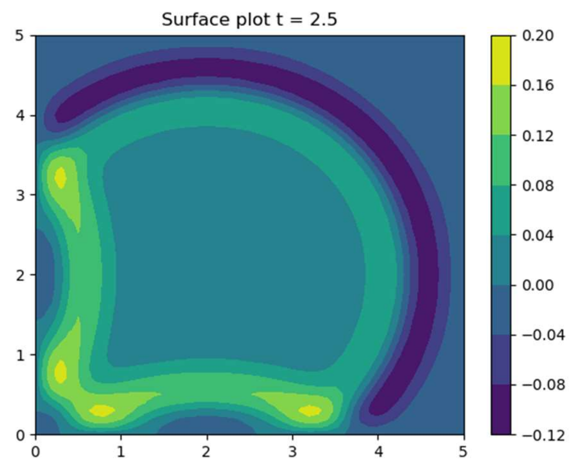
$$Hence\ U_{i,j}^{-1} = U_{i,j}^1 - 2\Delta t\ g_{ij}$$

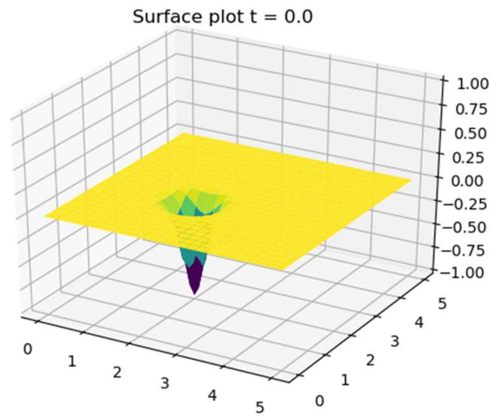
Using the expression above, for the first time step the method is going to be:

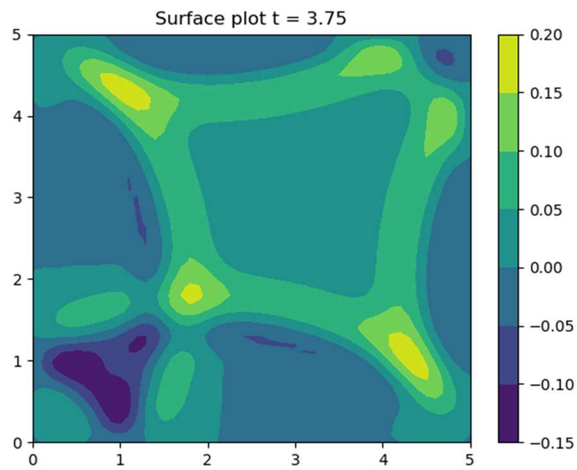
$$U_{i,j}^1 = U_{i,j}^0(1 - s_x - s_y) + \Delta t g_i + \frac{s_x}{2}\left(U_{i+1,j}^0 + U_{i-1,j}^0\right) + \frac{s_y}{2}\left(U_{i,j+1}^0 + U_{i,j-1}^0\right)$$

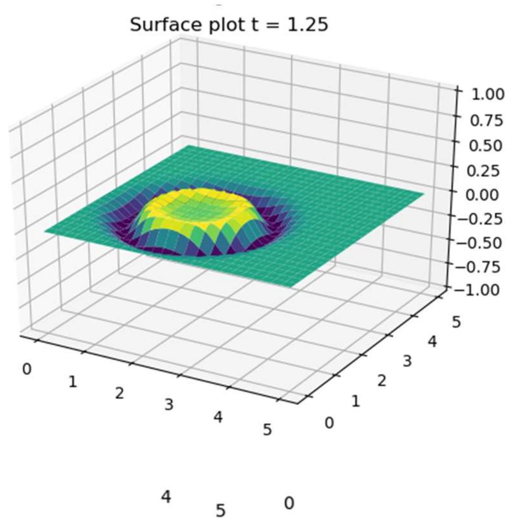
F) Plot of results and comments (discuss how the results describe the physics and comment on any discrepancies or unexpected behaviours):

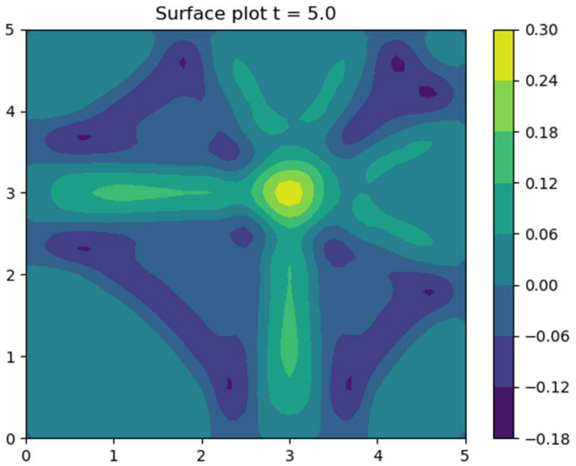
Below you can find snapshots of the results at different timesteps, as both 3D plots and 2D colour bar plots.

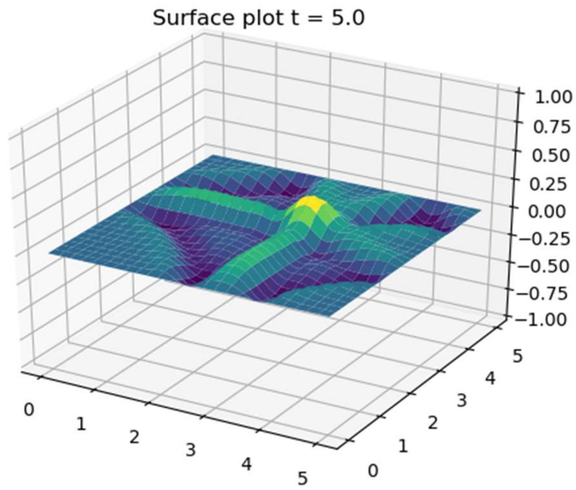












The results are inline without expectations. We expected the peak near the middle to propagate throughout the surface, hit the walls, then bounce back. The 3D plots show clearly that this is the case.

G) Other remarks (limits of the model, convergence problems, possible alternative approaches, anything you find relevant and important to mention):

The stability and convergence condition for x and y are:

$$\frac{\Delta t^2}{\Delta x^2} \leq c^2 \ and \ \frac{\Delta t^2}{\Delta y^2} \leq c^2$$

An unconditionally stable, albeit computationally more expensive, implicit method may also be used to solve the PDE. But for typical c^2 values around 1 the time step doesn't have to be prohibitively small to achieve stability, and so the function converges satisfactorily in most cases.

One limit of the model is that the finite difference method employed does not incorporate damping. Thus, the waves in the trampoline will propagate without eventually coming to a rest.