ME2 Computing- Session 2: Sets of Linear Equations and Gauss Elimination

Learning outcomes:

- Being able to solve a set of linear equations numerically
- Being able to implements Gauss elimination
- Being able to express engineering problems in matrix-vector terms

Before you start

In your H drive create a folder *H:\ME2MCP\Session2*and work within it.

Introduction

A set of linear equations can be written numerically in the matrix-vector form as:

$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$	**Eq. 1**	$\begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & \mathbf{a_{14}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{24}} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} & \mathbf{a_{34}} \\ \mathbf{a_{41}} & \mathbf{a_{42}} & \mathbf{a_{43}} & \mathbf{a_{44}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ $A \cdot x = b$

Two types of numerical methods are possible for solving sets of linear equations: **direct** and **iterative**. In direct methods, the solution is computed manipulating the equations. In iterative methods, an initial solution is guessed and then used iteratively to obtain more accurate solutions. In this session we will implement a direct method only, namely the Gauss Elimination.

Task A: Gauss elimination

If the given set of equations can be manipulated into the form:

$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ d_{22}x_2 + d_{23}x_3 + d_{24}x_4 = d_2 \\ e_{33}x_3 + e_{34}x_4 = e_3 \\ f_{44}x_4 = f_4 \end{cases}$	$\begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & \mathbf{a_{14}} \\ \mathbf{0} & \mathbf{d_{22}} & \mathbf{d_{23}} & \mathbf{d_{24}} \\ \mathbf{0} & \mathbf{0} & \mathbf{e_{33}} & \mathbf{e_{34}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f_{44}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_2 \\ e_3 \\ f_4 \end{bmatrix}$

then it is straightforward to find the solutions, starting from x_4 and proceeding backwards by substitution to find x_3, x_2 and x_1.

This manipulation can be done in repeated steps. At each step *i*, the variable x_i is eliminated from the $(i+1)^{th}$ row downwards, (i.e. all the coefficients in column *i* are set to zero, starting from the row below *i*).

For example, at step 1, starting from Eq. 1, the variable x_1 is eliminated from the second row downwards, (i.e. all the coefficients in column 1 are set to zero starting from row 2):

row 1 is unchanged new row 2 = row 2 - $a_{21}/\boldsymbol{a_{11}}$ · row 1 new row 3 = row 3 - $a_{31}/\boldsymbol{a_{11}}$ · row 1 new row 4 = row 4 - $a_{41}/\boldsymbol{a_{11}}$ · row 1	$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & d_{32} & d_{33} & d_{34} \\ 0 & d_{42} & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$	**Eq. 2**

The common coefficient, $\boldsymbol{a_{11}}$, is also knows as the **pivot**.

At step 2, starting from Eq. 2, the variable x_2 is eliminated from the third row downwards, (i.e. all the coefficients in column 2 are set to zero starting from row 3):

row 1 is unchanged new row 2 is unchanged new row 3 = new row 3 - $d_{32}/\boldsymbol{d_{22}}$ · new row 2 new row 4 = new row 4 - $d_{42}/\boldsymbol{d_{22}}$ · new row 2	$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 0 & e_{43} & e_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_2 \\ e_3 \\ e_4 \end{bmatrix}$	**Eq. 3**

The pivot at this step is the coefficient $\boldsymbol{d_{22}}$.

These steps are repeated, until the matrix A is reduced into an upper triangular form.

If you find difficult to understand this logic, read the supplemental document on Gauss elimination (on BB), by paying particular attention to the colour code adopted in it.

1. Write a function, *MyGauss*, that receives a set of *n* linear equations, in the form of a matrix *A* and a vector *b*, and outputs the vector solution *x*.

2. Test the function to solve the set of equations:

$$\begin{cases} 8x_1 - 2x_2 + x_3 + 3x_4 = 9 \\ x_1 - 5x_2 + 2x_3 + x_4 = -7 \\ -x_1 + 2x_2 + 7x_3 + 2x_4 = -1 \\ 2x_1 - x_2 + 3x_3 + 8x_4 = 5 \end{cases}$$

   Solution: [1.3232 1.5657 -0.6061 0.7172]

Task B: Further skills

Sometimes the set of equations does not have a solution, even though the numerical method may still provide a solution, which is obviously wrong. If we consider the matrix-vector form:

$$A \cdot x = b$$

the solution can be expressed as:

$$x = A^{-1} \cdot b$$

where A^{-1} is the inverse of matrix A. Therefore, there will exist a solution if the matrix *A* is invertible. This is equivalent of saying that the determinant of matrix *A* is non-zero.

1. For this set of linear equations:

$$\begin{cases} 4x + 3y = 2 \\ 8x + 6y = 1 \end{cases}$$

   Plot the two equations, each as a line *y* vs *x*, on the same graph.

Solve them numerically: what does it happen?
Evaluate the determinant of matrix A with the Numpy function numpy.linalg.det().

2. There are cases, when the solution exists, where numerical methods can provide completely inaccurate results. These cases are said to be **ill conditioned**.

   Consider these two sets of linear equations:

   $$\begin{cases} 400x - 201y = 200 \\ -800x + 401y = -200 \end{cases} \qquad \begin{cases} \mathbf{401}\boldsymbol{x} - \mathbf{201}\boldsymbol{y} = \mathbf{200} \\ -\mathbf{800}\boldsymbol{x} + \mathbf{401}\boldsymbol{y} = -\mathbf{200} \end{cases}$$

   Solve the two sets of equations separately. How do the numerical solutions differ from each other?
   This is a case of an ill conditioned problem: with a modest change in one of the coefficients (often induced by a numerical error) one would expect only a small change in the solution. However, in ill conditioned cases the change is quite significant: the solution is very sensitive to the values of the coefficients.

Task C: Electrical linear circuits: ME2-MTX tutorial 1.10.6 (resistor network 5)

1. Consider the ME2 Mechatronics exercise 1.10.6. After writing (in the Mechatronics tutorial) the set of KVL and KCL equations, solve the circuit with the numerical values given for resistors. Make use of the function *MyGauss* to find out the numerical value of the total resistance between terminals **a** and **b**.