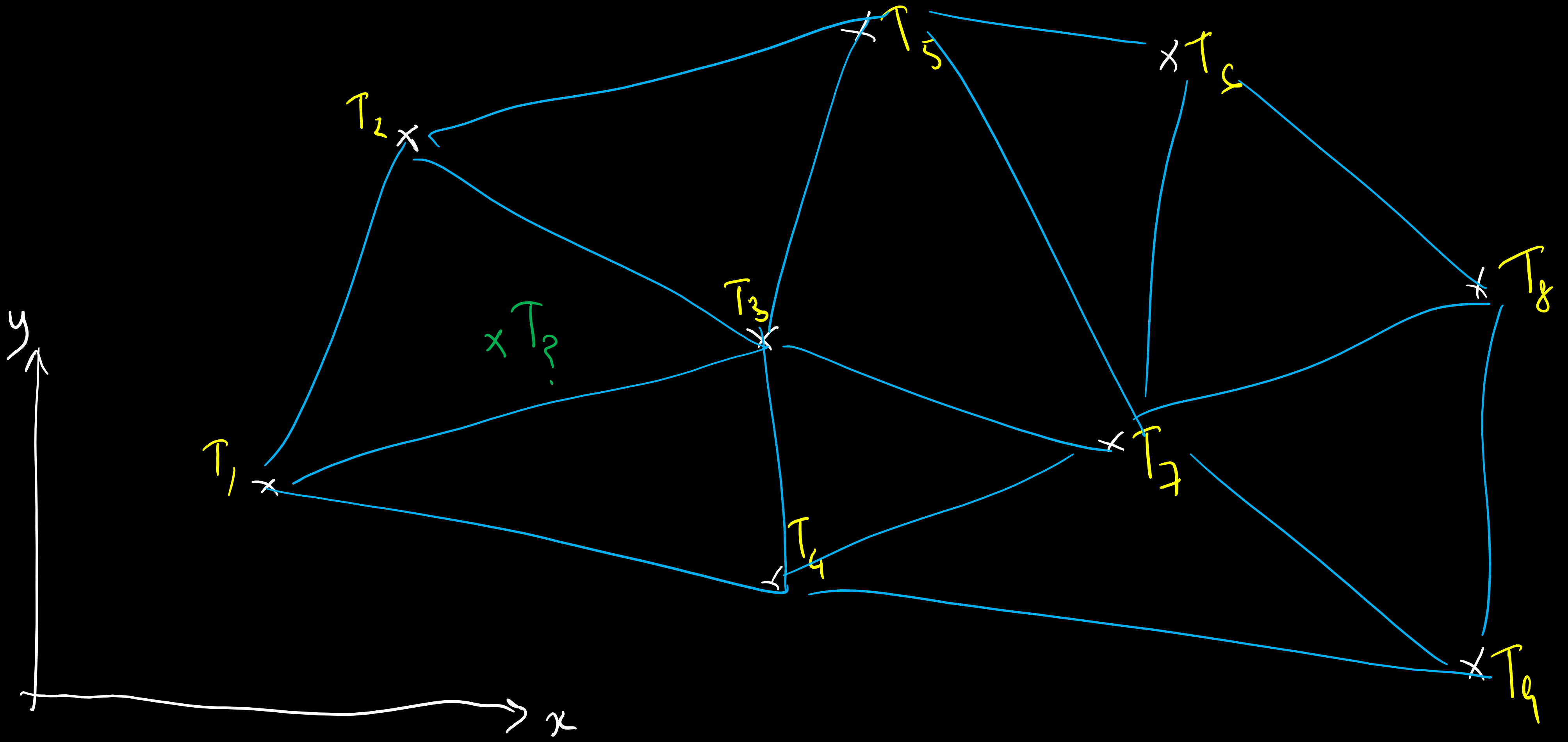
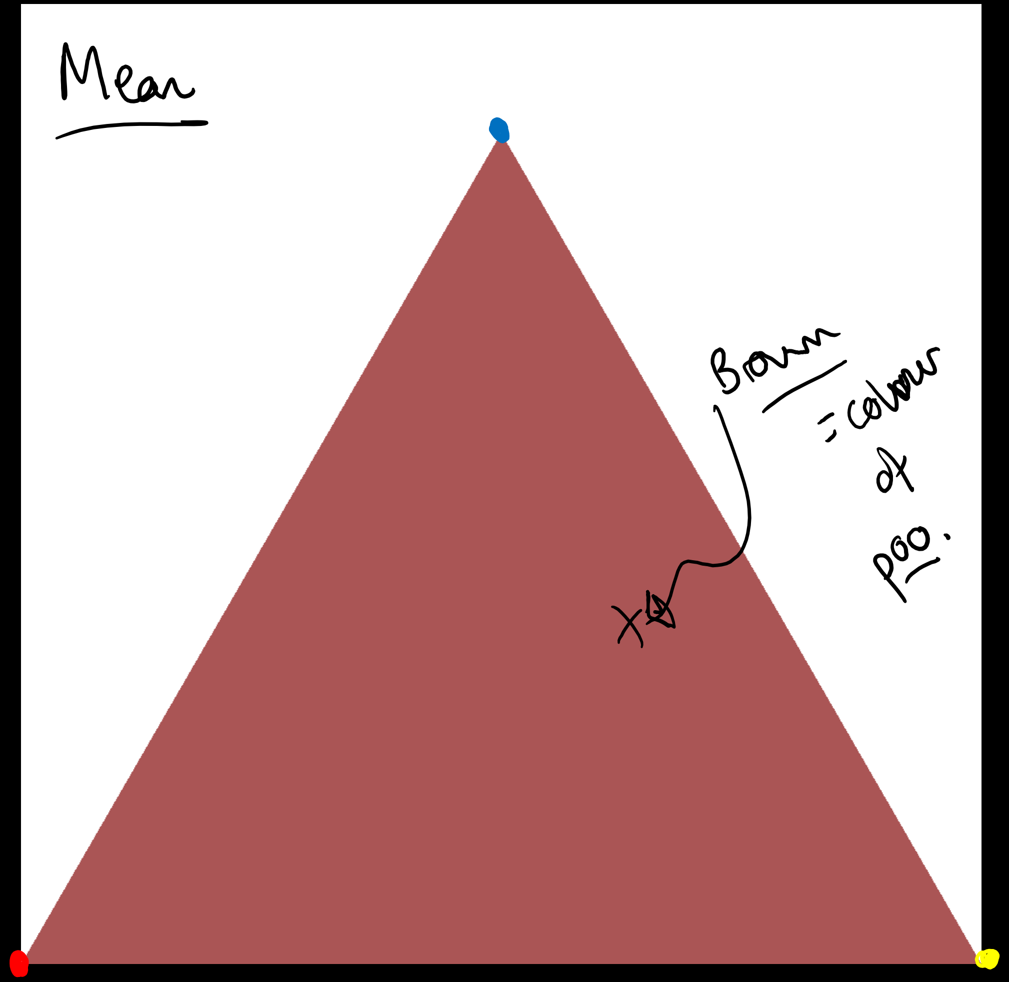
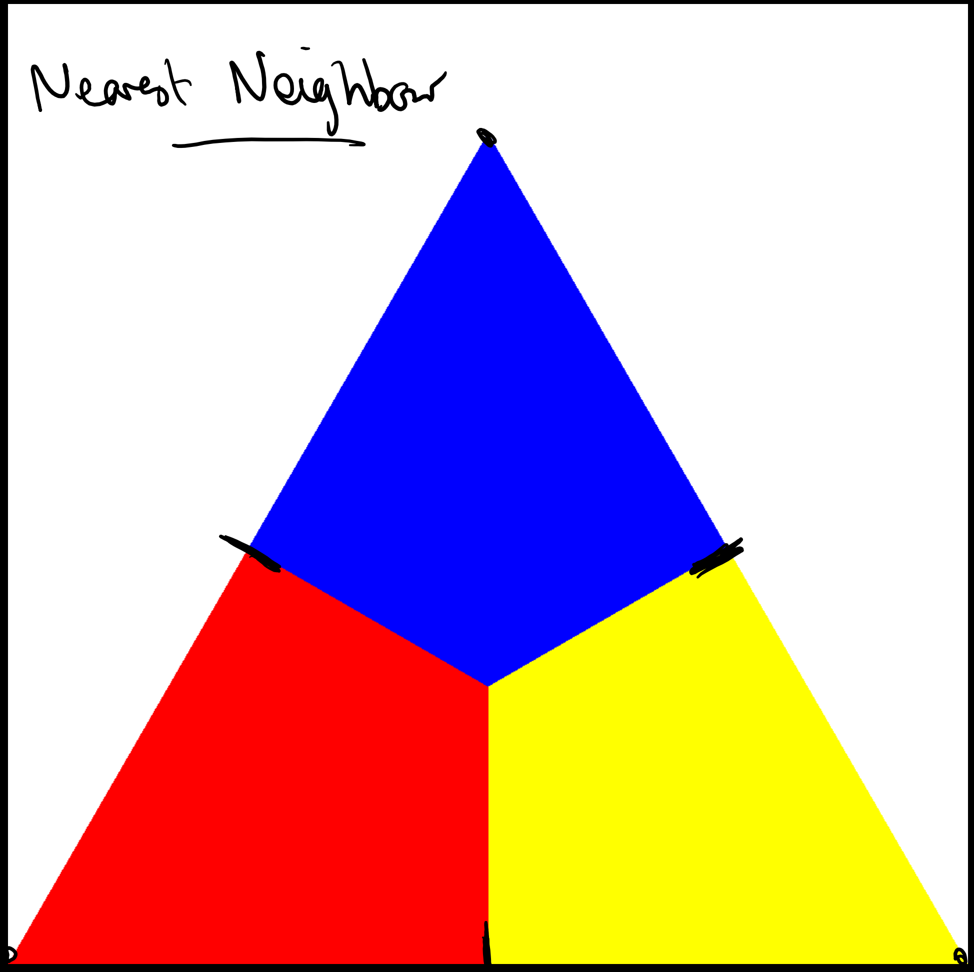
# 2D Interpolation for Unstructured Grids (Fields with Irregularly Spaced Nodes)



# Worked Example 11: Interpolating in a Triangle (lazy methods)



# Worked Example 11: Interpolating in a Triangle (elegant but flawed)

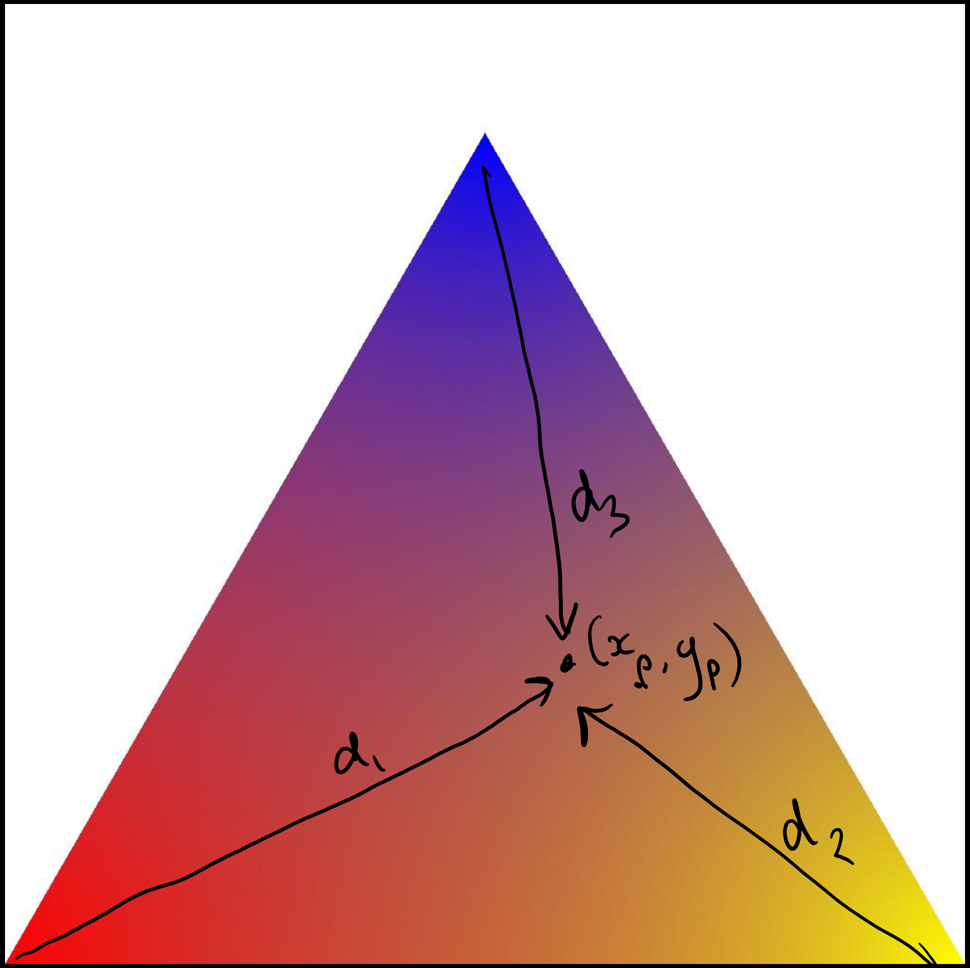
- It is more elegant (but still flawed) to interpolate proportional to inverse distance weightings:

$$d_1 = \sqrt{(x_1 - x_p)^2 + (y_1 - y_p)^2}, \quad w_1 = {}^1/_{d_1}$$

$$d_2 = \sqrt{(x_2 - x_p)^2 + (y_2 - y_p)^2}, \quad w_2 = {}^1/_{d_2}$$

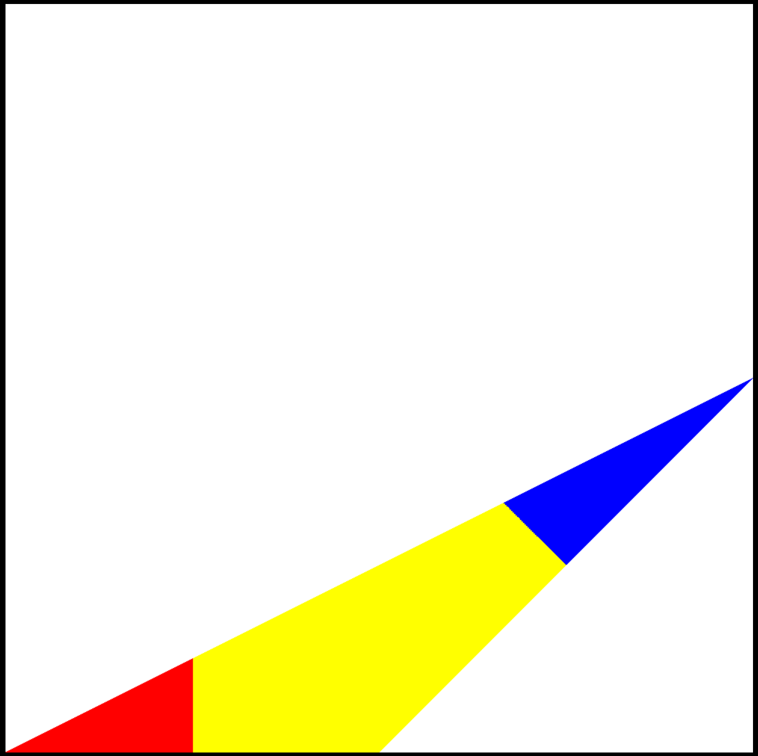
$$d_3 = \sqrt{(x_3 - x_p)^2 + (y_3 - y_p)^2}, \quad w_3 = {}^1/_{d_3}$$

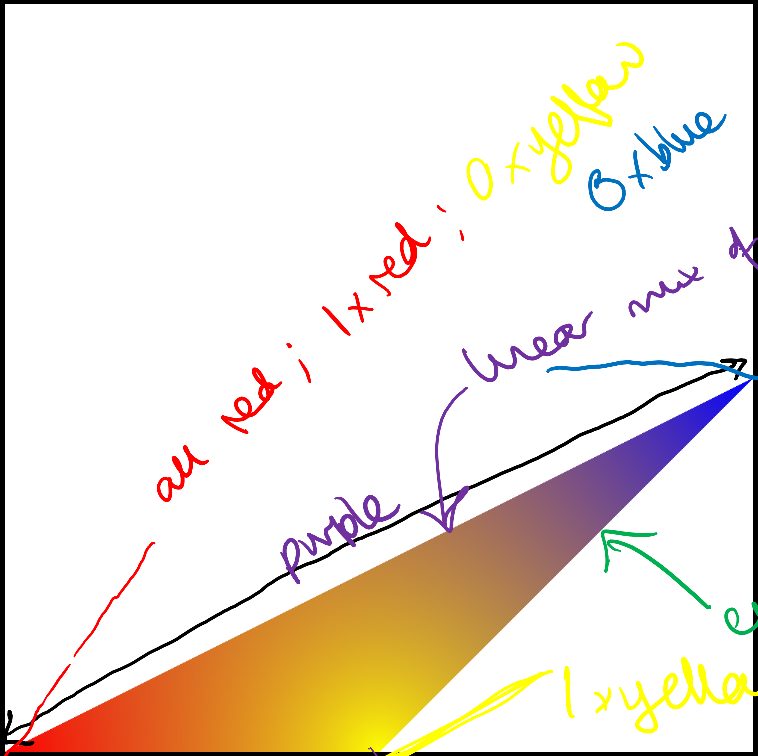
$$Colour = \frac{w_1 Red + w_2 Yellow + w_3 Blue}{w_1 + w_2 + w_3}$$



# Worked Example 11: Interpolating in a Triangle (the problem)

Nearest neighbour and inverse distance do not work intuitively for obtuse triangles:






What we need is a weighting variable that is one at the node, and always zero at the opposite edge.

# The Solution: Barycentric Coordinates

- Consider a triangle with vertices located at $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$. Each point $\boldsymbol{r}$ inside the triangle can be described as a linear combination of the vertices:

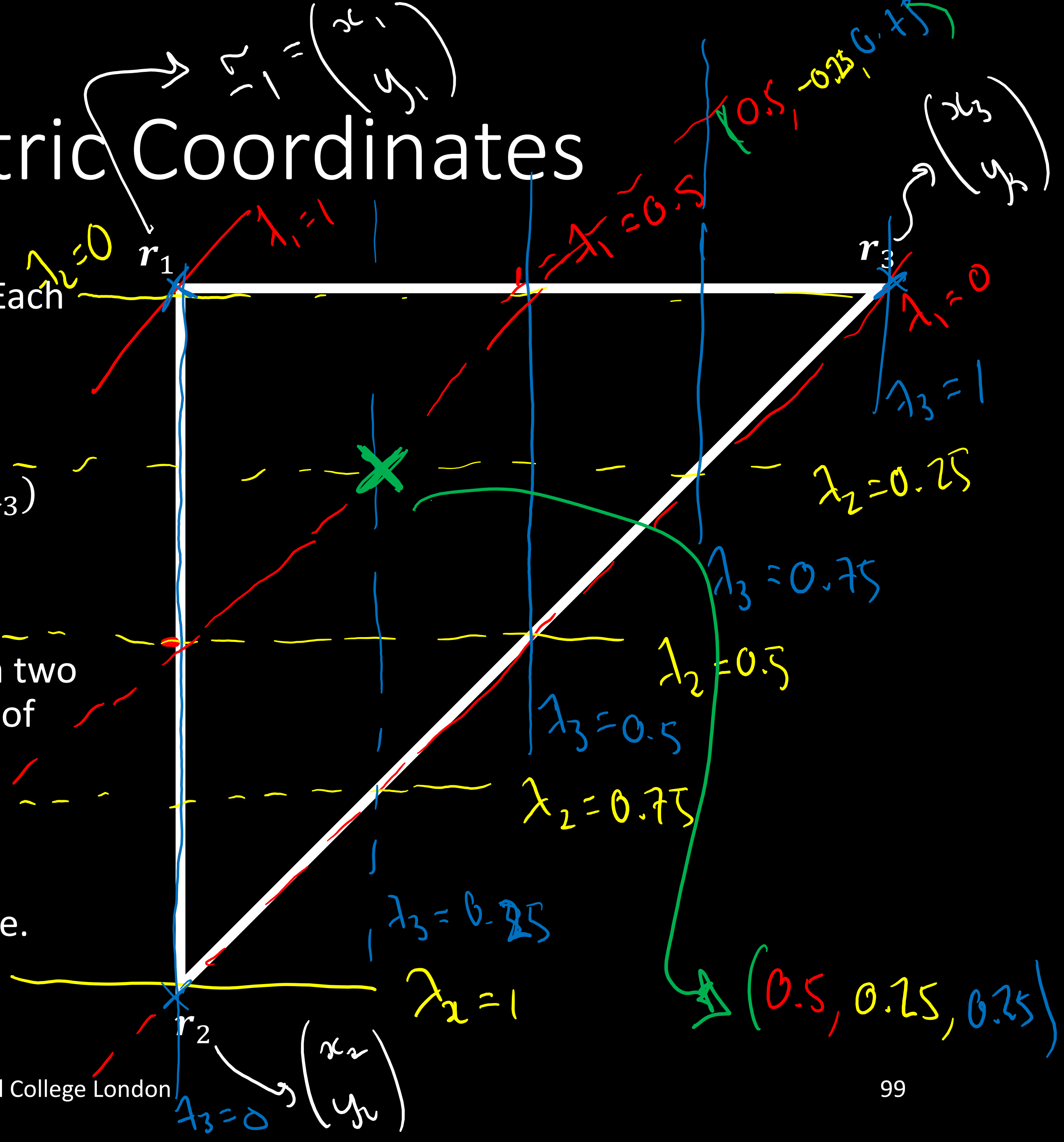
$$\boldsymbol{r} = \lambda_1 \boldsymbol{r}_1 + \lambda_2 \boldsymbol{r}_2 + \lambda_3 \boldsymbol{r}_3$$

  Alternative notation: $\boldsymbol{r} = (\lambda_1, \lambda_2, \lambda_3)$ or $\boldsymbol{r} = (\lambda_1 : \lambda_2 : \lambda_3)$

- Each barycentric centric coordinate (each $\lambda$) is 1 at its corresponding vertex, and 0 at the opposite edge.
- However, now we have three coordinates, describing a two dimensional triangle and hence have a surplus degree of freedom. The coordinates are thus normalised:

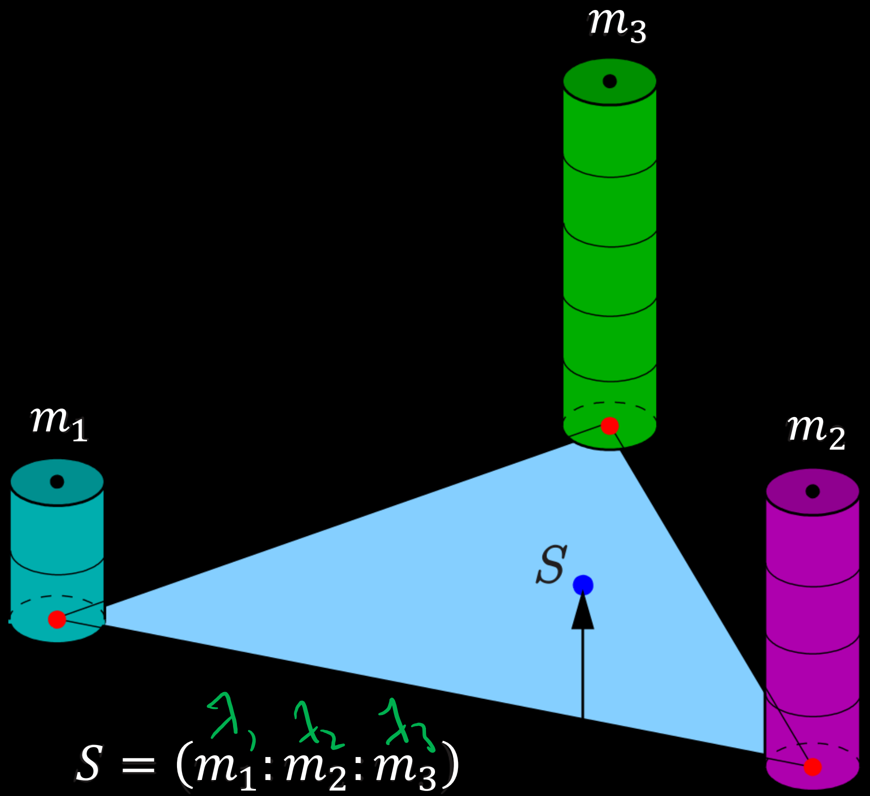
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

- Barycentric coordinates continue outside the triangle:
  - One/two barycentric coordinates must be negative.
  - Thus, testing if any $\lambda_i < 0$ is a simple method to determine if a point is inside a triangle or not.



# For a Triangle: Barycentric = Areal Coordinates

- Relationship to masses (barycentric):



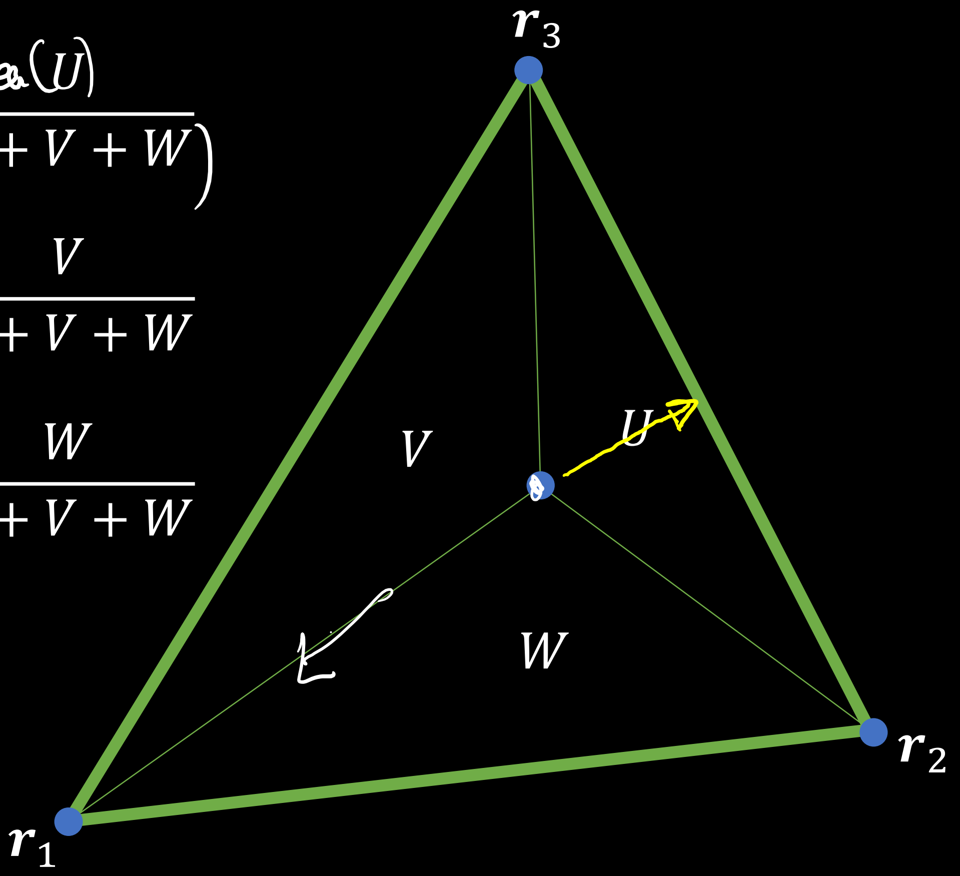

$$S = (\overset{\lambda_1}{m_1} : \overset{\lambda_2}{m_2} : \overset{\lambda_3}{m_3})$$

Relationship to areas (areal):

$$\lambda_1 = \frac{\text{area}(U)}{\text{area}(U + V + W)}$$

$$\lambda_2 = \frac{V}{U + V + W}$$

$$\lambda_3 = \frac{W}{U + V + W}$$



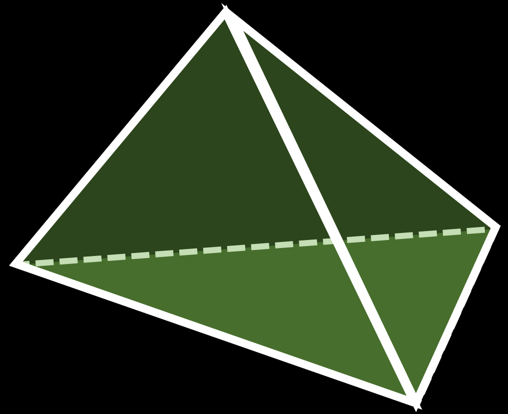

Left image from Ag2gaeh, Own Work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=93193242

# Barycentric Interpolation

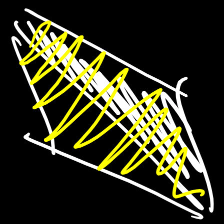
$T(\underline{r}) = \lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3$

$\lambda_1 R + \lambda_2 Y + \lambda_3 B$

- Barycentric coordinates make interpolation inside a triangle simple:
$$f(\boldsymbol{r}) = \lambda_1 f(\boldsymbol{r}_1) + \lambda_2 f(\boldsymbol{r}_2) + \lambda_3 f(\boldsymbol{r}_3)$$
  - Interpolation can only be done <u>inside</u> the triangle, so first use the barycentric coordinates to test if inside the triangle and only interpolate if so.
  - To both test whether inside the triangle, and to interpolate, we thus need to find the values for $\lambda_i$ (next slide).
- The same approach works in 3D as well using tetrahedra rather than triangles and four barycentric coordinates $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.



# Cartesian to Barycentric – Area Method



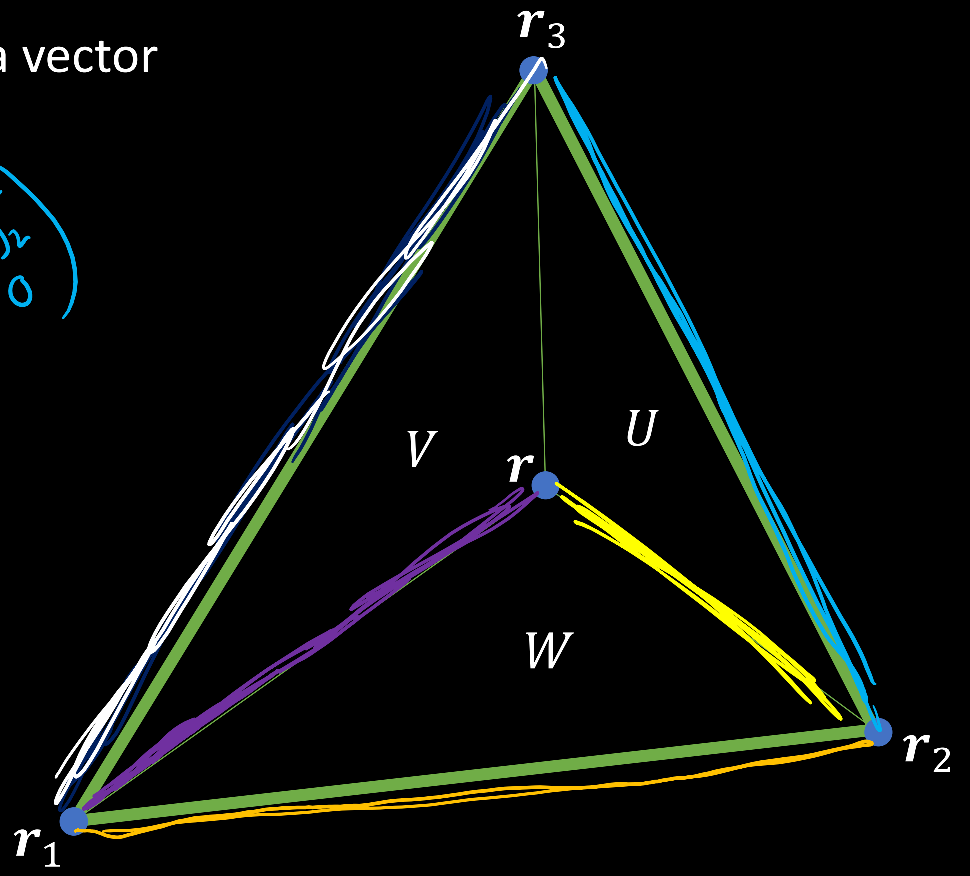
- The area of a triangle is half the area of a parallelogram.
- The area of a parallelogram is equivalent to the modulus of a vector cross product $|\boldsymbol{a} \times \boldsymbol{b}|$ (as described in your first year notes).
- Therefore:

$$\lambda_1 = \frac{U}{U+V+W} = \frac{|(r - r_2) \times (r_3 - r_2)| \cancel{2}}{\cancel{2}|(r_1 - r_2) \times (r_3 - r_2)|}$$

$$r_2 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}$$

$$\lambda_2 = \frac{V}{U+V+W} = \frac{|(r - r_1) \times (r_3 - r_1)|}{\quad}$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$



$$\underline{r} = \lambda_1 \underline{r}_1 + \lambda_2 \underline{r}_2 + \lambda_3 \underline{r}_3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

# Cartesian to Barycentric – Matrix Method

- If the vertices are known in cartesian coordinates, then $\boldsymbol{r}_i = (x_i, y_i)$ and hence:

$$\begin{array}{ll} \text{row 1} & x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \\ \text{row 2} & y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \\ \text{row 3} & 1 = \lambda_1 + \lambda_2 + \lambda_3 \end{array}$$

- These three equations can be written in matrix form:

$$\underbrace{\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}}_{\underline{\lambda}} = \underbrace{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}}_{\underline{d}}$$

$$\underline{\underline{A}}\,\underline{\lambda} = \underline{d}$$

$$\underline{\lambda} = \underline{\underline{A}}^{-1} \underline{d}$$

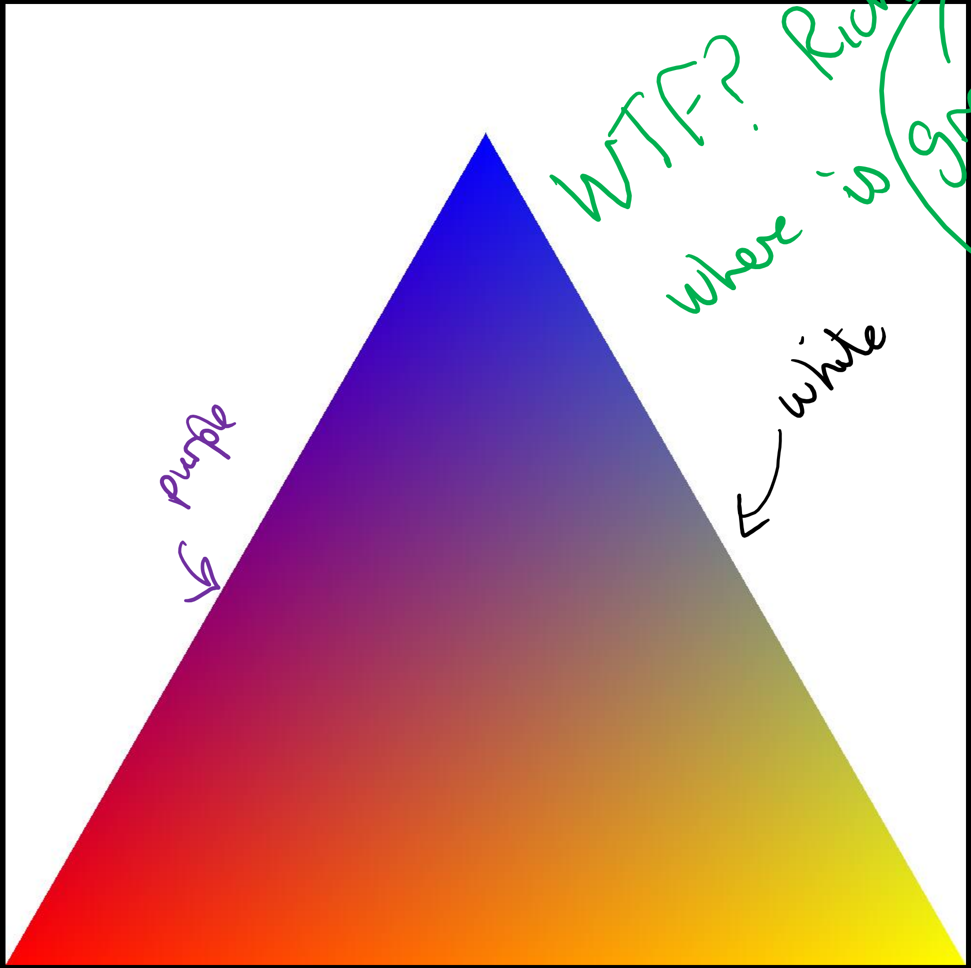
- and solved to find $\lambda_1, \lambda_2$ and $\lambda_3$ (the latter is more easily found from $\lambda_3 = 1 - \lambda_1 - \lambda_2$) for any $\boldsymbol{r}$:

$$\lambda_1 = \frac{(y_2 - y_3)(x - x_3) + (x_3 - x_2)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

$$\lambda_2 = \frac{(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

# Worked Example 11: Interpolating in a Triangle (Barycentric Method)

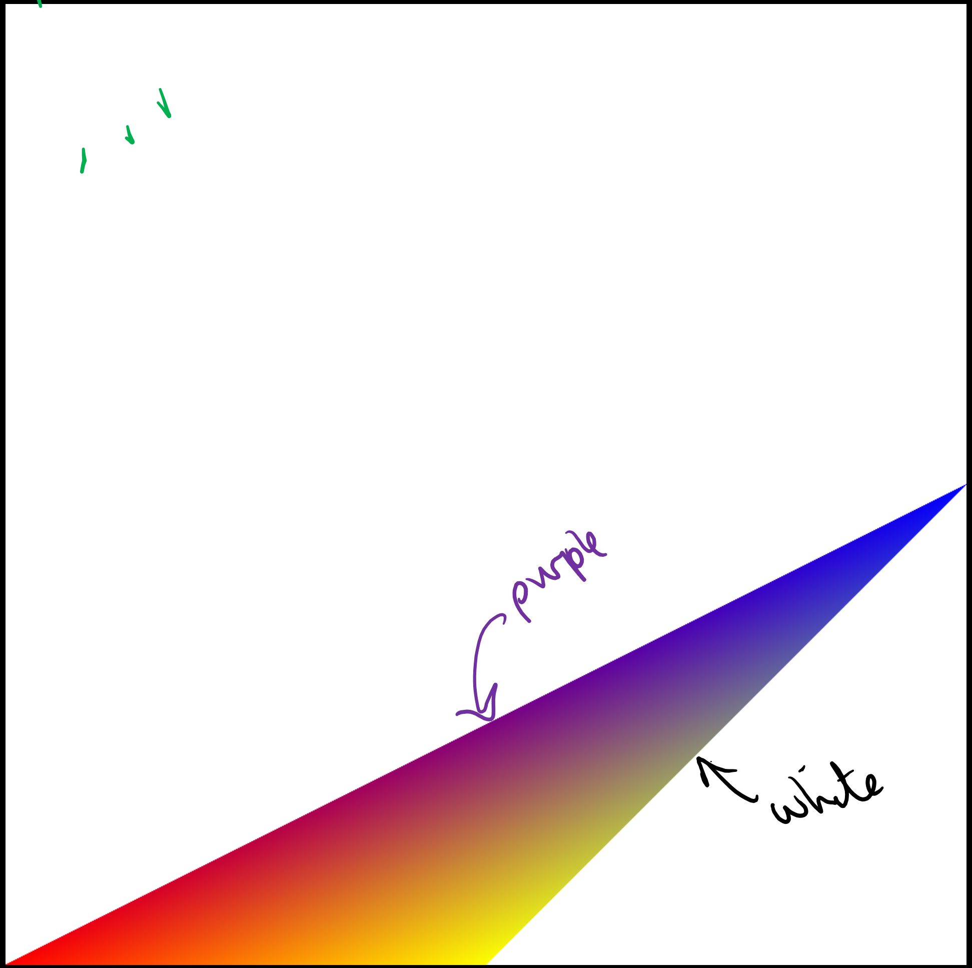
WTF? Rich where is green?



purple

white

orange



purple

white

orange