Relativity: A Modern Primer

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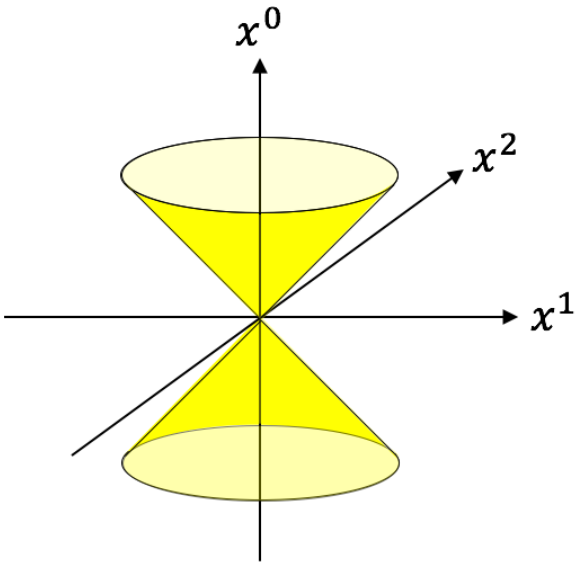

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Introduction

This is the textbook I wish I had when self-studying relativity. I aim to combine the best aspects of my favorite textbooks, from the clarity of Dirac's *General Theory of Relativity* to the elegance of Landau and Lifshitz's *Course of Theoretical Physics*. Some features include:

- **Concise**: Only about 100 pages long.
- **Deep**: Unlike a popular physics book, we will dive into the math. Keep a pencil and paper handy.
- **Broad**: In addition to special and general relativity, it covers Lagrangian mechanics, a bit of electromagnetism, and an introduction to cosmology.
- **Pedagogical**: Clear explanations with many figures and exercises. Exercises are marked by difficulty with 1-3 stars (*). I have tried to find the most intuitive way to understand each concept, and I anticipate common points of confusion based on my own experience.

Prerequisites: vector calculus and classical mechanics. Basically, a standard freshman or sophomore physics curriculum in the US. Some knowledge of linear algebra and electromagnetism will also help, although Appendix A develops the necessary linear algebra.

Relativity in a nutshell

Relativity says that physics happens on a *spacetime manifold*, a 4D surface analogous to a sphere or a disk, but in four dimensions instead of two. Just as the earth looks like a 2D plane as you stand at a point, this manifold

looks like a 4D "plane" locally around each point. At each point on the earth (except the poles), one direction is associated with changing latitude and one with changing longitude. Similarly, in spacetime, one direction is associated with time and three with space. The curvature of the manifold affects how matter (including light) propagates on it. In turn, matter itself curves spacetime, causing nearby matter to become attracted. This attraction is interpreted as the force of gravity.

First, let's discuss spacetime in the absence of curvature, known as *flat space* or *Minkowski space*. This theory is called *special relativity*.

Chapter 1

Special relativity and the nature of time

Special relativity deals with *events*, things that occur at a specific position and time. Position is measured with physical rulers of a standard length. *Time is defined as what clocks measure*.

First of all, what is a clock? Clocks are all around us: watches, smartphones, wall clocks, etc. In general, a clock is any physical system that undergoes change. When we say that a clock somewhere runs faster or slower, we mean that any physical process placed there runs faster or slower. Also, the ideal clock considered here is *point-like*, meaning it is much smaller than the standard rulers used to measure distance.

A system of clocks with rulers separating them is called a *reference frame* (Fig. 1.1)[*]. Such a system defines coordinate axes (t, x) so that t is the time read by the clock at position x. We will consider only one spatial dimension x for now. When the clocks and rulers are freely moving (no forces acting on them), the system is called an *inertial reference frame (IRF)*. By definition, any two IRFs move with constant relative velocity, since they cannot be accelerating ($F = ma = 0$).

[*]In relativity, time is typically drawn on the vertical axis.

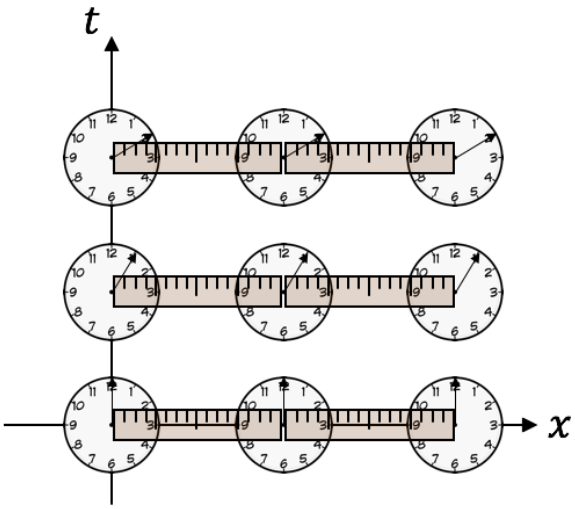


Figure 1.1: A reference frame that defines the coordinate axes t and x.

From now on, we will only draw IRFs as coordinate axes, instead of drawing all the clocks and rulers as in Fig. 1.1.

1.1 Postulates of special relativity

The fundamental postulates of special relativity are:

1. **The laws of physics are the same in all IRFs.**

   Consider some physical equation that uses the coordinates (t, x) defined by an IRF. It must remain the same when written using the coordinates (t', x') defined by a different IRF. This property is called *Lorentz covariance*. Don't worry if this seems vague — we will see many examples later.

2. **The speed of light is a constant ($c \approx 3 \times 10^8$ m/s) in all IRFs.**

   What is light? Light can be thought of as either an electromagnetic wave or a particle (the photon). The particle description is better for our current purpose. Then the second postulate simply says there is some particle that always travels at speed c in all IRFs. We will

sometimes call this particle a *light ray*, borrowing the classical optics term.

1.2 Locality

So far, IRFs seem clunky and useless. What do we need so many clocks for? Conceptually, we need clocks at every position because relativity is based on *local measurements*: we can only measure time at position x_1 using a clock at x_1, not a faraway clock at position x_2. In classical (non-relativistic)* physics, we can use any clock in any IRF to measure time, since time is globally shared among all objects.

When two clocks are at the same location, we can set them to the same time, and they remain synchronized. However, in order to set up an IRF, we must then move a clock from one location to another. How do we guarantee they remain synchronized? More precisely: how do we define synchronization between clocks in different locations? The constant speed of light comes to the rescue here. We can synchronize clocks at different locations by sending light rays between them and using $\Delta t = \Delta x/c$, where Δx is the known distance between clocks. For example, the clock at the origin ($t = 0, x = 0$) could send light rays in both directions. When another clock at x receives this, it could adjust its time to $|x|/c$. This proceeds until all clocks are synchronized and the IRF is completely "formed".

All interactions between particles and fields must also be local in space-time. More on this in Sec. 2.6.

1.3 Natural units

Given the constant speed of light, we will set $c \approx 3 \times 10^8$ m/s $= 1$ from now on (so-called *natural units*). This allows us to omit factors of c from all formulas, so we avoid having to keep track of it. If this makes you uneasy, you can think of c as merely a conversion factor between distance and time, which can be restored at the end of the calculation to obtain the right units. For example, a time calculation may give 3×10^8 m, then dividing by c gives

*We will use "classical" to mean "non-relativistic". Elsewhere, "classical" often means "non-quantum". Since we do not cover quantum physics here, we do not need to make this distinction.

1 s. There is nothing special about this factor other than convenience. We could also set 1 m/s $= 1$, for example. Then $c \approx 3 \times 10^8$, which would be an ugly number to include in our formulas.

*** Exercise 1.1**

As we will see later, the relativistic energy E and momentum $\vec{p}$ of a particle are related to its mass m as:

$$E^2 - \vec{p}^2 = m^2. \tag{1.1}$$

Restore the factors of c in this equation. Recall that energy in S.I. units is measured in $[J] = [kg \cdot m^2/s^2]$, momentum in $[kg \cdot m/s]$, and mass in $[kg]$.

In general, for a system of n independent units, we can set $n-1$ independent constants to 1 and still convert a quantity to the right unit at the end. For example, if we add mass to our unit system, we have mass, length, and time. Then, in quantum physics, we typically also set the reduced Planck's constant $\hbar \approx 1.05 \times 10^{-34}$ J $\cdot$ s $= 1$. (We will not do so here.)

1.4 Galilean transformations

Consider an IRF I and another IRF I' moving with velocity $v > 0$ relative to I. I has coordinates (t, x), and I' has coordinates (t', x'). Since all the clocks and rulers in an IRF are identical, we are free to choose the origin, so we take I and I' to share a common origin $t = t' = 0, x = x' = 0$. Given an event (t, x) measured in I, what are the time and position (t', x') measured in I'?

Let us use the notation $X \equiv (t, x)^T$ for the column vector of coordinates. Write $X = f(X')$ for the function relating coordinates X to X'. Since $f(X')$ must take straight lines into straight lines, it must be linear:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}, \tag{1.2}$$

or in matrix-vector notation:

$$X = \Lambda(v) X'. \tag{1.3}$$

We are trying to find the matrix $\Lambda(v) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$, which only depends on the relative velocity v.

First, let's answer this in classical mechanics. The clock with constant $x' = 0$ moves along the path $x = vt$. We simply have: $t = t'$, $x = x' + vt'$. In matrix form:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}. \tag{1.4}$$

This is called a *Galilean transformation* (Fig. 1.2).

In relativity, it turns out that t will also depend on x' and v, so time is no longer a globally shared coordinate among IRFs.

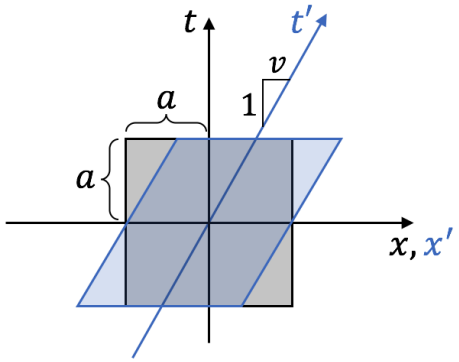


Figure 1.2: Relation between coordinates (t, x) and (t', x'), in *classical mechanics*. The grey shaded region is the square $\{|t| < a, |x| < a\}$, for some constant a. The blue shaded region is $\{|t'| < a, |x'| < a\}$.

1.5 Lorentz transformations

In relativity, the formula relating (t, x) to (t', x') is called a *Lorentz transformation*, or also a *boost*. It is graphically shown in Fig. 1.3. We will derive it by finding the eigenvectors and eigenvalues (Sec. A.3) of $\Lambda(v)$, using the constant speed of light. Appendix B contains an alternative derivation that may be easier to follow for some readers.

A light ray sent from the origin follows $t = \pm x$ in IRF I. In I', it must also follow $t' = \pm x'$ since the speed of light is constant. Thus, the eigenvectors

of $\Lambda(v)$ are

$$\hat{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.5}$$

They are normalized so that $\hat{w}_1^T\hat{w}_1 = \hat{w}_2^T\hat{w}_2 = 1$. $\hat{w}_1$ is the "forward-going" ray going the same direction as the moving IRF I'. Call its eigenvalue λ_f. Likewise, $\hat{w}_2$ is the "backward-going" ray with eigenvalue λ_b (Fig. 1.3*). Then we have:

$$\Lambda(v) = \lambda_f \hat{w}_1\hat{w}_1^T + \lambda_b \hat{w}_2\hat{w}_2^T, \tag{1.6}$$

which is easily verified by finding $\Lambda(v)\hat{w}_1$ or $\Lambda(v)\hat{w}_2$, and noting that $\hat{w}_1$ and $\hat{w}_2$ are orthonormal (see also (A.10)).

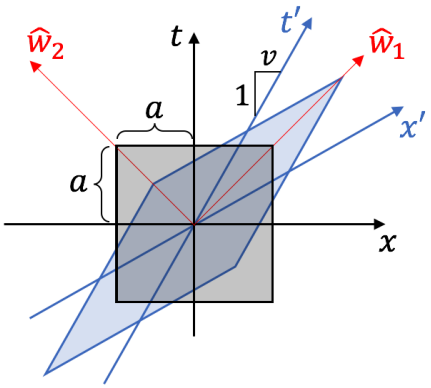


Figure 1.3: Relation between coordinates (t, x) and (t', x'), in relativity. The grey shaded region is the square $\{|t| < a, |x| < a\}$, for some constant a. The blue shaded region is $\{|t'| < a, |x'| < a\}$. Red lines show light rays in $\pm x$ direction.

Now invert Eq. (1.3):

$$X' = \Lambda^{-1}(v)X. \tag{1.7}$$

IRF I' is moving with velocity v relative to I, so I is moving with velocity $-v$ relative to I'. This gives:

$$\Lambda^{-1}(v) = \Lambda(-v). \tag{1.8}$$

*We will always scale spacetime diagrams so that ct and x have the same length on the page. Light rays then propagate at 45-degree angles.

$\Lambda(-v)$ can also represent the boost for IRF I' going in the $-x$ direction instead of the $+x$ direction in the original scenario. This swaps the forward-going and backward-going eigenvectors:

$$\Lambda(-v) = \lambda_b \hat{w}_1 \hat{w}_1^T + \lambda_f \hat{w}_2 \hat{w}_2^T \tag{1.9}$$

On the other hand, inverting a transformation simply inverts its eigenvalues:

$$\Lambda^{-1}(v) = \frac{1}{\lambda_f} \hat{w}_1 \hat{w}_1^T + \frac{1}{\lambda_b} \hat{w}_2 \hat{w}_2^T, \tag{1.10}$$

which you may verify by finding $\Lambda(v)\Lambda^{-1}(v) = I$.

Equating (1.9) and (1.10), we obtain:

$$\lambda_f = \frac{1}{\lambda_b}. \tag{1.11}$$

Then evaluating (1.6) explicitly gives:

$$\Lambda(v) = \frac{1}{2} \begin{pmatrix} \lambda + \frac{1}{\lambda} & \lambda - \frac{1}{\lambda} \\ \lambda - \frac{1}{\lambda} & \lambda + \frac{1}{\lambda} \end{pmatrix}, \tag{1.12}$$

where we define $\lambda \equiv \lambda_f$ for brevity.

To relate λ to v, note that the clock at $x' = 0$ in IRF I' moves along the path $x = vt$ in IRF I (Fig. 1.3). Plugging $X' = (t', 0)^T$ into (1.3) and using $x/t = v$ gives:

$$v = \frac{\lambda - 1/\lambda}{\lambda + 1/\lambda}. \tag{1.13}$$

Solving for λ:

$$\lambda = \sqrt{\frac{1+v}{1-v}}. \tag{1.14}$$

Finally, plugging into (1.12) gives the Lorentz transformation:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}. \tag{1.15}$$

where we define

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}. \tag{1.16}$$

As $v \to 1$, $\gamma \to \infty$, so the speed of light $c = 1$ is the speed limit for relative motion of IRFs.

Let us restore the factors of c to compare with classical physics:

$$\begin{aligned} t &= \gamma t' + \frac{\gamma v x'}{c^2} \\ x &= \gamma v t' + \gamma x'. \end{aligned} \tag{1.17}$$

For low velocities $v \ll c$, $\gamma \approx 1$, and this reduces to the Galilean transformation

$$\begin{aligned} t &\approx t' \\ x &\approx v t' + x', \end{aligned} \tag{1.18}$$

as expected.

Finally, because the Lorentz transformation is linear, it also applies to time and length differences:

$$\begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix}. \tag{1.19}$$

1.6 Time dilation

Fig. 1.4 shows the clock C_v at $x' = 0$ in IRF I'. Plugging $X' = (t', 0)^T$ into (1.15) gives $t = \gamma t'$. Since $\gamma > 1$, we have $t > t'$. Moving clocks run slower than the clocks in an IRF. This is called *time dilation*. For any stationary clock in IRF I', we can plug in $\Delta X' = (\Delta t', 0)^T$ into (1.19) to get $\Delta t = \gamma \Delta t'$.

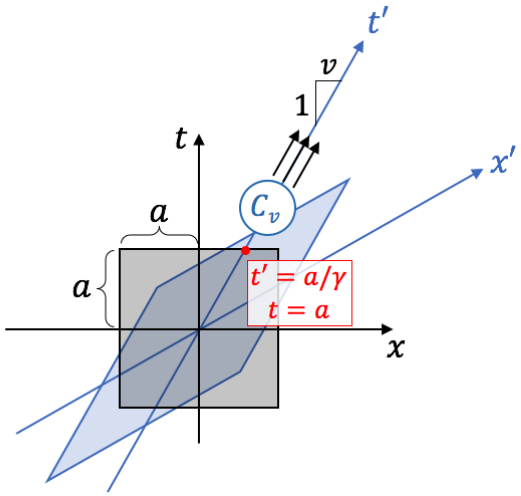


Figure 1.4: Same as Fig. 1.3, but showing the clock at $x' = 0$. It displays a lower time $t' = t/\gamma$ than the clocks in I (red point).

** Exercise 1.2

Twin paradox. Alice travels to the moon with constant velocity $+v$, then travels back to Earth with constant velocity $-v$. Her twin Eve stays on Earth. From Eve's perspective, Alice is always moving with speed $|v|$, so Alice's clock is slower. However, from Alice's perspective, Eve is also moving with speed $|v|$, so Eve's clock is slower. Whose clock is behind when Alice returns to Earth? (Ignore the rotation of the moon around the Earth, and the rotation of the Earth around the sun, etc.)

Hint 1: draw a spacetime diagram showing their paths, from an IRF where Eve is at rest (called her *rest frame*). Note that there is no single IRF where Alice is always at rest, since she must accelerate to get from velocity $+v$ to $-v$.

Hint 2: it may help to read Sec. 1.8.

* Exercise 1.3

Doppler shift. In the wave description of light, the peaks and troughs of light waves travel at the speed of light c. Consider a source at rest sending light at frequency f to a passing receiver moving at constant velocity $+v$. The time interval between successive peaks is $T = 1/f$. See Fig. 1.5. The receiver perceives a different interval between peaks $T' = 1/f'$. What is f' when the receiver is moving towards the source? Away from the source? This is called the relativistic *Doppler shift*, or *Doppler effect*.

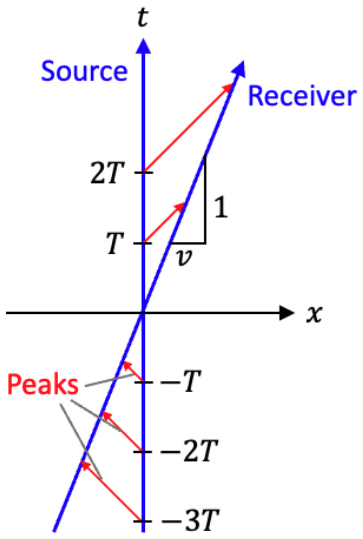


Figure 1.5: Source sending light at frequency $f = 1/T$ to a receiver moving at velocity $+v$.

1.7 Length contraction

Just as moving clocks tick slower, moving rulers also appear shorter. This phenomenon is called *length contraction*. First, what does it mean to measure a length? Length is measured from two ends of a standard ruler at the same time, just like a time interval is measured at two times by a standard clock at the same position.

Consider the ruler of length a in IRF I' between $x' = 0$ and $x' = a$ (Fig. 1.6). From (1.15), the endpoint of the ruler $X' = (0, a)^T$ corresponds to the point $X = (\gamma v a, \gamma a)^T$: the green point in Fig. 1.6. The path of this endpoint is:

$$x = \gamma a + v(t - \gamma v a). \tag{1.20}$$

Plugging in $t = 0$, the two ends of the ruler at $t = 0$ are at $x = 0$ and $x = \gamma a - \gamma v^2 a = a/\gamma$: the red point in Fig. 1.6. Thus, the observed length in I is $a/\gamma < a$.

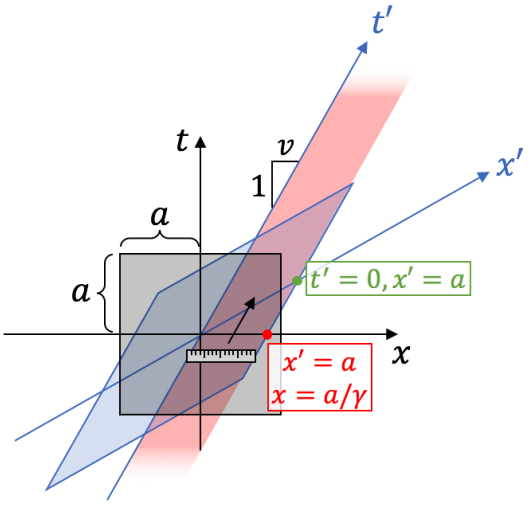


Figure 1.6: Length contraction. Red shaded region is the path of the ruler between $x' = 0$ and $x' = a$. As measured in IRF I, its length is $a/\gamma < a$.

Although length contraction is "merely" the spatial version of time dilation, it is conceptually somewhat different. Time dilation involves two events along the same path of a moving clock. On the other hand, length involves two events at different locations, so it is not possible to measure for a point-like observer. Of course, we may translate between space and time by arranging light to be sent from the faraway position (Fig. 1.7). Then we may either measure Δt and use $\Delta x = c\Delta t$, or use the spatial information from the light signal, e.g. see the object with our eyes.

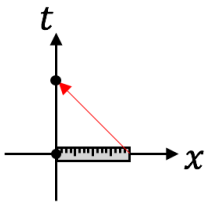


Figure 1.7: Translating a length measurement into a time measurement at $x = 0$, with a light ray (red line).

** Exercise 1.4

Ladder paradox. This apparent paradox is similar to the twin paradox, but for length contraction instead of time dilation. Consider a ladder passing through a barn with open front and back doors (Fig. 1.8). The ladder has length L at rest, but is moving with velocity v with respect to the barn, so appears contracted to length $L/\gamma(v)$. The barn at rest is size $L/\gamma(v)$, so it is able to close its front and back doors exactly when the ladder is fully inside. The doors then open and the ladder exits.

Now from the ladder's rest frame, the barn is moving with velocity $-v$ and appears contracted to length $L/\gamma(v)^2$, so it is far too small to fit the ladder of length L: the doors cannot close!

Which scenario is correct?

Hint: the two events "front door closes" and "back door closes" occur at the same time in the barn's rest frame. Do they occur at the same time in the ladder's rest frame?

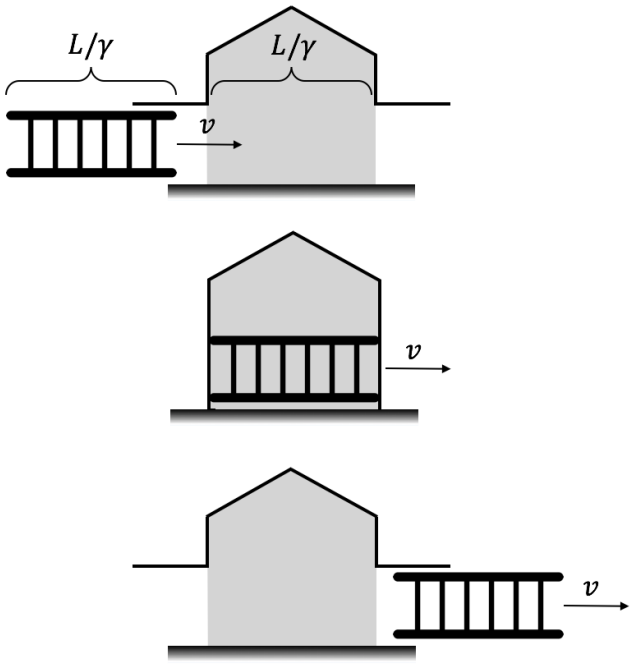


Figure 1.8: From top to bottom: (1) ladder with contracted length L/γ enters barn of size L/γ, (2) doors close, (3) doors open and ladder exits.

Fig. 1.6 and the ladder paradox illustrate the principle of *relativity of simultaneity*: the notion of two events being simultaneous depends on the IRF. The green point in Fig. 1.6 has coordinates $X' = (0, a)^T$ in IRF I', so it occurs simultaneously with the origin $X' = (0, 0)^T$. However, in IRF I, the green point clearly has $t \neq 0$, so it is not simultaneous with the origin.

1.8 Proper time

Consider a clock moving on an arbitrary path $x(t)$ in spacetime. As before, x and t refer to position and time measured in an IRF I. The path of an object is also called its *worldline*. It can be broken into an infinite number of small segments. For each segment (dt, dx), we can set the origin of the IRF to the start of the segment, and form another IRF I' with the t' axis along the segment (Fig. 1.9). Then dt is the time difference measured in the IRF, and $d\tau \equiv dt'$ is the difference in the clock's reading over the segment. (We use the symbol $d\tau$ instead of dt' since the t' axis changes for each segment.)

We may relate $d\tau$ and dt by plugging in $X' = (d\tau, 0)^T$ into (1.15):

$$\begin{aligned} d\tau &= \frac{dt}{\gamma(\dot{x})} \\ &= dt\sqrt{1-\dot{x}^2} \\ &= \sqrt{dt^2 - dx^2}, \end{aligned} \tag{1.21}$$

where $\dot{x} = dx/dt$ is the instantaneous velocity of the segment, and $\gamma(\dot{x}) = 1/\sqrt{1-\dot{x}^2}$. The dot ($\dot{}$) will always denote the time derivative.

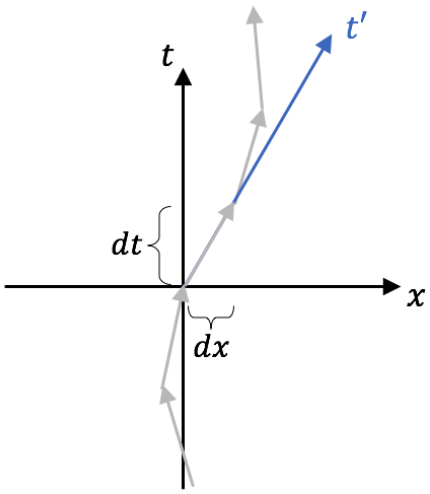


Figure 1.9: Worldline of a clock broken up into infinitesimal segments (gray). We set up IRFs I and I' around one segment (dt, dx).

We may integrate Eq. (1.21) to obtain a finite difference in clock reading τ:

$$\tau(t_1, t_2) \equiv \int_{t_1}^{t_2} dt\sqrt{1-\dot{x}(t)^2}. \tag{1.22}$$

τ is known as the *proper time*, from the French *propre*, meaning *own*. It measures the time difference displayed on a moving clock as it moves from t_1 to t_2: its "own" time.

*** Exercise 1.5**

Consider a clock attached to an oscillating spring or pendulum, so that it moves on the path $x = x_0 \sin(\omega t)$, with the maximum velocity $v_{\text{max}} = x_0\omega \ll 1$. What is the ratio of the time $\Delta\tau$ displayed on the clock to the time Δt displayed on a stationary clock, for $\Delta t \gg 1/\omega$? Find it to order v_{max}^2. Hint: use the Taylor expansion $(1+\epsilon)^p \approx 1 + p\epsilon$, for $\epsilon \ll 1$.

1.9 Velocity addition

How does a particle's velocity $\dot{x} = dx/dt$ transform under Lorentz transformations? It is a ratio between differentials dx and dt. Using (1.19), we have:

$$\begin{aligned} \dot{x} = \frac{dx}{dt} &= \frac{\gamma v dt' + \gamma dx'}{\gamma dt' + \gamma v dx'} \\ &= \frac{v + \dot{x}'}{1 + \dot{x}' v}. \end{aligned} \tag{1.23}$$

On the second line, we divide top and bottom by $\gamma dt'$. This is the *velocity addition rule* in special relativity. As a quick check, an object moving with speed $\dot{x} = v$ would appear stationary in IRF I', so $\dot{x}' = 0$. As another quick check, as $\dot{x}' \to 1$, $\dot{x} \to 1$, so the object can never reach the speed of light in any IRF. In non-relativistic physics, velocities add as $\dot{x} = v + \dot{x}'$, so objects can move at arbitrarily high speed.

1.10 Causality

While physical objects are limited to $|\dot{x}| < 1$, some phenomena can travel faster than light (called *superluminal* propagation). For example, a moving flashlight shining on a distant screen produces a moving spot (Fig. 1.10). This spot can travel faster than c across the screen if the flashlight is rotated fast enough or the screen is far enough away.

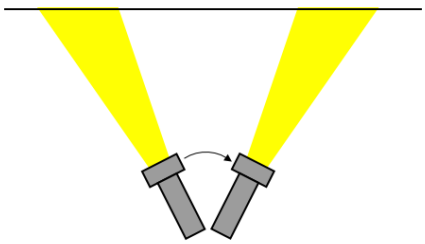


Figure 1.10: A moving flashlight illuminating a distant screen.

However, this spot cannot be used to send information: an observer on one end of the screen cannot control the spot to send a signal to another observer across the screen. Thus, it is really *information* that cannot travel faster than light, since this would violate causality.

We will define *causality* as the property that all inertial observers agree on the direction of cause and effect. Specifically, an event A can only cause another event B if all inertial observers agree on the time ordering between them: $t_A < t_B$. As usual, consider IRFs I and I' with coordinates X and X'. Let A have coordinates $X = X' = (0,0)^T$, and B have coordinates $X' = (a, ua)^T$ in IRF I', with $u, a > 0$. $u = \Delta x'/\Delta t' > 1$ corresponds to superluminal propagation. From (1.15), we have:

$$t_B = \gamma a(1 + vu). \tag{1.24}$$

We see that $t_B < t_A$ for sufficiently negative $v < -1/u$. Since v is limited to $-1 < v$, observers may disagree on the time ordering only for $u > 1$. Thus, A can only cause B by sending a signal traveling at the speed of light or slower.

Causality is a fundamental property of physical theories. Note that classical mechanics also satisfies causality while allowing superluminal signal propagation: time is a globally shared coordinate, so observers always agree on time ordering.

1.11 Four-vectors

Now let us finally move to four spacetime dimensions, and introduce some new notation. Call the time coordinate x^0 and the spatial coordinates x^1, x^2, x^3. Greek indices $\mu, \nu, \cdots$ are used for spacetime coordinates, and

Latin indices $i, j, \cdots$ for spatial coordinates. We also use $\vec{x}$ for the spatial position vector, and simply x for the full spacetime vector (instead of X). The Lorentz transformation for a boost in the x^1 direction becomes*:

$$x^\mu = \Lambda^\mu_\nu(v) x'^\nu, \tag{1.25}$$

where

$$\Lambda(v) = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.26}$$

Here, we have introduced the *Einstein summation convention*, where all repeated indices in an expression are implicitly summed over, or *contracted*. In this case, the index ν is summed from 0 to 3. The upper index of a matrix (μ here) is always the row index, and the lower index (ν here) is the column index (not that it matters because Λ is symmetric).

The proper time formula (1.21) becomes:

$$d\tau = \sqrt{dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2} = \sqrt{dt^2 - d\vec{x}^2}. \tag{1.27}$$

Since we have unified time and space coordinates into x^μ, let's try to define a 4-component velocity u^μ_{bad} as

$$u^\mu_{\text{bad}} \equiv \frac{dx^\mu}{dt} = \begin{pmatrix} 1 \\ \dot{\vec{x}} \end{pmatrix}. \tag{1.28}$$

As its name implies, this is a bad definition. From (1.23), it transforms non-linearly under Lorentz transformations due to the $\dot{x}'$ in the denominator. Ideally, it would transform in the same way as x^μ:

$$u^\mu = \Lambda^\mu_\nu(v) u'^\nu \tag{1.29}$$

Fortunately, we know of a differential that is invariant under Lorentz transformations: the proper time $d\tau$. It is the difference in clock reading over a path segment, so it is only a property of the path segment and is independent of the IRF used to measure time and distance dx^μ. Indeed, you

*We can always rotate our spatial axes so that a boost is in the x^1 direction. We will not need the general formula for a boost in an arbitrary direction.

may verify that the proper time (1.21) does not change under a Lorentz transformation (1.19)*. Such a quantity is called a *Lorentz invariant*.

Thus, let us define the *four-velocity*:

$$u^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{1.30}$$

It evidently satisfies the correct transformation law (1.29).

The path of a particle can be parametrized by t or τ. They are related by:

$$d\tau = \sqrt{dt^2 - d\vec{x}^2} = dt\sqrt{1 - \dot{\vec{x}}^2} = \frac{dt}{\gamma}, \tag{1.31}$$

where $\gamma = 1/\sqrt{1 - \dot{\vec{x}}^2}$. Thus, we have:

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{dt}\frac{dt}{d\tau} \\ &= \gamma \begin{pmatrix} 1 \\ \dot{\vec{x}} \end{pmatrix}, \end{aligned} \tag{1.32}$$

using the chain rule (Sec. C.2). We may take further derivatives $d/d\tau$ to obtain the four-acceleration, etc., which all transform linearly under boosts. In general, any 4-component quantity V^μ that transforms as

$$V^\mu = \Lambda^\mu_\nu(v) V'^\nu \tag{1.33}$$

under a boost is called a *four-vector*.

Four-vectors are important because they allow us to form other Lorentz invariants. Define the quantity ds^2 as

$$\begin{aligned} ds^2 \equiv -d\tau^2 &= d\vec{x}^2 - dt^2 \\ &= \eta_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \tag{1.34}$$

where η is a 4×4 matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.35}$$

*This can be seen from the eigenvectors $\hat{w}_1 = (1,1)^T/\sqrt{2}$ and $\hat{w}_2 = (1,-1)^T/\sqrt{2}$ (1.5). Using these as a basis, a displacement vector dv has coordinates $dv^1 = (dt + dx)/\sqrt{2}$ and $dv^2 = (dt - dx)/\sqrt{2}$. Since the eigenvalues of $\hat{w}_1$ and $\hat{w}_2$ are λ and $1/\lambda$ respectively, the product $dv^1 dv^2 = (dt^2 - dx^2)/2$ is constant under a boost.

ds^2 is called the *interval*, and $\eta_{\mu\nu}$ is called the *Minkowski metric*, or simply *metric*. More terminology: two spacetime points separated by dx^μ are *timelike-separated* when $ds^2 < 0$, *null-separated* when $ds^2 = 0$, or *spacelike-separated* when $ds^2 > 0$. Relating to the previous section, spacelike-separated events are *causally disconnected*: they cannot cause each other.

Clearly, we may replace dx^μ and/or dx^ν in (1.34) with any four-vector, and the result will also be Lorentz invariant. For example, the four-velocity squared is simply a constant, since we are dividing ds^2 by $d\tau^2$:

$$u^2 \equiv \eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1. \tag{1.36}$$

Finally, you may have noticed that we always sum over an upper and a lower index, and never upper/upper or lower/lower. This is because we would like upper and lower indices to transform in different ways. Define

$$V_\mu \equiv \eta_{\mu\nu} V^\nu \tag{1.37}$$

for any four-vector V^ν. In components: $V_0 = -V^0, V_i = V^i$. From the requirement that $V^\mu V_\mu$ is Lorentz invariant, V_μ transforms as:

$$V_\mu = (\Lambda^{-1}(v))^\nu_\mu V'_\nu \tag{1.38}$$

under a boost.

Eq. (1.37) is just a matrix-vector multiplication. We may invert the matrix η and use it to raise indices:

$$V^\mu = \eta^{\mu\nu} V_\nu. \tag{1.39}$$

$\eta^{\mu\nu}$ is called the *inverse metric*, denoted by the same symbol but with upper indices. Of course, it is the same matrix as the metric in this case, but we will later replace the metric with a more general matrix.

For now, upper and lower indices are just a convenient notation. We will see later that the two types of indices have a geometric interpretation in general relativity.

* Exercise 1.6

Following similar steps as in (1.23), how do the velocities in the x^2 and x^3 directions $\dot{x}^2 = dx^2/dt$ and $\dot{x}^3 = dx^3/dt$ transform under a boost in the x^1 direction?

*** Exercise 1.7**

Show that $\eta^{\mu\nu}V_\mu W_\nu = \eta_{\mu\nu}V^\mu W^\nu$ for any four-vectors V^μ and W^μ.

***** Exercise 1.8**

1. Derive the formula for the *four-acceleration* $a^\mu \equiv du^\mu/d\tau$ in terms of the ordinary velocity $\dot{\vec{x}}$ and acceleration $\ddot{\vec{x}}$.
2. Calculate the Lorentz invariant $a^\mu a_\mu$.
3. Consider an observer uniformly accelerating in the x^1 direction, so that $a^\mu a_\mu = a_0^2$, where a_0 is constant. Assume it starts at rest at $t = 0, \vec{x} = \vec{0}$. Find its path $x^1(t)$.

1.12 Energy and momentum

Notation alert: since we are running out of nice symbols for velocity, and $\dot{\vec{x}}$ is rather clumsy, we will use $\vec{v} = \dot{\vec{x}}$ for a particle's velocity instead of the relative velocity between IRFs, unless otherwise indicated.

We can multiply u^μ by the mass m to get the *four-momentum* p^μ:

$$p^\mu = mu^\mu = m\gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}. \tag{1.40}$$

The corresponding Lorentz invariant is $p^2 = -m^2$. At small velocity $v \ll 1$, the spatial components become $\vec{p}_{nr} = m\vec{v}$, the usual non-relativistic momentum. What about the time component? Taylor expand $\gamma(v)$ around $v = 0$[*]:

$$\gamma(v) = 1 + \frac{1}{2}v^2 + O(v^4) \tag{1.41}$$

We see that

$$E \equiv p^0 = m + \frac{1}{2}m\vec{v}^2 + O(\vec{v}^4). \tag{1.42}$$

E is the *energy*. It includes the familiar kinetic energy $m\vec{v}^2/2$ plus the *rest energy* m. This is the famous mass-energy equivalence $E_{\text{rest}} = mc^2$, upon restoring c.

[*]The notation $O(x^n)$ means terms of order x^n or higher (x^{n+1}, x^{n+2}, etc).

What about particles that travel at the speed of light, like light itself (the photon)? Their change in proper time $d\tau$ is always zero, so they do not have a four-velocity. However, consider Eq. (1.40) as $|\vec{v}| \to 1$. We have $\gamma(v) \to \infty$, but the energy $E = m\gamma(v)$ can remain finite if $m \to 0$ at the same time. Thus, massless particles move at the speed of light. Their four-momentum is:

$$p^\mu_{\text{massless}} = E \begin{pmatrix} 1 \\ \hat{n} \end{pmatrix} \tag{1.43}$$

where $\hat{n}$ is a unit vector.

Just as energy and momentum are conserved in non-relativistic physics, four-momentum is conserved* in all interactions. For example, consider a mass M particle decaying into two mass m particles 1 and 2. We have $p^\mu_M = p^\mu_1 + p^\mu_2$, $(p_M)^2 = -M^2$, and $(p_1)^2 = (p_2)^2 = -m^2$. In the rest frame of M, we get:

$$\begin{aligned} p_M &= \left(M, \vec{0}\right)^T \\ p_1 &= \left(\frac{M}{2}, \hat{n}\sqrt{\frac{M^2}{4} - m^2}\right)^T \\ p_2 &= \left(\frac{M}{2}, -\hat{n}\sqrt{\frac{M^2}{4} - m^2}\right)^T , \end{aligned} \tag{1.44}$$

where $\hat{n}$ is a unit vector. Note that this decay is allowed if $0 \leq m \leq M/2$. The extra mass $M - 2m$ is converted to kinetic energy of the products.

** Exercise 1.9

1. Find the final momenta p_1 and p_2 for the decay of M into two *unequal* masses m_1 and m_2, in the rest frame of M. Use coordinates where m_2 moves in the $+x^3$ direction.

2. Show that this decay is only allowed for $m_1 + m_2 \leq M$.

*Some students get confused about *conserved* versus *invariant* quantities, since they both involve the notion of staying the same. A conserved quantity stays the same over time, while an invariant quantity is the same under some transformation. (You could say that a conserved quantity is invariant under time translation.) Something can be both, one, or neither. p^μ is conserved but transforms as a four-vector under boosts so is not invariant. $\vec{x}^2(\tau)$ is obviously not conserved but is invariant under spatial rotations.

1.13 Lightcones

In the remainder of this chapter, we introduce some useful but nonessential concepts.

All the light rays that intersect a given point x_a form a region called the *lightcone* at x_a, since this region can be visualized as a cone in 3D spacetime (Fig. 1.11). The upper cone with $x^0 > x_a^0$ is called the *future lightcone*, and the lower cone with $x^0 < x_a^0$ is called the *past lightcone*. The worldline of any object intersecting x_a always lies within the lightcone at x_a. Under a boost, any event within the lightcone at x_a stays within the lightcone at x'_a, since it remains timelike-separated from x_a ($d\tau^2 = dt^2 - d\vec{x}^2 > 0$). Likewise, any event outside of the lightcone stays outside, since it remains spacelike-separated.

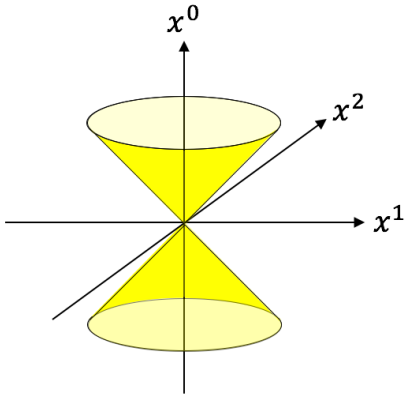


Figure 1.11: Lightcone at the origin for a 3D spacetime.

1.14 Wick rotation

Finally, let's discuss another way to think of Lorentz transformations that may be more intuitive. The interval (1.34) looks just like the formula for distance squared in Euclidean space $d\vec{x}^2$, but with the time dimension $-dt^2$ added on with the wrong sign. We can make it look exactly like Euclidean space using a trick called *Wick rotation*. Define an imaginary time variable

as*:

$$t_E \equiv it. \tag{1.45}$$

Then t_E is just another spatial dimension and (1.34) becomes:

$$ds^2 = d\vec{x}^2 + dt_E^2 \tag{1.46}$$

In Euclidean space, we know that rotations leave distances invariant. For example, the rotation around the x^3 axis†

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} \tag{1.47}$$

leaves $\vec{x}^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ the same. Since t_E is another spatial dimension, the "spacetime" rotation

$$\begin{pmatrix} it \\ x^1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} it' \\ x'^1 \end{pmatrix} \tag{1.48}$$

clearly leaves ds^2 invariant. However, we see that the angle θ cannot be real, or t will be complex when t' and x'^1 are real. Using the identities

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}, \end{aligned} \tag{1.49}$$

we see that if θ is imaginary, $\cos\theta$ is real and $\sin\theta$ is imaginary, which is just what we need so that t and x stay real. Thus, we define

$$\theta = i\beta \tag{1.50}$$

where the real number β is called the *rapidity*. We have

$$\begin{aligned} \cos\theta &= \frac{e^{\beta} + e^{-\beta}}{2} = \cosh\beta \\ \sin\theta &= i\left(\frac{e^{\beta} - e^{-\beta}}{2}\right) = i\sinh\beta, \end{aligned} \tag{1.51}$$

*$t_E \equiv -it$ would work just as well here. In quantum field theory, Wick rotation is associated with rotating a contour integral in the complex plane. There, one choice is better than the other.

†See Ex. A.2.

so that

$$\begin{pmatrix} t \\ x^1 \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} t' \\ x'^1 \end{pmatrix}. \tag{1.52}$$

Comparing with (1.15), we see that β is related to v as

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta} = v. \tag{1.53}$$

Note that $\tanh\beta$ is bounded by ± 1, as expected. Thus, Lorentz transformations are just rotations between space and imaginary time (by an imaginary angle).

Chapter 2

The action principle

We have set the stage for physics in flat space. Now let us discuss how particles and fields propagate in this spacetime more systematically. For example, particles in non-relativistic physics follow Newton's second law $\vec{F} = m\vec{a}$, and electromagnetic fields $\vec{E}(t, \vec{x})$ and $\vec{B}(t, \vec{x})$ follow Maxwell's equations. These are both examples of *equations of motion* that are derived from the *principle of stationary action*, or the *action principle*.

The action principle takes slightly different forms for particles and fields (Table 2.1). For simplicity, we consider particles in classical mechanics first. Those familiar with Lagrangians in classical mechanics can skip to Sec. 2.6.

2.1 The action

Consider a single particle in empty space. At any time t, it has a position $\vec{x}(t)$ and velocity $\vec{v}(t) = d\vec{x}/dt$. Define a real-valued quantity $S_{if}\{\vec{x}(t)\}$ that depends on the path of the particle $\vec{x}(t)$ from time t_i to t_f. The action principle states that the path the particle actually takes is one where the action is stable to small perturbations in the path $\vec{x}(t) \to \vec{x}(t) + \delta\vec{x}(t)$ (Fig. 2.1).

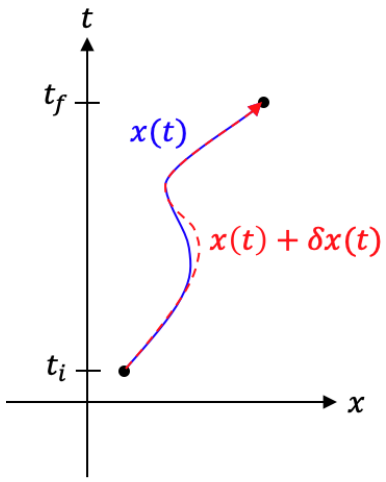


Figure 2.1: $x(t)$ is the path of a particle from t_i to t_f, and $\delta x(t)$ is a small perturbation. The action principle applies to any $\delta x(t)$ that vanishes at the boundaries t_i and t_f.

To elaborate, consider dividing the time interval from t_i to t_f into N segments, and take $N \to \infty$ in the end. You may think of S_{if} as a function of all the positions and times of each segment:

$$S_{if}\{\vec{x}(t_i), t_i, \vec{x}(t_i + \Delta t), t_i + \Delta t, \cdots, \vec{x}(t_f), t_f\}, \tag{2.1}$$

where $\Delta t = (t_f - t_i)/N$. (Note that the velocity $\vec{v}(t) = \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t}$, so it is not an independent variable here.) Such a function of infinitely many variables, or "function of a function", is called a *functional*. The principle of stationary action is then

$$\frac{\delta S_{if}}{\delta x^i(t)} = 0, \tag{2.2}$$

i.e. the partial derivative of S_{if} with respect to any component of the position x^i at any time t is zero. The δ symbol is used instead of ∂ for functional derivatives.

Instead of the derivative notation $\frac{\delta S_{if}}{\delta x^i(t)}$, we will mainly think of δS as a small change in S coming from a small change in $\vec{x}(t)$:

$$\delta S \equiv S\{\vec{x}(t) + \delta\vec{x}(t)\} - S\{\vec{x}(t)\} = \int dt\, \frac{\delta S_{if}}{\delta x^i(t)} \delta x^i(t) = 0, \tag{2.3}$$

where we expand to first order in $\delta x^i(t)$. As usual, the summation over i is implied. This is just the functional version of

$$\Delta f(\vec{x}) \equiv f(\vec{x} + \Delta\vec{x}) - f(\vec{x}) = \frac{df}{dx^i}\Delta x^i, \tag{2.4}$$

where $f(\vec{x})$ is some function of multiple variables x^i. In (2.3), the sum over all times t becomes an integral since it is a continuous variable, while the index i is discrete.

Eqs. (2.2) and (2.3) are equivalent. If we require $\frac{\delta S_{if}}{\delta x^i(t)} = 0$, then $\delta S = 0$. Conversely, if we require $\delta S = 0$ for any $\delta x^i(t)$, then $\frac{\delta S_{if}}{\delta x^i(t)} = 0$.

Finally, the action principle only applies to perturbations that are zero at the boundaries: $\delta\vec{x}(t_i) = \delta\vec{x}(t_f) = 0$. This will become important later.

2.2 The Lagrangian

Consider the action S_{12} for time t_1 to t_2, and the action S_{23} for time t_2 to t_3, with $t_1 < t_2 < t_3$. We require locality in time, meaning that a perturbation in the first interval only affects S_{12} and not S_{23}. Also, we require additivity of the action: $S_{12} + S_{23} = S_{13}$. These conditions imply that S_{if} can be written as an integral from t_i to t_f of some quantity:

$$S_{if} = \int_{t_i}^{t_f} L(\vec{x}(t), \vec{v}(t), t)\, dt. \tag{2.5}$$

$L(\vec{x}(t), \vec{v}(t), t)$ is known as the *Lagrangian*. In general, it may depend on the position and velocity at time t, as well as the time t itself*.

Note that we may add a total time derivative $\frac{df}{dt}(\vec{x}, t)$ to the Lagrangian without affecting the action principle. Such a term produces the action:

$$\int_{t_i}^{t_f} dt \frac{df}{dt}(\vec{x}, t) = f(\vec{x}(t_f), t_f) - f(\vec{x}(t_i), t_i) \tag{2.6}$$

by the fundamental theorem of calculus. This is irrelevant for the action principle since the perturbation $\delta\vec{x}(t)$ is zero at the boundaries by definition.

Let us now derive the form of the Lagrangian based on some other fundamental principles:

*It also cannot depend on higher time derivatives due to the Ostrogradsky instability.

- **Homogeneity of space and time**. No point in space or time is any different from any other, so the Lagrangian cannot depend on $\vec{x}$ or t explicitly.
- **Isotropy of space**. No direction in space is different from any other, so the Lagrangian can only depend on the magnitude (squared) of the velocity $\vec{v}(t)^2$.
- **Galilean invariance**. The theory should be invariant under shifts by a constant velocity, $\vec{x} \to \vec{x} + \vec{v}_0 t$ (a Galilean transformation, from Sec. 1.4). This is the non-relativistic version of the Lorentz transformation*. Taking the time derivative, this is $\vec{v} \to \vec{v} + \vec{v}_0$. To first order in $\vec{v}_0$, the Lagrangian changes as

$$L(\vec{v}^2) \to L(\vec{v}^2 + 2\vec{v}\cdot\vec{v}_0) = L(\vec{v}^2) + 2\frac{\delta L}{\delta \vec{v}^2}(\vec{v}^2)\vec{v}\cdot\vec{v}_0 + O(\vec{v}_0^2) \tag{2.7}$$

The term $2\frac{\delta L}{\delta \vec{v}^2}(\vec{v}^2)\vec{v}\cdot\vec{v}_0$ will not affect the physics if it is a total time derivative of the form above. This only occurs if $\frac{\delta L}{\delta \vec{v}^2}(\vec{v}^2)$ is a constant. Call this constant $\frac{1}{2}m$. Thus, the Lagrangian for a single particle in free space is:

$$L = \frac{1}{2}m\vec{v}^2. \tag{2.8}$$

The constant m is, of course, the mass.

To summarize, we derived the unique action and Lagrangian (up to a total time derivative) for a single particle from the following postulates:

1. Locality in time
2. Additivity of the action
3. Homogeneity of space and time
4. Isotropy of space
5. Galilean invariance

We used the first two to introduce the Lagrangian L, and the last three to constrain the form of L for a single particle. We can repeat this procedure for the n-particle case.

*It implies there is no universal stationary frame of reference, even in non-relativistic mechanics. The difference in relativity is that the speed of light is constant in all inertial frames.

2.3 Multiple particles

For n particles, we start again with a Lagrangian L which may depend on all the positions and velocities $\{\vec{x}_1, \vec{v}_1, \cdots, \vec{x}_n, \vec{v}_n\}$. We will use letters $a, b, \cdots$ to index the particles.

Due to spatial homogeneity, L can only depend on position differences $\vec{x}_{ab} \equiv \vec{x}_a - \vec{x}_b$ instead of absolute positions $\vec{x}_a$. Since L is a scalar and not a vector, we must form scalars from $\vec{x}_{ab}$ and $\vec{v}_a$ by taking dot products. This gives three types of terms: $\vec{x}_{ab} \cdot \vec{x}_{cd}$, $\vec{x}_{ab} \cdot \vec{v}_c$, and $\vec{v}_a \cdot \vec{v}_b$. We cannot take dot products with some other vector $\vec{a}$, since this would single out a particular direction and violate spatial isotropy. Therefore, we have $L(\vec{x}_{ab} \cdot \vec{x}_{cd}, \vec{x}_{ab} \cdot \vec{v}_c, \vec{v}_a \cdot \vec{v}_b)$.

Under a Galilean transformation, $\vec{v}_a \to \vec{v}_a + \vec{v}_0$, while $\vec{x}_{ab}$ is unchanged since it is a difference of positions. This gives:

$$L \to L + \frac{\delta L}{\delta(\vec{x}_{ab} \cdot \vec{v}_c)} \vec{x}_{ab} \cdot \vec{v}_0 + \frac{\delta L}{\delta(\vec{v}_a \cdot \vec{v}_b)} (\vec{v}_a + \vec{v}_b) \cdot \vec{v}_0 + O(\vec{v}_0^2). \tag{2.9}$$

The term $\frac{\delta L}{\delta(\vec{x}_{ab} \cdot \vec{v}_c)} \vec{x}_{ab} \cdot \vec{v}_0$ cannot be a total time derivative, so L cannot depend on $\vec{x}_{ab} \cdot \vec{v}_c$. On the other hand, the term $\frac{\delta L}{\delta(\vec{v}_a \cdot \vec{v}_b)} (\vec{v}_a + \vec{v}_b) \cdot \vec{v}_0$ is a total time derivative if the $\frac{\delta L}{\delta(\vec{v}_a \cdot \vec{v}_b)}$ are constant. This means the term $T_{ab}(\vec{v}_a \cdot \vec{v}_b)$ is allowed in L by the symmetries, for some constants T_{ab}. However, if T_{ab} is nonzero for $a \neq b$, a change in velocity of particle a affects particle b, no matter how far away they are. This violates common sense and does not correspond to any known interaction*, so we will assume that $T_{ab} = 0$ for $a \neq b$. Then the constants $T_{aa} \equiv m_a/2$, where m_a are the masses of the particles. Since $\vec{x}_{ab}$ is not affected by a Galilean transformation, we can also add a function $-U(\vec{x}_{ab})$ that only depends on all the separations between the particles. The total Lagrangian is then:

$$L = \left(\sum_{a=1}^{n} \frac{1}{2} m_a \vec{v}_a^2 \right) - U(\vec{x}_{ab}). \tag{2.10}$$

*If we do allow such cross-terms, we could eliminate them by redefining the positions $\vec{x}_a = A_{ab} \vec{x}'_b$, such that $A^T T A$ is a diagonal matrix. However, then spatial homogeneity and isotropy do not hold for the new position variables $\vec{x}'_a$. The potential U generally depends on the absolute positions $\vec{x}'_a$, and not just position differences $\vec{x}'_a - \vec{x}'_b$. We would have no way to explain why fundamental interactions (electromagnetic and gravitational) only depend on position differences.

For example, the Coulomb interaction between two charges q_1 and q_2 is:

$$U(\vec{x}_1 - \vec{x}_2) = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|}. \tag{2.11}$$

2.4 Euler-Lagrange equations

Let us now apply the principle of stationary action to the action (2.5), repeated here:

$$S_{if} = \int_{t_i}^{t_f} L(\vec{x}(t), \vec{v}(t), t)\, dt. \tag{2.12}$$

Now take $\vec{x}(t) \to \vec{x}(t) + \delta\vec{x}(t)$. The time derivative gives* $\vec{v}(t) \to \vec{v}(t) + \frac{d\delta\vec{x}}{dt}(t)$. The change in the action is:

$$\begin{aligned} \delta S_{if} &= \int_{t_i}^{t_f} \left(\frac{\delta L}{\delta x^i(t)} \delta x^i(t) + \frac{\delta L}{\delta v^i(t)} \delta v^i(t) \right) dt \\ &= \int_{t_i}^{t_f} \left(\frac{\delta L}{\delta x^i(t)} - \frac{d}{dt} \frac{\delta L}{\delta v^i(t)} \right) \delta x^i(t)\, dt. \end{aligned} \tag{2.13}$$

We use the Einstein summation convention of the previous chapter, where the index i is summed over. On the second line, we use $\delta\vec{v} = \frac{d\delta\vec{x}}{dt}$ and integrate by parts. Note that we can discard the boundary term $\frac{\delta L}{\delta v^i(t)} \delta x^i(t)$ since $\delta\vec{x} = 0$ at the boundaries.

Since this must equal zero for any variation $\delta\vec{x}(t)$, we obtain the *Euler-Lagrange equations*:

$$\frac{\delta L}{\delta x^i(t)} = \frac{d}{dt} \frac{\delta L}{\delta v^i(t)}. \tag{2.14}$$

For multiple particles, this becomes:

$$\frac{\delta L}{\delta x_a^i(t)} = \frac{d}{dt} \frac{\delta L}{\delta v_a^i(t)} \tag{2.15}$$

for each particle a.

Applying this to the multi-particle Lagrangian (2.10) gives Newton's law for a conservative potential:

$$\vec{F} \equiv -\nabla_a U = m_a \vec{a}_a \tag{2.16}$$

for each particle a, where $\vec{a}_a = d\vec{v}_a/dt$ and ∇_a is the gradient with respect to $\vec{x}_a$.

*Thus, the variation "operator" δ commutes with the derivative d/dt: $\delta \frac{d\vec{x}}{dt} = \frac{d\delta\vec{x}}{dt}$.

2.5 Noether's theorem

Noether's theorem is a simple but profound result that relates symmetries of the Lagrangian to conserved quantities. Consider a perturbation $\delta\vec{x}(t)$ that only changes the Lagrangian by a total time derivative df/dt:

$$\delta L = \frac{df}{dt}. \tag{2.17}$$

On the other hand, the Lagrangian depends on $\vec{x}(t)$ and $\vec{v}(t)$, so we have

$$\begin{aligned}\delta L &= \frac{\delta L}{\delta x^i}\delta x^i + \frac{\delta L}{\delta v^i}\delta v^i \\ &= \frac{d}{dt}\left(\frac{\delta L}{\delta v^i}\right)\delta x^i + \frac{\delta L}{\delta v^i}\delta v^i \\ &= \frac{d}{dt}\left(\frac{\delta L}{\delta v^i}\delta x^i\right)\end{aligned} \tag{2.18}$$

where we have used the Euler-Lagrange equations (2.14) on the second line. Equating these two, we see that

$$\frac{d}{dt}\left(\frac{\delta L}{\delta v^i}\delta x^i - f\right) = 0. \tag{2.19}$$

Thus, the quantity

$$j \equiv \frac{\delta L}{\delta v^i}\delta x^i - f \tag{2.20}$$

is conserved. j is called the *Noether charge*.

This may feel like circular reasoning, since the Euler-Lagrange equations were derived by considering perturbations $\delta\vec{x}(t)$. The difference is that $\delta\vec{x}(t)$ here does *not* necessarily vanish at the boundaries.

The extension to multiple particles is again straightforward; simply index by a as well as i:

$$j \equiv \frac{\delta L}{\delta v_a^i}\delta x_a^i - f. \tag{2.21}$$

Let's test this out. The multi-particle Lagrangian (2.10) does not change under a global translation $\vec{x}_a \to \vec{x}_a + \vec{b}$. We have $\delta\vec{x}_a = \vec{b}$ and $\delta L = 0$ (so $f = 0$). Plugging into (2.21):

$$\begin{aligned}j_{\vec{b}} &= \frac{\delta L}{\delta v_a^i}\delta x_a^i \\ &= m_a v_a^i b^i\end{aligned} \tag{2.22}$$

Since $dj_{\vec{b}}/dt = 0$ for any $\vec{b}$, this implies that

$$\vec{p}_{tot} \equiv m_a \vec{v}_a \tag{2.23}$$

is conserved. This is, of course, the total momentum.

Now consider time translation $t \to t + \delta t$. We have $\delta \vec{x}_a = \vec{v}_a \delta t$. The Lagrangian depends on time implicitly through $\vec{x}(t)$ and $\vec{v}(t)$, so changes by

$$\delta L = \frac{dL}{dt} \delta t. \tag{2.24}$$

Note the distinction between the total derivative d/dt and the partial derivative $\partial/\partial t$. $\partial L/\partial t = 0$ since L does not depend on time explicitly, but $dL/dt \neq 0$. Thus, $f(t) = L\delta t$. Plugging into (2.21):

$$\begin{aligned} j_{\delta t} &= \frac{\delta L}{\delta v_a^i} \delta x_a^i - f \\ &= m_a \vec{v}_a^2 \delta t - L \delta t \\ &= \left(\frac{1}{2} m_a \vec{v}_a^2 + U(\vec{x}_{ab}) \right) \delta t. \end{aligned} \tag{2.25}$$

Again, because this is conserved for all δt, the *energy*

$$E \equiv \frac{1}{2} m_a \vec{v}_a^2 + U(\vec{x}_{ab}) \tag{2.26}$$

is conserved. As you know, the first term is the kinetic energy and the second is the potential energy.

** Exercise 2.1

Due to spatial isotropy, the Lagrangian (2.10) is invariant under rotations. The change in x_a^i for an infinitesimal rotation around the axis $\vec{\theta}$ by an angle $\left|\vec{\theta}\right|$ is:

$$\delta x_a^i = \left(\vec{\theta} \times \vec{x}_a \right)^i \tag{2.27}$$

as you may verify using a diagram and the right-hand rule. Show that the total *angular momentum*

$$\vec{L}_{tot} \equiv \vec{x}_a \times \vec{p}_a \tag{2.28}$$

is conserved. You may find Eq. (C.5) useful.

2.6 Relativistic particles

Let's go back to relativity and start with a single (massive) particle again. Since its worldline can still be parametrized by t, the Euler-Lagrange equations (2.14) still hold. However, the Lagrangian is different. Because the laws of physics should take the same form in all IRFs, the action must be an integral of a Lorentz invariant. The only differential Lorentz invariant that characterizes the path is the proper time $d\tau$. Thus, the point-particle action is:

$$S_{pp} = -mc^2 \int d\tau \tag{2.29}$$

where we have temporarily restored c and introduced a constant m with units of mass, to match the units of the non-relativistic action (mass $\times$ length^2/time)*. Using the proper time formula (1.27):

$$S_{pp} = -m \int \sqrt{1 - \vec{v}(t)^2}\, dt \tag{2.30}$$

so the Lagrangian is

$$L_{pp} = -m\sqrt{1 - \vec{v}(t)^2}. \tag{2.31}$$

Plugging into the Euler-Lagrange equations gives:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(m\frac{\vec{v}}{\sqrt{1-\vec{v}^2}}\right) \\ &= \frac{d}{dt}\left(m\gamma(v)\vec{v}\right) \\ &= \frac{d\vec{p}}{dt} \end{aligned} \tag{2.32}$$

where $\vec{p}$ is the spatial part of the four-momentum (1.40). Thus, the velocity is a constant and particles propagate in straight lines, as expected. This also holds for massless particles, although we started with a massive Lagrangian.

Unlike non-relativistic mechanics, it is difficult to couple multiple particles through a direct interaction as in (2.10). This is because non-relativistic

*The negative sign is so that the action is minimized when the proper time is maximized. It is always possible to connect two timelike-separated points with multiple null vectors so that the proper time is minimized at zero, but this is not a stationary path and obviously not the path the particle takes.

physics allows *action-at-a-distance*: particles far away can affect each other instantaneously in time. However, relativistic interactions must be local in spacetime while preserving Lorentz invariance. This only permits delta-function terms* like

$$\int \int d\tau_a d\tau_b \, \delta^4 \left(x_a(\tau_a) - x_b(\tau_b) \right) \tag{2.33}$$

in the action, which do not correspond to any known interaction. Indeed, this term can be integrated explicitly, giving a term $\propto \delta^2(\cdots)$ which is either infinite or zero depending on whether the paths of the two particles intersect. The problem is that there are only two variables of integration $d\tau_a$ and $d\tau_b$ but we must use a four-dimensional delta function for Lorentz invariance†.

2.7 From particle to field

Although direct particle interaction has issues, particles can be coupled to fields quite easily. A *field* is simply a quantity that varies in spacetime. For example, a *scalar field* $\phi(x)$ assigns a real number to each point in spacetime. Under a boost (1.25), it transforms as:

$$\phi(x) = \phi'(x'(x)) \tag{2.34}$$

where $x'^\mu = (\Lambda^{-1}(v))^\mu_\nu x^\nu$. This equation can be a little confusing, so let's explain it in words. We have a field $\phi'(x')$, which is a function from spacetime point x' to a real number. We also have the boost equation $x'^\mu(x) = (\Lambda^{-1}(v))^\mu_\nu x^\nu$. To evaluate the field $\phi(x)$, first find $x'(x)$ and then evaluate ϕ' there. Fig. 2.2 shows this visually for a rotation instead of a boost.

A four-vector field $A_\mu(x)$ assigns a four-vector to each point in spacetime. Under a boost, it transforms as:

$$A_\mu(x) = (\Lambda^{-1}(v))^\nu_\mu A'_\nu(x'(x)). \tag{2.35}$$

*See Appendix D for an introduction to delta functions.

†We can show $\delta^4(x)$ is Lorentz invariant using the identity $\int d^D x \, \delta^D(\vec{x}) = 1$ for general dimension D. Here, $d^D x \equiv dx^0 dx^1 \cdots dx^{D-1}$. Under a linear change of variables $\vec{x} = A\vec{x}'$, $d^D x = d^D x' \left|\det A\right|$ (A.22), so $\delta^D(\vec{x}) = \delta^D(\vec{x}')/\left|\det A\right|$. For the Lorentz transformation (1.15), $\det \Lambda = 1$. See also (D.17).

Since A_μ is a four-vector, it transforms according to Eq. (1.38) in addition to the transformation of the argument x. This is shown in Fig. 2.2 for a rotation instead of a boost.

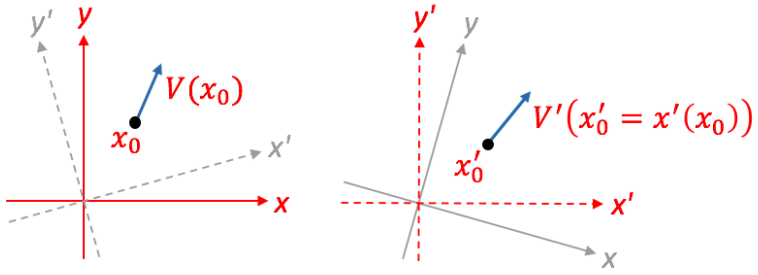


Figure 2.2: A vector's components V^i and position (x, y) look different under a change of coordinates. To visualize a scalar field transformation (2.34), simply remove the blue arrow.

Let's try to couple a field $A_\mu(x)$ to a particle. The simplest Lorentz invariant action is

$$\begin{aligned} S_A &= q \int A_\mu(t, \vec{x}) dx^\mu \\ &= q \int \left(A_0(t, \vec{x}(t)) + v^i A_i(t, \vec{x}(t)) \right) dt, \end{aligned} \tag{2.36}$$

where q is some scaling factor, just like the mass in (2.29). We have explicitly indicated the dependence of A_μ on t and $\vec{x}(t)$. The total Lagrangian is

$$\begin{aligned} L &= L_{pp} + L_A \\ &= -m\sqrt{1 - v^2} + q(A_0 + v^i A_i). \end{aligned} \tag{2.37}$$

The Euler-Lagrange equations are:

$$\begin{aligned} q\frac{\partial A_0}{\partial x^i} + qv^j \frac{\partial A_j}{\partial x^i} &= \frac{d}{dt}(p_i + qA_i) \\ &= \frac{dp_i}{dt} + q\frac{\partial A_i}{\partial t} + q\frac{\partial A_i}{\partial x^j} v_j, \end{aligned} \tag{2.38}$$

where on the second line we use the chain rule on $\frac{dA_i(t,\vec{x}(t))}{dt}$, since A_μ depends on t through $\vec{x}(t)$ as well as t explicitly (Sec. C.2). Rearranging, we

get:

$$\frac{dp_i}{dt} = q(\partial_i A_0 - \partial_0 A_i) + qv_j(\partial_i A_j - \partial_j A_i), \tag{2.39}$$

using an abbreviated notation $\partial_\mu \equiv \partial/\partial x^\mu$. Now use the vector calculus identity (C.6) to write the second term in a more familiar form:

$$\frac{dp_i}{dt} = q\left(\partial_i A_0 - \partial_0 A_i\right) + q\left(\vec{v} \times \left(\nabla \times \vec{A}\right)\right)_i. \tag{2.40}$$

This is the *Lorentz force law*[*] for a charge q:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}, \tag{2.41}$$

upon defining the electric and magnetic fields

$$\begin{aligned} E_i &\equiv \partial_i A_0 - \partial_0 A_i \\ B_i &\equiv \left(\nabla \times \vec{A}\right)_i = \epsilon_{ijk}\partial_j A_k, \end{aligned} \tag{2.42}$$

using (C.3). You may recognize $\vec{A}$ as the vector potential and $A_0 = -A^0 = -V$ as the electric potential of electromagnetism. We have "discovered" electromagnetism by simply postulating a four-vector field A_μ and writing a Lorentz-invariant coupling to a particle!

**** Exercise 2.2**

Show that equations (2.42) imply two of Maxwell's equations:

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}. \end{aligned} \tag{2.43}$$

You may find the identities in Appendix C useful.

2.8 Gauge invariance

Let us discuss an additional symmetry principle that constrains the Lagrangian. Examine the action S_A again (2.36). The transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \phi(x) \tag{2.44}$$

[*]Note that $\vec{p}$ here is still the spatial part of the four-momentum, which only reduces to the non-relativistic momentum $m\vec{v}$ at low velocity.

for some scalar field $\phi(x)$ produces a total time derivative in the Lagrangian:

$$\begin{aligned} S_A &\to S_A + q\int \partial_\mu \phi dx^\mu \\ &= S_A + q\int (\partial_t \phi + v^i \partial_i \phi) dt \\ &= S_A + q\int \frac{d\phi}{dt} dt. \end{aligned} \tag{2.45}$$

As we have repeated many times, a total time derivative does not affect the physics. Indeed, you can verify that $\vec{E}$ and $\vec{B}$ (2.42) are left invariant by this transformation, as you may recall from your electromagnetism courses.

Also, note that $\partial_\mu \phi(x)$ transforms as a four-vector with lower index (1.38) under a boost:

$$\begin{aligned} \partial_\mu \phi(x) &= \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \phi'(x') \\ &= (\Lambda^{-1}(v))^\nu_\mu \partial'_\nu \phi'(x') \end{aligned} \tag{2.46}$$

using the chain rule. Thus, the new $A_\mu(x)$ is still a four-vector.

The transformation (2.44) is called a *gauge transformation*. We will require all our Lagrangians to be gauge-invariant (up to a total time derivative). This eliminates terms like

$$\int A^2(x) d\tau \tag{2.47}$$

that we could have added to the particle Lagrangian.

Gauge invariance (in a more general form) is a fundamental principle in physics. It is the basis of all forces in the Standard Model of particle physics: electromagnetic, strong, and weak. Even gravity can be thought of as a type of gauge theory, although we will not explore that here.

2.9 Fields in motion

We have been treating the field $A_\mu(x)$ as fixed and deriving equations of motion for particles propagating on this background field. We will now treat the field as dynamic and derive its equations of motion. The dynamical variable $x^\mu(t)$ becomes a free parameter, and $A_\mu(x)$ is now the dynamical variable. The change is summarized in the simple translation table:

	Particle	Field
Dynamical variable	x	A
Free parameter	t	x

Table 2.1: Objects in particle and field theory.

The action is given by an integral over space and time:

$$S = \int \mathcal{L}(A_\mu, \partial_\nu A_\mu)\, d^4x \tag{2.48}$$

where $d^4x \equiv dx^0 dx^1 dx^2 dx^3$. We use $\mathcal{L}$ for field Lagrangians instead of L. Since Eq. (2.48) looks like Eq. (2.5) if we define $L \equiv \int dx^1 dx^2 dx^3 \mathcal{L}$, $\mathcal{L}$ is sometimes called the *Lagrangian density*, but we will not do so. $\mathcal{L}$ now depends on the field A_μ and all its partial derivatives. It cannot depend on x explicitly because of spacetime homogeneity. Note that d^4x is already Lorentz invariant, using (A.22) and $\det \Lambda = 1$ for a boost matrix Λ. Thus, $\mathcal{L}$ must be Lorentz invariant.

The action integral is now over all of spacetime. Just as the particle action was defined between times t_i and t_f, we will pretend the spacetime V has a boundary ∂V. The action principle only applies to field variations $\delta A_\mu(x)$ away from the boundary. A term in the Lagrangian $\partial_\mu f^\mu(x)$ for some vector field $f^\mu(x)$ becomes a boundary term:

$$\int_V \partial_\mu f^\mu d^4x = \int_{\partial V} f^\mu n_\mu d^3s, \tag{2.49}$$

where n_μ is an outward normal vector. This is the divergence theorem from vector calculus*, written in familiar vector notation as:

$$\int_V \nabla \cdot \vec{f}\, d^4x = \int_{\partial V} \vec{f} \cdot d^3\vec{s}. \tag{2.50}$$

This boundary term does not affect the physics, just as a total time derivative did not affect the particle Lagrangian.

Following a similar derivation as Sec. 2.4, the Euler-Lagrange equations (2.14) become:

$$\frac{\delta \mathcal{L}}{\delta A_\mu(x)} = \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta \partial_\nu A_\mu(x)} \right). \tag{2.51}$$

*Unlike Stoke's theorem, the divergence theorem holds in any number of dimensions. In this case, a 4D spacetime integral becomes a 3D boundary integral.

Note that when ∂_μ is applied to expressions involving Lagrangians, it actually means the total derivative, since Lagrangians do not depend on x^μ explicitly. For example, if the Lagrangian for a scalar field is $\mathcal{L}(\phi) = \phi^2$, then $\partial_\mu \mathcal{L} = \partial_\mu(\phi^2) = 2\phi\partial_\mu\phi$.

** Exercise 2.3

Noether's theorem for fields. Consider a Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi)$ for the scalar field $\phi(x)$. Assume it changes as $\mathcal{L} \to \mathcal{L} + \partial_\mu f^\mu(x)$ under a field transformation $\phi(x) \to \phi(x) + \delta\phi(x)$. Following Sec. 2.5, show that the *Noether current*

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi - f^\mu \tag{2.52}$$

is conserved*:

$$\partial_\mu j^\mu = 0. \tag{2.53}$$

In vector notation, this is the continuity equation:

$$\frac{dj^0}{dt} = -\nabla\cdot\vec{j}. \tag{2.54}$$

j^0 is the charge density, and $\vec{j}$ is the current density. Integrating over volume V:

$$\begin{aligned}\int_V \frac{dj^0}{dt}\,d^3x = \frac{dQ}{dt} &= -\int_V \nabla\cdot\vec{j}\,d^3x \\ &= -\int_{\partial V}\vec{j}\cdot d\vec{S} \\ &= 0.\end{aligned} \tag{2.55}$$

$Q \equiv \int_V j^0 d^3x$ is the total charge. On the second line, we use the divergence theorem. We take V large enough so that the current density vanishes at the boundary ∂V. Thus, the total charge is conserved.

2.10 The Maxwell Lagrangian

Let us try to construct a Lagrangian $\mathcal{L}_A$ for $A_\mu(x)$. Due to gauge invariance, it must be constructed from the $\vec{E}$ and $\vec{B}$ fields, but it is hard to see

*When we say a four-vector field A^μ is "conserved", we mean that $\partial_\mu A^\mu = 0$.

how these transform under Lorentz transformations using the traditional 3D vector notation. Looking back to (2.42), note how both $\vec{E}$ and $\vec{B}$ are related to the two-index quantity

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.56}$$

as

$$\begin{aligned} E_i &= F_{i0} \\ B_i &= \frac{1}{2}\epsilon_{ijk}F_{jk}. \end{aligned} \tag{2.57}$$

$F_{\mu\nu}$ is sometimes called the *field strength*. It can be written as a matrix, where μ is the row index and ν is the column index:

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{2.58}$$

Because the derivative ∂_μ transforms as a four-vector (2.46), both indices of $F_{\mu\nu}$ transform as lower indices under a boost:

$$F_{\mu\nu}(x) = (\Lambda^{-1}(v))^\rho_\mu (\Lambda^{-1}(v))^\sigma_\nu F'_{\rho\sigma}(x') \tag{2.59}$$

as you may verify. Thus, we may form the Lorentz-invariant quantity

$$\mathcal{L}_A = -\frac{\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu} = \frac{\epsilon_0}{2}\left(\vec{E}^2 - \vec{B}^2\right) \tag{2.60}$$

where indices are raised using the inverse metric (1.39): $F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}F_{\sigma\rho}$. This is the Maxwell Lagrangian. $\epsilon_0 \approx 8.854 \times 10^{-12}$ F/m is the vacuum permittivity*. Note that the simpler term $F^\mu_\mu = \eta^{\mu\nu}F_{\mu\nu}$ vanishes since $F_{\mu\nu}$ is antisymmetric†.

*Since we have a new unit (electric charge), we can add to our system of natural units by setting another constant to one. The usual choice is $4\pi\epsilon_0 = 1$. The Lagrangian becomes: $\mathcal{L}_A = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$. Since we will only briefly discuss electromagnetism, we do not do this here.

†We can also add the term $\epsilon^{\mu\nu\sigma\lambda}F_{\mu\nu}F_{\sigma\lambda}$ to the Lagrangian, where $\epsilon^{\mu\nu\sigma\lambda}$ is the totally antisymmetric symbol, with $\epsilon^{0123} = 1$. You may show that this term is Lorentz-invariant using $\det(\Lambda(v)) = 1$ and the definition of the determinant (A.11). This gives the equation of motion $\epsilon^{\mu\nu\sigma\lambda}\partial_\nu F_{\sigma\lambda} = 0$. This is trivially satisfied since $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$.

The Euler-Lagrange equations (2.51) give:

$$\begin{aligned} 0 &= \partial_\nu \left(\frac{\delta \mathcal{L}_A}{\delta \partial_\nu A_\mu(x)} \right) \\ &= -\frac{\epsilon_0}{4} \partial_\nu \left(\frac{\delta}{\delta \partial_\nu A_\mu} \left[F_{\sigma\lambda} F_{\alpha\beta} \eta^{\sigma\alpha} \eta^{\lambda\beta} \right] \right) \\ &= -\frac{\epsilon_0}{4} \partial_\nu \left(\frac{\delta}{\delta \partial_\nu A_\mu} \left[(\partial_\sigma A_\lambda - \partial_\lambda A_\sigma)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \eta^{\sigma\alpha} \eta^{\lambda\beta} \right] \right) \\ &= \epsilon_0 \partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) \\ &= \epsilon_0 \partial_\nu F^{\mu\nu}. \end{aligned} \tag{2.61}$$

The third line is evaluated using:

$$\begin{aligned} \frac{\delta}{\delta \partial_\nu A_\mu} (\partial_\sigma A_\lambda \partial_\alpha A_\beta) &= \frac{\delta \partial_\sigma A_\lambda}{\delta \partial_\nu A_\mu} \partial_\alpha A_\beta + \partial_\sigma A_\lambda \frac{\delta \partial_\alpha A_\beta}{\delta \partial_\nu A_\mu} \\ &= \delta^\nu_\sigma \delta^\mu_\lambda \partial_\alpha A_\beta + \delta^\nu_\alpha \delta^\mu_\beta \partial_\sigma A_\lambda. \end{aligned} \tag{2.62}$$

Using (2.57), you may show that (2.61) is equivalent to the other two Maxwell equations:

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 \\ \nabla \times \vec{B} &= \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{2.63}$$

in the absence of sources.

Maxwell's equations imply the speed of light is a constant. To see this, take a plane wave:

$$A_\mu = A_{0\mu} \sin(k_\mu x^\mu) \tag{2.64}$$

for some constant k_μ and $A_{0\mu}$. $k^0 = \omega$ is the frequency and $\vec{k}$ is the wavevector. This A_μ satisfies the equation of motion (2.61) if

$$k^2 = k_\mu k^\mu = -\omega^2 + \vec{k}^2 = 0, \tag{2.65}$$

$$k_\mu A^\mu_0 = 0. \tag{2.66}$$

In the wave description of light, the speed of light is the *phase velocity*: how fast the peaks and troughs of the wave propagate. For a plane wave, this is given by $v_p = \omega/|\vec{k}|$. Eq. (2.65) implies the phase velocity is constant: $v_p = 1$. This holds in all IRFs since Maxwell's equations come from a Lorentz invariant Lagrangian.

Finally, we may write the Lorentz force law (2.39) in a clearly Lorentz covariant way using $F_{\mu\nu}$:

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu, \tag{2.67}$$

as you may verify.

2.11 Charges and currents

We may easily incorporate a four-vector source $J^\mu(x)$ by adding a source term $J^\mu(x)A_\mu(x)$ to the Lagrangian:

$$\mathcal{L}_{\text{EM}} = \mathcal{L}_A + J^\mu A_\mu = -\frac{\epsilon_0}{2}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu. \tag{2.68}$$

$\rho = J^0$ is the electric charge density and $\vec{J}$ is the current density. For example,

$$J^\mu_{pp} = q\delta^3(\vec{x} - \vec{x}_p(t)) \begin{pmatrix} 1 \\ \frac{d\vec{x}_p}{dt}(t) \end{pmatrix} \tag{2.69}$$

for a point charge q moving along the path $\vec{x}_p(t)$. Indeed, you can check that $\int J^\mu_{pp}A_\mu\, d^4x$ gives the action for a point charge $S_A\{\vec{x}_p\}$ (2.36), upon doing the spatial integral over $d^3x \equiv dx^1dx^2dx^3$.

The Euler-Lagrange equations give:

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu}{\epsilon_0}. \tag{2.70}$$

This is equivalent to Maxwell's equations with sources:

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \vec{B} &= \frac{\vec{J}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{2.71}$$

Any source that obeys the equation of motion (2.70) also satisfies:

$$\partial_\mu J^\mu = \epsilon_0 \partial_\mu \partial_\nu F^{\mu\nu} = 0 \tag{2.72}$$

due to the antisymmetry of $F^{\mu\nu}$. This means the electric charge is conserved (2.55).

Chapter 3

The geometry of spacetime

We now move from flat space to curved space. This involves first developing the machinery of differential geometry on manifolds. Unfortunately, the usual treatment using abstract manifolds is quite unintuitive. We will instead pretend d-dimensional curved spacetime is embedded in D-dimensional flat space, called the *ambient space*. Here, $d < D$, and we are most interested in $d = 4$, not caring much what D is[*].

3.1 Submanifolds of flat space

We use y^I for the coordinates in the ambient space, $0 \leq I \leq D-1$, and x^μ for the coordinates in spacetime, $0 \leq \mu \leq d-1$. As before, y^0 is the time coordinate and $y^1, \cdots, y^{D-1}$ are the spatial coordinates. Also, $x = (x^0, \cdots, x^{d-1})^T$ is the coordinate vector, and $\partial_\mu \equiv \partial/\partial x^\mu$ is the partial derivative.

Spacetime is defined by the D functions f^I:

$$y^I = f^I(x). \tag{3.1}$$

This is called a *submanifold* of the ambient Minkowski space.

[*]There is a more abstract definition of spacetime that does not involve embedding it into an ambient space. Such an object is called a *manifold*. We will not discuss this, and instead will use "submanifold" and "manifold" interchangeably. Every curved spacetime can in fact be embedded in higher-dimensional flat space, so we lose no generality here. For some interesting discussion on the dimension D needed, see here.

Note that the x^μ here do not necessarily have an interpretation as space or time, but are simply used to parametrize the manifold, just like how spherical coordinates (θ, ϕ) parametrize the sphere embedded in 3D space (Ex. 3.2). This is unlike the x^μ in the previous chapter, which correspond to physical times and lengths measured in an IRF.

At each point x in spacetime, there is a vector space spanned by the d tangent vectors

$$e^I_{(\mu)} \equiv \frac{\partial f^I}{\partial x^\mu}, \tag{3.2}$$

called the *tangent space* at x (Fig. 3.1).

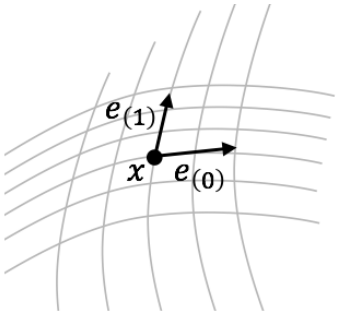


Figure 3.1: Tangent space at x for a 2D submanifold. Side note: you don't need to visualize 4D spacetime to understand relativity. When I think of a manifold, I just have something like this image in mind.

Now reparametrize spacetime using new coordinates x'^μ, so that $y^I = f'^I(x') \equiv f^I(x(x'))$. The basis vectors become:

$$\begin{aligned} e'^I_{(\mu)} &= \frac{\partial f'^I}{\partial x'^\mu}(x') \\ &= \frac{\partial f^I}{\partial x^\nu}(x(x'))\frac{\partial x^\nu}{\partial x'^\mu}(x') \\ &= e^I_{(\nu)}\frac{\partial x^\nu}{\partial x'^\mu}(x') \end{aligned} \tag{3.3}$$

using the chain rule. We show the function arguments for clarity. Since what we call "new" and "old" coordinates is arbitrary, we also have:

$$e^I_{(\mu)} = e'^I_{(\nu)}\frac{\partial x'^\nu}{\partial x^\mu}(x). \tag{3.4}$$

A given tangent vector V can be written as a linear combination of basis vectors:

$$V^I = v^\mu e^I_{(\mu)} \tag{3.5}$$

where v^μ are the components of the vector. Because this tangent vector exists in the ambient space, it does not depend on the parametrization of the submanifold:

$$\begin{aligned} V^I = v^\mu e^I_{(\mu)} &= v'^\mu e'^I_{(\mu)} \\ &= v'^\mu \frac{\partial x^\nu}{\partial x'^\mu} e^I_{(\nu)}, \end{aligned} \tag{3.6}$$

using (3.3). Comparing the components on both sides, we obtain:

$$v^\mu = v'^\nu \frac{\partial x^\mu}{\partial x'^\nu} \tag{3.7}$$

upon relabeling indices. Any object with one index that transforms as (3.7) under a reparametrization is called a *contravariant vector*, or simply *vector*[*]. It is called contravariant because it transforms oppositely to the basis vectors (3.4). An example of a vector is a coordinate displacement dx^μ. The corresponding tangent vector is simply a displacement in the ambient space:

$$dy^I = dx^\mu \frac{\partial f^I}{\partial x^\mu}. \tag{3.8}$$

Conversely, any object that transforms as

$$v_\mu = v'_\nu \frac{\partial x'^\nu}{\partial x^\mu} \tag{3.9}$$

is called a *covariant vector*, or *covector*, since it transforms in the same way as the basis vectors. An easy way to remember the transformation properties (3.7) and (3.9) is that indices are always summed top with bottom, and primed with primed. (A ∂x^μ in the denominator acts as a bottom index.) An example of a covector is the gradient $\partial_\mu \phi(x)$ of any function $\phi(x)$ defined on the submanifold[†]. It transforms as:

$$\partial_\mu \phi(x) = \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \phi(x'), \tag{3.10}$$

[*]The tangent vector V in the ambient space is also called a "vector". We will always capitalize such vectors to avoid confusion.

[†]Such as the embedding functions themselves $f^I(x)$. However, we do not call the basis vectors themselves covectors, hence the parentheses around the index $e^I_{(\mu)}$. Also, some texts define the basis vectors more abstractly as the partial derivative operators $\partial/\partial x^\mu$. There is no particular advantage to doing so here.

using the chain rule.

The transformation laws for vectors and covectors (3.7) and (3.9) generalize those of four-vectors (1.33) and (1.37) under boosts. However, they mean slightly different things. As mentioned above, the coordinates x^μ in Chapter 1 correspond to physically measured times and distances in an IRF, so Eqs. (1.33) and (1.37) relate physical coordinates. On the other hand, the x^μ here in general have no physical significance, so Eqs. (3.7) and (3.9) are simply mathematical statements of how vector components transform under a change of coordinates.

Finally, let us emphasize that each point x has its *own* tangent space. If we add vectors at two different points x and z, the result will *not* transform as a vector:

$$v^\mu(x) + w^\mu(z) = v'^\nu(x')\frac{\partial x^\mu}{\partial x'^\nu}(x') + w'^\nu(z')\frac{\partial x^\mu}{\partial x'^\nu}(z'). \tag{3.11}$$

We use (3.7) with the arguments restored. Since $\frac{\partial x^\mu}{\partial x'^\nu}(x') \neq \frac{\partial x^\mu}{\partial x'^\nu}(z')$, we cannot factor it out.

3.2 The metric

We may write the interval ds^2 (1.34) in terms of displacements dx^μ on the spacetime manifold:

$$\begin{aligned} ds^2 &= \eta_{IJ}dy^I dy^J \\ &= \eta_{IJ}\frac{\partial f^I}{\partial x^\mu}\frac{\partial f^J}{\partial x^\nu}dx^\mu dx^\nu \\ &= g_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \tag{3.12}$$

using (3.8). We have defined the *metric tensor*, or simply *metric*:

$$g_{\mu\nu}(x) \equiv \eta_{IJ}\frac{\partial f^I}{\partial x^\mu}(x)\frac{\partial f^J}{\partial x^\nu}(x). \tag{3.13}$$

It is a symmetric $d \times d$ matrix. Since it is made of two basis vectors, both indices transform covariantly under reparametrization:

$$g_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g'_{\rho\sigma}. \tag{3.14}$$

In matrix notation:

$$\bar{g} = J^T\bar{g}'J, \tag{3.15}$$

where $J_{\sigma\nu} = \partial x'^{\sigma}/\partial x^{\nu}$ is the Jacobian. As usual, the first (second) index is the row (column) index. We use $\bar{g}$ for the matrix since g is typically used for the determinant of $\bar{g}$:

$$g \equiv \det \bar{g}. \tag{3.16}$$

We assume that the manifold is embedded such that one eigenvalue of $\bar{g}$ is negative and three are positive, just like the Minkowski metric $\eta_{\mu\nu}$. The respective eigenvectors correspond to timelike ($ds^2 < 0$) and spacelike ($ds^2 > 0$) directions. For an eigenvector dx with eigenvalue λ, Eq. (3.12) in matrix form reads: $ds^2 = dx^T \bar{g} dx = \lambda dx^T dx$, which has the same sign as λ.

The number of positive and negative eigenvalues is called the *signature* of the metric. For a manifold in Minkowski space, it is denoted $(-,+,+,+)$. It is invariant under a change of coordinates. To see this, fix $\bar{g}'$ and start from the identity transformation, $J = I$. As J is continuously deformed, the eigenvalues of $\bar{g}$ change continuously. The metric must stay full rank ($g \neq 0$), so the eigenvalues cannot cross zero*.

We can always choose coordinates to make the metric Minkowski at any given point x (so-called *inertial coordinates* at x). This is done in two steps. First, change coordinates so the metric is diagonal. Then, change coordinates to scale the diagonal entries to $\eta_{\mu\nu}$. From (A.8), we can write:

$$\bar{g}(x) = X(x)\Lambda(x)X^{-1}(x), \tag{3.17}$$

where $X(x)$ is the matrix whose columns are the eigenvectors of $\bar{g}(x)$ (starting with the timelike one), and $\Lambda(x)$ is the diagonal matrix of eigenvalues. We will omit the argument x from now on. Since $\bar{g}$ is symmetric, we have:

$$X^T = X^{-1}, \tag{3.18}$$

so X defines a coordinate change with Jacobian $J = X^T$ and $\bar{g}' = \Lambda$. Then we can simply apply the scaling† $K = \mathrm{diag}\left(|\lambda_0|^{1/2}, \cdots, |\lambda_{d-1}|^{1/2}\right)$:

$$\Lambda = K^T \eta K, \tag{3.19}$$

so the overall transformation is:

$$\bar{g} = J^T \eta J, \tag{3.20}$$

*Parity-reversing transformations like $x^1 = -x'^1$ cannot be obtained by a continuous deformation of the coordinates, but they also do not change the signature.

†Note that the first transformation is a change of basis, while the second is not, since $K^{-1} \neq K^T$.

where the Jacobian $J = KX^T$.

You may think we can repeat this procedure at every point and make the metric Minkowski over all of spacetime. This is not possible: a good thing indeed, since then curved space would be the same as flat space. This is because we cannot always integrate the Jacobian $J_{\mu\nu}(x) = \partial x'^{\mu}/\partial x^{\nu}(x)$ to get the full coordinate change $x'(x)$. However, it is possible to choose coordinates so that the metric is Minkowski within the neighborhood of a worldline, as explained in Sec. 4.3.

Similar to the Minkowski metric in flat space, we can lower indices using the metric. Given a vector v^{μ}, the quantity

$$v_{\mu} \equiv g_{\mu\nu} v^{\nu} \tag{3.21}$$

transforms as a covector. Define the *inverse metric* $g^{\mu\nu}$ as the matrix inverse of the metric, so that

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{3.22}$$

This must stay the matrix inverse under reparametrization, so both indices transform contravariantly:

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} g'^{\rho\sigma} \tag{3.23}$$

The inverse metric can be used to raise indices. Given a covector v_{μ}, the quantity

$$v^{\mu} \equiv g^{\mu\nu} v_{\nu} \tag{3.24}$$

transforms as a vector.

* Exercise 3.1

Derive (3.23) using $g^{\mu\nu} g_{\nu\rho} = g'^{\mu\nu} g'_{\nu\rho} = \delta^{\mu}_{\rho}$. Use the identity

$$\frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} \tag{3.25}$$

coming from the chain rule.

You can think of the metric $g_{\mu\nu}$ as the dynamical field of spacetime, like the vector field A_{μ} is the field of electromagnetism*. Electric charges and

*Actually, when gravity is developed as a gauge theory, A_{μ} is analogous to the Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$, and $F_{\mu\nu}$ is analogous to the curvature tensor $R_{\mu\nu\sigma\lambda}$. However, for most purposes, $g_{\mu\nu}$ is more similar to A_{μ} since we vary $g_{\mu\nu}$ in the field Lagrangian.

currents produce electromagnetic fields, while mass and energy produce a curved metric.

3.3 A Euclidean analogy

To understand vectors and covectors more intuitively, let's pretend we are embedding spacetime in Euclidean space instead of Minkowski space. As mentioned in Sec. 1.14, Minkowski space is just like Euclidean space with an imaginary coordinate. The Minkowski metric η_{IJ} becomes simply δ_{IJ}, the Kronecker delta function:

$$\delta_{IJ} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases} \tag{3.26}$$

and the interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ measures distance squared on the manifold. The metric $g_{\mu\nu}$ is then the dot product of two basis vectors:

$$g_{\mu\nu} = \delta_{IJ}e^I_{(\mu)}e^J_{(\nu)} = e_{(\mu)} \cdot e_{(\nu)}. \tag{3.27}$$

It has signature $(+,+,+,+)$. The covariant components of a vector v_μ are:

$$v_\mu = g_{\mu\nu}v^\nu = e_{(\mu)} \cdot e_{(\nu)}v^\nu = e_{(\mu)} \cdot V \tag{3.28}$$

using (3.5). Thus, covariant components are dot products with the basis vectors, while contravariant components are the weights of the basis vectors (3.5). The two types of components are identical in orthonormal bases, but they differ in skew bases (Fig. 3.2) or with basis vectors of non-unit length.

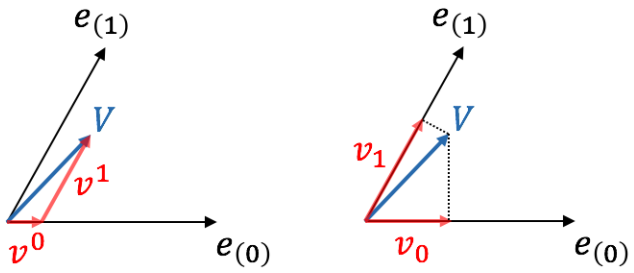


Figure 3.2: Contravariant (left) and covariant (right) components of vector V, where $|e_{(0)}| = |e_{(1)}| = 1$.

Note that covariant components naturally exist even for vectors V outside of the tangent space, using the same formula (3.28). On the other hand, contravariant components depend on the choice of basis vectors $\{e_{(d)}, \cdots, e_{(D-1)}\}$ outside of the tangent space. For example, in Fig. 3.2, let's say $e_{(0)}$ is the basis vector of a 1D tangent space with 2D ambient space. Then v_0 does not depend on $e_{(1)}$, but v^0 does. Going back to ambient Minkowski space, (3.28) becomes

$$v_\mu = \eta_{IJ} \frac{\partial f^I}{\partial x^\mu}(x) V^J \tag{3.29}$$

for *any* vector V at the point x. This will be crucial when we discuss parallel transport and the covariant derivative.

*** Exercise 3.2**

3D Euclidean space with coordinates (x^0, x^1, x^2) can be reparametrized with spherical coordinates (r, θ, ϕ) as:

$$\begin{aligned} x^0 &= f^0(r,\theta,\phi) = r\sin\theta\cos\phi \\ x^1 &= f^1(r,\theta,\phi) = r\sin\theta\sin\phi \\ x^2 &= f^2(r,\theta,\phi) = r\cos\theta. \end{aligned} \tag{3.30}$$

Show that the metric is*:

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.31}$$

By fixing r so that $dr = 0$, we also get the metric for a sphere of radius r parametrized by (θ, ϕ):

$$ds^2_{\text{sphere}} = r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.32}$$

3.4 General tensors

Now that we have seen 1-index objects (vectors and covectors) and 2-index objects (the metric and inverse metric), it is natural to construct objects

*We will often give the metric by writing out (3.12) explicitly like this. You can read out the metric components as the coefficients of the $dx^\mu dx^\nu$. For example, $g_{\theta\theta} = r^2$.

with any number of indices, where each upper index transforms as a vector, and each lower index transforms as a covector. These are called *tensors*. Specifically, a (p, q)-tensor $T^{\mu\nu\cdots}{}_{\sigma\rho\cdots}$ is an object with p upper indices and q lower indices that transforms as:

$$T^{\mu\nu\cdots}{}_{\sigma\rho\cdots} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\cdots\frac{\partial x'^\gamma}{\partial x^\sigma}\frac{\partial x'^\delta}{\partial x^\rho}\cdots T'^{\alpha\beta\cdots}{}_{\gamma\delta\cdots} \tag{3.33}$$

under reparametrizations. The *rank* r of the tensor is $p + q$. It is called *contravariant* if it has only upper indices, *covariant* if it has only lower indices, and *mixed* otherwise. A rank-0 tensor has no indices and is called a *scalar*. An example is the interval ds^2. Scalars are like the Lorentz invariants of Chapter 1, but invariant under any reparametrizations.

The easiest way to form new tensors is to place them next to each other. For example, we can define the tensor $U_\mu{}^\nu \equiv V_\mu W^\nu$, where V_μ is a covector and W^ν is a vector. It transforms as:

$$U_\mu{}^\nu = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\sigma}U'^\sigma_\rho{}. \tag{3.34}$$

Given a tensor with upper and lower indices, we can form a new tensor by summing over one pair, e.g. $A_\mu \equiv S_{\mu\nu}{}^\nu$. This is called index *contraction*. Since upper and lower indices transform oppositely, the result of contraction still transforms as a tensor with two fewer indices*. For example, A_μ transforms as a covector:

$$\begin{aligned} S_{\mu\nu}{}^\nu &= \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\lambda}S'_{\sigma\rho}{}^\lambda \\ &= \frac{\partial x'^\sigma}{\partial x^\mu}\delta^\rho_\lambda S'_{\sigma\rho}{}^\lambda \\ &= \frac{\partial x'^\sigma}{\partial x^\mu}S'_{\sigma\rho}{}^\rho, \end{aligned} \tag{3.35}$$

using (3.25) on the second line.

We can also form new tensors by raising and lowering indices with the metric, e.g. $T^\mu{}_\nu \equiv g^{\mu\rho}T_{\rho\nu}$. You can think of this as placing two tensors next to each other, then contracting: $g^{\mu\rho}T_{\sigma\nu} \to g^{\mu\rho}T_{\rho\nu}$. Of course, we could do

*If at any time it feels like you are drowning in indices, just relax and let it happen. It is a rite of passage for any relativist. You will get used to it after some practice. Using Latin instead of Greek indices would probably smooth the process, but unfortunately Greek indices are mostly standard.

this with any rank-2 tensor $g^{\mu\rho}$, but when it is the metric, the new tensor is denoted by the same symbol T. This means that the order of the indices matters: note the μ is written to the left of ν in $T^{\mu}{}_{\nu}$. If we simply wrote T^{μ}_{ν}, we don't know whether the first or second index was raised: $g^{\mu\rho}T_{\rho\nu}$ or $g^{\mu\rho}T_{\nu\rho}$.

Since the coordinate system has no physical significance, any laws of physics must hold in all coordinate systems. This property is called *general covariance*. It is a simple but important aspect of relativity. Here we see where tensors shine. Any tensor equation is generally covariant as long as the upper and lower indices match on both sides, since both sides transform in the same way under a change of coordinates.

Tensors can be confusing since they are not as easily visualized as vectors, although rank-2 tensors are often helpfully written as matrices, like we did with the metric. The best way to understand tensors is to work with various examples, as we will do.

3.5 Parallel transport

When moving a tangent vector V from x to $x + \delta x$, it may not stay in the tangent space of $x+\delta x$, due to the curvature of the manifold (Fig. 3.3). We would like to define a notion of transporting a vector from point to point that makes it stay as "parallel" as possible while remaining in the tangent space. This is done by simply projecting the vector V at $x + \delta x$ onto the tangent space so it becomes $\bar{V}$, with covariant components $\bar{v}_\mu$.

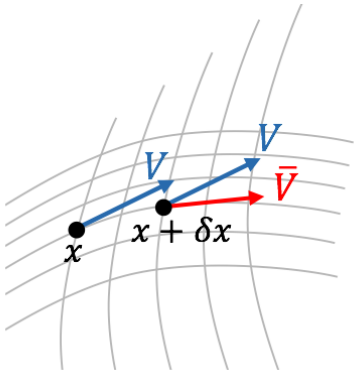


Figure 3.3: The tangent vector V at x does not stay in the tangent space when transported to $x + \delta x$. We can project it onto the tangent space to get $\bar{V}$.

We have:

$$\begin{aligned}
\bar{v}_\mu &= \eta_{IJ}\frac{\partial f^I}{\partial x^\mu}(x+\delta x)V^J \\
&= \eta_{IJ}\frac{\partial f^I}{\partial x^\mu}(x+\delta x)v^\nu\frac{\partial f^J}{\partial x^\nu}(x) \\
&= \eta_{IJ}\left(\frac{\partial f^I}{\partial x^\mu}(x)+\frac{\partial^2 f^I}{\partial x^\mu \partial x^\sigma}(x)\delta x^\sigma\right)v^\nu\frac{\partial f^J}{\partial x^\nu}(x) \\
&= v_\mu + \Gamma_{\nu,\mu\sigma}(x)v^\nu\delta x^\sigma.
\end{aligned} \tag{3.36}$$

On the first line, we use (3.29). On the third line, we expand $\frac{\partial f^I}{\partial x^\mu}(x+\delta x)$ to first order in δx. On the last line, we define the *Christoffel symbols of the first kind*:

$$\Gamma_{\nu,\mu\sigma}(x) \equiv \eta_{IJ}\frac{\partial^2 f^I}{\partial x^\mu \partial x^\sigma}(x)\frac{\partial f^J}{\partial x^\nu}(x). \tag{3.37}$$

Note that it is symmetric in the last two indices.

Thus, the change in the covariant vector v_μ as it is parallel transported from x to $x+\delta x$ is:

$$\delta v_\mu = \bar{v}_\mu - v_\mu = \Gamma_{\nu,\mu\sigma}(x)v^\nu\delta x^\sigma. \tag{3.38}$$

We may lower the index of v^ν so the formula is purely in terms of covariant components:

$$\delta v_\mu = \Gamma^\nu_{\mu\sigma}(x)v_\nu\delta x^\sigma \tag{3.39}$$

where the *Christoffel symbols of the second kind* are:

$$\Gamma^{\nu}_{\mu\sigma}(x) \equiv g^{\nu\rho}\Gamma_{\rho,\mu\sigma} = g^{\nu\rho}\eta_{IJ}\frac{\partial^2 f^I}{\partial x^\mu \partial x^\sigma}(x)\frac{\partial f^J}{\partial x^\rho}(x). \tag{3.40}$$

The Christoffel symbols are related to the partial derivatives of the metric:

$$\begin{aligned}\partial_\sigma g_{\mu\nu} &= \eta_{IJ}\left(\frac{\partial^2 f^I}{\partial x^\sigma \partial x^\mu}\frac{\partial f^J}{\partial x^\nu} + \frac{\partial f^I}{\partial x^\mu}\frac{\partial^2 f^J}{\partial x^\sigma \partial x^\nu}\right) \\ &= \Gamma_{\nu,\mu\sigma} + \Gamma_{\mu,\nu\sigma}.\end{aligned} \tag{3.41}$$

We would like to invert this formula to express $\Gamma_{\mu,\nu\sigma}$ in terms of the metric. Take all permutations of the indices $(\mu\nu\sigma)$ to get three equations total (since $g_{\mu\nu}$ is symmetric). Now add two of these equations and subtract the other, to get*:

$$\Gamma_{\mu,\nu\sigma} = \frac{1}{2}\left(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\nu\sigma}\right). \tag{3.42}$$

** Exercise 3.3

1. Derive the parallel transport equation for contravariant vectors:

$$\delta v^\mu = -\Gamma^\mu_{\nu\sigma} v^\nu \delta x^\sigma. \tag{3.43}$$

   Hint: expand $\bar{v}^\nu$ to first order in δx:

$$\bar{v}^\nu = v^\nu + C^\nu_{\mu\sigma} v^\mu \delta x^\sigma \tag{3.44}$$

   for some quantity $C^\nu_{\mu\sigma}$. Then use $\bar{v}_\mu = g_{\mu\nu}(x+\delta x)\bar{v}^\nu$ with (3.39) and (3.41), and expand to first order in δx, to show that $C^\nu_{\mu\sigma} = -\Gamma^\nu_{\mu\sigma}$.

2. Show that this implies the product $a_\mu(x)b^\mu(x)$ is constant as vectors a and b are parallel transported.

Finally, we can also parallel transport tensors, since they just act like products of vectors. For example, the tensor $T_{\mu\nu} \equiv v_\mu w_\nu$ becomes:

$$\begin{aligned}\bar{T}_{\mu\nu} &= \bar{v}_\mu \bar{w}_\nu \\ &= v_\mu w_\nu + \Gamma^\rho_{\mu\sigma} v_\rho w_\nu \delta x^\sigma + \Gamma^\rho_{\nu\sigma} v_\mu w_\rho \delta x^\sigma \\ &= T_{\mu\nu} + \Gamma^\rho_{\mu\sigma} T_{\rho\nu}\delta x^\sigma + \Gamma^\rho_{\nu\sigma} T_{\mu\rho}\delta x^\sigma,\end{aligned} \tag{3.45}$$

*This equation comes up often and is worth memorizing. I remember it as adding all permutations $(\mu\nu\sigma)$ of $\partial_\sigma g_{\mu\nu}$, with the $\nu \leftrightarrow \sigma$ symmetric term $\partial_\mu g_{\nu\sigma}$ having a minus sign.

Any rank-2 tensor $A_{\mu\nu}$ can be written as a sum of such vector products, so (3.45) holds for any rank-2 tensor since it is linear in T. The generalization to contravariant indices and tensors of any rank is obvious: add a term like (3.39) for each lower index and (3.43) for each upper index.

3.6 Covariant derivative

Note that $\delta v_\mu = \bar{v}_\mu(x+\delta x) - v_\mu(x)$ does not transform as a covector, since it involves subtracting covectors at two different points. Indeed, you can verify that the Christoffel symbols $\Gamma_{\mu,\nu\sigma}$ do *not* transform as a tensor, using (3.42) and (3.14).

Now instead of a covector at a single point x, consider a covector field $v_\mu(x)$. Expand it at a point $x+\delta x$ as:

$$v_\mu(x+\delta x) = v_\mu(x) + \partial_\sigma v_\mu(x)\delta x^\sigma. \tag{3.46}$$

The directional derivative $\partial_\sigma v_\mu(x)$ also does not transform as a tensor, since $\partial_\sigma v_\mu(x)\delta x^\sigma = v_\mu(x+\delta x) - v_\mu(x)$ involves subtracting covectors at two different points. This inspires us to write:

$$\begin{aligned} v_\mu(x+\delta x) &= v_\mu(x) + \partial_\sigma v_\mu(x)\delta x^\sigma \\ &= \bar{v}_\mu(x+\delta x) - \Gamma^\nu_{\mu\sigma}(x)v_\nu(x)\delta x^\sigma + \partial_\sigma v_\mu(x)\delta x^\sigma \\ &= \bar{v}_\mu(x+\delta x) + (\nabla_\sigma v_\mu)(x)\delta x^\sigma \end{aligned} \tag{3.47}$$

using (3.39). On the last line, we define the *covariant derivative*

$$\nabla_\sigma v_\mu \equiv \partial_\sigma v_\mu - \Gamma^\nu_{\sigma\mu} v_\nu. \tag{3.48}$$

Since $\nabla_\sigma v_\mu \delta x^\sigma$ involves subtracting two covectors at the same point $x+\delta x$, $\nabla_\sigma v_\mu$ is manifestly a rank-2 tensor, which you should verify explicitly.

Going through the same procedure for a vector field $v^\mu(x)$, we have

$$\nabla_\sigma v^\mu \equiv \partial_\sigma v^\mu + \Gamma^\mu_{\sigma\nu} v^\nu. \tag{3.49}$$

as the covariant derivative of a vector field*.

*I remember these equations as: the Christoffel symbol $\Gamma^\mu_{\sigma\nu}$ "steals" the index from the (co)vector and contracts with it. It comes in with a positive sign for "usual" vectors and a negative sign for "unusual" covectors.

Another way to think of the covariant derivative is as the projection of the directional derivative $\partial_\nu V^I(x)$ onto the tangent space. We have:

$$\begin{aligned}\partial_\nu V^I &= \partial_\nu \left(v^\mu \frac{\partial f^I}{\partial x^\mu} \right) \\ &= \partial_\nu v^\mu \frac{\partial f^I}{\partial x^\mu} + v^\mu \frac{\partial^2 f^I}{\partial x^\mu \partial x^\nu}.\end{aligned} \tag{3.50}$$

The covariant components of this (non-tangent) vector are (3.29):

$$\begin{aligned}(\nabla_\nu v)_\sigma &\equiv \eta_{IJ} \frac{\partial f^I}{\partial x^\sigma} \left(\partial_\nu v^\mu \frac{\partial f^J}{\partial x^\mu} + v^\mu \frac{\partial^2 f^J}{\partial x^\mu \partial x^\nu} \right) \\ &= g_{\sigma\mu} \partial_\nu v^\mu + \Gamma_{\sigma,\mu\nu} v^\mu.\end{aligned} \tag{3.51}$$

Raising the index σ gives:

$$(\nabla_\nu v)^\sigma = \partial_\nu v^\sigma + \Gamma^\sigma_{\mu\nu} v^\mu, \tag{3.52}$$

which is the same as (3.49).

The covariant derivative of a scalar field $\phi(x)$ is defined as the ordinary directional derivative:

$$\nabla_\mu \phi \equiv \partial_\mu \phi, \tag{3.53}$$

since this is already a covector.

Finally, just as we can parallel transport a tensor, we can take the covariant derivative of a general tensor by contracting $\Gamma^\mu_{\nu\sigma}$ with each index as appropriate. For example,

$$\nabla_\sigma T^\mu{}_\nu = \partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\sigma\rho} T^\rho{}_\nu - \Gamma^\rho_{\sigma\nu} T^\mu{}_\rho. \tag{3.54}$$

This implies that the covariant derivative follows the same product rule as the ordinary derivative:

$$\nabla(TU) = (\nabla T)U + T(\nabla U), \tag{3.55}$$

for tensors T, U, suppressing the indices.

* Exercise 3.4

Show that the metric is covariantly constant:

$$\nabla_\sigma g_{\mu\nu} = 0. \tag{3.56}$$

This is because $g_{\mu\nu}$ can be simply viewed as the constant Minkowski metric η^{IJ} projected onto the tangent space.

3.7 Curvature: Riemann and Ricci

How do we measure the curvature of a manifold? We know that parallel transport is affected by curvature, since a transported vector can lie outside the tangent space when the manifold is curved. Thus, you may think the Christoffel symbols can measure curvature. However, as we saw above, the Christoffel symbols are not tensors. We would like some tensor quantity that measures curvature and only depends on the metric and its derivatives. One idea is to parallel transport a vector along two different paths to the same destination. The final vector depends on the path taken, when the manifold is curved (Fig. 3.4). We can take the difference of the two final vectors as a measure of curvature.

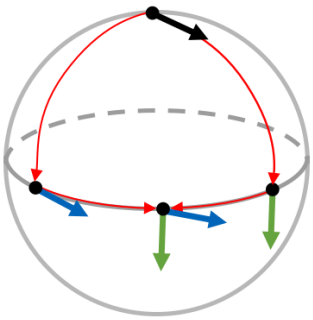


Figure 3.4: Parallel transport of a vector (black) along two different paths can give two different vectors (blue, green) when the manifold is curved, like a sphere.

Mathematically, it is easiest to work with infinitesimal paths. Let v^μ be the vector at point x, and transport it along the two paths $x \to x + \delta x_1 \to x + \delta x_1 + \delta x_2$ and $x \to x + \delta x_2 \to x + \delta x_2 + \delta x_1$. Call the first vector v_{12} and the second v_{21} (Fig. 3.5).

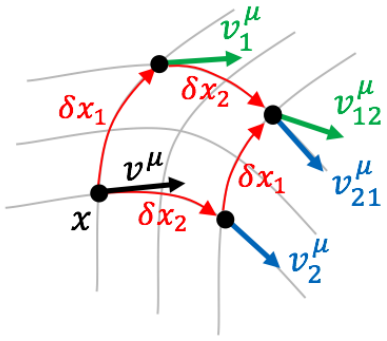


Figure 3.5: Vector v^μ at x transported along two infinitesimal paths to $x+\delta x_1+\delta x_2$.

Before evaluating this difference, let us just state the result: it is given by the *commutator* of two covariant derivatives:

$$v_{21}^\mu - v_{12}^\mu = [\nabla_\nu, \nabla_\sigma] v^\mu(x) \delta x_1^\nu \delta x_2^\sigma \equiv (\nabla_\nu \nabla_\sigma - \nabla_\sigma \nabla_\nu) v^\mu(x) \delta x_1^\nu \delta x_2^\sigma. \tag{3.57}$$

We have:

$$\begin{aligned} \nabla_\nu \nabla_\sigma v^\mu &= \nabla_\nu (\partial_\sigma v^\mu + \Gamma^\mu_{\sigma\lambda} v^\lambda) \\ &= \partial_\nu \partial_\sigma v^\mu + \partial_\nu \Gamma^\mu_{\sigma\lambda} v^\lambda + \Gamma^\mu_{\sigma\lambda} \partial_\nu v^\lambda \\ &\quad - \Gamma^\lambda_{\nu\sigma} \partial_\lambda v^\mu - \Gamma^\rho_{\nu\sigma} \Gamma^\mu_{\rho\lambda} v^\lambda + \Gamma^\mu_{\nu\lambda} \partial_\sigma v^\lambda + \Gamma^\mu_{\nu\rho} \Gamma^\rho_{\sigma\lambda} v^\lambda, \end{aligned} \tag{3.58}$$

using the definition of covariant derivative for vectors (3.49) and 2-tensors (3.54). Now exchange the indices $\nu \leftrightarrow \sigma$ and subtract, to get:

$$[\nabla_\nu, \nabla_\sigma]\, v^\mu = R^\mu{}_{\lambda\nu\sigma} v^\lambda \delta x_1^\nu \delta x_2^\sigma, \tag{3.59}$$

where we define the *Riemann curvature tensor*

$$R^\mu{}_{\lambda\nu\sigma} \equiv \partial_\nu \Gamma^\mu_{\lambda\sigma} - \partial_\sigma \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\rho\nu} \Gamma^\rho_{\lambda\sigma} - \Gamma^\mu_{\rho\sigma} \Gamma^\rho_{\lambda\nu}. \tag{3.60}$$

Note that it is antisymmetric in its last two indices. It also satisfies the *first Bianchi identity*:

$$R^\mu{}_{\rho\lambda\nu} + R^\mu{}_{\lambda\nu\rho} + R^\mu{}_{\nu\rho\lambda} = 0, \tag{3.61}$$

which is easily verified from (3.60).

Let us show that this commutator indeed gives the difference between the two transported vectors. Recall that the covariant derivative at x involves the difference of a vector field at $x + \delta x$ and a vector transported from x to $x + \delta x$ (3.47). Write this as:

$$(\nabla_\nu w^\mu)_x \delta x_1^\nu = w^\mu_{x+\delta x_1} - w^\mu_{x;\delta x_1} \tag{3.62}$$

for a contravariant vector w^μ. The first subscript on a vector denotes where it starts, and the following subscripts denote parallel transports, e.g. $w^\mu_{x;\delta x_1}$ means $w^\mu(x)$ transported to $x+\delta x_1$. Now let w^μ itself be a covariant derivative contracted with a displacement δx_2:

$$w^\mu_x \equiv (\nabla_\sigma v^\mu)_x \delta x_2^\sigma = v^\mu_{x+\delta x_2} - v^\mu_{x;\delta x_2}. \tag{3.63}$$

We have:

$$\begin{aligned} w^\mu_{x+\delta x_1} &= v^\mu_{x+\delta x_1+\delta x_2} - v^\mu_{x+\delta x_1;\delta x_2} \\ w^\mu_{x;\delta x_1} &= v^\mu_{x+\delta x_2;\delta x_1} - v^\mu_{x;\delta x_2;\delta x_1}. \end{aligned} \tag{3.64}$$

Plug this into (3.62) to get:

$$(\nabla_\nu \nabla_\sigma v^\mu)_x \delta x_1^\nu \delta x_2^\sigma = v^\mu_{x+\delta x_1+\delta x_2} - v^\mu_{x+\delta x_1;\delta x_2} - v^\mu_{x+\delta x_2;\delta x_1} + v^\mu_{x;\delta x_2;\delta x_1}. \tag{3.65}$$

Exchanging the indices $\nu \leftrightarrow \sigma$ is the same as exchanging $\delta x_1 \leftrightarrow \delta x_2$. Exchanging and subtracting, we get:

$$\begin{aligned} [\nabla_\nu, \nabla_\sigma]\, v^\mu(x) \delta x_1^\nu \delta x_2^\sigma &= v^\mu_{x;\delta x_2;\delta x_1} - v^\mu_{x;\delta x_1;\delta x_2} \\ &= v^\mu_{21} - v^\mu_{12}. \end{aligned} \tag{3.66}$$

** Exercise 3.5

Show that a covariant vector v_μ satisfies:

$$v_{21\mu} - v_{12\mu} = [\nabla_\nu, \nabla_\sigma] v_\mu \delta x_1^\nu \delta x_2^\sigma = -R^\lambda{}_{\mu\nu\sigma} v_\lambda \delta x_1^\nu \delta x_2^\sigma \tag{3.67}$$

upon parallel transport along two different paths.

Since the metric is covariantly constant (3.56), we also have:

$$\begin{aligned} [\nabla_\nu, \nabla_\sigma]\, v_\mu &= g_{\mu\lambda} [\nabla_\nu, \nabla_\sigma] v^\lambda \\ &= g_{\mu\lambda} R^\lambda{}_{\rho\nu\sigma} v^\rho \\ &= R_{\mu\rho\nu\sigma} v^\rho, \end{aligned} \tag{3.68}$$

using (3.59). Since this equals $-R_{\rho\mu\nu\sigma} v^\rho$ (3.67), $R_{\mu\rho\nu\sigma}$ is antisymmetric in its first two indices:

$$R_{\mu\rho\nu\sigma} = -R_{\rho\mu\nu\sigma}. \tag{3.69}$$

As a rank-4 tensor, $R^\rho{}_{\nu\sigma\alpha}$ has $d^4 = 256$ elements for $d = 4$. However, most of these elements are not independent due to the symmetry conditions found above:

$$\begin{aligned} &R_{\rho\nu\sigma\alpha} = -R_{\rho\nu\alpha\sigma} = -R_{\nu\rho\sigma\alpha} \\ &R_{\rho\nu\sigma\alpha} + R_{\rho\sigma\alpha\nu} + R_{\rho\alpha\nu\sigma} = 0. \end{aligned} \tag{3.70}$$

After applying these conditions, there are only $d^2(d^2-1)/12$ independent components (Appendix E). For $d = 4$, this is 20.

We can contract indices of the Riemann tensor to form a simpler rank-2 tensor. Due to the antisymmetries of $R^\rho{}_{\nu\sigma\alpha}$, the only unique contraction is:

$$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\alpha\rho} \Gamma^\alpha_{\mu\nu} - \Gamma^\rho_{\alpha\nu} \Gamma^\alpha_{\mu\rho}, \tag{3.71}$$

known as the *Ricci tensor*. From the above symmetry conditions, it is symmetric: $R_{\mu\nu} = R_{\nu\mu}$. Finally, we can also contract the indices of the Ricci tensor to form the *Ricci scalar*, or the *scalar curvature*:

$$R \equiv R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{3.72}$$

As we will see later, the Ricci scalar is essentially the Lagrangian of general relativity.

* Exercise 3.6

Show that $[\nabla_\mu, \nabla_\nu]\,\phi = 0$, for a scalar field $\phi(x)$.

** Exercise 3.7

1. Recall the product rule for the covariant derivative (3.55). Show that the commutator $[\nabla_\mu, \nabla_\rho]$ also satisfies this:

$$[\nabla_\mu, \nabla_\rho](v_\nu w_\sigma) = ([\nabla_\mu, \nabla_\rho]v_\nu)w_\sigma + v_\nu([\nabla_\mu, \nabla_\rho]w_\sigma). \tag{3.73}$$

   Since a 2-tensor $T_{\nu\sigma}$ acts like a product of vectors $v_\nu w_\sigma$, this implies that

$$[\nabla_\mu, \nabla_\rho]T_{\nu\sigma} = -R^\lambda{}_{\nu\mu\rho}T_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\rho}T_{\nu\lambda}, \tag{3.74}$$

   using (3.67).

2. By direct computation, prove the *Jacobi identity*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{3.75}$$

   for any operators A, B, C. As above, $[A, B] \equiv AB - BA$.

3. Calculate $[\nabla_\mu, [\nabla_\nu, \nabla_\sigma]]v_\rho$ using (3.67) and (3.74), then use the Jacobi identity to derive the *second Bianchi identity*:

$$\nabla_\mu R^\lambda{}_{\rho\nu\sigma} + \nabla_\nu R^\lambda{}_{\rho\sigma\mu} + \nabla_\sigma R^\lambda{}_{\rho\mu\nu} = 0. \tag{3.76}$$

* Exercise 3.8

1. Using the metric for a 2D sphere you derived in Ex. 3.2, find the Christoffel symbols $\Gamma^\mu_{\nu\rho}$. In two dimensions, there are only six independent components: Γ^1_{11}, Γ^1_{12}, Γ^1_{22}, Γ^2_{11}, Γ^2_{12}, Γ^2_{22}.

2. Find the Riemann curvature tensor $R^\rho{}_{\nu\sigma\alpha}$. In two dimensions, there is only one independent component: R_{1212}.

3. Find the Ricci tensor $R_{\mu\nu}$ and scalar R.

* Exercise 3.9

Consider the *Schwarzschild metric*:

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{1}{1 - \frac{r_0}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.77}$$

The coordinates are (t, r, θ, ϕ). t is a time coordinate, r is a radial coordinate, and the piece $r^2 d\Omega^2 \equiv r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$ is the metric for a sphere that you derived in Ex. 3.2.

As we will show in Sec. 4.9, this is the metric outside of a stationary, spherical body of mass M and radius R, centered at $r = 0$. r_0 is related to M as $r_0 = 2GM/c^2$, where $G \approx 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is Newton's constant.

The radius R of the body must be greater than r_0, and this metric is only valid outside of the body ($r > R$).

1. Find the non-zero Christoffel symbols $\Gamma^\mu_{\nu\rho}$. Hint: there are nine of them: $\Gamma^t_{tr}, \Gamma^r_{tt}, \Gamma^r_{rr}, \Gamma^r_{\theta\theta}, \Gamma^r_{\phi\phi}, \Gamma^\theta_{r\theta}, \Gamma^\phi_{r\phi}, \Gamma^\theta_{\phi\phi}, \Gamma^\phi_{\theta\phi}$.
2. Show that $R_{\mu\nu} = 0$.

* Exercise 3.10

Consider 3D Minkowski space with coordinates (t, x, y). Let H^2 be the 2D manifold defined by all points a constant proper time $ds^2 = -R^2$ away from the origin:

$$t^2 - x^2 - y^2 = R^2. \tag{3.78}$$

This is known as 2D *hyperbolic space* (Fig. 3.6). From the figure, we see that any tangent vector is spacelike, so the signature is $(+, +)$.

1. We may parametrize H^2 using coordinates (x, y). Find the metric $g_{\mu\nu}$. Remember that the interval is

   $$ds^2 = dx^2 + dy^2 - dt^2 \tag{3.79}$$

   since we are in ambient Minkowski space.

2. Find the Riemann tensor $R^{\rho}{}_{\nu\sigma\alpha}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R.

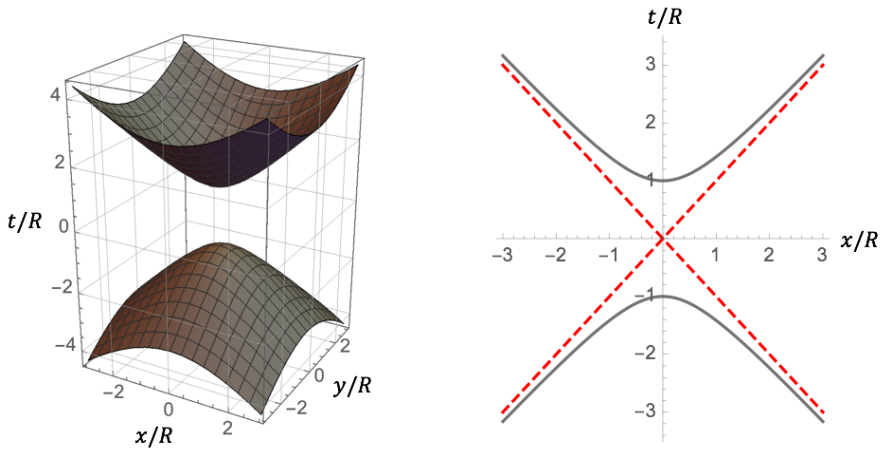


Figure 3.6: Left: hyperbolic space H^2 embedded in 3D Minkowski space. Right: cross section at $y = 0$. Red lines are $t = \pm x$.

Chapter 4

General relativity

We have covered the mathematical description of curved spacetime. Now we get to the physics. This essentially follows the development of particle and field dynamics in Chapter 2, but with general covariance instead of Lorentz invariance as a guiding principle. As before, let us start with the equation of motion for a massive particle in curved spacetime.

4.1 The geodesic equation

The particle Lagrangian in general relativity is the same as special relativity (2.29):

$$S_{pp} = -m \int d\tau = -m \int \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu} \tag{4.1}$$

but using (3.12) for $d\tau^2 = -ds^2$. Now that x^0 is not necessarily a time coordinate, it cannot be used to parametrize the path. Instead, we parametrize the path by some quantity* λ. Given a path in spacetime, we can always associate a value of λ to each point (Fig. 4.1).

*We cannot use the proper time τ to parametrize the path in the action, since we cannot independently vary $x^\mu(\tau_0)$ at a given τ_0 without violating $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$. Indeed, this would give a trivial Lagrangian $L_{pp} = -m$.

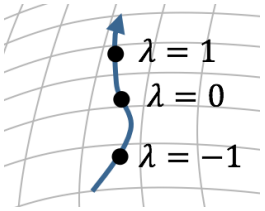


Figure 4.1: Path of an object parametrized by λ.

The action is:

$$S_{pp} = -m \int \sqrt{-g_{\mu\nu}(x(\lambda))U^\mu(\lambda)U^\nu(\lambda)}d\lambda \tag{4.2}$$

where $U^\mu \equiv dx^\mu/d\lambda$, and we show the arguments explicitly. It is invariant under reparametrizations $\lambda'(\lambda)$, as it must be. The Euler-Lagrange equations (2.14) now read:

$$\frac{\delta L}{\delta x^\mu(\lambda)} = \frac{d}{d\lambda}\frac{\delta L}{\delta U^\mu(\lambda)}. \tag{4.3}$$

The Lagrangian is:

$$L_{pp} = -m\sqrt{-g_{\mu\nu}U^\mu U^\nu}. \tag{4.4}$$

First, L_{pp} only depends on x through $g_{\mu\nu}(x)$, so we have:

$$\frac{\delta L}{\delta x^\mu} = \frac{m}{2\sqrt{-U^2}}\partial_\mu g_{\nu\sigma}U^\nu U^\sigma, \tag{4.5}$$

where $U^2 = g_{\mu\nu}U^\mu U^\nu$. The right-hand side of (4.3) involves a long calculation, and we do not show all the details. We have:

$$\frac{\delta L}{\delta U^\mu} = -\frac{m}{\sqrt{-U^2}}g_{\mu\nu}U^\nu. \tag{4.6}$$

This depends on λ through $g_{\mu\nu}(x(\lambda))$ and $U^\mu(\lambda)$. We get:

$$\begin{aligned}\frac{d}{d\lambda}\frac{\delta L}{\delta U^\mu} = &-m(-U^2)^{-3/2} \times \\ &\left[\frac{1}{2}U^\mu\left(\partial_\sigma g_{\rho\lambda}U^\sigma U^\rho U^\lambda + 2U_\rho\frac{dU^\rho}{d\lambda}\right) - U^2(\partial_\sigma g_{\mu\nu}U^\sigma U^\nu)\right].\end{aligned} \tag{4.7}$$

Equating this to (4.5) and collecting U^2 terms, we get:

$$\begin{aligned} 0 &= U^2\left(\frac{1}{2}\partial_\mu g_{\nu\sigma}U^\nu U^\sigma - \partial_\sigma g_{\mu\nu}U^\nu U^\sigma - g_{\mu\nu}\frac{dU^\nu}{d\lambda}\right) \\ &\qquad + U_\mu\left(\frac{1}{2}\partial_\sigma g_{\rho\lambda}U^\sigma U^\rho U^\lambda + U_\rho\frac{dU^\rho}{d\lambda}\right) \\ &= U^2\left(-\Gamma_{\mu,\nu\sigma}U^\nu U^\sigma - g_{\mu\nu}\frac{dU^\nu}{d\lambda}\right) + U_\mu\left(\Gamma_{\rho,\sigma\lambda}U^\rho U^\sigma U^\lambda + U_\rho\frac{dU^\rho}{d\lambda}\right) \\ &= \left(\frac{dU^\nu}{d\lambda} + \Gamma^\nu_{\rho\sigma}U^\rho U^\sigma\right)(U_\mu U_\nu - U^2 g_{\mu\nu}). \end{aligned} \tag{4.8}$$

On the second line, we use (3.41) and (3.42), along with the identity:

$$\partial_\sigma g_{\mu\nu}U^\nu U^\sigma = \frac{1}{2}(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma})U^\nu U^\sigma, \tag{4.9}$$

since $U^\nu U^\sigma$ is a symmetric tensor. This "symmetrization trick" is worth remembering. On the last line, we relabel indices and factorize.

Eq. (4.8) is a complicated equation of motion for general λ. We may choose the parametrization $\lambda = \tau$ to simplify it. Then we have $U^\mu = u^\mu = dx^\mu/d\tau$ and $u^2 = g_{\mu\nu}u^\mu u^\nu = -1$. Taking the derivative:

$$\begin{aligned} \frac{d}{d\tau}(g_{\mu\nu}u^\mu u^\nu) = 0 &= \partial_\sigma g_{\mu\nu}u^\sigma u^\mu u^\nu + 2g_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu \\ &= 2\Gamma_{\sigma,\mu\nu}u^\sigma u^\mu u^\nu + 2g_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu \\ &= 2u_\mu\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma}u^\nu u^\sigma\right), \end{aligned} \tag{4.10}$$

using (3.41) on the second line. We can then eliminate the $U_\mu U_\nu$ term in (4.8), so that the term in the left parentheses must be zero:

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{4.11}$$

This is called the *geodesic equation*. The resulting worldline is called a *timelike geodesic*.

There is an easier way to remember the geodesic equation: it is the result of parallel transporting the velocity vector $u^\mu = dx^\mu/d\tau$ along the velocity vector itself. Indeed, take the parallel transport equation (3.43)

with $v^\mu = u^\mu$, and divide by $\delta\tau$, to directly obtain (4.11). Intuitively, a particle wants to take the straightest path where its velocity changes the least, which is accomplished by parallel transport.

For massless particles, τ cannot be used as a parameter since $d\tau = 0$. Instead, the geodesic equation still holds for a particular parametrization $\bar{\lambda}$:

$$\frac{d^2x^\nu}{d\bar{\lambda}^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\bar{\lambda}}\frac{dx^\sigma}{d\bar{\lambda}} = 0. \tag{4.12}$$

The resulting worldline is called a *null geodesic*. It is easy to show that this equation is only invariant under reparametrizations $\bar{\lambda}'(\bar{\lambda})$ where $d^2\bar{\lambda}'/d\bar{\lambda}^2 = 0$, or $\bar{\lambda}' = a\bar{\lambda} + b$ with a, b constant. This is an affine transformation, so $\bar{\lambda}$ is also called an *affine parameter*. Similarly, (4.11) is only invariant under a reparametrization $\tau' = a\tau + b$.

As in classical mechanics, we solve the equation of motion using initial conditions $x^\mu(\tau_0) = x_0^\mu$ and $U^\mu(\tau_0) = U_0^\mu$. Due to the affine freedom, we may scale U_0^μ by any constant without affecting the resulting path of the particle, although only one choice corresponds to the four-velocity $U_0^2 = -1$. U^2 remains constant on the path since it is preserved by parallel transport (Ex. 3.3). For massless particles, replace τ with $\bar{\lambda}$ and use $U^\mu = dx^\mu/d\bar{\lambda}$. $U^2 = 0$ always, so there is no preferred scaling of U_0^μ.

**** Exercise 4.1**

Consider a test mass in a circular orbit at radius R around a spherical body of mass M. As mentioned in Ex. 3.9, the metric outside of this body is the Schwarzschild metric (3.77) with $r_0 = 2GM$.

1. Show that the period is $T = 2\pi\sqrt{\frac{R^3}{GM}}$ in Newtonian gravity.
2. Show that the period (proper time) experienced by the test mass is shorter by a factor
$$\sqrt{1 - \frac{3GM}{R}} \tag{4.13}$$
in relativity. Hint: we are looking for geodesics at constant $r = R$ and $\theta = \pi/2$, so we can assume $dr/d\tau = d\theta/d\tau = 0$. Use the geodesic equation (4.11) together with $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$ to solve for $d\phi/d\tau$.
3. Show that light in a circular orbit around the body can only occur at radius $R = 3r_0/2 = 3GM$. This is sometimes called the *photon sphere*.

4.2 The equivalence principle

While the four-acceleration $d^2x^\mu/d\tau^2$ is a four-vector in flat space, it is *not* a vector in curved space, since $\delta u^\mu = u^\mu(x + \delta x) - u^\mu(x)$ subtracts vectors at two different points. To find the acceleration vector, note that we may write:
$$\frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = (\partial_\nu u^\mu)u^\nu, \tag{4.14}$$
using the chain rule*. We can make this transform like a vector by replacing ∂_ν with ∇_ν. Thus, we define the acceleration vector
$$a^\mu \equiv (\nabla_\nu u^\mu)u^\nu. \tag{4.15}$$

*This might seem fishy, since $\partial_\nu u^\mu$ treats $u^\mu(x)$ as a field (function of x) although it is only defined on the path of the particle $x^\mu(\tau)$. However, it is contracted with u^ν, so $(\partial_\nu u^\mu)u^\nu$ only depends on u^ν along the path.

What is the significance of making this a vector? Due to general covariance, only scalar quantities are measurable, since all other quantities depend on the choice of coordinates. Thus, the perceived acceleration (squared) is $a^2 = a^\nu a_\nu$.

Now let us compare the geodesic equation (4.11) with the equation of motion in flat space (2.32), which is simply*:

$$\frac{d^2x^\nu}{d\tau^2} = 0. \tag{4.16}$$

The geodesic equation can be written in a similar form:

$$\begin{aligned} 0 &= \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma \\ &= (\partial_\nu u^\mu) u^\nu + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma \\ &= (\nabla_\nu u^\mu) u^\nu \\ &= a^\mu. \end{aligned} \tag{4.17}$$

Objects on geodesics experience no acceleration. This is the *equivalence principle*: moving freely in a gravitational field is locally indistinguishable from moving freely in flat space. Since the geodesic equation is independent of the mass or composition of the object, the equivalence principle applies to all objects equally.

This implies that objects on non-geodesic paths experience acceleration. Consider an object constrained to move on a path where

$$\frac{d^2x^\mu}{d\tau^2} = 0 \tag{4.18}$$

with respect to a particular choice of coordinates x^μ. Then the perceived acceleration $a^2 \neq 0$ in general. As we will see in the next section, one example is when an object is stationary with respect to a mass, such as when one stands on the surface of the Earth (neglecting the rotation of the Earth). This is indistinguishable from standing in an accelerating rocket in flat space (Fig. 4.2).

*Eq. (2.32) only contains the spatial components, but it is easy to show $du^0/d\tau = 0$ using $u^2 = \vec{u}^2 - (u^0)^2 = -1$.

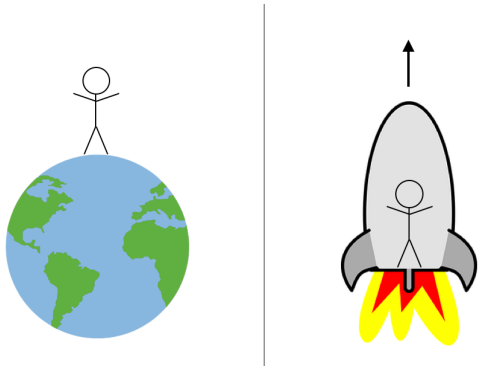


Figure 4.2: A non-geodesic path (left) is locally indistinguishable from acceleration in flat space (right).

4.3 Fermi normal coordinates

The equivalence principle implies that given a timelike geodesic γ, we may choose coordinates such that the metric is Minkowski along a geodesic: $g_{\mu\nu}(\gamma) = \eta_{\mu\nu}$. Physically, this is easy to understand: as you move along the geodesic, simply construct an IRF using freely moving clocks and rulers around you (Fig. 4.3). If the clocks are sufficiently close by, they remain spaced out by the rulers as they move. This defines an inertial coordinate system, called *Fermi normal coordinates*.

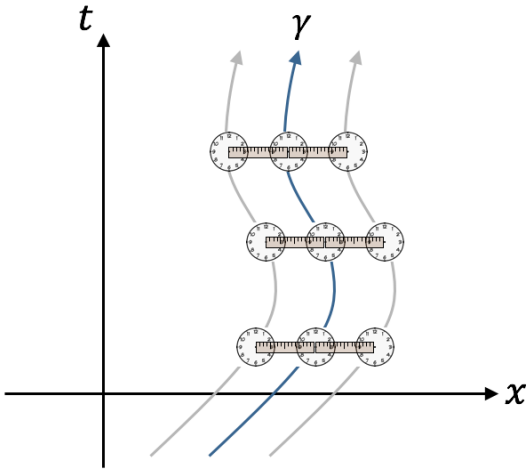


Figure 4.3: Fermi normal coordinates defined by an IRF around the geodesic γ.

In fact, in these coordinates the Christoffel symbols all vanish along the geodesic: $\Gamma^{\mu}_{\nu\sigma}(\gamma) = 0$. However, the derivatives $\partial_\rho \Gamma^{\mu}_{\nu\sigma}(\gamma)$ are nonzero in general, so the curvature tensors are nonzero, as required for a curved spacetime.

We will not cover the mathematical details of constructing Fermi normal coordinates here.

4.4 Local measurements

As we have emphasized, only inertial coordinates have physical significance. Likewise, tensor components only have physical significance in an inertial coordinate system. The transformation law of the metric (3.14) from general coordinates to inertial coordinates can be written as:

$$\eta_{ab} = e^{\rho}_{a}(x) e^{\sigma}_{b}(x) g_{\rho\sigma}(x), \tag{4.19}$$

where $e^{\rho}_{a} \equiv \partial x^{\rho}/\partial x^{a}$. Indices $a, b, \cdots$ denote inertial coordinates while Greek indices denote general coordinates. We will omit the argument x from now on. Likewise, we have:

$$\eta^{ab} = e^{a}_{\rho} e^{b}_{\sigma} g^{\rho\sigma}, \tag{4.20}$$

where $e^{a}_{\rho} \equiv \partial x^{a}/\partial x^{\rho}$. We also have $e^{a}_{\mu} e^{\nu}_{a} = \delta^{\nu}_{\mu}$ and $e^{a}_{\mu} e^{\mu}_{b} = \delta^{a}_{b}$, from (3.25).

Given a tensor in general coordinates $T^{\mu\nu\cdots}{}_{\sigma\rho\cdots}$, we can find the components in inertial coordinates by multiplying by factors of e^{μ}_{a} and e^{a}_{μ}.

Now instead of thinking of the e^{μ}_{a} as a coordinate transformation, we can think of them as a set of four orthonormal vectors $e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}$. They are orthonormal in the sense of (4.19). These form the coordinate axes of the IRF. $e_{(0)}$ is the time axis, which is also the four-velocity of a stationary observer in the IRF. The $e_{(i)}$ are the orthonormal spacelike vectors defined by the rulers, $i = 1, 2, 3$. We can see this by finding the components of the vector $e_{(\mu)}$ in inertial coordinates:

$$e^{a}_{(\nu)} = \frac{\partial x^{a}}{\partial x^{\mu}} e^{\mu}_{(\nu)} = e^{a}_{\mu} e^{\mu}_{(\nu)} = \delta^{a}_{(\nu)}. \tag{4.21}$$

Thus, in inertial coordinates, $e_{(0)} = (1, 0, 0, 0)^{T}$, and so on.

For example, we will encounter the *energy-momentum tensor* $T^{\mu\nu}$ in Sec. 4.11. The component T^{00} is the energy density in flat space, and the momentum density in the i direction is T^{0i}. Then, the energy density measured by an observer with four-velocity u^μ is: $T^{00}_{\text{inertial}} = e^0_\mu e^0_\nu T^{\mu\nu} = u_\mu u_\nu T^{\mu\nu}$. If the observer erects an IRF defining the i direction with a vector $e^\mu_{(i)}$, the momentum density is $T^{0i}_{\text{inertial}} = u_\mu e^i_\nu T^{\mu\nu}$.

4.5 Static spacetimes

A stationary mass distribution such as a non-rotating planet generates a static spacetime. In this case, we may choose time and space coordinates $x = (t, x^1, x^2, x^3)^T$ such that a worldline with constant position $\vec{x} \equiv (x^1, x^2, x^3)^T = \vec{x}_0$ corresponds to a stationary object with respect to the mass distribution. Then the metric does not depend on t, and $g_{0i} = g^{0i} = 0$ due to time reversal invariance. As before, Latin indices $i, j, \cdots = 1, 2, 3$. $g_{\mu\nu}$ takes the form:

$$g_{\mu\nu} = \begin{pmatrix} g_{00}(\vec{x}) & 0 \\ 0 & g_{ij}(\vec{x}) \end{pmatrix} \tag{4.22}$$

where the spatial part of the metric g_{ij} is a 3×3 matrix. The inverse metric is:

$$g^{\mu\nu} = \begin{pmatrix} 1/g_{00}(\vec{x}) & 0 \\ 0 & g^{ij}(\vec{x}) \end{pmatrix} \tag{4.23}$$

where g^{ij} is the matrix inverse of g_{ij}.

For slow objects where $|\vec{u}| \ll 1$, $|u^0| \gg |u^i|$. We have $u^2 = -1 \approx u^0 u_0 = g_{00}(u^0)^2$. The spatial acceleration vector (4.15) becomes approximately:

$$\begin{aligned} a^i &\approx \frac{du^i}{d\tau} + \Gamma^i_{00}(u^0)^2 \\ &\approx \frac{du^i}{d\tau} + \frac{1}{2} g^{ij} \frac{\partial_j g_{00}}{g_{00}}, \end{aligned} \tag{4.24}$$

using (3.42) on the second line. The a^0 component becomes:

$$a^0 \approx \frac{du^0}{d\tau} \tag{4.25}$$

since $\Gamma^0_{00} = 0$.

Let us apply this to a stationary, spherical body of mass M and radius R. As mentioned in Ex. 3.9, the metric around this body is given by the Schwarzschild metric (3.77) with $r_0 = 2GM$.

Plugging the metric (3.77) into (4.24), we obtain:

$$\begin{aligned} a^r &\approx \frac{du^r}{d\tau} + \frac{1}{2} g^{rr} \frac{\partial_r g_{00}}{g_{00}} \\ &= \frac{du^r}{d\tau} + \frac{GM}{r^2} + O(r^{-3}), \end{aligned} \tag{4.26}$$

so the acceleration experienced by a stationary object at radius r is

$$\sqrt{a^2} \approx \sqrt{g_{rr}(a^r)^2} \approx \frac{GM}{r^2}. \tag{4.27}$$

In classical physics, this is generated by the normal force counteracting the "force" of gravity:

$$\vec{F}_n = \frac{GMm}{r^2} \hat{r}, \tag{4.28}$$

where m is the mass of the object. Due to the equivalence principle, the force of gravity cannot be detected, so only the normal force is experienced.

4.6 Gravitational redshift

The static metric (4.22) has the property that time differences Δt between null worldlines are preserved from place to place. Consider a light wave sent by an oscillating charge q at $\vec{x}_1$ (Fig. 4.4), from $(t_1, \vec{x}_1)$ to $(t_2, \vec{x}_2)$.

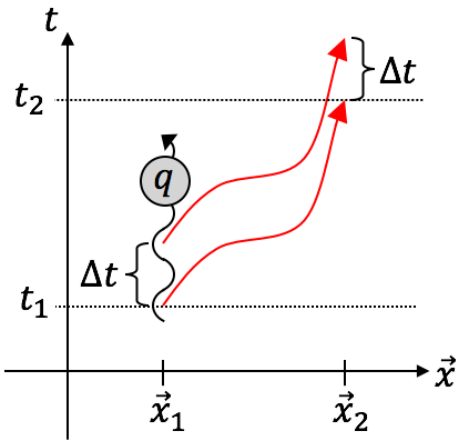


Figure 4.4: A light wave being sent from $\vec{x}_1$ to $\vec{x}_2$. Red lines indicate null worldlines corresponding to constant phase of the electromagnetic field. The lines are not straight since the metric $g_{\mu\nu}$ varies with position $\vec{x}$.

The peaks of the electromagnetic field propagate along null worldlines:

$$ds^2 = 0 = g_{00}(\vec{x})dt^2 + g_{ij}(\vec{x})dx^i dx^j. \tag{4.29}$$

Solving for dt and integrating from position $\vec{x}_1$ to $\vec{x}_2$:

$$t_2 - t_1 = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{-\frac{g_{ij}(\vec{x})dx^i dx^j}{g_{00}(\vec{x})}}. \tag{4.30}$$

Since g_{00} and g_{ij} are independent of t, $t_2 - t_1$ is purely a function of space. Thus, the next peak occurring a time $\Delta t = 1/f$ later at $\vec{x}_1$ will also occur Δt later at $\vec{x}_2$. However, the physical time difference at $\vec{x}$ is the proper time:

$$\Delta\tau = \sqrt{-g_{00}(\vec{x})}\Delta t. \tag{4.31}$$

The physical frequency is then:

$$f_{\text{phys}} = \frac{1}{\Delta\tau} = \frac{f}{\sqrt{-g_{00}(\vec{x})}} \tag{4.32}$$

Thus, we have:

$$f = \sqrt{-g_{00}(\vec{x}_1)}f_{\text{phys},1} = \sqrt{-g_{00}(\vec{x}_2)}f_{\text{phys},2}. \tag{4.33}$$

Far away from any gravitating bodies, space is nearly flat and $g_{00} \approx -1$. Close to a massive object, $g_{00} > -1$ (3.77). Thus, $f_{\text{phys,far}} < f_{\text{phys,near}}$. This is the phenomenon of *gravitational redshift*: the frequency of light decreases as it moves away from a gravitating body, and vice versa*.

4.7 Field Lagrangians

Once again, we move from particles to fields. Let us construct an action for the metric field $g_{\mu\nu}$. From general covariance, it must be a scalar. We immediately run into a problem: the volume element

$$d^4x = dx^0 dx^1 dx^2 dx^3 \tag{4.34}$$

is not a scalar, since it transforms as (A.22):

$$d^4x = \frac{d^4x'}{|\det J|} \tag{4.35}$$

under a coordinate change $J_{ij} \equiv \frac{\partial x'_i}{\partial x_j}$. Since the metric is used to convert coordinate lengths dx^μ into physical scalar lengths ds^2, we can try to multiply d^4x by some function of the metric to produce a scalar. Using the transformation law (3.15) and the properties (A.13) and (A.17) of determinants, we see that

$$g = |\det J|^2 \, g', \tag{4.36}$$

so the quantity

$$\sqrt{-g}\, d^4x \tag{4.37}$$

is a scalar (recall $g < 0$). It can be interpreted as a physical spacetime "volume".

We can now form the simplest action

$$S_\Lambda = -\frac{\Lambda}{8\pi G} \int \sqrt{-g}\, d^4x \tag{4.38}$$

for some constant Λ known as the *cosmological constant*.

*Generally, *redshift/blueshift* refers to a decrease/increase in frequency, since red is a low frequency of the optical spectrum and blue is a high frequency.

Instead of using the Euler-Lagrange equations, it is easier to find the change in the action δS given a variation $\delta g_{\mu\nu}$, and setting $\delta S = 0$. We will need the variations of certain quantities. First,

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{4.39}$$

This is derived as follows. The determinant g is a sum of terms, each one a product of $d = 4$ matrix elements $g_{\mu\nu}$ (A.11). Taking the variation of this product is equivalent to taking the variation of the first item and multiplying by d, from (A.11). We may write

$$g = g_{\mu\nu} C^{\mu\nu} \tag{4.40}$$

where $C^{\mu\nu}$ represents a product of $d - 1$ elements. Then $\delta g = d \delta g_{\mu\nu} C^{\mu\nu}$. From inspection of (4.40), $C^{\mu\nu} = g g^{\mu\nu}/d$, giving (4.39).

Now we can easily find $\delta\sqrt{-g}$:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}. \tag{4.41}$$

and

$$\delta S_\Lambda = -\frac{\Lambda}{16\pi G} \int g^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g}\, d^4x. \tag{4.42}$$

This must equal zero for any $\delta g_{\mu\nu}(x)$, so the equation of motion is simply $g^{\mu\nu} = 0$, which is clearly nonsensical. We started with this simple example to show the calculational procedure. While not useful by itself, the cosmological constant term with $\Lambda > 0$ is important in cosmology as the source of dark energy, as explained in Chapter 5.

* Exercise 4.2

Consider the metric for the sphere from Ex. 3.2 again:

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{4.43}$$

Show that

$$\int_0^{2\pi} \int_0^{\pi} \sqrt{g}\, d\theta d\phi \tag{4.44}$$

gives the familiar surface area of the sphere: $4\pi r^2$. We use $\sqrt{+g}$ here since we are in Euclidean space, with $(+,+)$ signature.

**** Exercise 4.3**

Derive the following useful identities:

$$\partial_\mu(\sqrt{-g}A^\mu) = \sqrt{-g}\nabla_\mu A^\mu \tag{4.45}$$

$$\partial_\mu(\sqrt{-g}B^{\mu\nu\cdots}) = \sqrt{-g}\nabla_\mu B^{\mu\nu\cdots} \tag{4.46}$$

where A^μ is a vector field, and $B^{\mu\nu\cdots}$ is a totally antisymmetric tensor field (it is negated under the exchange of any two indices). Hint: from (4.41), we have:

$$\begin{aligned}\partial_\sigma\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\sigma g_{\mu\nu} \\ &= \sqrt{-g}\Gamma^\mu_{\sigma\mu}.\end{aligned} \tag{4.47}$$

On the second line, we use the identity

$$\Gamma^\mu_{\sigma\mu} = \frac{1}{2}g^{\mu\nu}\partial_\sigma g_{\mu\nu}, \tag{4.48}$$

which comes from contracting (3.41) with $g^{\mu\nu}$.

4.8 Einstein-Hilbert action

The next simplest scalar involving the metric is the Ricci scalar R (3.72). This gives the standard action of general relativity, known as the *Einstein-Hilbert action*:

$$S_{\text{EH}} = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x, \tag{4.49}$$

We have:

$$\begin{aligned}\delta S_{\text{EH}} &= \frac{1}{16\pi G}\int \delta\left(R\sqrt{-g}\right)d^4x \\ &= \frac{1}{16\pi G}\int \delta R\sqrt{-g} + R\delta\sqrt{-g}\,d^4x \\ &= \frac{1}{16\pi G}\int \left(\delta R + \frac{1}{2}Rg^{\mu\nu}\delta g_{\mu\nu}\right)\sqrt{-g}\,d^4x \\ &= \frac{1}{16\pi G}\int \left(\delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}\delta g_{\mu\nu}\right)\sqrt{-g}\,d^4x.\end{aligned} \tag{4.50}$$

On the third line, we use (4.41). On the fourth line, we use $R = R_{\mu\nu}g^{\mu\nu}$.

As usual, we must find $\delta R_{\mu\nu}$ and $\delta g^{\mu\nu}$ in terms of $\delta g_{\mu\nu}$, so that we can factor out $\delta g_{\mu\nu}$. Let's start with $\delta g^{\mu\nu}$. $g^{\mu\nu}$ can be thought of as raising both indices of $g_{\mu\nu}$:

$$\begin{aligned}\delta g^{\mu\nu} &= \delta(g^{\mu\sigma}g^{\nu\lambda}g_{\sigma\lambda}) \\ &= 2\delta g^{\mu\nu} + g^{\mu\sigma}g^{\nu\lambda}\delta g_{\sigma\lambda} \\ &= -g^{\mu\sigma}g^{\nu\lambda}\delta g_{\sigma\lambda}.\end{aligned} \tag{4.51}$$

On the third line, we move the $2\delta g^{\mu\nu}$ to the other side and negate both sides.

Finding $\delta R_{\mu\nu}$ involves a long calculation. We will bypass it using dimensional analysis and general covariance.

First, consider $\delta\Gamma^{\mu}_{\nu\sigma}$. Although $\Gamma^{\mu}_{\nu\sigma}$ is not a tensor, $\delta\Gamma^{\mu}_{\nu\sigma}$ must be a tensor. To see this, write the parallel transport equation (3.39) as:

$$v^{(g)}_{\mu}(x+\delta x) - v_{\mu}(x) = \Gamma^{\nu(g)}_{\mu\sigma}(x)v_{\nu}(x)\delta x^{\sigma}, \tag{4.52}$$

where $v^{(g)}_{\mu}(x+\delta x)$ denotes the vector $v_{\mu}(x)$ transported to $x+\delta x$ using the metric $g_{\mu\nu}$. Then we have:

$$\begin{aligned}v^{(g+\delta g)}_{\mu}(x+\delta x) - v^{(g)}_{\mu}(x+\delta x) &= \left[v^{(g+\delta g)}_{\mu}(x+\delta x) - v_{\mu}(x)\right] - \left[v^{(g)}_{\mu}(x+\delta x) - v_{\mu}(x)\right] \\ &= \left[\Gamma^{\nu(g+\delta g)}_{\mu\sigma}(x) - \Gamma^{\nu(g)}_{\mu\sigma}(x)\right] v_{\nu}(x)\delta x^{\sigma} \\ &= \delta\Gamma^{\nu(g)}_{\mu\sigma}(x)v_{\nu}(x)\delta x^{\sigma}.\end{aligned} \tag{4.53}$$

Since the left-hand side (LHS) subtracts vectors at the same point, the right-hand side (RHS) must be a vector, so $\delta\Gamma^{\nu(g)}_{\mu\sigma}(x)$ must be a tensor. The most general form it can take is:

$$\delta\Gamma^{\mu}_{\nu\sigma} = B^{\mu\lambda\rho\alpha}_{\nu\sigma}\nabla_{\lambda}\delta g_{\rho\alpha} \tag{4.54}$$

for some tensor $B^{\mu\lambda\rho\alpha}_{\nu\sigma}$. This is seen as follows. Since $\Gamma^{\mu}_{\nu\sigma}$ contains one derivative, the RHS also contains one derivative. We must use the covariant derivative instead of the ordinary derivative so that the RHS is a tensor. Since $\nabla_{\sigma}g_{\mu\nu} = 0$ (3.56), we can only take the derivative of $\delta g_{\mu\nu}$.

$B^{\mu\lambda\rho\alpha}_{\nu\sigma}$ contains no derivatives and only depends on the metric. Indeed, an explicit calculation gives:

$$\delta\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(\nabla_{\nu}\delta g_{\lambda\sigma} + \nabla_{\sigma}\delta g_{\lambda\nu} - \nabla_{\lambda}\delta g_{\nu\sigma}). \tag{4.55}$$

Note that $\delta\Gamma^\mu_{\nu\sigma}$ only depends on derivatives of $\delta g_{\mu\nu}$, and not $\delta g_{\mu\nu}$ directly. The same is true for $\delta R_{\mu\nu}$, since $R_{\mu\nu}$ only involves products and derivatives of $\Gamma^\mu_{\nu\sigma}$ (3.60). Then, we may write the most general form of $\delta R_{\mu\nu}$:

$$\delta R_{\mu\nu} = C^{\alpha\beta\sigma\lambda}_{\mu\nu}\nabla_\alpha\nabla_\beta\delta g_{\sigma\lambda} \tag{4.56}$$

for some tensor $C^{\alpha\beta\sigma\lambda}_{\mu\nu}$. Again, since $R_{\mu\nu}$ contains two derivatives, so must the RHS. We must use covariant derivatives so that the RHS is a tensor. There are no terms like $\nabla_\sigma\delta g_{\mu\nu}$ since there are no one-derivative tensors to contract it with ($\nabla_\sigma g_{\mu\nu} = 0$). Finally, $C^{\alpha\beta\sigma\lambda}_{\mu\nu}$ contains no derivatives and only depends on the metric.

Now we can evaluate the first term in the integral (4.50). Since $C^{\alpha\beta\sigma\lambda}_{\mu\nu}$ only involves the metric, which is covariantly constant, we can factor out the covariant derivative:

$$\begin{aligned}\int \delta R_{\mu\nu}g^{\mu\nu}\sqrt{-g}\,d^4x &= \int g^{\mu\nu}C^{\alpha\beta\sigma\lambda}_{\mu\nu}\nabla_\alpha\nabla_\beta\delta g_{\sigma\lambda}\sqrt{-g}\,d^4x \\ &= \int \nabla_\alpha A^\alpha\sqrt{-g}\,d^4x \\ &= \int \partial_\alpha\left(A^\alpha\sqrt{-g}\right)d^4x.\end{aligned} \tag{4.57}$$

where we define the vector $A^\alpha \equiv g^{\mu\nu}C^{\alpha\beta\sigma\lambda}_{\mu\nu}\nabla_\beta\delta g_{\sigma\lambda}$. On the third line, we use (4.45). Since this is a total derivative, it can be converted to a boundary term $\propto \nabla_\beta\delta g_{\sigma\lambda}$, using the divergence theorem (2.49). We are only interested in physics away from the boundary, so we assume that this variation vanishes fast enough that the boundary integral equals zero.

Putting it all together, we have:

$$\delta S_{\text{EH}} = \frac{1}{16\pi G}\int\left(-R^{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}\,d^4x. \tag{4.58}$$

Since this must vanish for any $\delta g_{\mu\nu}$, we arrive at the *Einstein field equations* in vacuum:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 0. \tag{4.59}$$

We can contract this with $g_{\mu\nu}$, giving $R = 0$. Plugging this back into (4.59), we get the simpler form of Einstein's equations in vacuum:

$$R^{\mu\nu} = 0. \tag{4.60}$$

The metrics that satisfy this equation are called *vacuum solutions*. The simplest one is, of course, flat space. In inertial coordinates, $g_{\mu\nu} = \eta_{\mu\nu}$, so all the Christoffel symbols and curvature tensors are zero. In arbitrary coordinates, the Christoffel symbols are not necessarily zero, but the curvature tensors remain zero since they transform as tensors. Another solution is the Schwarzschild metric (3.77) describing a black hole*, which we will derive in Sec. 4.9.

Finally, let's add the cosmological constant term back in:

$$S = S_\Lambda + S_{\text{EH}} = \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\, d^4x. \tag{4.61}$$

The variation is

$$\delta S = \frac{1}{16\pi G}\int \left(-\Lambda g^{\mu\nu} - R^{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}\, d^4x, \tag{4.62}$$

so the equation of motion is

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = 0. \tag{4.63}$$

4.9 The Schwarzschild solution

Let us look for static, spherically symmetric vacuum solutions. The general form of such a metric is:

$$ds^2 = F(R)dt^2 + H(R)dR^2 + I(R)d\Omega^2, \tag{4.64}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$, as before. R is a radial coordinate, and $F(R), H(R), I(R)$ are arbitrary functions. Spherical symmetry means that (θ, ϕ) only appear in the combination $d\Omega^2$. Static means that F, H, and I only depend on R, and that we cannot have $dt\, dR$ terms, since this would break time reversal $t \to -t$. Now define a new radial coordinate r by:

$$I(R) = r^2 \tag{4.65}$$

so that

$$ds^2 = f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2. \tag{4.66}$$

*Historically, it was also the first vacuum solution found after flat space.

By differentiating (4.65), one can show that

$$f(r) = F(R(r)), \qquad h(r) = \frac{4r^2}{\left(\frac{dI}{dR}(R(r))\right)^2} H(R(r)), \tag{4.67}$$

where $R(r)$ is implicitly defined by (4.65). r is a better radial coordinate, since a sphere at radius r centered at the origin has surface area $4\pi r^2$. This is seen by fixing t and r, so that the metric becomes (4.43). Thus, we will use the form (4.66) as our starting point.

Now we plug and chug* to find $R_{\mu\nu}$. I will simply show the results but you should go through the algebra for practice. First, the non-vanishing Christoffel symbols are (3.42):

$$\Gamma^t_{tr} = \frac{1}{2} f^{-1} f' \quad \Gamma^r_{tt} = -\frac{1}{2} f' h^{-1} \quad \Gamma^r_{rr} = \frac{1}{2} h^{-1} h' \tag{4.68}$$

$$\Gamma^r_{\theta\theta} = -r h^{-1} \quad \Gamma^r_{\phi\phi} = -r h^{-1} \sin^2\theta \quad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = r^{-1} \tag{4.69}$$

$$\Gamma^\theta_{\phi\phi} = -\cos\theta \sin\theta \quad \Gamma^\phi_{\theta\phi} = \cot\theta \tag{4.70}$$

where $f' = df/dr$, $h' = dh/dr$. Plugging into (3.71), we obtain:

$$R_{tt} = \frac{1}{4} f' h' h^{-2} - \frac{1}{2} f'' h^{-1} + \frac{1}{4} f'^2 f^{-1} h^{-1} - r^{-1} f' h^{-1} = 0 \tag{4.71}$$

$$R_{rr} = \frac{1}{4} f' f^{-1} h' h^{-1} - \frac{1}{2} f'' f^{-1} + \frac{1}{4} f'^2 f^{-2} + r^{-1} h' h^{-1} = 0. \tag{4.72}$$

We see that $f R_{rr}$ is identical to $h R_{tt}$ except for the last term. We have:

$$f R_{rr} - h R_{tt} = r^{-1} \left(\frac{h' f}{h} + f' \right) = 0, \tag{4.73}$$

or

$$\frac{f'}{f} = -\frac{h'}{h}. \tag{4.74}$$

Plugging this back into $h R_{tt}$ to eliminate h, we get:

$$\frac{1}{2} f'' + \frac{f'}{r} = 0. \tag{4.75}$$

*Tedious calculations like this one can be automated using symbolic algebra software such as Mathematica. A good Mathematica package for general relativity is GREATER2.

This can be solved using the substitution $y(r) = f'(r)$ followed by separation of variables. We get:

$$f(r) = \frac{c_1}{r} + c_2 \tag{4.76}$$

for constants c_1, c_2.

To solve for h, we may write (4.74) as:

$$\frac{df}{f} = -\frac{dh}{h}, \tag{4.77}$$

since the dr in the denominator cancels out. Integrating both sides, we get:

$$h(r) = \frac{c_3}{f(r)} \tag{4.78}$$

for some constant c_3.

To determine the constants c_1, c_2, c_3, we require that the metric be flat as $r \to \infty$:

$$ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2. \tag{4.79}$$

This condition is called *asymptotic flatness*. The spatial part is the metric of 3D Euclidean space (3.31). This gives:

$$c_2 = c_3 = -1. \tag{4.80}$$

Thus, the final metric becomes the *Schwarzschild solution* (3.77), upon identifying $c_1 = 2GM$.

4.10 Black holes

As mentioned in Ex. 3.9, the Schwarzschild solution (3.77) gives the metric outside of a spherically symmetric body of radius R and mass M, with $R > 2GM$. To see why, note that there is no matter outside $r = R$, so the metric must obey the vacuum Einstein's equations $R_{\mu\nu} = 0$ there. It must also be static, spherically symmetric, and asymptotically flat. From the derivation above, the Schwarzschild solution is the only such metric.

However, we may also regard (3.77) as a vacuum solution valid for *all* values of r (except $r = 2GM$ and $r = 0$, where the metric components go to zero or infinity). This solution is called the *Schwarzschild black hole*.

In general, a *black hole* is any object with an *event horizon*: a boundary through which light and matter can only pass one way. The event horizon for the Schwarzschild solution is at $r = r_0 \equiv 2GM$. r_0 is called the *Schwarzschild radius*. Once an object enters the $r < r_0$ region, it can only travel towards the origin $r = 0$ (we will prove this shortly). There, it encounters a *gravitational singularity*, where spacetime itself becomes undefined.

You may suspect that spacetime also becomes undefined at $r = r_0$ since $g_{tt} \to 0$ and $g_{rr} \to \infty$. However, this is merely a *coordinate singularity* caused by a poor choice of coordinates. This can be seen by calculating a scalar quantity such as $K \equiv R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ (the *Kretschmann scalar*), since this doesn't depend on the coordinate system. For the Schwarzschild solution, $K \propto 1/r^6$, with no unusual behavior* near $r = r_0$. Conversely, $r = 0$ contains a true gravitational singularity, since K blows up there.

To understand the event horizon, we can make a coordinate change to eliminate the coordinate singularity at $r = r_0$. First, write (3.77) as:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{4.81}$$

where $f(r) = 1 - r_0/r$. The singularity comes from $f(r_0) = 0$. Define a new time coordinate

$$T = t + l(r) \tag{4.82}$$

where $l(r)$ will be chosen to eliminate the singularity. We have:

$$dT = dt + l'dr, \tag{4.83}$$

where the prime denotes d/dr, as before. The metric in (T, r, θ, ϕ) coordinates becomes:

$$ds^2 = -f dT^2 + 2fl'dTdr - (fl'^2 - \frac{1}{f})dr^2 + r^2 d\Omega^2. \tag{4.84}$$

By choosing

$$l' = \frac{1}{f}, \tag{4.85}$$

*Although K is usually a good indicator, some exotic spacetimes have a finite K where there is still a real singularity. In any case, we can show there is no singularity at $r = r_0$ using an explicit coordinate change, as done below.

we both eliminate the dr^2 term and make the coefficient of $dTdr$ constant. Solving this differential equation gives:

$$l = r + r_0 \ln \left| \frac{r - r_0}{r_1} \right| \tag{4.86}$$

where r_1 is a constant of integration. The metric becomes:

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dT^2 + 2dTdr + r^2 d\Omega^2. \tag{4.87}$$

These are called *Eddington–Finkelstein* coordinates. At $r = r_0$, although the dT^2 term vanishes, the remaining term $2dTdr$ is a perfectly valid metric for the (T, r) plane.

For null worldlines moving in the radial direction ($d\theta = d\phi = 0$), we have:

$$ds^2 = dT(2dr - fdT) = 0. \tag{4.88}$$

This gives the two equations:

$$dT = 0, \tag{4.89}$$

$$\frac{dT}{dr} = \frac{2}{f} = 2l', \tag{4.90}$$

corresponding to ingoing and outgoing light rays. The solutions are:

$$T = T_0, \tag{4.91}$$

$$T = 2l + T_1 = 2r + 2r_0 \ln \left| \frac{r - r_0}{r_1} \right|, \tag{4.92}$$

for constants T_0, r_1. We have absorbed the constant T_1 into r_1, since changing r_1 just corresponds to an additive shift. These are plotted in Fig. 4.5 for various T_0, r_1. The curves intersecting a given (T, r) form the "lightcone" in the T–r plane that bounds the radial motion of particles (Fig. 4.5). We see that for $r > r_0$, particles can escape to $r = \infty$, while for $r < r_0$, they can only go in the $-r$ direction.

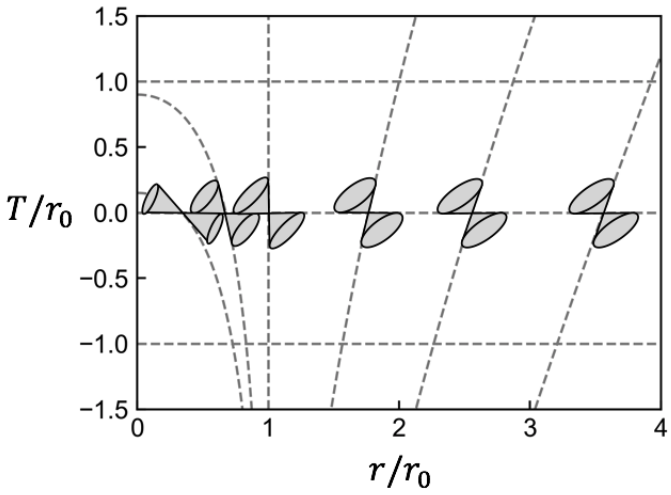


Figure 4.5: Null worldlines (dashed lines) and lightcones for varying r at constant $T = 0$.

To an observer at constant $r = r_{obs} > r_0$, an object falling into the black hole appears to take an infinite amount of time to reach the horizon. From the future lightcones in Fig. 4.5, an outgoing light ray emitted from an object at $T = 0, r = r_{emit} > r_0$ follows (4.92). Solving for r_1 gives:

$$r_1 = (r_{emit} - r_0)e^{r_{emit}/r_0}. \tag{4.93}$$

Then plugging in $r = r_{obs}$ into (4.92) gives the detection time T_{obs}:

$$T_{obs} = 2(r_{obs} - r_{emit}) + 2r_0 \ln\left(\frac{r_{obs} - r_0}{r_{emit} - r_0}\right). \tag{4.94}$$

As $r_{emit} \to r_0$, $T_{obs} \to \infty$. Note that T is a good time coordinate for the observer, since the proper time of the observer $\Delta\tau$ is simply proportional to ΔT: $\Delta\tau = \sqrt{f(r_{obs})}\Delta T$.

The gravitational redshift (4.33) also becomes infinite at the horizon, since $g_{tt} \to 0$ there. To an outside observer, objects appear to slow down and become redder and dimmer as they approach the horizon.

The Schwarzschild solution is the simplest black hole. It is non-rotating and uncharged. Rotating black holes are described by the *Kerr metric*, black holes with electric charge are described by the *Reissner–Nordström metric*, and black holes that are both rotating and charged are described by the

Kerr-Newman metric. Real astrophysical black holes closely follow the Kerr solution, since they are generally rotating and carry negligible charge. We will not discuss the details of these other black holes here.

4.11 The energy-momentum tensor

Just as electric charges and currents J^μ produce an electromagnetic field, matter causes spacetime to curve. Let us add a general matter action S_m to S_{EH} and S_Λ:

$$S = S_{\text{EH}} + S_\Lambda + S_m. \tag{4.95}$$

The variation is:

$$\delta S = \int \left(\frac{1}{16\pi G} \left(-\Lambda g^{\mu\nu} - R^{\mu\nu} + \frac{1}{2} R g^{\mu\nu} \right) \sqrt{-g} + \frac{\delta S_m}{\delta g_{\mu\nu}} \right) \delta g_{\mu\nu} \, d^4x. \tag{4.96}$$

The variation of S_m becomes an integral over x; compare with (2.3). The equation of motion is then:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -\Lambda g^{\mu\nu} + 8\pi G T^{\mu\nu}, \tag{4.97}$$

where we define the *energy-momentum tensor* or *stress-energy tensor*

$$T^{\mu\nu}(x) \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}(x)}. \tag{4.98}$$

It is clearly symmetric: $T^{\mu\nu} = T^{\nu\mu}$.

In cases where we know $T^{\mu\nu}$ but not $R^{\mu\nu}$, it is helpful to write (4.97) in terms of $T = T^\mu{}_\mu$ instead of R. First, contract the indices of (4.97):

$$-R = -4\Lambda + 8\pi G T. \tag{4.99}$$

Then we have:

$$R^{\mu\nu} = \Lambda g^{\mu\nu} + 8\pi G \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right). \tag{4.100}$$

** Exercise 4.4

Electromagnetism in curved space. The Maxwell action (2.60) in curved space becomes*:

$$S_{\text{EM}} = -\frac{\epsilon_0}{4}\int g^{\mu\sigma}g^{\nu\rho}F_{\mu\nu}F_{\sigma\rho}\sqrt{-g}\,d^4x. \tag{4.101}$$

We contract indices with the metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$, and use $\sqrt{-g}\,d^4x$ instead of d^4x.

1. Show that the electromagnetic stress-energy tensor is:

$$T^{\mu\nu}_{\text{EM}} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{EM}}}{\delta g_{\mu\nu}} = \epsilon_0\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}g^{\mu\nu}F^{\sigma\rho}F_{\sigma\rho}\right). \tag{4.102}$$

   Hint: use (4.41) and (4.51).

2. Show that $T^{\mu\nu}_{\text{EM}}$ is traceless: $g_{\mu\nu}T^{\mu\nu}_{\text{EM}} = 0$.

3. In flat space, show that

$$\begin{aligned} T^{00}_{\text{EM}} &= \frac{1}{2}\epsilon_0\vec{E}^2 + \frac{1}{2}\epsilon_0\vec{B}^2 \\ T^{0i}_{\text{EM}} &= \epsilon_0(\vec{E}\times\vec{B})_i. \end{aligned} \tag{4.103}$$

   You may recognize T^{00}_{EM} as the energy density of the electric and magnetic fields, and T^{0i}_{EM} as the energy flux, or *Poynting vector* $\vec{S}$.

4.12 Energy-momentum conservation

*We must write S_{EM} in terms of $F_{\mu\nu}$ and not $F^{\mu\nu}$, since $F_{\mu\nu}$ inherently has lower indices, not upper indices (2.56). In other words, $\delta F_{\sigma\rho}/\delta g_{\mu\nu} = 0$, but $\delta F^{\sigma\rho}/\delta g_{\mu\nu} = \delta(g^{\sigma\alpha}g^{\rho\beta}F_{\alpha\beta})/\delta g_{\mu\nu} \neq 0$.

**** Exercise 4.5**

Recall the second Bianchi identity (3.76). Contract the λ and ν indices, multiply by $g^{\mu\rho}$, and raise the index σ, to obtain:

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = 0. \tag{4.104}$$

Thus, using (4.97), the stress-energy is covariantly conserved:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.105}$$

In flat space with inertial coordinates*, $\Gamma^\mu_{\nu\sigma} = 0$, so we have $\partial_\mu T^{\mu\nu} = 0$. Then $T^{\mu\nu}$ defines four currents, one for each ν, that are conserved in the sense of (2.53). $T^{\mu 0}$ is the energy current: $\partial_\mu T^{\mu 0} = 0$ means the change in energy density equals the incoming energy flux. Likewise, $T^{\mu i}$ is the momentum current in direction i. Importantly, the total energy and momentum are conserved (2.55). In matrix form:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} = \text{Energy density} & T^{0i} = \text{Momentum density} \\ T^{i0} = \text{Energy flux} & T^{ij} = \text{Momentum flux} \end{pmatrix}. \tag{4.106}$$

Because $T^{\mu\nu}$ is symmetric, the momentum density equals the energy flux. We will see how this works for a mass density in Sec. 4.14.

However, when $\Gamma^\mu_{\nu\sigma} \neq 0$, there is no ordinary conservation law:

$$\partial_\mu T^{\mu\nu} \neq 0, \tag{4.107}$$

and thus the derivation of charge conservation (2.55) does not hold, since we cannot convert the spatial integral into a boundary term. In other words, there is *no globally conserved energy or momentum* in general spacetimes.

*We assume that the energy-momentum is small enough that it does not curve the metric, analogous to a test particle in an electromagnetic field.

4.13 $T^{\mu\nu}$ for particles

Let us find $T^{\mu\nu}$ for a single particle of mass m. The point-particle action is (4.2):

$$S_{pp} = \int L(x_p, U_p, \lambda)\, d\lambda \tag{4.108}$$

where $L(x_p, U_p, \lambda) = -m\sqrt{-g_{\mu\nu}(x_p)U_p^\mu U_p^\nu}$. Since we are varying with respect to $g_{\mu\nu}(x)$ and not x, we write this in terms of a fixed path $x_p(\lambda)$. As before, $U_p^\mu = dx_p^\mu/d\lambda$.

We would like the functional derivative $\delta S_m/\delta g_{\mu\nu}(x)$ to be nonzero only when x is on the path of the particle $x_p(\lambda)$. This can be done using a delta function:

$$\begin{aligned}
\frac{\delta S_{pp}}{\delta g_{\mu\nu}(x)} &= \int \frac{\delta L}{\delta g_{\mu\nu}}(x_p, U_p, \lambda)\delta^4(x - x_p)\, d\lambda \\
&= \frac{1}{2}m \int \frac{U_p^\mu U_p^\nu}{\sqrt{-U_p^2}}\delta^4(x - x_p)\, d\lambda \\
&= \frac{1}{2}m \int \frac{(dx^\mu/d\lambda)(dx^\nu/d\lambda)}{d\tau/d\lambda}\delta^4(x - x_p)\, d\lambda \\
&= \frac{1}{2}m \int \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta^4(x - x_p)\, d\tau
\end{aligned} \tag{4.109}$$

On the third line, we use $-dx^\mu dx_\mu = d\tau^2$. The differentials $d\lambda$ on top and bottom cancel out. The stress-energy tensor is:

$$T_{pp}^{\mu\nu}(x) = m \int u_p^\mu u_p^\nu \frac{\delta^4(x - x_p)}{\sqrt{-g}}\, d\tau, \tag{4.110}$$

where $u_p^\mu = dx_p^\mu/d\tau$.

4.14 $T^{\mu\nu}$ for mass densities

Now instead of a point particle, consider a distributed mass (Fig. 4.6). Starting with (4.110), the quantity $\delta^4(x)/\sqrt{-g}$ gets smeared out into an inverse spacetime volume:

$$\frac{\delta^4(x)}{\sqrt{-g}} \to \frac{H(x)}{d^4x\sqrt{-g}} = \frac{H(x)}{dV\, d\tau}. \tag{4.111}$$

Recall that we may convert a coordinate volume d^4x to a physical volume by multiplying by $\sqrt{-g}$ (4.37). Thus, dV is a physical *spatial* volume that contains the mass, and $d\tau$ is a proper time. $H(x)$ is 1 when x is within this spacetime volume centered at the origin, and 0 otherwise. For multiple masses indexed by i, we have:

$$T^{\mu\nu}_{\rho}(x) = \sum_i \frac{m_i H(x - x_i)}{dV_i} u_i^\mu u_i^\nu \tag{4.112}$$

$$\to \rho(x) u^\mu(x) u^\nu(x). \tag{4.113}$$

The integral becomes a sum over spacetime locations x_i that contain each volume dV_i with mass m_i. In the infinitesimal limit, this becomes a mass density $\rho(x)$: a mass per unit volume. $u^\mu(x)$ is the local four-velocity at x.

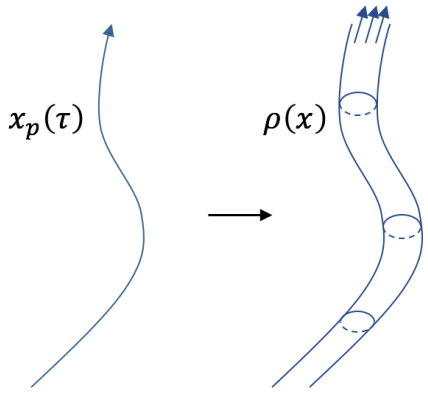


Figure 4.6: Particle worldline $x_p(\tau)$ and mass density $\rho(x)$.

Energy-momentum conservation gives:

$$\nabla_\mu T^{\mu\nu}_{\rho} = \nabla_\mu(\rho u^\mu) u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0. \tag{4.114}$$

The second term is zero since the mass moves along geodesics (4.17). The vanishing of the first term then implies that the "mass current" ρu^μ is covariantly conserved.

In flat space with inertial coordinates, we have (1.32):

$$\begin{aligned} u^0 = \gamma &\approx 1 + \frac{1}{2}\vec{v}^2 \\ \vec{u} = \gamma\vec{v} &\approx \vec{v}, \end{aligned} \tag{4.115}$$

for $v \ll c$ and expanding up to order $\vec{v}^2$. Then we have:

$$\begin{aligned} T^{00}_{\rho} &= \rho\gamma^2 \approx \rho + \rho\vec{v}^2 \\ T^{0i}_{\rho} &= T^{i0} = \rho\gamma^2 v^i \approx \rho v^i \\ T^{ij}_{\rho} &= T^{ji} = \rho\gamma^2 v^i v^j \approx \rho v^i v^j. \end{aligned} \tag{4.116}$$

As mentioned above, T^{00}_{ρ} is the energy density. It consists of the rest energy density $\rho_{\text{rest}} = \rho c^2$, plus the kinetic energy density $\rho\vec{v}^2$. We see that the kinetic energy is twice as large as that of a point particle (1.42). This is because a moving volume holding a mass m gets length contracted in the direction of motion (Fig. 4.7). Thus, the density ρ acquires another factor of γ.

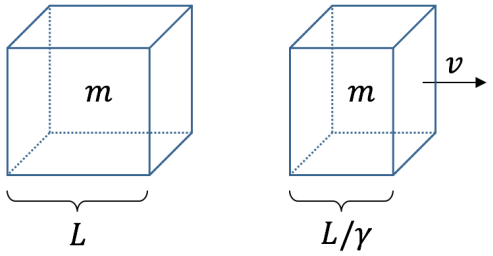


Figure 4.7: A box at rest containing mass m (left) gets length contracted when moving with velocity v (right).

T^{0i}_{ρ} is the momentum density, since if we take $\rho = m/V$, we have $\rho\vec{v} = m\vec{v}/V$. It is also the energy flux (power per unit area), since the rest energy density ρc^2 $[J/m^3]$ is multiplied by the velocity $\vec{v}$ $[m/s]$.

Finally, T^{ij}_{ρ} is evidently the momentum flux, from the conservation law $\partial_\mu T^{\mu j} = 0$. More intuitively, it is the pressure tensor (force per unit area). It can be written as a 3×3 matrix of rank 1:

$$Q = \rho\vec{v}\vec{v}^T \tag{4.117}$$

so that the pressure in any direction $\hat{n}$ is given by:

$$\hat{n}^T Q \hat{n} = \rho(\hat{n} \cdot \vec{v})^2 = \rho v^2 \cos^2\theta, \tag{4.118}$$

where $\hat{n}$ is a unit vector and θ is the angle between $\hat{n}$ and $\vec{v}$. One factor of $|\hat{n} \cdot \vec{v}|$ comes from the increased area (lower flux) seen by the surface

perpendicular to $\hat{n}$, and the other factor comes from the angle between $\hat{n}$ and $\vec{v}$ (Fig. 4.8).

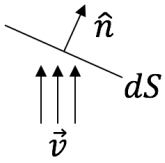


Figure 4.8: Matter with velocity $\vec{v}$ impinging on a surface dS with normal vector $\hat{n}$.

For massless matter, the four-velocity is a null vector t^μ, with $t^\mu t_\mu = 0$. The stress-energy tensor is:

$$T_k^{\mu\nu} = k(x)t^\mu(x)t^\nu(x), \tag{4.119}$$

where $k(x)$ is just some scalar field instead of a mass density.

In flat space, we have:

$$\begin{aligned} T_k^{00} &= k(t^0)^2 \equiv \rho_E \\ T_k^{0i} &= kt^0t^i = \rho_E\hat{t}^i \\ T_k^{ij} &= kt^it^j = \rho_E\hat{t}^i\hat{t}^j, \end{aligned} \tag{4.120}$$

where $\hat{t}^i \equiv t^i/t^0$ is a unit vector. Note that the momentum density T_k^{0i} has the same magnitude as the energy density $T_k^{00} = \rho_E$, as expected from (1.43).

4.15 $T^{\mu\nu}$ for ideal fluids

Now consider an ensemble of particles in thermal equilibrium. We may treat this as a mass distribution (4.113), where $\rho(\vec{x})$ is the time-averaged mass density and u^μ is a statistical quantity. Let us find $T^{\mu\nu}$. From (4.116), we have:

$$T^{00} = \rho\langle\gamma^2\rangle \tag{4.121}$$

$$T^{0i} = \rho\langle\gamma^2 v^i\rangle = 0 \tag{4.122}$$

$$T^{ij} = \rho\langle\gamma^2 v^i v^j\rangle = \rho\langle\gamma^2 (v^i)^2\rangle\delta^{ij} \equiv P\delta^{ij}, \tag{4.123}$$

where we define the *pressure* $P \equiv \rho\langle\gamma^2(v^i)^2\rangle$. Here, $\langle A\rangle$ is the time-averaged value of the quantity A. Since the velocities are equally distributed between the $+i$ and $-i$ direction, the average $\langle\gamma^2 v^i\rangle = 0$, and $\langle\gamma^2 v^i v^j\rangle = 0$ for the components in different directions ($i \neq j$). In matrix form:

$$T_f^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{4.124}$$

Let us show that P is indeed the usual pressure in the nonrelativistic limit. Assume the particles are enclosed in a cubic box with linear dimension L. A single particle with mass m and velocity v^i in the x^i direction imparts a momentum change $\Delta p^i = 2mv^i$ on a wall during a round trip, since it hits the wall with velocity $+v^i$ and leaves the wall with velocity $-v^i$. The round trip time is $\Delta t = 2L/v^i$. Thus, the time-averaged pressure is $P = \Delta p^i/(\Delta t L^2) = m(v^i)^2/V$. Here, L^2 is the area of the wall, and $V = L^3$ is the volume. Summing over all N particles gives:

$$P = \frac{Nm}{V}\langle(v^i)^2\rangle = \rho\langle(v^i)^2\rangle, \tag{4.125}$$

where $\rho = Nm/V$ is the mass density.

Now let us try to write a tensor expression for $T_f^{\mu\nu}$, in terms of $g^{\mu\nu}$ and $U^\mu(x) \equiv \langle u^\mu(x)\rangle$, the average four-velocity. It must take the form:

$$T_f^{\mu\nu} = AU^\mu U^\nu + Bg^{\mu\nu} \tag{4.126}$$

for scalars A and B, since these are the only 2-tensors we can form from U^μ and $g^{\mu\nu}$. This must agree with (4.124) in flat space with zero net velocity, where $U^\mu = (1,0,0,0)^T$ and $g^{\mu\nu} = \eta^{\mu\nu}$. We get:

$$T_f^{\mu\nu} = (\rho + P)U^\mu U^\nu + Pg^{\mu\nu}. \tag{4.127}$$

Matter with $T^{\mu\nu}$ given by (4.127) is also called an *ideal fluid*. In general, ρ, P, and U^μ can depend on x.

Any equation relating ρ and P using macroscopic (thermodynamic) quantities is called an *equation of state*. Different types of matter will have different equations of state. For example, if the particles are coupled to a

heat bath at temperature T, they follow the ideal gas law (which we will not derive here):

$$P = \frac{NkT}{V} = \left(\frac{Nm}{V}\right)\left(\frac{kT}{m}\right) = \rho\left(\frac{kT}{m}\right) \tag{4.128}$$

where V is the volume and k is Boltzmann's constant.

For an ensemble of massless particles in flat space, we have (4.120):

$$T^{00} = \rho_E \tag{4.129}$$

$$T^{0i} = \rho_E \langle \hat{t}^i \rangle = 0 \tag{4.130}$$

$$T^{ij} = \rho_E \langle \hat{t}^i \hat{t}^j \rangle = \frac{1}{3}\rho_E \delta^{ij}. \tag{4.131}$$

The $1/3$ factor comes from the average value of $\left(\hat{t}^i\right)^2$ for a randomly oriented unit vector. To find this, integrate $(x^3)^2 = \cos^2\theta$ over the unit sphere in spherical coordinates, and divide by the total surface area:

$$\frac{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi}{\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi} = \frac{1}{3}. \tag{4.132}$$

Thus, $T_f^{\mu\nu}$ follows the same tensor expression (4.127), with $P = \rho/3$. Although each individual particle does not have a four-velocity, we can still define the average four-velocity as $U^\mu = (1, 0, 0, 0)^T$ in flat space, since the particle velocities in different directions cancel out.

Chapter 5

Cosmology and the expanding universe

We conclude with a basic introduction to cosmology and the history of the universe. This will apply many of the concepts that we have learned. We will omit most of the intermediate calculations, but you should go through them for practice, since they are not particularly onerous.

On a large scale, the universe is homogeneous and isotropic in space $\vec{x}$, but evolves in time T. The general form of the metric under these assumptions is:

$$ds^2 = -f(T)dT^2 + H_{ij}(T, \vec{x})dx^i dx^j, \tag{5.1}$$

for some 3×3 matrix H_{ij}. $f(T)$ is some function of time but not of space, due to spatial homogeneity. We may change to another time coordinate $t(T)$ so that the coefficient of dt^2 is 1:

$$f(T)dT^2 = dt^2. \tag{5.2}$$

There are no $dt\, dx^i$ terms due to isotropy of space. We additionally assume that H_{ij} is separable into a function of time $a(t)^2$ and a function of space $h_{ij}(\vec{x})$:

$$ds^2 = -dt^2 + a(t)^2 h_{ij}(\vec{x})dx^i dx^j. \tag{5.3}$$

This is called the *Friedmann–Lemaître–Robertson–Walker (FLRW)* metric. $a(t)$ is known as the *scale factor*. At fixed time t, it scales all lengths dx to get the physical distance ds. We will also scale the coordinates $\vec{x}$ so that $a(t_0) = 1$, where t_0 is the current time. This can be done since $a(t)$ is

essentially constant within human timescales. In matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2 h_{ij} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2} h^{ij} \end{pmatrix}, \tag{5.4}$$

where h^{ij} is the inverse matrix of h_{ij}. An increasing a over time corresponds to an expanding universe. This can be visualized as the size of lightcones decreasing over time in coordinate space (Fig. 5.1).

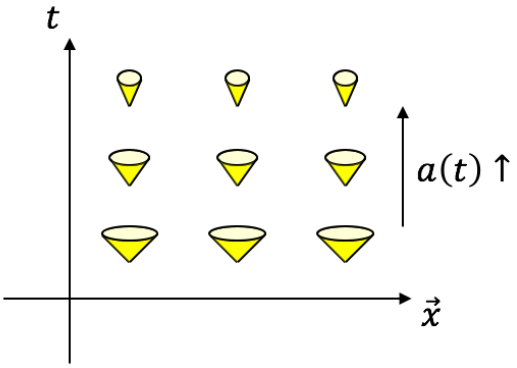


Figure 5.1: Lightcones over time for an expanding universe ($da/dt > 0$).

The nonzero Christoffel symbols are:

$$\Gamma^t_{ij} = a\dot{a}h_{ij}, \quad \Gamma^i_{tj} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \tag{5.5}$$

where the dot means d/dt, and $\tilde{\Gamma}^i_{jk}$ is the Christoffel symbol calculated with the spatial metric h_{ij}: $\tilde{\Gamma}^i_{jk} = (1/2)h^{il}(\partial_j h_{lk} + \partial_k h_{jl} - \partial_l h_{jk})$. In what follows, $\tilde{A}$ will mean the quantity A calculated with the metric h_{ij} and only summing over spatial components.

5.1 Matter, radiation, and dark energy

The matter content is described by a homogeneous ideal fluid (4.127):

$$T^{\mu\nu}_m = (\rho_m(t) + P_m(t))U^\mu U^\nu + P_m(t)g^{\mu\nu}. \tag{5.6}$$

Due to isotropy, the velocity vector is in the t direction: $U^\mu = (1, 0, 0, 0)$. It is correctly normalized so that $U^\mu U_\mu = -1$.

The full equation of motion is (4.97). We use the subscript m in (5.6) because we will absorb the cosmological constant term in (4.97) into $T^{\mu\nu}$, so that the total energy-momentum is:

$$T^{\mu\nu} = T_m^{\mu\nu} - \frac{\Lambda}{8\pi G} g^{\mu\nu} = (\rho + P) U^\mu U^\nu + P g^{\mu\nu}, \tag{5.7}$$

and the equation of motion in terms of T (4.100) becomes:

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{5.8}$$

where we also lower the indices for later convenience.

From (5.7), $T^{\mu\nu}$ and $T_{\mu\nu}$ in matrix form are:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & a^{-2} P h^{ij} \end{pmatrix} \qquad T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & a^2 P h_{ij} \end{pmatrix}. \tag{5.9}$$

The total effective ρ and P are:

$$\begin{aligned} \rho &= \rho_m + \frac{\Lambda}{8\pi G} \\ P &= P_m - \frac{\Lambda}{8\pi G}. \end{aligned} \tag{5.10}$$

Finally, the trace $T = T^\mu_{\ \mu} = 3P - \rho$.

The total energy density and pressure are composed of several sources $\rho = \sum_i \rho_i$ and $P = \sum_i P_i$. Each has an equation of state:

$$P_i = x_i \rho_i. \tag{5.11}$$

- For massive particles at low temperature ($kT \ll m$), $x_i \approx 0$ (4.128). This is commonly just called *matter*. In our universe, ordinary matter makes up only ∼16% of this, while so-called *dark matter* makes up the rest. Dark matter is a hypothetical form of matter that only interacts gravitationally.

- For massless particles, $x_i = 1/3$ (4.131). This is commonly called *radiation*. It is composed of photons and neutrinos (which are nearly massless). Sometimes, only the photons are called radiation.

- For a cosmological constant, $x_i = -1$ (5.10). This is known as *dark energy*. Specifically, dark energy is any substance with positive energy and negative pressure that is highly uniform in space. The cosmological constant with $\Lambda > 0$ is the simplest model of this substance that agrees with observation. The fundamental origins of dark energy and dark matter remain mysterious.

Typically, one source dominates at a time, so we will simply write $P = x\rho$ and consider each x separately.

Let's start by applying energy-momentum conservation (4.105). This will relate ρ and P to the scale factor a. We have:

$$\nabla_\mu T^{\mu 0} = \dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0. \tag{5.12}$$

$$\nabla_\mu T^{\mu i} = \tilde{\nabla}_j T^{ji} = 0, \tag{5.13}$$

where the dot denotes $\partial/\partial t$. Note that (5.13) is automatically satisfied since $\tilde{\nabla}_j h^{ji} = 0$.

Rewrite (5.12) as:

$$\frac{\dot{\rho}}{\dot{a}} = \frac{d\rho}{da} = -3\left(\frac{\rho + P}{a}\right), \tag{5.14}$$

since the dt cancels out. Thus, ρ and P are functions of a. Using $P = x\rho$ gives:

$$\frac{d\rho}{\rho} = -3(1 + x)\frac{da}{a}. \tag{5.15}$$

Integrating, we get:

$$\rho = \rho_0 a^{-3(1+x)}, \tag{5.16}$$

where ρ_0 is the energy density at the current time t_0. For matter, $\rho \propto a^{-3}$, since a constant amount of energy is divided by the volume $V \propto a^3$. For radiation, $\rho \propto a^{-4}$. One way to understand this is as follows. In quantum physics, each field mode i contributes an energy $\hbar\omega_i$, so the total energy is: $E = \sum_i \hbar\omega_i N_i$, where N_i is the number of particles in mode i. Since $\omega \sim |\vec{k}| \propto 1/a$, the energy density E/V acquires another factor of $1/a$. Finally, for the cosmological constant, ρ is constant.

At small a, the universe was dominated by radiation. As a increased, matter began to dominate, followed by dark energy (Fig. 5.2).

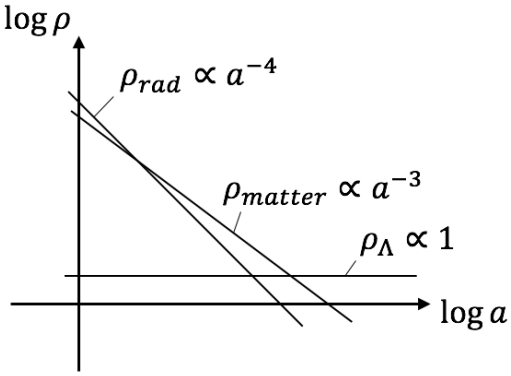


Figure 5.2: Schematic log-log plot of energy density ρ versus a, for various forms of energy.

5.2 Shape of the universe

Plugging (5.5) into (3.71), the Ricci tensor is:

$$R_{tt} = -\frac{3\ddot{a}}{a} \tag{5.17}$$

$$R_{ij} = \tilde{R}_{ij} + (2\dot{a}^2 + a\ddot{a})h_{ij}. \tag{5.18}$$

Plugging these into (5.8), we get:

$$-\frac{3\ddot{a}}{a} = 4\pi G\left(\rho + 3P\right) \tag{5.19}$$

$$\tilde{R}_{ij} + (2\dot{a}^2 + a\ddot{a})h_{ij} = 4\pi G a^2\left(\rho - P\right)h_{ij}. \tag{5.20}$$

From the second equation, we see that

$$\tilde{R}_{ij} = c_2 h_{ij} \tag{5.21}$$

for some constant c_2. In fact, this is already required by spatial homogeneity and isotropy. Intuitively, for a space where every point looks the same, $\tilde{R}_{ij}$ can only be formed from h_{ij} and constants. Similarly, the spatial Riemann tensor $\tilde{R}_{ijkl}$ must only involve h_{ij} while satisfying the symmetries (3.70). This gives:

$$\tilde{R}_{ijkl} = c_1(h_{ik}h_{jl} - h_{il}h_{jk}) \tag{5.22}$$

for some constant c_1. By contracting indices to form $\tilde{R}_{ij}$, we have: $2c_1 = c_2$.

Mathematically, a homogeneous and isotropic space is called *maximally symmetric*. There are three maximally symmetric spaces of dimension n, one for each sign of c_2: the n-sphere S^n ($c_2 > 0$), Euclidean space E^n ($c_2 = 0$), and hyperbolic space H^n ($c_2 < 0$). We have already encountered the 2-sphere S^2 and hyperbolic space H^2 in Exs. 3.8 and 3.10. You can verify that (5.22) and (5.21) hold for these spaces with $c_1 = c_2 = \pm 1/R^2$, where R is the characteristic length. In general, for S^n and H^n, $c_1 = \pm 1/R^2$ and $c_2 = \pm(n-1)/R^2$. For our 3D space, we have $c_2 = 2k/R^2$ for $k = -1, 0,$ or 1:

$$\tilde{R}_{ij} = \frac{2k}{R^2} h_{ij}. \tag{5.23}$$

A more rigorous discussion of maximally symmetric spaces involves isometries and isometry groups. We will not cover this here for brevity.

In cosmology, $k = 1$ is called a *closed* universe, since a sphere has a finite volume. $k = 0$ is called a *flat* universe. $k = -1$ is called an *open* universe, since a hyperbolic space has infinite volume. Observations suggest our universe is nearly flat, but a very small curvature cannot be ruled out.

5.3 Time evolution and fate of the universe

Let us finally solve for $a(t)$. First, solve for $\ddot{a}$ in (5.19). Then plug into (5.20) and use (5.23) to get:

$$\dot{a} = \sqrt{\frac{8\pi G}{3} a^2 \rho(a) - \frac{k}{R^2}}. \tag{5.24}$$

We take the positive square root $\dot{a} > 0$ for an expanding universe. $\rho(a)$ is found using (5.14) and the equations of state (5.11). This equation can have many different behaviors depending on k and the composition of ρ. We will only consider cases where one source of energy dominates.

First, consider the flat case $k = 0$. Set $\rho = \rho_0 a^{-p}$, with $p = 3$ in the matter-dominated case and $p = 4$ in the radiation-dominated case. After separation of variables and integrating, we get:

$$a = \left(\frac{p}{2} H_0 (t - t_0) + 1\right)^{2/p}, \tag{5.25}$$

where $H_0 \equiv \sqrt{8\pi G\rho_0/3}$. Recall that $a(t_0) = 1$ by definition. As $t \to \infty$, $a \propto t^{2/3}$ for matter and $a \propto t^{1/2}$ for radiation. We also have $a = 0$ at time $t_s = t_0 - 2/(pH_0)$, when space ceased to exist. This is called the *big bang*.

For a cosmological constant, $\rho = \rho_0$, and we have

$$a = e^{H_0(t-t_0)}. \tag{5.26}$$

We are currently in the dark-energy-dominated era with such an exponential expansion. In this case, H_0 is also related to Λ as: $H_0 = \sqrt{\Lambda/3}$, using (5.10).

Next, consider the open universe with $k = -1$. If the curvature term $1/R^2$ in (5.24) dominates, then we simply have a linear growth:

$$a = \frac{t - t_0}{R} + 1. \tag{5.27}$$

If there is matter and radiation but no cosmological constant, the curvature term eventually dominates, since matter and radiation get diluted as a^{-3} or a^{-4}. If there is a cosmological constant $\Lambda > 0$, it will eventually dominate.

Finally, the most interesting case is the closed universe with $k = 1$. From (5.24), we see that ρ must be greater than the minimum density

$$\rho_{\text{min}} = \frac{3}{8\pi G a^2 R^2}. \tag{5.28}$$

For matter and radiation, ρ decreases as the universe expands. Once it reaches ρ_{min}, the expansion stops and the universe starts to contract, eventually reaching $a = 0$. This is called a *big crunch*. For radiation with $\rho = \rho_0 a^{-4}$, we can solve (5.24) exactly to obtain:

$$a_{\text{rad}} = \sqrt{(H_0 R)^2 - \left(H_0 R - \frac{t}{R}\right)^2}, \tag{5.29}$$

where we shift time so the big bang is at $t = 0$. This is plotted in Fig. 5.3. In order to get $a \geq 1$, we must have $H_0 R \geq 1$, which is the same as the condition $\rho_0 \geq \rho_{\text{min}}$ at $a = 1$. For matter with $\rho = \rho_0 a^{-3}$, (5.24) cannot be solved analytically for $a(t)$, but the overall behavior is similar. For the cosmological constant with $\rho = \rho_0$, the expansion accelerates and there is no big crunch.

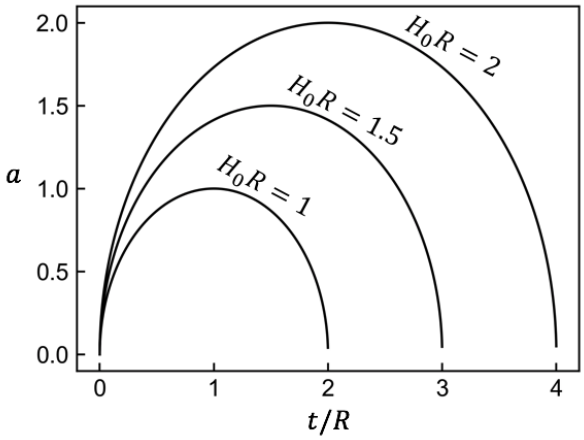


Figure 5.3: $a(t)$ in a radiation-dominated closed universe, for different values of H_0R.

You may wonder why the universe expands in the first place, since matter should cause it to contract by gravitational attraction. It comes down to the difference between velocity and acceleration*. If we assume the universe is expanding in the first place, which we indeed observe, then matter causes the expansion to decelerate, as seen in the sublinear time dependence of (5.25). More directly, from (5.19), the expansion decelerates if the pressure $P > -\rho/3$, and accelerates otherwise. Thus, the important feature of dark energy that causes accelerating expansion is not the energy density itself, but rather the negative pressure.

*Zee, *Einstein Gravity in a Nutshell*, p. 500.

Conclusion

In this primer, I have tried to explain Einstein's theory of relativity as simply and deeply as possible. By keeping it short, I had to leave out some fascinating topics: rotating/charged black holes, causal structure/Penrose diagrams, isometries, de Sitter/anti-de Sitter space, and differential forms, to name a few. Nevertheless, I hope that it provides a solid foundation for further learning. Of the textbooks I have read, I highly recommend the following two:

- Dirac, Paul. *General Theory of Relativity*.
- Zee, Anthony. *Einstein Gravity in a Nutshell*.

Appendix A

Linear algebra

Here, we review some basic linear algebra.

Following the mathematician's style, we start with vector spaces instead of matrices and linear equations. This approach is more elegant but may be unfamiliar for some readers. To guide intuition, keep in mind the idea of a vector as an arrow in 2D or 3D space: something with a magnitude and direction. These are added "tip to tail", placing the tail of one vector on the tip of another. As you read, check how each definition applies to such vectors.

A.1 Vector spaces

A *vector space* consists of a set of vectors V as well as vector addition and scaling operations. The vector space must be closed under vector addition and scaling[*]: $\vec{v} + \vec{w} \in V$ for all $\vec{v}, \vec{w} \in V$, and $a\vec{v} \in V$ for all $\vec{v} \in V$ and a a real number[†]. It must also contain a zero vector $\vec{0}$ that satisfies $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$, and $0\vec{v} = \vec{0}$ for all $\vec{v} \in V$. We only consider real vector spaces here. For complex vector spaces, a is a complex number.

[*]The "$\in$" symbol means "is an element of".

[†]Vector addition and scaling must also satisfy some boring and obvious properties like commutativity ($\vec{v} + \vec{w} = \vec{w} + \vec{v}$), etc. For a full definition, see Wikipedia.

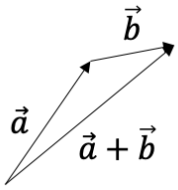
Example

$\mathbb{R}^2$ is the set of 2-tuples of real numbers $\vec{v} = (v_1, v_2)$. This is a vector space if we define

- Addition $\vec{u} = \vec{v} + \vec{w}$ as $u_1 = v_1 + w_1, u_2 = v_2 + w_2$
- Scaling $\vec{u} = a\vec{v}$ as $u_1 = av_1, u_2 = av_2$.

It is clearly closed under these operations: multiplying and adding real numbers gives real numbers. You may also easily check the zero vector $\vec{0} = (0, 0)$ satisfies the required properties.

As mentioned above, vectors in $\mathbb{R}^2$ can be visualized as arrows in the plane. Vectors are added by placing them "tip to tail":



We may easily generalize to $\mathbb{R}^d$, the set of d-tuples of real numbers $(v_1, v_2, \cdots, v_d)$.

A *linear combination* of the vectors $\{\vec{v}_i\}$ is any weighted sum:

$$\sum_i c^i \vec{v}_i \tag{A.1}$$

where the $\{c^i\}$ are real numbers that are not all zero.

A set of vectors is *linearly dependent* if some linear combination of them equals zero. Otherwise they are linearly independent.

A *basis* is a set of linearly independent vectors $\{\vec{e}_i\}$ in V such that all vectors $\vec{v}$ in V can be formed from a linear combination of basis vectors:

$$\vec{v} = \sum_i v^i \vec{e}_i. \tag{A.2}$$

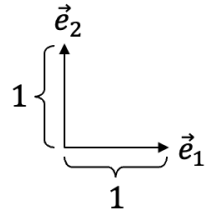
v^i are the *components* of the vector $\vec{v}$. The *dimension* d of the vector space is the number of basis vectors. As a shorthand, we may write $\vec{v}$ as a column

vector of its components:

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^d \end{pmatrix}. \tag{A.3}$$

Of course, this representation depends on the chosen basis.

For example, the *standard basis* of $\mathbb{R}^2$ is the set $\{\vec{e}_1 = (1,0), \vec{e}_2 = (0,1)\}$:



Of course, this is not the only possible basis; another one is $\{\vec{e}_1 = (1,0), \vec{e}_2 = (1,1)\}$. Why is $\{\vec{e}_1 = (1,0), \vec{e}_2 = (2,0)\}$ not a basis? Why is $\{\vec{e}_1 = (1,0), \vec{e}_2 = (0,1), \vec{e}_3 = (-1,1)\}$ not a basis?

The *span* of a set of vectors is the vector space formed from taking all linear combinations of the vectors. Thus, we may also define a basis as a set of linearly independent vectors that span the whole space V.

A *subspace* W of a vector space V is a subset of vectors in V that also form a vector space.

Every d-dimensional vector space V has a one-to-one correspondence with $\mathbb{R}^d$: simply choose a basis of V, then the components of any vector $\vec{v} \in V$ are a vector in $\mathbb{R}^d$. One reason we do not simply define a real vector space as $\mathbb{R}^d$ is that this correspondence depends on the chosen basis of V.

* Exercise A.1

1. Show that the vectors $(1,1,1)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$ are linearly dependent.

2. If V is a d-dimensional vector space, show that any set of vectors with more than d elements is linearly dependent.

A.2 Linear functions and matrices

A *linear* function $f(\vec{x})$ is any function that satisfies the distributive and scaling properties: $f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$ and $f(a\vec{x}) = af(\vec{x})$, for vectors $\vec{x}, \vec{y}$ and real number a.

Consider a linear function between vector spaces $f : V \to W$, where V has dimension d_V and W has dimension d_W. This notation means that f takes vectors in V and outputs vectors in W. Let V have a basis $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_{d_V}\}$, and W have a basis $\{\vec{g}_1, \vec{g}_2, \cdots, \vec{g}_{d_W}\}$. Then f is completely specified by its action on basis vectors $\{\vec{e}_i\}$, since all vectors $\vec{v}$ in V can be expanded in this basis:

$$\begin{aligned} f(\vec{v}) &= f\left(\sum_{i=1}^{d_V} v^i \vec{e}_i\right) \\ &= \sum_{i=1}^{d_V} f\left(v^i \vec{e}_i\right) \\ &= \sum_{i=1}^{d_V} v^i f\left(\vec{e}_i\right), \end{aligned} \tag{A.4}$$

using the distributive and scaling properties of f. Let $\vec{w}_i \equiv f(\vec{e}_i)$. Each $\vec{w}_i$ has components $(w_i^1, w_i^2, \cdots, w_i^{d_W})^T$ written as a column vector. Then f can be represented by a matrix A with the $\{\vec{w}_i\}$ as the columns:

$$\begin{aligned} f(\vec{v}) &= \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_{d_V} \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^{d_V} \end{pmatrix} \\ &= \begin{pmatrix} w_1^1 & w_2^1 & & w_{d_V}^1 \\ w_1^2 & w_2^2 & \cdots & w_{d_V}^2 \\ \vdots & \vdots & & \vdots \\ w_1^{d_W} & w_2^{d_W} & & w_{d_V}^{d_W} \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^{d_V} \end{pmatrix} \\ &= A\vec{v}. \end{aligned} \tag{A.5}$$

A has d_W rows and d_V columns and has components $A_i^j \equiv w_i^j$, where the upper (lower) index is the row (column) index. Then $f(\vec{v})$ can also be

written:

$$(f(\vec{v}))^j = \sum_{i=1}^{d_V} A_i^j v^i, \quad 1 \leq j \leq d_W \tag{A.6}$$

In short, matrices represent linear functions between vector spaces.

*** Exercise A.2**

Consider the rotation matrix $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{A.7}$$

for some real angle θ.

1. Using basic trigonometry, show that it rotates the standard basis vectors of $\mathbb{R}^2$ clockwise by θ.
2. Since any vector can be expanded in the standard basis, $T(\theta)$ rotates any vector clockwise by θ. Draw a picture to show this.

A.3 Eigenvectors and eigenvalues

Let A be a $d \times d$ matrix. An *eigenvector* $\vec{v} \neq \vec{0}$ of A satisfies $A\vec{v} = \lambda\vec{v}$ for some real number λ, called the *eigenvalue*. Clearly, $\vec{v}$ remains an eigenvector with the same eigenvalue when it is scaled by any nonzero real number. The *rank* of A is $r = d - n_z$, where n_z is the number of linearly independent zero eigenvectors (eigenvectors with $\lambda = 0$).

Eigenvectors are useful because A acts on them in a simple way: scaling by λ. Intuitively, λ measures how "strongly" A acts in a particular direction $\vec{v}$:

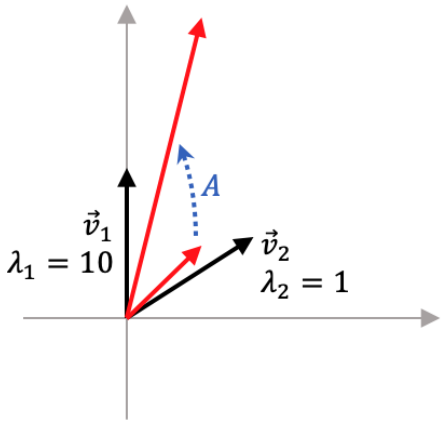


Figure A.1: A has eigenvectors $\vec{v}_1$ and $\vec{v}_2$ with eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 1$. It acts much more strongly in the $\vec{v}_1$ direction than $\vec{v}_2$.

This picture is most useful when A has all d eigenvectors. This is not always the case (Ex. A.3).

We now prove some other facts about eigenvectors. Let $h(\vec{v}) = A\vec{v}$ be the linear function represented by A. h is invertible (meaning the inverse function h^{-1} exists) when $h(\vec{v}) \neq h(\vec{w})$ for all $\vec{v} \neq \vec{w}$. Otherwise, it is non-invertible:

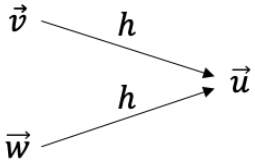


Figure A.2: An example of a *non-invertible* function h. The inverse h^{-1} does not exist because it cannot be defined on $\vec{u}$.

If $h(\vec{v}) = h(\vec{w})$ for some $\vec{v} \neq \vec{w}$, then $h(\vec{v} - \vec{w}) = 0$, so $\vec{v} - \vec{w}$ is a zero eigenvector. Conversely, any zero eigenvector can be written as a difference $\vec{v} - \vec{w}$ for $\vec{v} \neq \vec{w}$. Then $h(\vec{v} - \vec{w}) = 0$ implies $h(\vec{v}) = h(\vec{w})$, so h is non-invertible.

Thus, the rank equals d if and only if* h is invertible. The inverse func-

*"if and only if" is a common phrase in higher math. The statement "A if and only if B" means that A implies B AND B implies A.

tion h^{-1} is represented by the *inverse matrix* A^{-1}, so that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix, $I_i^j = \delta_i^j$.

If A has d eigenvectors $\{\vec{v}_i\}$, it can be written as:

$$A = X\Lambda X^{-1}, \tag{A.8}$$

where X is the matrix whose columns are the $\{\vec{v}_i\}$:

$$X = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_d \end{pmatrix}, \tag{A.9}$$

and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d)$ is the diagonal matrix of eigenvalues. To show this, we have to show that $A\vec{v}_i = \lambda_i \vec{v}_i$ for every eigenvector $\vec{v}_i$ with eigenvalue λ_i. This can be seen by acting A on X. First, X^{-1} acts on each column of X separately, and $X^{-1}X = I$. Thus, for column $\vec{v}_i$, $X^{-1}\vec{v}_i$ picks out the i'th column of the identity matrix. Λ then multiplies by the corresponding eigenvalue λ_i, and X turns it back into $\lambda_i \vec{v}_i$.

Two column vectors $\vec{v}, \vec{w}$ are *orthogonal* if $\vec{v}^T\vec{w} = \sum_i v_i w_i = 0$. A set of vectors $\{\vec{v}_i\}$ is *orthonormal* if $\vec{v}_i^T\vec{v}_j = \delta_{ij}$. If the d eigenvectors of A are orthonormal, then $X^{-1} = X^T$ (show this!). We have:

$$A = X\Lambda X^T = \sum_i \lambda_i \vec{v}_i \vec{v}_i^T. \tag{A.10}$$

*** Exercise A.3**

What is the rank of the rotation matrix $T(\theta)$ in the previous exercise? Does it have any real eigenvectors when $\theta \neq \{0, \pi\}$?

A.4 Determinants and volumes

In this section, we will use the Einstein summation convention, where repeated indices are summed over (p. 19).

The determinant of an $n \times n$ matrix A is:

$$\det A \equiv \frac{1}{n!}\epsilon_{i_1 i_2 \cdots i_n}\epsilon_{j_1 j_2 \cdots j_n} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n}, \tag{A.11}$$

where $\epsilon_{i_1 i_2 \cdots i_n}$ is the totally antisymmetric symbol: it is negated under exchange of any two indices, and we define $\epsilon_{12\cdots n} = 1$. For example, for

$n = 3$:

$$\begin{aligned}\det A &= \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}A_{il}A_{jm}A_{kn} \\ &= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} + A_{12}A_{23}A_{31} \\ &\quad - A_{12}A_{21}A_{33} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}.\end{aligned} \tag{A.12}$$

As you can see, the $1/n!$ in the definition cancels out the $n!$ permutations of indices for each combination of matrix elements. From the definition, we also have:

$$\det A = \det A^T. \tag{A.13}$$

The determinant can be viewed as a multilinear, antisymmetric function of the column vectors in the matrix: $\det(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$. Multilinear means linear in all arguments:

$$\begin{aligned}\det(a\vec{v} + b\vec{w}, \vec{v}_2, \cdots, \vec{v}_n) &= a\det(\vec{v}, \vec{v}_2, \cdots, \vec{v}_n) + b\det(\vec{w}, \vec{v}_2, \cdots, \vec{v}_n) \\ \det(\vec{v}_1, a\vec{v} + b\vec{w}, \cdots, \vec{v}_n) &= a\det(\vec{v}_1, \vec{v}, \cdots, \vec{v}_n) + b\det(\vec{v}_1, \vec{w}, \cdots, \vec{v}_n) \\ &\vdots \\ \det(\vec{v}_1, \vec{v}_2, \cdots, a\vec{v} + b\vec{w}) &= a\det(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}) + b\det(\vec{v}_1, \vec{v}_2, \cdots, \vec{w})\end{aligned} \tag{A.14}$$

Multilinearity comes from the fact that each term in the sum (A.11) contains exactly one element from each column. Antisymmetry means it is negated under exchange of any two arguments. In particular, if $\vec{v}_i = \vec{v}_j$ for any $i \neq j$, the determinant is zero. Antisymmetry comes from the antisymmetry of $\epsilon_{j_1 j_2 \cdots j_n}$.

In fact, the determinant is the unique multilinear, antisymmetric function of n vectors that also satisfies:

$$\det I = \det(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n) = 1. \tag{A.15}$$

where $\hat{e}_i$ is the standard orthonormal basis. This is because every argument can be expanded in this basis, and the determinant reduced to $(\text{constant}) \times \det(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n)$ using antisymmetry and multilinearity. Thus, these three conditions are enough to define the function.

Note that this definition only agrees with the matrix definition when the vector arguments are expanded in the standard basis. Then the $\hat{e}_i$ have components $\hat{e}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$, $\hat{e}_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \end{pmatrix}^T$, etc. In another basis, the $\hat{e}_i$ have different components, so the identity matrix I does not equal the matrix $(\hat{e}_1 \hat{e}_2 \cdots \hat{e_n})$.

* Exercise A.4

Evaluate $\det(3\hat{e}_1 + 2\hat{e}_2, 5\hat{e}_1 - \hat{e}_2)$ using its definition as a multilinear, antisymmetric function satisfying (A.15). Show that it agrees with the matrix determinant of

$$\begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}. \tag{A.16}$$

The determinant of a product of matrices satisfies:

$$\det(AB) = \det(A)\det(B). \tag{A.17}$$

To see this, expand it as:

$$\det(AB) = \det(A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_n), \tag{A.18}$$

where the $\vec{b}_i$ are the column vectors of B. Viewed as a function of the $\vec{b}_i$, this is antisymmetric and multilinear, so must equal $c(A)\det B$ for some constant $c(A)$ depending on A. By a similar argument, this also equals $c(B)\det A$ for some constant $c(B)$ depending on B. Thus, it must equal $\det(A)\det(B)$, with the overall constant fixed by taking $A = B = I$, for example.

The determinant also gives the *oriented volume* $\text{Vol}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ of the parallelpiped spanned by the column vectors (Fig. A.3). Oriented volume is defined as the usual volume but antisymmetric under exchange of vectors. We normalize it by defining $\text{Vol}(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n) = 1$.

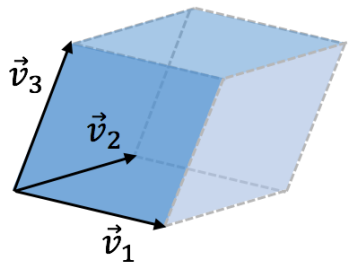


Figure A.3: The oriented parallelpiped spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. It has oriented volume $V = \det(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

Then, in order to show the determinant is the same function, we simply have to show that Vol is multilinear. First, observe that scaling any vector by c also scales the volume by c. For the distributive property, we can decompose any argument vector as $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$, where $\vec{v}_\parallel$ is parallel to the subspace spanned by the remaining vectors and $\vec{v}_\perp$ is perpendicular to the subspace. Then we have:

$$\begin{aligned} \text{Vol}(\vec{v}+\vec{w}, \vec{v}_2, \cdots, \vec{v}_n) &= \text{Vol}(\vec{v}_\parallel + \vec{v}_\perp + \vec{w}_\parallel + \vec{w}_\perp, \vec{v}_2, \cdots, \vec{v}_n) \\ &= \text{Vol}(\vec{v}_\perp + \vec{w}_\perp, \vec{v}_2, \cdots, \vec{v}_n) \\ &= \text{Vol}(\vec{v}_\perp, \vec{v}_2, \cdots, \vec{v}_n) + \text{Vol}(\vec{w}_\perp, \vec{v}_2, \cdots, \vec{v}_n) \\ &= \text{Vol}(\vec{v}, \vec{v}_2, \cdots, \vec{v}_n) + \text{Vol}(\vec{w}, \vec{v}_2, \cdots, \vec{v}_n) \end{aligned} \tag{A.19}$$

On the second line, shifting $\vec{v}_\perp + \vec{w}_\perp$ by a parallel vector $\vec{v}_\parallel + \vec{w}_\parallel$ does not affect the volume, so the parallel part vanishes (Fig. A.4a). On the third line, the perpendicular part distributes since $\vec{v}_\perp$ and $\vec{w}_\perp$ go in the same direction (Fig. A.4b). On the last line, we restore $\vec{v}_\parallel$ and $\vec{w}_\parallel$.

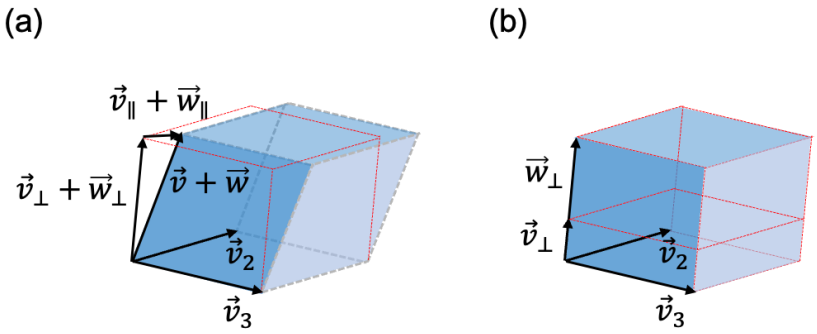


Figure A.4: **(a)** Decomposing $\vec{v}+\vec{w}$ into $\vec{v}_\perp + \vec{w}_\perp$ and $\vec{v}_\parallel + \vec{w}_\parallel$. $\vec{v}_\perp + \vec{w}_\perp$ is perpendicular to $\vec{v}_2$ and $\vec{v}_3$. **(b)** Volume distributes since $\vec{v}_\perp$ and $\vec{w}_\perp$ are parallel.

An infinitesimal volume is defined by the displacement vectors $\{d\vec{x}_{(1)}, d\vec{x}_{(2)}, \cdots, d\vec{x}_{(n)}\}$ as:

$$d^n x \equiv \left|\text{Vol}(d\vec{x}_{(1)}, d\vec{x}_{(2)}, \cdots, d\vec{x}_{(n)})\right|. \tag{A.20}$$

Under a coordinate change $x \to x'$, we have:

$$dx^i_{(a)} = \frac{\partial x^i}{\partial x'^j} dx'^j_{(a)} \tag{A.21}$$

for each a, so that

$$\begin{aligned} d^n x &= \left|\text{Vol}(d\vec{x}_{(1)}, d\vec{x}_{(2)}, \cdots, d\vec{x}_{(n)})\right| \\ &= \left|\text{Vol}(Jd\vec{x}'_{(1)}, Jd\vec{x}'_{(2)}, \cdots, Jd\vec{x}'_{(n)})\right| \\ &= |\det J| \left|\text{Vol}(d\vec{x}'_{(1)}, d\vec{x}'_{(2)}, \cdots, d\vec{x}'_{(n)})\right| \\ &= |\det J|\, d^n x', \end{aligned} \tag{A.22}$$

where $J^i_j \equiv \frac{\partial x_i}{\partial x'_j}$ is the Jacobian. We use (A.18) and the product rule (A.17) on the third line.

Finally, the determinant is non-zero if and only if the matrix A is invertible. To see this, assume A has a zero eigenvector $\vec{v}$, so is not invertible. The equation $A\vec{v} = 0$ gives a linear combination of column vectors of A that equals zero, from (A.5). Thus, we can rewrite one of the column vectors in this linear combination in terms of the others*. Then by antisymmetry and multilinearity, the determinant equals zero. Conversely, if A has no zero eigenvectors, the column vectors are all linearly independent. Then the volume spanned by the column vectors is non-zero.

To summarize, the following conditions on an $n \times n$ matrix A are all equivalent:

- A has rank n
- A is invertible
- The column vectors of A are linearly independent
- $\det A \neq 0$

*Or, if there is only one column vector in this linear combination, it equals zero and the determinant is trivially zero.

Appendix B

Lorentz transformation from moving clocks

We can arrive at the Lorentz transformation by first deriving time dilation. Consider two clocks C and C' moving with velocity $-v/2$ and $+v/2$ respectively, starting from the origin. They are synchronized to read 0 at the origin. They send light signals to each other when they each read time τ_s. These signals are received when they each read time τ_r. Now view this situation from an IRF where C is stationary at $x = 0$ (Fig. B.1). C' now moves on the path $x = vt$. The t-axis in this IRF now corresponds to the times read by C, but *not* necessarily the times read by the moving clock C'. Instead, C' now sends a signal at time $t = t_s$ and receives a signal at time $t = t_r$, *as measured in the IRF*.

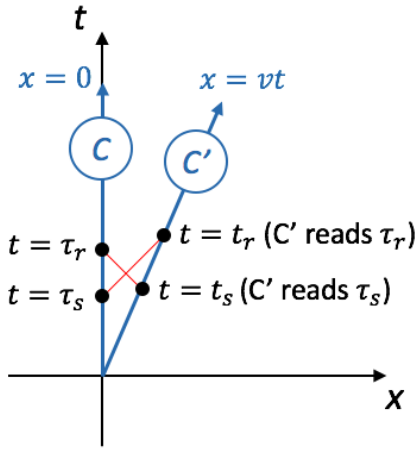


Figure B.1: Clocks C and C' each send signals (red) when they read τ_s and receive signals when they read τ_r. In the IRF shown, C is stationary at $x = 0$.

From Fig. B.1, we have:

$$\begin{aligned} \tau_r &= t_s + vt_s \\ t_r &= \tau_s + vt_r, \end{aligned} \tag{B.1}$$

using the speed of light $c = 1$. Because τ_s is arbitrary and time is homogeneous, t_s (the time in the IRF) is related to τ_s (the time on C') by a constant factor: $t_s = \gamma\tau_s$. Likewise, $t_r = \gamma\tau_r$. Combine these four equations and solve for γ. After some algebra, we obtain:

$$\gamma = \frac{1}{\sqrt{1 - v^2}}. \tag{B.2}$$

Since $\gamma > 1$, the time on the moving clock (τ_s) is smaller than the time in the IRF (t_s). This is called *time dilation*. Note that as $v \to 1$, $\gamma \to \infty$. If we take t_s as finite, τ_s is "frozen" at 0. Thus, the speed of light $c = 1$ is the speed limit for relative motion of IRFs.

Back to the Lorentz transformation. C' is also the clock at $x' = 0$ in IRF I' (Fig. 1.3). It reads time t'. For the point $X' = (t', 0)^T$, we now have $t = \gamma t'$ and $x = vt = v\gamma t'$. This gives two of the matrix elements of $\Lambda(v)$:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \Lambda_{12} \\ \gamma v & \Lambda_{22} \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}. \tag{B.3}$$

To find the other two matrix elements, invert Eq. (1.3):

$$X' = \Lambda^{-1}(v)X. \tag{B.4}$$

IRF I' is moving with velocity v relative to I, so I is moving with velocity $-v$ relative to I'. This gives:

$$\Lambda^{-1}(v) = \Lambda(-v). \tag{B.5}$$

The inverse of a 2×2 matrix is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{B.6}$$

as you may verify. We have:

$$\Lambda^{-1}(v) = \frac{1}{\gamma(d - bv)} \begin{pmatrix} d & -b \\ -\gamma v & \gamma \end{pmatrix}, \tag{B.7}$$

where $b = \Lambda_{12}(v)$, $d = \Lambda_{22}(v)$. Plugging this into (B.5), we get:

$$\frac{1}{\gamma(d - bv)} \begin{pmatrix} d & -b \\ -\gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \bar{b} \\ -\gamma v & \bar{d} \end{pmatrix}, \tag{B.8}$$

where $\bar{b} = \Lambda_{12}(-v)$, $\bar{d} = \Lambda_{22}(-v)$. The bottom left equation gives:

$$\gamma(d - bv) = 1, \tag{B.9}$$

or

$$b = \frac{\gamma d - 1}{\gamma v}. \tag{B.10}$$

Then the top left equation gives:

$$d = \gamma. \tag{B.11}$$

Plugging back into (B.10), we get:

$$b = \frac{\gamma^2 - 1}{\gamma v} = \gamma v, \tag{B.12}$$

using the definition of γ (B.2).

Thus, we arrive at Eq. (1.15).

I personally find such derivations harder to remember than the one in the main text, because linear transformations are best understood using their eigenvectors and eigenvalues. You may have a different view!

Appendix C

Vector calculus

C.1 Identities

In this section, we derive some 3D vector calculus identities using index notation, which may be unfamiliar for some readers but is more flexible than traditional vector notation. Here,

- $\partial_j \equiv \partial/\partial x^j$ is the partial derivative operator
- ϵ_{ijk} is the totally antisymmetric symbol. Indices i, j, k go from 1 to 3, so it has $3^3 = 27$ components total. It is negated under exchange of any two indices, and $\epsilon_{123} \equiv 1$. Explicitly: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$, with all other entries 0.

We also use the Einstein summation convention where repeated indices are summed over.

Product of antisymmetric symbols

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{C.1}$$

This is shown as follows. Since k is the same index on both ϵ's, (ij) must be the same indices as (lm) in some order. There are only two orderings, corresponding to the two terms with different signs.

Div, curl and cross product

$$\nabla \cdot \vec{A} \equiv \partial_i A_i \tag{C.2}$$

$$\left(\nabla \times \vec{A}\right)_i \equiv \epsilon_{ijk}\partial_j A_k \tag{C.3}$$

$$\left(\vec{A} \times \vec{B}\right)_i \equiv \epsilon_{ijk}A_j B_k \tag{C.4}$$

Dot product of cross product

$$\begin{aligned}\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) &= \epsilon_{ijk}A_i B_j C_k \\ &= \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) \\ &= \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)\end{aligned} \tag{C.5}$$

Cross product of curl

$$\begin{aligned}\left(\vec{A} \times \left(\nabla \times \vec{B}\right)\right)_i &= \epsilon_{ijk}A_j\epsilon_{klm}\partial_l B_m \\ &= A_j\partial_l B_m(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &= A_j\partial_i B_j - A_j\partial_j B_i\end{aligned} \tag{C.6}$$

Div of curl

$$\left(\nabla \cdot \left(\nabla \times \vec{A}\right)\right)_i = \partial_i\epsilon_{ijk}\partial_j A_k = 0 \tag{C.7}$$

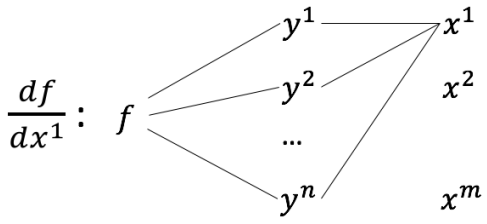
due to the antisymmetry of ϵ_{ijk}.

C.2 Chain rule

The multidimensional chain rule is:

$$\frac{df}{dx^j}(\vec{y}(\vec{x})) = \sum_i \frac{\partial f}{\partial y^i}(\vec{y}(\vec{x}))\frac{\partial y^i}{\partial x^j}(\vec{x}) \tag{C.8}$$

for a function $f(\vec{y}(\vec{x}))$. You can visualize this using lines connecting all possible paths to a variable:



This extends to further nested functions:

$$\frac{df}{dx^j}(\vec{z}(\vec{y}(\vec{x}))) = \sum_{i,k} \frac{\partial f}{\partial z^k}(\vec{z}(\vec{y}(\vec{x}))) \frac{\partial z^k}{\partial y^i}(\vec{y}(\vec{x})) \frac{\partial y^i}{\partial x^j}(\vec{x}) \tag{C.9}$$

for a function $f(\vec{z}(\vec{y}(\vec{x})))$.

Appendix D

Delta functions

This appendix introduces Dirac delta functions in 1D and higher dimensions. Delta functions are useful in many areas of science and engineering. For example, they are used to describe point masses and point charges, as in Sec. 2.11.

The *Dirac delta function* $\delta(x)$ is heuristically defined as an infinitely high and narrow peak at $x = 0$:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{D.1}$$

for x a real number. It is normalized by defining

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{D.2}$$

With these properties, it is clearly not a function in the normal sense. It can be thought of as the limit of a sequence of functions that grow to infinity at $x = 0$ while going to zero at $x \neq 0$. For example, consider the functions

$$f_l(x) \equiv \frac{1}{\pi l}\left(\frac{1}{1 + (x/l)^2}\right), \tag{D.3}$$

parametrized by $l > 0$. This looks like a peak of width πl and height $1/(\pi l)$, centered at zero (Fig. D.1). As you may verify, $\int_{-\infty}^{\infty} f_l(x)\, dx = 1$, for any l. As $l \to 0$, the peak becomes infinitely high and narrow:

$$\lim_{l \to 0^+} f_l(x) = \delta(x). \tag{D.4}$$

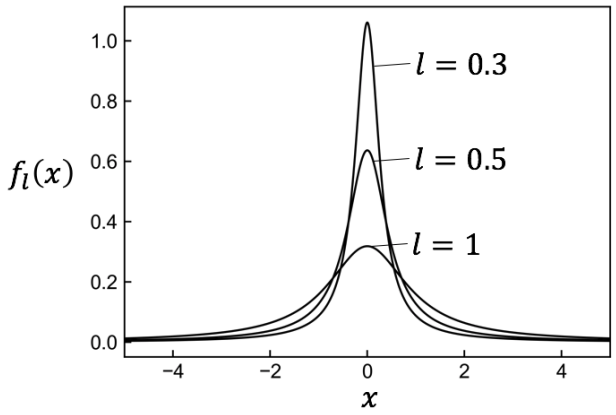


Figure D.1: Functions $f_l(x)$ for $l = 1$, $l = 0.5$, and $l = 0.3$.

In an integral, $\delta(x)$ has the property of picking out the value at zero of any function $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0). \tag{D.5}$$

This is because the integrand is zero away from $x = 0$, so the result can only be proportional to $f(0)$. The proportionality constant must be 1, to agree with (D.2) when $f(x) = 1$.

This property can be regarded as a more rigorous definition of $\delta(x)$: it is a functional* of $f(x)$ that returns the real number $f(0)$. However, the heuristic definition above is more useful, since thinking of $\delta(x)$ as a normal function allows us to perform the usual manipulations inside integrals.

For example, define the function $g(x) \equiv f(x + a)$. Then we have

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\delta(x)\,dx &= \int_{-\infty}^{\infty} f(x+a)\delta(x)\,dx \\ &= \int_{-\infty}^{\infty} f(y)\delta(y-a)\,dy \\ &= g(0) \\ &= f(a).\end{aligned} \tag{D.6}$$

On the second line, we change variables $y = x + a$. On the third line, we

*Function of a function, see Ch. 2.

use (D.5). This gives the so-called *sifting property*

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{D.7}$$

for any function $f(x)$. The delta function $\delta(x-a)$ "sifts out" the value $f(a)$ from the integral.

Another useful identity is:

$$\int_{-\infty}^{\infty} \delta(f(x))\,dx = \sum_{x\,:\,f(x)=0} \frac{1}{|f'(x)|}, \tag{D.8}$$

where $f(x)$ is any function, and the sum is over all values x such that $f(x)=0$. To prove this, expand $f(x)$ around some zero x_0 as:

$$y \equiv f(x) \approx f'(x_0)(x-x_0). \tag{D.9}$$

Since the integrand is only nonzero at x_0, we only need the first order term, and can integrate over a small neighborhood $[x_0-\epsilon, x_0+\epsilon]$:

$$\begin{aligned}\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(f(x))\,dx &= \int_{-f'(x_0)\epsilon}^{f'(x_0)\epsilon} \frac{\delta(y)}{f'(x_0)}\,dy \\ &= \frac{1}{|f'(x_0)|}.\end{aligned} \tag{D.10}$$

On the first line, we change variables $y = f(x)$. Note that if $f'(x_0) < 0$, we are integrating y from a positive to a negative value. Therefore, the integral picks up an extra minus sign, turning $f'(x_0)$ into $|f'(x_0)|$. Finally, if $f(x)$ has multiple zeros, sum over them to obtain (D.8).

D.1 Higher dimensions

In n dimensions with coordinates $\vec{x} = (x^1, x^2, \cdots, x^n)^T$, the delta function $\delta^n(\vec{x})$ is defined as:

$$\delta^n(\vec{x}) = \delta(x^1)\delta(x^2)\cdots\delta(x^n). \tag{D.11}$$

The sifting property becomes:

$$\int f(\vec{x})\delta^n(\vec{x}-\vec{a})\,d^n x = f(\vec{a}) \tag{D.12}$$

for any function $f(\vec{x})$ and vector $\vec{a}$ (including $\vec{a} = \vec{0}$). $d^n x \equiv dx^1 dx^2 \cdots dx^n$ is the volume form.

Eq. (D.8) becomes:

$$\int \delta^n(\vec{f}(\vec{x}))\, d^n x = \sum_{\vec{x}\,:\,\vec{f}(\vec{x})=\vec{0}} \frac{1}{|\det(J(\vec{x}))|} \tag{D.13}$$

where $\vec{f}(\vec{x})$ is a vector-valued function, and $J(\vec{x})$ is the Jacobian matrix at $\vec{x}$, with entries $J_{ij} = \partial f^i / \partial x^j$. This can be derived in a similar way as (D.8). Around a zero $\vec{x}_0$, we have:

$$\vec{f}(\vec{x}) \approx J(\vec{x}_0)(\vec{x} - \vec{x}_0). \tag{D.14}$$

Under a change of coordinates $\vec{y} = \vec{f}(\vec{x})$, we have (A.22): $d^n y = |\det(J(\vec{x}_0))| d^n x$. This gives:

$$\int \delta^n(\vec{f}(\vec{x}))\, d^n x = \int \frac{\delta^n(\vec{y})}{|\det(J(\vec{x}_0))|}\, d^n y = \frac{1}{|\det(J(\vec{x}_0))|}. \tag{D.15}$$

Again, we sum over zeros of $\vec{f}(\vec{x})$ to get (D.13).

For a linear transformation $\vec{y} = A\vec{x}$, the Jacobian $J = A$ is constant, and we simply have:

$$\int \delta^n(A\vec{x})\, d^n x = \frac{1}{|\det(A)|}. \tag{D.16}$$

Therefore, we can also write:

$$\delta^n(\vec{x}) = |\det(A)| \delta^n(A\vec{x}). \tag{D.17}$$

Appendix E

Riemann tensor components

The Riemann tensor $R_{\rho\nu\sigma\alpha}$ satisfies:

$$\begin{aligned} &R_{\rho\nu\sigma\alpha} = -R_{\rho\nu\alpha\sigma} = -R_{\nu\rho\sigma\alpha} \quad \text{(Antisymmetry)} \\ &R_{\rho\nu\sigma\alpha} + R_{\rho\sigma\alpha\nu} + R_{\rho\alpha\nu\sigma} = 0 \quad \text{(First Bianchi identity)} \end{aligned} \tag{E.1}$$

The number of independent components of $R_{\rho\nu\sigma\alpha}$ in d dimensions is:

$$\frac{d^2(d^2-1)}{12}. \tag{E.2}$$

For $d = 4$, this is 20.

One derivation is as follows. First, any component with 3 or 4 of the same index is zero due to antisymmetry. The next largest number of common indices is 2 of one index and 2 of another, such as R_{1212}. Next are components with 3 different indices, such as R_{1213}. Finally, there are components with all different indices, such as R_{1234}. The number of ways to choose each index pattern is given in Table E.1 as a function of d. For example, for $d = 4$, there is only $\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 1$ way to choose 4 different indices. Here, $\begin{pmatrix} n \\ k \end{pmatrix} \equiv n!/(k!(n-k)!)$ is the choose function.

Index pattern	# choices	Antisymmetry	Antisymmetry+Bianchi
R_{1212}	$\binom{d}{2}$	1	1
R_{1213}	$\binom{d}{2}(d-2)$	2	1
R_{1234}	$\binom{d}{4}$	6	2

Table E.1: Number of ways to choose each index pattern, and number of components after imposing antisymmetry or antisymmetry + Bianchi.

For each index pattern, we can apply antisymmetry and the first Bianchi identity to any particular choice of indices. First, pattern R_{1212} has only 1 component due to antisymmetry. Pattern R_{1213} has 2: R_{1213} and R_{1312}. These are related by Bianchi, reducing the count to 1. Pattern R_{1234} has $\binom{4}{2} = 6$ by antisymmetry. Applying Bianchi to each index in the first position gives 4 independent equations, reducing the count to 2. These are summarized in Table E.1. Multiplying the number of choices per index pattern by the number of components per choice, and summing, gives:

$$\binom{d}{2} + \binom{d}{2}(d-2) + \binom{d}{4} \cdot 2 = \frac{d^2(d^2-1)}{12}. \tag{E.3}$$