Asch, Mark. *A toolbox for digital twins: from model-based to data-driven*. Society for Industrial and Applied Mathematics, 2022.

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Problem Statement

Bayesian filtering problem

Consider CO_2 monitoring as a Bayesian filtering problem:

$$\mathbf{x}_k \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1}) \qquad \mathbf{y}_k \sim p(\mathbf{y}_k \mid \mathbf{x}_k), \quad k = 0,1,2,\ldots$$

where

- $\mathbf{x}_k \in \mathbb{R}^n$ is the *state* (CO_2 saturation/pressure) vector at time $t = k\Delta t$
- $\mathbf{y}_k \in \mathbb{R}^m$ is the *observation* vector
- $p(\mathbf{x}_k \mid \mathbf{x}_{k-1}) = p(\mathbf{x}_k \mid \mathbf{x}_{1:k-1}, \mathbf{y}_{1:k-1})$ is the Markovian *transition* probability
- $p(\mathbf{y}_k \mid \mathbf{x}_k) = p(\mathbf{y}_k \mid \mathbf{x}_{1:k}, \mathbf{y}_{1:k-1})$ is the *likelihood* of the *measurement* model

Tatsis, Konstantinos E., Vasilis K. Dertimanis, and Eleni N. Chatzi. "Sequential Bayesian Inference for Uncertain Nonlinear Dynamic Systems: A Tutorial." arXiv preprint arXiv:2201.08180 (2022).

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Sequential Bayesian Inference

Calculate *posterior* $p\left(\mathbf{x}_k \mid \mathbf{y}_{1:k}\right)$ for the *state*, $\mathbf{x}_k$, recursively, via the *predictive* distribution:

$$p\left(\mathbf{x}_k \mid \mathbf{y}_{1:k-1}\right) = \int p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right) p\left(\mathbf{x}_{k-1} \mid \mathbf{y}_{1:k-1}\right) \mathrm{d}\mathbf{x}_{k-1}$$

$$= \mathbb{E}_{\mathbf{x}_k \sim p\left(\mathbf{x}_{k-1} \mid \mathbf{y}_{1:k-1}\right)} \left[p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right)\right]$$

followed by the *correction* step involving Bayes formula:

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) = \frac{\overbrace{p(\mathbf{y}_k \mid \mathbf{x}_k)}^{\text{likelihood}} \; \overbrace{p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1})}^{\text{"prior"}}}{\int p(\mathbf{y}_k \mid \mathbf{x}_k) p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1}) \mathrm{d}\mathbf{x}_k}$$

- **Marginalization over state $\mathbf{x}_{k-1}$ in Chapman-Kolmogorov integral, and**
- **Integral for evidence are both computationally unfeasible!**