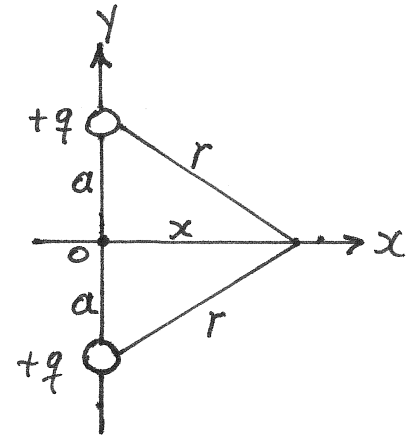
Taller 8

1. $\boxed{23.22}$

a)



b) El potencial para cargas puntuales: $V = \sum_{i=1}^{N} k \frac{q_i}{r_i}$.

En el origen: $V_0 = k\left(\frac{q}{a} + \frac{q}{a}\right)$

$$\boxed{V_0 = 2k\frac{q}{a}}$$

c) Sobre cualquie punto en el eje x: $V(x) = k\left(\frac{q}{r} + \frac{q}{r}\right)$

$$\boxed{V(x) = 2\frac{kq}{r} = \frac{2kq}{\sqrt{a^2+x^2}}}$$

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4. $\boxed{23.48}$

$$\boxed{V = \frac{kQ}{\sqrt{x^2+y^2+z^2}}}$$

$r = (x^2+y^2+z^2)^{1/2}$

a) Debido a que el campo electrostático es conservativo, entonces, éste debe ser igual a menos el gradiente de un potencial:

$\boxed{\vec{E} = -\vec{\nabla} V}$, donde $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\Rightarrow \vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

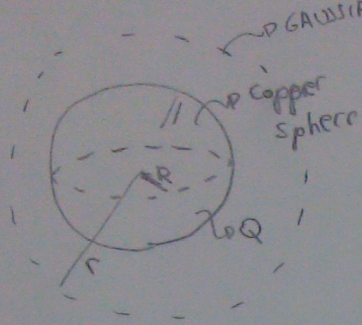
$$\Rightarrow \vec{E} = -\left(-\frac{kQx}{(x^2+y^2+z^2)^{3/2}}\hat{i} - \frac{kQy}{(x^2+y^2+z^2)^{3/2}}\hat{j} - \frac{kQz}{(x^2+y^2+z^2)^{3/2}}\hat{k}\right)$$

$$\vec{E} = \frac{kQ}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{kQ}{r^3}\vec{r} = \frac{kQ}{r^2}\underbrace{\left(\frac{\vec{r}}{r}\right)}_{\hat{r}}$$

$$\boxed{\vec{E} = \frac{kQ}{r^2}\hat{r}}$$

siendo $\frac{\vec{r}}{r} = \hat{r}$ (por definición de vector unitario).

PROBLEM 23.42



$$\oint_{S(r)} \vec{E}\cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0}$$

if $r<R$ the inner charge is 0, then $E=0$

ir $r>R$ the charge is $Q_{int}=Q$, hence

$$E\,4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow \boxed{\begin{array}{l}\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \text{ if } r>R \\ \vec{E}=0 \text{ if } r<R\end{array}}$$

Now $\vec{E} = -\vec{\nabla} V$ hence

$$\int_{r_0}^{r} \vec{E}\cdot d\vec{r} = -\int_{r_0}^{r} \vec{\nabla}\cdot V = -\int_{r_0}^{r} dV \Rightarrow V(r) - V_0 = -\int_{r_0}^{r} \vec{E}\cdot d\vec{r}$$

CASE $r<R$:

$$V(r) - V_0 = 0 \Rightarrow V(r) = V_0 \text{ if } r<R$$

CASE $r>R$:

$$V(r) - V_0 = -\frac{Q}{4\pi\varepsilon_0}\int_{r_0}^{r} r^{-2}\,dr = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\Big|_{r_0}^{r} \Rightarrow V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + \left(V_0 - \frac{Q}{4\pi\varepsilon_0 r_0^2}\right)$$

if r_0 is placed at the infinit $r_0 \to \infty$ then

$$\boxed{V(r) = V_0 + \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}}$$

If $r=R \Rightarrow V(R) - V_0 = \dfrac{Q}{4\pi\varepsilon_0 R} \Rightarrow$

$$\boxed{\begin{array}{l} Q = 4\pi\varepsilon_0[V(R)-V_0]R \\ Q = (9\times10^9\,Nm^2/c^2)(1.5\,kV)(0.25\,m) = 20.8\times10^{-} \end{array}}$$

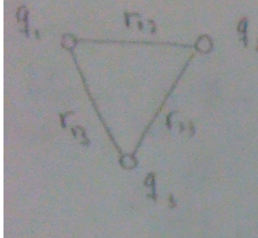
Anwer (a)

PROBLEM 23.57

The energy of a pair of charges is

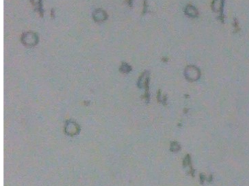
$$U_{12} = q_2 V_{12} = q_2\left(\frac{q_1}{4\pi\varepsilon_0 r_{12}}\right) = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{r_{12}}$$

The energy of a configuration of three point charges is



$$U_{123} = U_{12} + \frac{q_2 q_3}{4\pi\varepsilon_0 r_{23}} + \frac{q_1 q_3}{4\pi\varepsilon_0 r_{13}} = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i=1\\ i<j}}^{3}\sum_{j=1}^{3}\frac{q_i q_j}{r_{ij}}$$

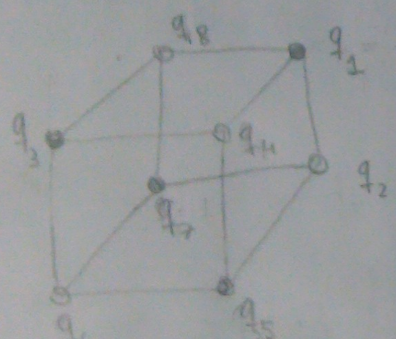
As a result, for $q_1, \dots, q_N$ charges we have



$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i<j}}^{N}\frac{q_i q_j}{r_{ij}}}$$

et us consider the following cristal model where $q_1 = q_3 = q_5 = q_7 = -q$

$q_2 = q_4 = q_6 = q_8 = q$



Therefore, $\boxed{q_i = (-1)^i q}$

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{1}{2}\left(\sum_{\substack{i=1\\ \text{odd } i}}^{N} + \sum_{\substack{i=1\\ \text{even } i}}^{N}\right)\left(\sum_{\substack{j=1\\ \text{odd } j}}^{N} + \sum_{\substack{j=1\\ \text{even } j}}^{N}\right)_{i\neq j}\frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{8\pi\varepsilon_0}\left(\underbrace{\sum_{\substack{i=1\\ \text{odd } i}}^{N}\sum_{\substack{j=1\\ \text{odd } j}}^{N}} + \sum_{\substack{i=1\\ \text{odd } i}}^{N}\sum_{\substack{j=1\\ \text{even } j}}^{N} + \underbrace{\sum_{\substack{i=1\\ \text{even } i}}^{N}\sum_{\substack{j=1\\ \text{odd } j}}^{N}} + \underbrace{\sum_{\substack{i=1\\ \text{even } i,\ \text{even } j}}^{N}\sum_{j=1}^{N}}\right)_{i\neq j}\frac{q_i q_j}{r_{ij}}$$

EQUAL

EQUAL

$$U = \frac{1}{4\pi\varepsilon_0}\left[\left.\sum_{\substack{i=1 \\ \text{odd } i}}^{N} \sum_{\substack{j=1 \\ \text{odd } j}}^{N} \frac{q_i q_j}{r_{ij}}\right|_{i\neq j} + \left.\sum_{\substack{i=1 \\ \text{odd } i}}^{N} \sum_{\substack{j=1 \\ \text{even } j}}^{N} \frac{q_i q_j}{r_{ij}}\right|_{i\neq j} \right]$$

$$U = \frac{q^2}{4\pi\varepsilon_0}\left[\sum_{\substack{i,j=1 \\ \text{odd } i,\ \text{odd } j}}^{N} \frac{1}{r_{ij}} \ \square \sum_{\substack{i,j=1 \\ \text{odd } i,\ \text{even } j}}^{N} \frac{1}{r_{ij}} \right]_{i\neq j}$$

This sign appears for $q_i q_j = -q^2$ if i is odd and j is even

$r_{ij} = \sqrt{d^2 + d^2}$ if i and j are odd

$r_{ij} = \sqrt{2}\, d$

Hence
$$\sum_{\substack{ij=1 \\ \text{odd } i,\ \text{odd } j \\ i\neq j}}^{N} \frac{1}{r_{ij}} = \frac{1}{\sqrt{2}d} \sum_{\substack{i,j=1 \\ \text{odd } i,\ \text{odd } j}}^{N} 1$$

$$= \frac{1}{\sqrt{2}d}\Big(\underset{(1,3)}{1} + \underset{(1,5)}{1} + \underset{(1,7)}{1} + \underset{(3,5)}{1} + \underset{(3,7)}{1} + \underset{(5,7)}{1} + \underset{(3,1)}{1} + \underset{(5,1)}{1} + \underset{(7,1)}{1} + \underset{(5,3)}{1} + \underset{(7,3)}{1} + \underset{(7,5)}{1}$$

$$\sum_{\substack{i,j=1 \\ \text{odd } i,\ \text{odd } j \\ i\neq j}}^{N} \frac{1}{r_{ij}} = \frac{1}{\sqrt{2}\,d}(12)$$

Now

$$\sum_{\substack{i,j=1 \\ \text{odd},\ \text{even} \\ i\neq j}}^{N} \frac{1}{r_{ij}} = \left[\left(\frac{1}{r_{12}} + \frac{1}{r_{18}} + \frac{1}{r_{14}}\right) + \left(\frac{1}{r_{38}} + \frac{1}{r_{34}} + \frac{1}{r_{36}}\right) + \left(\frac{1}{r_{52}} + \frac{1}{r_{54}} + \frac{1}{r_{56}}\right) + \left(\frac{1}{r_{72}} + \frac{1}{r_{78}} + \frac{1}{r_{76}}\right)\right] + \left[\left(\frac{1}{r_{16}} + \frac{1}{r_{58}} + \frac{1}{r_{74}} + \frac{1}{r_{32}}\right)\right]$$

Where $r_{12} = r_{18} = r_{14} = r_{38} = \dots = r_{76} = d$ and $r_{16} = r_{58} = r_{74} = r_{32} = \sqrt{d^2+d^2+d^2} = \sqrt{3}d$

Hence

$$\sum_{\substack{i,j=1 \\ \text{odd},\ \text{even} \\ i\neq j}}^{N} \frac{1}{r_{ij}} = \frac{12}{d} + \frac{4}{\sqrt{3}d}$$

finally

$$\boxed{U = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{2}d} \ \square \left(\frac{12}{d} + \frac{4}{\sqrt{3}d}\right)\right]}$$