Appendix C

Matrix Factorizations

1. $A = LU = \begin{pmatrix}\text{lower triangular } L\\ \text{1s on the diagonal}\end{pmatrix}\begin{pmatrix}\text{upper triangular } U\\ \text{pivots on the diagonal}\end{pmatrix}$

Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $A = LDU = \begin{pmatrix}\text{lower triangular } L\\ \text{1s on the diagonal}\end{pmatrix}\begin{pmatrix}\text{pivot matrix}\\ D \text{ is diagonal}\end{pmatrix}\begin{pmatrix}\text{upper triangular } U\\ \text{1s on the diagonal}\end{pmatrix}$

Requirements: No row exchanges. The pivots in D are divided out to leave 1s in U. If A is symmetric, then U is L^{T} and $A = LDL^{\mathrm{T}}$.

3. $PA = LU$ (permutation matrix P to avoid zeros in the pivot positions).

Requirements: A is invertible. Then P, L, U are invertible. P does the row exchanges in advance. Alternative: $A = L_1P_1U_1$.

4. $EA = R$ $(m \times m$ invertible $E)$ (any $A) = \mathrm{rref}(A)$.

Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The Last $m - r$ rows of E are a basis for the left nullspace of A. and the first r columns of E^{-1} are a basis for the column space of A.

5. $A = CC^{\mathrm{T}} = \Big(\text{lower triangular matrix } C\Big)\Big(\text{transpose is upper triangular}\Big)$

Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* has $C = L\sqrt{D}$.

6. $A = QR = \Big(\text{orthonormal columns in } Q\Big)\Big(\text{upper triangular } R\Big)$

Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt process. If A is square, then $Q^{-1} = Q^{\mathrm{T}}$.

7. $A = S\Lambda S^{-1} = \Big(\text{eigenvectors in } S\Big)\Big(\text{eigenvalues in } \Lambda\Big)\Big(\text{left eigenvectors in } S^{-1}\Big)$.

Requirements: A must have n linearly independent eigenvectors.

8. $A = Q\Lambda Q^{\mathrm{T}} = \Big(\text{orthogonal matrix } Q\Big)\Big(\text{real eigenvalue matrix } \Lambda\Big)\Big(Q^{\mathrm{T}} \text{ is } Q^{-1}\Big).$

Requirements: A is *symmetric*. This is the Spectral Theorem.

9. $A = MJM^{-1} = \Big(\text{generalized eigenvectors in } M\Big)\Big(\text{Jordan blocks in } J\Big)\Big(M^{-1}\Big).$

Requirements: A is any square matrix. *Jordan form* J has a block for each independent eigenvector of A. Each block has one eigenvalue.

10. $A = U\Sigma V^{\mathrm{T}} = \begin{pmatrix}\text{orthogonal}\\ U \text{ is } m\times m\end{pmatrix}\begin{pmatrix} m\times n \text{ matrix } \Sigma\\ \sigma_1,\ldots,\sigma_r \text{ on diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ V \text{ is } n\times n\end{pmatrix}.$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $A^+ = V\Sigma^+U^{\mathrm{T}} = \begin{pmatrix}\text{orthogonal}\\ n\times n\end{pmatrix}\begin{pmatrix}\text{diagonal } n\times m\\ 1/\sigma_1,\ldots,1/\sigma_r\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ m\times m\end{pmatrix}.$

Requirements: None. The *pseudoinverse* has A^+A = projection onto row space of A and AA^+ = projection onto column space. The shortest least-squares solution to $Ax = b$ is $\widehat{x} = A^+b$. This solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

12. $A = QH = \Big(\text{orthogonal matrix } Q\Big)\Big(\text{symmetric positive definite matrix } H\Big).$

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $A = U\Lambda U^{-1} = \Big(\text{unitary } U\Big)\Big(\text{eigenvalue matrix } \Lambda\Big)\Big(U^{-1} = U^H = \overline{U}^{\mathrm{T}}\Big).$

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$.

14. $A = UTU^{-1} = \Big(\text{unitary } U\Big)\Big(\text{triangular } T \text{ with } \lambda\text{'s on diagonal}\Big)\Big(U^{-1} = U^{\mathrm{H}}\Big).$

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular.

15. $F_n = \begin{bmatrix} I & D\\ I & -D\end{bmatrix}\begin{bmatrix} F_{n/2} & \\ & F_{n/2}\end{bmatrix}\begin{bmatrix}\text{even-odd}\\ \text{permutation}\end{bmatrix}$ = one step of the **FFT**.

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$, $w = e^{2\pi i/n}$. Then $F_n\overline{F}_n = nI$. D has $1, w, w^2, \ldots$ on its diagonal. For $n = 2^\ell$ the *Fast Fourier Transform* has $\frac{1}{2}n\ell$ multiplications from ℓ stages of D's.

Appendix D

Glossary: A Dictionary for Linear Algebra

Adjacency matrix of a graph Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ for an undirected graph.

Affine transformation $T(v) = Av + v_0$ = linear transformation plus shift.

Associative Law $(AB)C = A(BC)$ Parentheses can be removed to leave ABC.

Augmented matrix $[A\ \ b]$ $Ax = b$ is solvable when b is in the column space of A; then $[A\ \ b]$ has the same rank as A. Elimination on $[A\ \ b]$ keeps equations correct.

Back substitution Upper triangular systems are solved in reverse order x_n to x_1.

Basis for V Independent vectors $v_1, \ldots, v_d$ whose linear combinations give every v in **V**. A vector space has many bases!

Big formula for n by n determinants $\det(A)$ is a sum of $n!$ terms, one term for each permutation P of the columns. That term is the product $a_{1\alpha} \cdots a_{n\omega}$ down the diagonal of the reordered matrix, times $\det(P) = \pm 1$.

Block matrix A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns.

Block multiplication of AB is allowed if the block shapes permit (the columns of A and rows of B must be in matching blocks).

Cayley-Hamilton Theorem $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix*.

Change of basis matrix M The old basis vectors v_j are combinations $\sum m_{ij} w_i$ of the new basis vectors. The coordinates of $c_1 v_1 + \cdots + c_n v_n = d_1 w_1 + \cdots + d_n w_n$ are related by $d = Mc$. (For $n = 2$, set $v_1 = m_{11} w_1 + m_{21} w_2$, $v_2 = m_{12} w_1 + m_{22} w_2$.)

Characteristic equation $\det(A - \lambda I) = 0$ The n roots are the eigenvalues of A.

Cholesky factorization $A = CC^{\mathrm{T}} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1} S^{n-1}$. $Cx =$ **convolution** $c * x$. Eigenvectors in F.

Cofactor C_{ij} Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $Ax = b$ The vector b becomes a combination of the columns of A. The system is solvable only when b is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ Space of all combinations of the columns of A.

Commuting matrices $AB = BA$ If diagonalizable, they share n eigenvectors.

Companion matrix Put $c_1, \ldots, c_n$ in row n and put $n-1$ 1s along diagonal 1. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots)$.

Complete solution $x = x_p + x_n$ **to** $Ax = b$ (Particular x_p) + (x_n in nullspace).

Complex conjugate $\bar{z} = a - ib$ for any complex number $z = a + ib$.Then $z\bar{z} = |z|^2$.

Condition number $\text{cond}(A) = \kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$ In $Ax = b$, the relative change $\|\delta x\|/\|x\|$ is less than $\text{cond}(A)$ times the relative change $\|\delta b\|/\|b\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method A sequence of steps to solve positive definite $Ax = b$ by minimizing $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ over growing Krylov subspaces.

Covariance matrix Σ When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\bar{x}_i$, the matrix Σ = mean of $(x - \bar{x})(x - \bar{x})^{\mathrm{T}}$ is positive (semi)definite; it is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$ B_j has b replacing column j of A, and $x_j = |B_j|/|A|$.

Cross product $u \times v$ **in** $\mathbf{R}^3$ Vector perpendicular to u and v, length $\|u\|\|v\||\sin\theta|$ = parallelogram area, computed as the "determinant" of $[i\ \ j\ \ k;\ \ u_1\ \ u_2\ \ u_3;\ \ v_1\ \ v_2\ \ v_3]$.

Cyclic shift S Permutation with $s_{21} = 1$, $s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$ Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$, $|A^{-1}| = 1/|A|$, and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n-1$, volume of box $= |\det(A)|$.

Diagonal matrix D $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$ $\Lambda =$ eigenvalue matrix and $S =$ eigenvector matrix. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(\mathbf{V}) =$ number of vectors in any basis for $\mathbf{V}$.

Distributive Law $A(B+C) = AB+AC$ Add then multiply, or multiply then add.

Dot product $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$ Complex dot product is $\bar{x}^{\mathrm{T}}y$. Perpendicular vectors have zero dot product. $(AB)_{ij} =$ (row i of A)$\cdot$(column j of B).

Echelon matrix U The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ **and eigenvector** x $Ax = \lambda x$ with $x \neq 0$, so $\det(A - \lambda I) = 0$.

Eigshow Graphical 2 by 2 eigenvalues and singular values (MATLAB or Java).

Elimination A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = Elementary matrix E_{ij} The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$ A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$ If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT) A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so F_nx and $F_n^{-1}c$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers 0, 1, 1, 2, 3, 5,... satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1+\sqrt{5})/2$ the largest eigenvalue of the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$.

Four fundamental subspaces of A $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{T}})$, $\boldsymbol{N}(A^{\mathrm{T}})$.

Fourier matrix F Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}}F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A Columns without pivots; combinations of earlier columns.

Free variable x_i Column i has no pivot in elimination. We can give the $n-r$ free variables any values, then $Ax=b$ determines the r pivot variables (if solvable!).

Full column rank $r=n$ Independent columns, $\boldsymbol{N}(A)=\{0\}$, no free variables.

Full row rank $r=m$ Independent rows, at least one solution to $Ax=b$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements (perpendicular subspaces of $\mathbf{R}^n$ with dimensions r and $n-r$) from $Ax=0$. Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$.

Gauss-Jordan method Invert A by row operations on $[A \ \ I]$ to reach $[I \ \ A^{-1}]$.

Gram-Schmidt orthogonalization $A=QR$ Independent columns in A, orthonormal columns in Q. Each column q_j of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\operatorname{diag}(R)>0$.

Graph G Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n-1$ edges and no closed loops. A **directed graph** has a direction arrow specified on each edge.

Hankel matrix H Constant along each antidiagonal; h_{ij} depends on $i+j$.

Hermitian matrix $A^{\mathrm{H}}=\overline{A}^{\mathrm{T}}=A$ Complex analog of a symmetric matrix: $\overline{a_{ji}}=a_{ij}$.

Hessenberg matrix H Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix $\operatorname{hilb}(n)$ Entries $H_{ij}=1/(i+j-1)=\int_0^1 x^{i-1}x^{j-1}dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number.

Hypercube matrix P_L^2 Row $n+1$ counts corners, edges, faces, ..., of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**)** Diagonal entries $=1$, off-diagonal entries $=0$.

Incidence matrix of a directed graph The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $v_1,\dots,v_k$ No combination $c_1v_1+\cdots+c_kv_k=$ zero vector unless all $c_i=0$. If the v's are the columns of A, the only solution to $Ax=0$ is $x=0$.

Inverse matrix A^{-1} Square matrix with $A^{-1}A=I$ and $AA^{-1}=I$. No inverse if $\det A=0$ and $\operatorname{rank}(A)<n$, and $Ax=0$ for a nonzero vector x. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$ Cofactor formula $(A^{-1})_{ij}=C_{ji}/\det A$.

Iterative method A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$ If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1s on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector $(1, 0, \ldots, 0)$.

Kirchhoff's Laws *Current law*: net current (in minus out) is zero at each node. *Voltage law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$ Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, b)$ The subspace spanned by $b, Ab, \ldots, A^{j-1}b$. Numerical methods approximate $A^{-1}b$ by x_j with residual $b - Ax_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least-squares solution $\widehat{x}$ The vector $\widehat{x}$ that minimizes the error $\|e\|^2$ solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Then $e = b - A\widehat{x}$ is orthogonal to all columns of A.

Left inverse A^+ If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ Nullspace of A^{T} = "left nullspace" of A because $y^{\mathrm{T}}A = 0^{\mathrm{T}}$.

Length $\|x\|$ Square root of $x^{\mathrm{T}}x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ **or** $\sum c_j v_j$ Vector addition and scalar multiplication.

Linear transformation T Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = cT(v) + dT(w)$. Examples: Matrix multiplication Av, differentiation in function space.

Linearly dependent $v_1, \ldots, v_n$ A combination other than all $c_i = 0$ gives $\sum c_i v_i = 0$.

Lucas numbers $L = 2, 1, 3, 4, \ldots$, satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_n^2$, with eigenvalues $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ of the Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Compare $L_0 = 2$ with Fibonacci.

Markov matrix M All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady-state eigenvector $Ms = s > 0$.

Matrix multiplication AB The i, j entry of AB is (row i of A) $\cdot$ (column j of B) $= \sum a_{ik}b_{kj}$. By columns: column j of $AB = A$ times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times x equals A times Bx.

Minimal polynomial of A The lowest-degree polynomial with $m(A)$ = zero matrix. The roots of m are eigenvalues, and $m(\lambda)$ divides $\det(A - \lambda I)$.

Multiplication $Ax = x_1(\text{column } 1) + \cdots + x_n(\text{column } n)$ = combination of columns.

Multiplicities AM **and** GM The algebraic multiplicity AM of an eigenvalue λ is the number of times λ appears as a root of $\det(A-\lambda I)=0$. The geometric multiplicity GM is the number of independent eigenvectors (= dimension of the eigenspace for λ).

Multiplier ℓ_{ij} The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: $\ell_{ij} =$ (entry to eliminate)/(jth pivot).

Network A directed graph that has constants $c_1,\ldots,c_m$ associated with the edges.

Nilpotent matrix N Some power of N is the zero matrix, $N^k=0$. The only eigenvalue is $\lambda=0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$ **of a matrix** The "ℓ^2 norm" is the maximum ratio $\|Ax\|/\|x\|=\sigma_{\max}$. Then $\|Ax\|\le\|A\|\|x\|$, $\|AB\|\le\|A\|\|B\|$, and $\|A+B\|\le\|A\|+\|B\|$. **Frobenius norm** $\|A\|_F^2=\sum\sum a_{ij}^2$; ℓ^1 and ℓ^∞ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{x}=A^{\mathrm{T}}b$ Gives the least-squares solution to $Ax=b$ if A has full rank n. The equation says that (columns of A) $\cdot\,(b-A\widehat{x})=0$.

Normal matrix N $NN^{\mathrm{T}}=N^{\mathrm{T}}N$, leads to orthonormal (complex) eigenvectors.

Nullspace matrix N The columns of N are the $n-r$ special solutions to $As=0$.

Nullspace $\boldsymbol{N}(A)$ Solutions to $Ax=0$. Dimension $n-r=$ (# columns) $-$ rank.

Orthogonal matrix Q Square matrix with orthonormal columns, so $Q^{\mathrm{T}}Q=I$ implies $Q^{\mathrm{T}}=Q^{-1}$. Preserves length and angles, $\|Qx\|=\|x\|$ and $(Qx)^{\mathrm{T}}(Qy)=x^{\mathrm{T}}y$. All $|\lambda|=1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces Every v in $\mathbf{V}$ is orthogonal to every w in $\mathbf{W}$.

Orthonormal vectors $q_1,\ldots,q_n$ Dot products are $q_i^{\mathrm{T}}q_j=0$, if $i\neq j$ and $q_i^{\mathrm{T}}q_j=1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q=I$. If $m=n$, then $Q^{\mathrm{T}}=Q^{-1}$ and $q_1,\ldots,q_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $v=\sum(v^{\mathrm{T}}q_j)q_j$.

Outer product is uv^{T} column times row = rank-1 matrix.

Partial pivoting In elimination, the jth pivot is chosen as the largest available entry (in absolute value) in column j. Then all multipliers have $|\ell_{ij}|\le 1$. Roundoff error is controlled (depending on the *condition number* of A).

Particular solution x_p Any solution to $Ax=b$; often x_p has free variables $=0$.

Pascal matrix $P_S=\mathrm{pascal}(n)$ The symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S=P_LP_U$ all contain Pascal's triangle with det $=1$ (see index for more properties).

Permutation matrix P There are $n!$ orders of $1,\ldots,n$; the $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij}; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d The first nonzero entry when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$ Solutions to $a^{\mathrm{T}}x = 0$ give the plane (dimension $n-1$) perpendicular to $a \neq 0$.

Polar decomposition $A = QH$ Orthogonal Q, positive (semi)definite H.

Positive definite matrix A Symmetric matrix with positive eigenvalues and positive pivots. Definition: $x^{\mathrm{T}}Ax > 0$ unless $x = 0$.

Projection matrix P **onto subspace** S Projection $p = Pb$ is the closest point to b in $\mathbf{S}$, error $e = b - Pb$ is perpendicular to $\mathbf{S}$. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in $\mathbf{S}$ or $\mathbf{S}^{\perp}$. If columns of A = basis for $\mathbf{S}$, then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Projection $p = a(a^{\mathrm{T}}b/a^{\mathrm{T}}a)$ **onto the line through** a $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ has rank 1.

Pseudoinverse A^+ **(Moore-Penrose inverse)** The n by m matrix that "inverts" A from column space back to row space, with $\boldsymbol{N}(A^+) = \boldsymbol{N}(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. $\mathrm{rank}(A^+) = \mathrm{rank}(A)$.

Random matrix $\mathrm{rand}(n)$ **or** $\mathrm{randn}(n)$ MATLAB creates a matrix with random entries, uniformly distributed on $[0\ \ 1]$ for rand, and standard normal distribution for randn.

Rank 1 matrix $A = uv^{\mathrm{T}} \neq 0$ Column and row spaces = lines cu and cv.

Rank $r(A)$ Equals number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(x) = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ For $A = A^{\mathrm{T}}$, $\lambda_{\min} \leq q(x) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors x for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \mathrm{rref}(A)$ Pivots= 1; zeros above and below pivots; r nonzero rows of R give a basis for the row space of A.

Reflection matrix $Q = I - 2uu^{\mathrm{T}}$ The unit vector u is reflected to $Qu = -u$. All vectors x in the plane $u^{\mathrm{T}}x = 0$ are unchanged because $Qx = x$. The "Householder matrix" has $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+ If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the plane by θ, and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Orthogonal matrix, eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors $(1, \pm i)$.

Row picture of $Ax = b$ Each equation gives a plane in $\mathbf{R}^n$ planes intersect at x.

Row space $\boldsymbol{C}(A^{\mathrm{T}})$ All combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$ A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j =$ **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$ Appears in block elimination on $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Schwarz inequality $|v \cdot w| \leq \|v\| \|w\|$ Then $|v^{\mathrm{T}}Aw|^2 \leq (v^{\mathrm{T}}Av)(w^{\mathrm{T}}Aw)$ if $A = C^{\mathrm{T}}C$.

Semidefinite matrix A (Positive) semidefinite means symmetric with $x^{\mathrm{T}}Ax \geq 0$ for all vectors x. Then all eigenvalues $\lambda \geq 0$; no negative pivots.

Similar matrices A **and** B $B = M^{-1}AM$ has the same eigenvalues as A.

Simplex method for linear programming The minimum cost vector x^* is found by moving from corner to lower-cost corner along the edges of the feasible set (where the constraints $Ax = b$ and $x \geq 0$ are satisfied). Minimum cost at a corner!

Singular matrix A A square matrix that has no inverse: $\det(A) = 0$.

Singular Value Decomposition (SVD) $A = U\Sigma V^{\mathrm{T}} =$ **(orthogonal** U**) times (diagonal** Σ**) times (orthogonal** V^{T}**)** First r columns of U and V are orthonormal bases of $\boldsymbol{C}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$, with $Av_i = \sigma_i u_i$ and singular value $\sigma_i > 0$. Last columns of U and V are orthonormal bases of the nullspaces of A^{T} and A.

Skew-symmetric matrix K The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $Ax = b$ The right side b is in the column space of A.

Spanning set $v_1, \ldots, v_m$**, for V** Every vector in $\mathbf{V}$ is a combination of $v_1, \ldots, v_m$.

Special solutions to $As = 0$ One free variable is $s_i = 1$, other free variables $= 0$.

Spectral theorem $A = Q\Lambda Q^{\mathrm{T}}$ Real symmetric A has real λ_i and orthonormal q_i, with $Aq_i = \lambda_i q_i$. In mechanics, the q_i give the *principal axes*.

Spectrum of A The set of eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. **Spectral radius** $= |\lambda_{\max}|$.

Standard basis for $\mathbf{R}^n$ Columns of n by n identity matrix (written i, j, k in $\mathbf{R}^3$).

Stiffness matrix K When x gives the movements of the nodes in a discrete structure, Kx gives the internal forces. Often $K = A^{\mathrm{T}}CA$, where C contains spring constants from Hooke's Law and $Ax =$ stretching (strains) from the movements x.

Subspace S of V Any vector space inside $\mathbf{V}$, including $\mathbf{V}$ and $\mathbf{Z} = \{\text{zero vector}\}$.

Sum $\mathbf{V}+\mathbf{W}$ of subspaces Space of all (v in V)+(w in $\mathbf{W}$). **Direct sum**: $\dim(\mathbf{V}+\mathbf{W}) = \dim \mathbf{V} + \dim \mathbf{W}$, when $\mathbf{V}$ and $\mathbf{W}$ share only the zero vector.

Symmetric factorizations $A = LDL^{\mathrm{T}}$ **and** $A = Q\Lambda Q^{\mathrm{T}}$ The number of positive pivots in D and positive eigenvalues in Λ is the same.

Symmetric matrix A The transpose is $A^{\mathrm{T}} = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric. All matrices of the form $R^{\mathrm{T}}R$ and LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are symmetric. Symmetric matrices have real eigenvalues in Λ and orthonormal eigenvectors in Q.

Toeplitz matrix T Constant-diagonal matrix, so t_{ij} depends only on $j-i$. Toeplitz matrices represent linear time-invariant filters in signal processing.

Trace of A Sum of diagonal entries = sum of eigenvalues of A. $\mathrm{Tr}AB = \mathrm{Tr}BA$.

Transpose matrix A^{T} Entries $A^{\mathrm{T}}_{ij} = A_{ji}$. A^{T} is n by m, $A^{\mathrm{T}}A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^{\mathrm{T}}A^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$.

Triangle inequality $\|u+v\| \le \|u\| + \|v\|$ For matrix norms, $\|A+B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T $t_{ij} = 0$ if $|i-j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}} = U^{-1}$ Orthonormal columns (complex analog of Q).

Vandermonde matrix V $Vc = b$ gives the polynomial $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$ at n points. $V_{ij} = (x_i)^{j-1}$, and $\det V$ = product of $(x_k - x_i)$ for $k > i$.

Vector addition $v + w = (v_1 + w_1, \ldots, v_n + w_n)$ = diagonal of parallelogram.

Vector space V Set of vectors such that all combinations $cv + dw$ remain in $\mathbf{V}$. Eight required rules are given in Section 2.1 for $cv + dw$.

Vector v in $\mathbf{R}^n$ Sequence of n real numbers $v = (v_1, \ldots, v_n)$ = point in $\mathbf{R}^n$.

Volume of box The rows (or columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$ or vectors w_{jk} Rescale and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$. Vectors from $w_{00} = (1,1,-1,-1)$ would be $(1,-1,0,0)$ and $(0,0,1,-1)$.

Appendix E

MATLAB Teaching Codes

cofactor	Compute the n by n matrix of cofactors.
cramer	Solve the system $Ax = b$ by Cramer's Rule.
deter	Matrix determinant computed from the pivots in $PA = LU$.
eigen2	Eigenvalues, eigenvectors, and $\det(A - \lambda I)$ for 2 by 2 matrices.
eigshow	Graphical demonstration of eigenvalues and singular values.
eigval	Eigenvalues and their multiplicity as roots of $\det(A - \lambda I) = 0$.
eigvec	Compute as many linearly independent eigenvectors as possible.
elim	Reduction of A to row echelon form R by an invertible E.
findpiv	Find a pivot for Gaussian elimination (used by plu).
fourbase	Construct bases for all four fundamental subspaces.
grams	Gram-Schmidt orthogonalization of the columns of A.
house	2 by 12 matrix giving corner coordinates of a house.
inverse	Matrix inverse (if it exists) by Gauss-Jordan elimination.
leftnull	Compute a basis for the left nullspace.
linefit	Plot the least squares fit to m given points by a line.
lsq	Least-squares solution to $Ax = b$ from $A^{\mathrm{T}}A = A^{\mathrm{T}}b$.
normal	Eigenvalues and orthonormal eigenvectors when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$.
nulbasis	Matrix of special solutions to $Ax = 0$ (basis for null space).
orthcomp	Find a basis for the orthogonal complement of a subspace.
partic	Particular solution of $Ax = b$, with all free variables zero.

plot2d	Two-dimensional plot for the house figures.
plu	Rectangular $PA = LU$ factorization with row exchanges.
poly2str	Express a polynomial as a string.
project	Project a vector b onto the column space of A.
projmat	Construct the projection matrix onto the column space of A.
randperm	Construct a random permutation.
rowbasis	Compute a basis for the row space from the pivot rows of R.
samespan	Test whether two matrices have the same column space.
signperm	Determinant of the permutation matrix with rows ordered by p.
slu	LU factorization of a square matrix using *no row exchanges*.
slv	Apply slu to solve the system $Ax = b$ allowing no row exchanges.
splu	Square $PA = LU$ factorization *with row exchanges*.
splv	The solution to a square, invertible system $Ax = b$.
symmeig	Compute the eigenvalues and eigenvectors of a symmetric matrix.
tridiag	Construct a tridiagonal matrix with constant diagonals a, b, c.

These Teaching Codes are directly available from the Linear Algebra Home Page:
`http://web.mit.edu/18.06/www.`
They were written in MATLAB , and translated into Maple and Mathematica.

Appendix F

Linear Algebra in a Nutshell

(A is n by n)

Nonsingular	**Singular**
A is invertible.	A is not invertible.
The columns are independent.	The columns are dependent.
The rows are independent.	The rows are dependent.
The determinant is not zero.	The determinant is zero.
$Ax = 0$ has one solution $x = 0$.	$Ax = 0$ has infinitely many solutions.
$Ax = b$ has one solution $x = A^{-1}b$.	$Ax = b$ has no solution or infinitely many.
A has n (nonzero) pivots.	A has $r < n$ pivots.
A has full rank $r = n$.	A has rank $r < n$.
The reduced row echelon form is $R = I$.	R has at least one zero row.
The column space is all of $\mathbf{R}^n$.	The column space has dimension $r < n$.
The row space is all of $\mathbf{R}^n$.	The row space has dimension $r < n$.
All eigenvalues are nonzero.	Zero is an eigenvalue of A.
$A^{\mathrm{T}}A$ is symmetric positive definite.	$A^{\mathrm{T}}A$ is only semidefinite.
A has n (positive) singular values.	A has $r < n$ singular values.

Each line of the singular column can be made quantitative using r.

Solutions to Selected Exercises

Problem Set 1.2

1. The lines intersect at $(x, y) = (3, 1)$. Then $3(\text{column } 1) + 1(\text{column } 2) = (4, 4)$.

3. These "planes" intersect in a line in four-dimensional space. The fourth plane normally intersects that line in a point. An inconsistent equation like $u + w = 5$ leaves no solution (no intersection).

5. The two points on the plane are $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$.

7. Solvable for $(3, 5, 8)$ and $(1, 2, 3)$; not solvable for $b = (3, 5, 7)$ or $b = (1, 2, 2)$.

9. Column $3 = 2(\text{column } 2) - \text{column } 1$. If $b = (0, 0, 0)$, then $(u, v, w) = (c, -2c, c)$

11. Both $a = 2$ and $a = -2$ give a line of solutions. All other a give $x = 0$, $y = 0$.

13. The row picture has two lines meeting at $(4, 2)$. The column picture has $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right-hand side } (0, 6)$.

15. The row picture shows four *lines*. The column picture is in *four*-dimensional space. No solution unless the right-hand side is a combination of *the two columns*.

17. If x, y, z satisfy the first two equations, they also satisfy the third equation. The line $\mathbf{L}$ of solutions contains $v = (1, 1, 0)$, $w = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$, and $u = \frac{1}{2}v + \frac{1}{2}w$, and all combinations $cv + dw$ with $c + d = 1$.

19. Column $3 =$ column 1; solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $b = (4, 6, c)$ needs $c = 10$ for solvability.

21. The second plane and row 2 of the matrix and all columns of the matrix are changed. The solution is not changed.

23. $u = 0$, $v = 0$; $w = 1$, because $1(\text{column } 3) = b$.

Problem Set 1.3

1. Multiply by $\ell = \frac{10}{2} = 5$, and subtract to find $2x + 3y = 1$ and $-6y = 6$. Pivots $2, -6$.

3. Subtract $-\frac{1}{2}$ times equation 1 (or add $\frac{1}{2}$ times equation 1). The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right-hand side changes sign, so does the solution: $(x, y) = (-5, -1)$.

5. $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right-hand side is $2 \cdot 10 = 20$. Then all points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$.

7. If $a = 2$, elimination must fail. The equations have no solution. If $a = 0$, elimination stops for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

9. $6x - 4y$ is 2 times $(3x - 2y)$. Therefore, we need $b_2 = 2b_1$. Then there will be infinitely many solutions. The columns $(3, 6)$ and $(-2, -4)$ are on the same line