Math 63: Real Analysis

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Problem 1. Verify that the following are metric spaces:

1. all n-tuples of real numbers, with

$$d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|$$

2. all bounded infinite sequences $x = (x_1, x_2, x_3, \cdots)$ of elements of $\mathbb{R}$ with

$$d(x, y) = \text{l.u.b.}\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \cdots\}$$

Solution.

1. Let $E = \{\text{all n-tuples of real numbers}\}$, with the distance between two points x and y $d(x, y) = d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|$. To see if (E, d) defines a metric space, we check the following:

   (a) $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) \geq 0$
   Since, $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|$, and each term $|x_i - y_i| \geq 0$, the sum of these terms is also non-negative. Hence, we can say that $\forall x, y \in E$, $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) \geq 0$.

   (b) $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = 0 \iff (x_1, \cdots, x_n) = (y_1, \cdots, y_n)$
   $(\implies)$
   Assume $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = 0$. Since every term $|x_i - y_i|$ is non-negative, the sum being zero implies that every term is also zero. I.e. $|x_i - y_i| = 0 \implies x_i = y_i$, for all x_i, y_i. Hence, $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$.

   $(\impliedby)$
   Assume $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$. Then,
   $d((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - x_i| = \sum_{i=1}^{n} 0 = 0$

(c) $d(x, y) = d(y, x)$, $\forall x, y \in E$

For every term $|x_i - y_i|$ in the summation, we know that by definition, $|x_i - y_i| = |y_i - x_i|$. Hence, $d((x_1, \cdots, x_n)(y_1, \cdots, y_n)) = d((y_1, \cdots, y_n), (x_1, \cdots, x_n))$ or $d(x, y) = d(y, x)$.

(d) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$

$$
\begin{aligned}
d(x, y) + d(y, z) &= d((x_1, \cdots, x_n), (y_1, \cdots, y_n)) + d((y_1, \cdots, y_n), (z_1, \cdots, z_n)) \\
&= \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| \\
&= \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i|
\end{aligned}
$$

We know that,

$$
\begin{aligned}
& |x_i - y_i| + |y_i - z_i| \geq |x_i - y_i + y_i - z_i| \\
\implies & |x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|
\end{aligned}
$$

I.e.,

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i| &\geq \sum_{i=1}^{n} |x_i - z_i| \\
d((x_1, \cdots, x_n), (y_1, \cdots, y_n)) + d((y_1, \cdots, y_n), (z_1, \cdots, z_n)) &\geq d((x_1, \cdots, x_n), (z_1, \cdots, z_n)) \\
d(x, y) + d(y, z) &\geq d(x, z)
\end{aligned}
$$

Hence, (E, d) defines a metric space.

2. Let $E = \{$all bounded infinite sequences of real numbers$\}$ with the distance function d defined as $d(x, y) = \text{l.u.b.}\{|x_1 - y_1|, |x_2 - y_2|, \cdots\} = \{|x_i - y_i\}_{i \in \mathbb{N}}$. To see if (E, d) defines a metric space, we check the following:

   (a) $d(x, y) \geq 0$

$$d(x, y) = \text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}}$$

   with each element in the set $|x_i - y_i| \geq 0$, $\forall i \in \mathbb{N}$ and $\text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} \geq |x_j - y_j|$ for any $j \in \mathbb{N}$. On combining the inequalities we get,

$$
\begin{aligned}
& \text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} \geq |x_j - y_j| \geq 0 \\
& \text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} \geq 0
\end{aligned}
$$

(b) $d(x,y)=0 \iff x=y$

$(\implies)$

Assume $d(x,y)=0$. Then $\text{l.u.b.}\{|x_i-y_i|\}_{i\in\mathbb{N}}=0 \implies |x_i-y_i|\leq 0, \forall i\in\mathbb{N}$. But by definition, the absolute value is positive, i.e. $|x_i-y_i|\geq 0, \forall i\in\mathbb{N}$. On combining the inequalities, we get: $|x_i-y_i|=0, \forall i\in\mathbb{N}$.

I.e. $\forall i\in\mathbb{N}, x_i=y_i \implies x=y$.

$(\impliedby)$

Assume $x=y$.

Then $\forall i\in\mathbb{N}, x_i=y_i \implies x_i-y_i=0 \implies \text{l.u.b.}\{|x_i-y_i|\}_{i\in\mathbb{N}}=0=d(x,y)$.

(c) $d(x,y)=d(y,x)$

By definition, $|x_i-y_i|=|y_i-x_i|$, so $\text{l.u.b.}\{|x_i-y_i|\}_{i\in\mathbb{N}}=\text{l.u.b.}\{|y_i-x_i|\}_{i\in\mathbb{N}}$. Hence, $d(x,y)=d(y,x)$.

(d) $d(x,z)\leq d(x,y)+d(y,z)$

Since

$$|x_i-z_i|\leq|x_i-y_i|+|y_i-z_i|\,\forall i\in\mathbb{N},$$

we can see that because

$$\text{l.u.b.}\{|x_i-y_i|+|y_i-z_i|\}_{i\in\mathbb{N}}$$

is an upper bound on $\{|x_i-y_i|+|y_i-z_i|\}_{i\in\mathbb{N}}$ it is also an upper bound on $\{|x_i-z_i|\}_{i\in\mathbb{N}}$. I.e.

$$\{|x_i-z_i|\}_{i\in\mathbb{N}}\leq\{|x_i-y_i|+|y_i-z_i|\}_{i\in\mathbb{N}}\leq\text{l.u.b.}\{|x_i-y_i|+|y_i-z_i|\}_{i\in\mathbb{N}}$$

And (from pset 1, problem 9) we know for non-empty, bounded subsets of $\mathbb{R}$ that, $\text{l.u.b.}\{x+y : x\in S_1, y\in S_2\}=\text{l.u.b.}\,S_1+\text{l.u.b.}\,S_2$. I.e.,

$$\implies \text{l.u.b.}\{|x_i-y_i|+|y_i-z_i|\}_{i\in\mathbb{N}}=\text{l.u.b.}\{|x_i-y_i|\}+\text{l.u.b.}\{|y_i-z_i|\}_{i\in\mathbb{N}}$$

Combining and substituting gives us,

$$\implies \text{l.u.b.}\{|x_i-z_i|\}_{i\in\mathbb{N}}\leq\text{l.u.b.}\{|x_i-y_i|\}+\text{l.u.b.}\{|y_i-z_i|\}_{i\in\mathbb{N}}$$

Hence the triangle inequality holds, and (E,d) defines a metric space.

Problem 2. What are the open and closed balls in the metric space of example (4), § 1? Show that two balls of different centers and radii may be equal. What are the open sets in this metric space?

Example (4),§ 1: Let E be an arbitrary set and, for $p, q \in E$, define $d(p, q) = 0$ if $p = q$, $d(p, q) = 1$ if $p \neq q$.

Solution.

1. (a) $0 < r < 1$
    i. Open Balls:
    $$B_r(p) = \{d(p, q) < r; q \in E\} = \{p\}$$
    Since $d(p, q) < 1$ implies $p = q$.
    ii. Closed Balls:
    $$B_r(p) = \{d(p, q) \leq r; q \in E\} = \{p\}$$
    Since by definition $d(p, q) < 1$ implies $p = q$.

    (b) $r = 1$
    i. Open Balls:
    $$B_r(p) = \{d(p, q) < r; q \in E\} = \{p\}$$
    Since $d(p, q) < 1$ implies $p = q$.
    ii. Closed Balls:
    $$B_r(p) = \{d(p, q) \leq r; q \in E\} = E$$
    Since $d(p, q) \leq 1$ for all $q \in E$.

    (c) $r > 1$
    i. Open Balls:
    $$B_r(p) = \{d(p, q) \leq r; q \in E\} = E$$
    Since $d(p, q) \leq 1$ for all $q \in E$.
    ii. Closed Balls:
    $$B_r(p) = \{d(p, q) \leq r; q \in E\} = E$$
    Since $d(p, q) \leq 1$ for all $q \in E$.

2. To see that two balls with different centers and radii may be equal, consider two open balls $B_{r_1}(p), B_{r_2}(q)$ with $p \neq q$, $r_1 \neq r_2$, and $r_1, r_2 > 0$. Then, as we saw above,
$$B_{r_1}(p) = E, \qquad B_{r_2}(q) = E$$
I.e. two balls with different centers and radii may be equal.

3. Every singleton set $\{p\} \in E$ is open, as we can construct an open ball $B_r(p)$ with $0 < r < 1$ centered at p. Since unions of any collection of open sets are open, any subset of E is open.

Problem 3. Show that the subset of E^2 given by $\{(x_1, x_2) \in E^2 : x_1 > x_2\}$ is open.

Solution.

Let $S = \{(x_1, x_2) \in E^2 : x_1 > x_2\}$, with distance function defined as $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for any two points $(x_1, x_2), (y_1, y_2) \in E^2$. To show that S is an open subset we pick an arbitrary point in S and show that there exists an open ball centered at that point contained entirely inside S.

Let $B_{r/4}((x_1, x_2))$ be an open ball centered around an arbitrary point $(x_1, x_2) \in S$, with $r = x_1 - x_2$ (obviously, $r > 0$), and let $(y_1, y_2) \in B_{r/4}((x_1, x_2))$. Then,

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &< r/4 \\ \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &< r/4 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 &< r^2/16 \\ \implies (x_1 - y_1)^2 \leq (x_1 - y_1)^2 + (x_2 - y_2)^2 &< r^2/16 \\ |x_1 - y_1| &< r/4 \end{aligned}$$

Similarly, we get

$$|x_2 - y_2| < r/4$$

We can then write: $y_1 - y_2 = x_1 - x_2 - (x_1 - y_1) + (x_2 - y_2)$.

We know that, $x_1 - x_2 = r, |x_1 - y_1| < r/4, |x_2 - y_2| < r/4$. Then,

$$y_1 - y_2 > r/2 \implies y_1 - y_2 > 0 \implies y_1 > y_2 \implies (y_1, y_2) \in S$$

I.e. $B_{r/4}((x_1, x_2)) \in S$. Since, (x_1, x_2) was an arbitrary point, this works for any element of S. S is thus an open subset because of every point of S we can construct such a ball.

Problem 4. Prove that any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.

Solution.

Let S be a bounded open subset of $\mathbb{R}$. $\forall p \in S$ let S_p be the set of all open intervals in S containing p. We know that S_p is not empty because by definition of open sets there exists at least one ball centred at p in S. (All open balls $B_\epsilon(a)$ are open intervals in $\mathbb{R}$ because they contain all elements between $a - \epsilon$ and $a + \epsilon$).

Consider the set R_p of the right end point of each of the intervals in S_p. Again, we know that this set is not empty because there exists at least one interval in S_p. We also know that this set is bounded from above because S has an upper bound. Since, R_p is nonempty and bounded from above, there must exist a l.u.b. for R_p. Similarly, construct the set L_p of the left end points of each of the intervals in S_p. By the same logic, there exist a g.l.b. for L_p.

Now, consider the open interval $X_p = (\text{g.l.b.}\, L_p, \text{l.u.b.}\, R_p)$. To show that $X_p \subset S$, consider $x \in X_p$. Assume $x \notin S$ then since S is open one can find intervals in S_p arbitrarily close to x, contradicting the definitions of $\text{g.l.b.}\, L_p$, $\text{l.u.b.}\, R_p$. Hence, x must be in S. Additionally, note that X_p is the maximal open interval contained in S that contains p, because any open interval in S containing p has endpoints in L_p, R_p, and hence is contained in X_p.

Claim: If $X_q \cap X_p \neq \emptyset$ then $X_q = X_p$.

Suppose there exists $r \in X_p \cap X_q$. Then $r \in S$. Because X_p is an open interval containing both p and r and X_q is an open interval containing both q and r, the union $X_p \cup X_q$ itself an open interval in S containing p. But by construction, X_p is the maximal open interval contained in S that contains p. Hence,

$$X_p \cup X_q \subset X_p \implies X_q \subset X_p$$

Similarly, we also get,

$$X_p \cup X_q \subset X_q \implies X_p \subset X_q$$

Therefore, $X_p = X_q$.

So, either $X_q \cap X_p = \emptyset$ or $X_q = X_p$.

Then we can write $S = \bigcup_p X_p$. By the claim, these intervals are either equal or disjoint, so selecting all unique intervals gives us a union of disjoint open intervals.