Math 81: Abstract Algebra

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Problem 1. For $f(x) = x^4 - 1$ and $g(x) = 3x^2 + 3x$ find: the quotient and remainder after dividing f by g; the gcd of f and g; and the expression of this gcd in the form $af + bg$ for some $a, b \in \mathbb{Q}[x]$. For the last two, you'll need to recall the Euclidean Algorithm and the Bezout Identity.

Solution.

Quotinent: $\frac{1}{3}(x^2 - x + 1)$
Remainder: $-x - 1$
Using Euclid's Algorithm:

$$\begin{aligned} x^4 - 1 &= (3x^2 + 3x)(\frac{1}{3}(x^2 - x + 1)) + (-x - 1) \\ (3x^2 + 3x) &= (-x - 1)(-3x) + 0 \end{aligned}$$

$gcd(x^4 - 1, 3x^2 + 3x) = -x - 1$

Using Bezout's Idenity:

$$\begin{aligned} (x^4 - 1, 3x^2 + 3x) &= af + bg \\ -x - 1 &= f - (\frac{1}{3}(x^2 - x + 1))g \\ -x - 1 &= 1(x^4 - 1) + (-(\frac{1}{3}(x^2 - x + 1)))(3x^2 + 3x) \end{aligned}$$

$a = 1, b = -(\frac{1}{3}(x^2 - x + 1))$

Problem 2. Prove that two polynomials $f, g \in \mathbb{Z}[x]$ are relatively prime in $\mathbb{Q}[x]$ (i.e., they share no common nonconstant factor) if and only if the ideal $(f, g) \subset \mathbb{Z}[x]$ contains a nonzero integer.

Solution.

$(\implies)$

Assume the polynomials f, g are relatively prime in $\mathbb{Q}[x]$.
I.e. $(f, g) = (gcd(f, g)) = (1) = \mathbb{Q}[x]$. Since we are in a euclidean domain,

$$1 = af + bg$$

for some a, b with rational coefficients. Let k be the product of the denominators of the coefficients of the terms in a, b. Then

$$k = kaf + kbg$$

has integer coefficients. I.e. kaf, $kbg \in \mathbb{Z}[x]$, and since k can be expressed as a linear combination of f and g, $k \in (f, g) \subset \mathbb{Z}[x]$. Hence, the ideal $(f, g) \subset \mathbb{Z}[x]$ contains a nonzero integer.

$(\impliedby)$

Assume the ideal $(f, g) \subset \mathbb{Z}[x]$ contains a non-zero integer k.
Since this ideal is a subset of the ideal generated by f, g in $\mathbb{Q}[x]$, $k \in (f, g) \subset \mathbb{Q}[x]$. But all integers are units in $\mathbb{Q}[x] \implies 1 \in (f, g) \subset \mathbb{Q}[x]$. I.e. for some polynomials $a, b \in \mathbb{Q}[x]$,

$$1 = af + bg$$

Hence, the polynomials f, g are relatively prime in $\mathbb{Q}[x]$.

Problem 3. Decide whether each of the following polynomials is irreducible, and if not, then find the factorization into monic irreducibles.

1. $x^4 + 1 \in \mathbb{R}[x]$
2. $x^4 + 1 \in \mathbb{Q}[x]$
3. $x^7 + 66x^6 - 77x + 737 \in \mathbb{Q}[x]$
4. $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$
5. $x^3 + 5x^2 - 9x + 3 \in \mathbb{Q}[x]$

Solution.

1. $x^4 + 1 \in \mathbb{R}[x]$
$$(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$
2. $x^4 + 1 \in \mathbb{Q}[x]$ Let $f(x) = x^4 + 1$. Then,
$$f(y+1) = (y+1)^4 + 1 = y^4 + 4y^3 + 6y^2 + 4y + 2$$
We can see that $2|4, 2|6, 2|2$, and $4 \nmid 2$. Then by Eisentein's Criterion, the polynomials of the form $f(x)$ are irreducible in $\mathbb{Q}[x]$.
3. $x^7 + 66x^6 - 77x + 737 \in \mathbb{Q}[x]$
We can see that $11|66$, $11|-77$, $11|737$, and $121 \nmid 737$. Then by Eisentein's Criterion, the polynomial is irreducible in $\mathbb{Q}[x]$.
4. $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$
Let $f(x) = x^4 + x^3 + x^2 + 1$. Then,
$$f(y+1) = (y+1)^4 + (y+1)^3 + (y+1)^2 + 1 = y^4 + 5y^3 + 10y^2 + 10y + 5$$
We can see that $5|5, 5|10$, and $25 \nmid 5$. Then by Eisentein's Criterion, the polynomials of the form $f(x)$ are irreducible in $\mathbb{Q}[x]$.
5. $x^3 + 5x^2 - 9x + 3 \in \mathbb{Q}[x]$
Assume $\frac{r}{s}$ is a root of the polynomial in the lowest terms. From proposition 11 we know that $r \mid a_n$ and $s \mid a_0$. I.e. $r \mid 1$, $s \mid 3$. The only such candidate is $\frac{1}{3}$. Checking,
$$\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^2 - 9\left(\frac{1}{3}\right) + 3 = \frac{16}{27}$$
Hence, $\frac{16}{27}$ is not a root of the polynomial. By proposition 10, we know that this polynomial (degree 3) is irreducible in $\mathbb{Q}[x]$ (over a field).

Problem 4. *Irreducible polynomials over finite fields.* Let $\mathbb{F}_3$ be the field with three elements.

1. Determine all the monic irreducible polynomials of degree ≤ 3 in $\mathbb{F}_3[x]$.
2. Determine the number of monic irreducible polynomials of degree 4 in $\mathbb{F}_3[x]$.
   Hint. This is easier than determining all of them.

Solution.

1. (a) Linear Irreducible Polynomials
   By definition all monic linear polynomials are irreducible.

$$\begin{aligned} x &= 0 \\ x+1 &= 0 \\ x+2 &= 0 \end{aligned}$$

   (b) Quadratic Irreducible Polynomials
   All quadratic polynomials are of the form $x^2 + ax + b = 0$, where $a, b \in \mathbb{F}_3$. There are 9 such polynomials. By Propopsition 10, we know that polynomials of degree two over a field is reducible if and only if it has a root in the field.
   Upon checking, we are left with:

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 + 1x + 2 &= 0 \\ x^2 + 2x + 2 &= 0 \end{aligned}$$

   (c) Cubic Irreducible Polynomials All cubic polynomials are of the form $x^3 + ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{F}_3$. There are 27 such polynomials. By Propopsition 10, we know that polynomials of degree three over a field is reducible if and only if it has a root in the field.

Upon checking, we are left with:

$$
\begin{aligned}
x^3 + 2x + 1 &= 0 \\
x^3 + 2x + 2 &= 0 \\
x^3 + 1x^2 + 2 &= 0 \\
x^3 + 1x^2 + 2x + 1 &= 0 \\
x^3 + 1x^2 + 1x + 2 &= 0 \\
x^3 + 2x^2 + 1 &= 0 \\
x^3 + 2x^2 + 1x + 1 &= 0 \\
x^3 + 2x^2 + 2x + 2 &= 0
\end{aligned}
$$

2. Quartic Irreducible Polynomials

   All cubic polynomials are of the form $x^4 + ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in \mathbb{F}_3$. There are 81 such polynomials. To the irreducibles we first count the reducibles. The reducibles can be classified by the degrees of their factors that is partitions of 4.

   - 3 + 1

     There are 8 irreducible cubics and 3 irreducible linear polynomials, the number of quartics factored as such are: $8 \cdot 3 = 24$.
   - 2 + 2

     There are 3 irreducible quadratics, the number of quartics factored as such are: $3! = 6$
   - 2 + 1 + 1

     There are 3 irreducible quadratics, and 3 irreducible linear polynomials, the number of quartics factored as such are: $3 \cdot (3!) = 18$.
   - 1 + 1 + 1 + 1

     There are 3 irreducible linear polynomials and 4 places to fill, so by stars and bars the number of quartics that can be factored as such are: $\binom{6}{2} = 15$.

   Then the number of irreducible quartics is

$$81 - 24 - 6 - 18 - 15 = 18.$$

Problem 5(a). *Symmetric polynomials.* Let R be a commutative ring with 1 and $R[x_1, \dots, x_n]$ the ring of polynomials in the variables $x_1, \dots, x_n$ with coefficients in R. Consider the symmetric group S_n acting on the set $\{x_1, \dots, x_n\}$ by permutations. Extend this action linearly to $R[x_1, x_2, \dots, x_n]$; for example, if $\sigma = (123) \in S_3$, then

$$\sigma \cdot (x_1x_2 - 6x_3^2 + 7x_2x_3^2) = x_2x_3 - 6x_1^2 + 7x_3x_1^2.$$

Then this action satisfies $\sigma \cdot (f + g) = \sigma \cdot f + \sigma \cdot g$ and $\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$ for all $\sigma \in S_n$ and all $f, g \in R[x_1, \dots, x_n]$.
Let $S \subset R[x_1, \dots, x_n]$ be the subset fixed under the action of S_n. Prove that S is a subring with 1. This is called the **ring of symmetric polynomials**.

Solution.

To prove that S is a subring with 1:

1. Contains 1

   Since 1 is a constant polynomial and S_n is acting on the set of variables $\{x_1, \dots, x_n\}$,

   $$\sigma(1) = 1$$

   Hence, $1 \in S$.

2. Closed under multiplication

   $\forall f, g \in S$ by definition, $\sigma(f) = f, \sigma(g) = g$ and again by definition,

   $$\sigma(fg) = \sigma(f) \cdot \sigma(g) = f \cdot g$$

   Hence, $f \cdot g \in S$. Since f, g were arbitrary polynomials this holds for any elements in S and S is closed under multiplication.

Problem 5(b). For each $n \geq 0$, define polynomials $e_i \in R[x_1, \dots, x_n]$ by $e_0 = 1$ and

$$e_1 = x_1 + \cdots + x_n, \quad e_2 = \sum_{1 \leq i < j \leq n} x_i x_j, \quad \dots, \quad e_n = x_1 \cdots x_n$$

and $e_k = 0$ for $k > n$. In words, e_k is the sum of all distinct products of subsets of k distinct variables. Prove that each e_k is a symmetric polynomial. These are called the **elementary symmetric polynomials**.

Solution.

For a given k, $1 \leq k \leq n$,

$$e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Let A be the set of all terms in e_k.

$$A = \{x_{i_1} x_{i_2} \cdots x_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

Since σ as a n-cycle is a bijection on $\{i_1, i_2, \cdots, i_n\}$, $\forall a \in A$, $\sigma(a)$ is also a product of k distinct variables. And by definition of A all distinct multiples of subsets of k distinct variables, $\sigma(a) \in A$. Hence, $\sigma : A \to A$.

Also, for any $a \in A$ we can find $b \in A$ such that $\sigma^{-1}(b) = a$. Hence σ is surjective. A surjective mapping from A to A is bijective.

Hence, $\sigma(e_k)$ only permutes the terms of e_k.

$$\sigma(e_k) = \sum_{a \in A} \sigma(a) = \sum_{a \in A} a = e_k$$

e_k is invariant under the action of σ, and hence is a symmetric polynomial.

Problem 5(c). The **generic polynomial** of degree n is the polynomial

$$f(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$$

in the ring $R[x_1, \ldots, x_n][x]$ of polynomials in x with coefficients in $R[x_1, \ldots, x_n]$. Prove (by induction) that

$$\begin{aligned} f(x) &= (x - x_1)(x - x_2)\cdots(x - x_n) = x^n - e_1x^{n-1} + e_2x^{n-2} + \cdots + (-1)^n e_n \\ &= \sum_{j=0}^{n} (-1)^{n-j} e_{n-j} x^j. \end{aligned}$$

Solution.

Base case: $n = 1, R[x_1][x]$

Then, by definition,

$$f(x) = x - x_1$$

In $R[x_1][x]$, e_1 is the sum of all distinct products of subsets of 1 distinct variables. I.e. $e_1 = x_1$. Upon substitution.

$$\begin{aligned} f(x) &= x - x_1 \\ &= x - e_1 \\ &= x + (-1)^1 e_1 \\ &= (-1)^0 e_0 x + (-1)^1 e_1 x^0 \\ &= (-1)^1 e_1 x^0 + (-1)^0 e_0 x \\ &= \sum_{j=0}^{1} (-1)^{1-j} e_{1-j} x^j \end{aligned}$$

Hence, base case holds.

Notation: $e_{a,k}$, where a refers to the elementary symmetric polynomials in $R[x_1, \cdots, x_a][x]$, or in a ring with a adjoined variables, and k refers to the number of variables in the subset. For example, in $R[x_1 \cdots x_n][x]$, the elementary symmetric polynomial with j elements in the subset as $e_{n,j}$.

Inductive Hypothesis: Assume the $n-1$ the case holds. I.e. in $R[x_1, \cdots x_{n-1}][x]$,

$$
\begin{aligned}
f(x) = (x-x_1)(x-x_2)\cdots(x-x_{n-1}) &= x^{n-1} - e_1 x^{n-2} + e_2 x^{n-3} + \cdots + (-1)^{n-1} e_{n-1} \\
&= \sum_{j=0}^{n-1} (-1)^{n-1-j} e_{(n-1),(n-1-j)} x^j
\end{aligned}
$$

Inductive Step: $n = n, R[x_1, \cdots, x_n][x]$

Then, by definition,

$$f'(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$$

Upon substitution,

$$
\begin{aligned}
f'(x) &= (x-x_1)(x-x_2)\cdots(x-x_n) \\
f'(x) &= f(x)(x-x_n) \qquad \text{where } f(x) \text{ is from IH} \\
&= \left(\sum_{j=0}^{n-1} (-1)^{n-1-j} e_{(n-1),(n-1-j)} x^j\right)(x-x_n) \\
&= x\left(\sum_{j=0}^{n-1} (-1)^{n-1-j} e_{(n-1),(n-1-j)} x^j\right) - x_n\left(\sum_{j=0}^{n-1} (-1)^{n-1-j} e_{(n-1),(n-1-j)} x^j\right)
\end{aligned}
$$

(Reindexing j)

$$
\begin{aligned}
&= \left(x^n + \sum_{j=0}^{n-1} (-1)^{n-j} e_{(n-1),(n-j)} x^j\right) - \left(\sum_{j=0}^{n-1} (-1)^{n-1-j} x_n e_{(n-1),(n-1-j)} x^j\right) \\
&= x^n + \sum_{j=0}^{n-1} x^j \left((-1)^{n-j} e_{(n-1),(n-j)} + (-1)^{n-j} x_n e_{(n-1),(n-1-j)}\right) \\
&= \left(x^n + \sum_{j=0}^{n-1} x^j (-1)^{n-j} \left(e_{(n-1),(n-j)} + x_n e_{(n-1),(n-1-j)}\right)\right)
\end{aligned}
$$

$x_n e_{(n-1),(n-1-j)}$ represents all the elements in $e_{n,(n-1-j)}$ that contain x_n as $e_{(n-1),(n-1-j)}$ contains all combinations of terms from $x_1, \cdots, x_{n-1}$. And $e_{(n-1),(n-1-j)}$ all elements in $e_{n,(n-1-j)}$ that don't. So, their sum equals $e_{(n),(n-j)}$. Overall this gives,

$$
\begin{aligned}
f'(x) &= \left(x^n + \sum_{j=0}^{n-1} x^j (-1)^{n-j} e_{(n),(n-j)}\right) \\
&= \left(\sum_{j=0}^{n} x^j (-1)^{n-j} e_{(n),(n-j)}\right)
\end{aligned}
$$

where the last equation follows as $(-1)^{n-n} e_{(n),n-n} = 1$.

Problem 5(d). For each $k \geq 1$, define the **power sums** $p_k = x_1^k + \cdots + x_n^k$ in $R[x_1, \ldots, x_n]$. Clearly, the power sums are symmetric. Verify the following identities by hand:

$$p_1 = e_1, \quad p_2 = e_1 p_1 - 2e_2, \quad p_3 = e_1 p_2 - e_2 p_1 + 3e_3$$

In general **Newton's identities** in $R[x_1, \ldots, x_n]$ are (recall that $e_k = 0$ for $k > n$):

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0.$$

Prove Newton's identities whenever $k \geq n$.

Hint. For each i, consider the equation in part (c) for $f(x_i)$ and sum all these equations together. This gives Newton's identity for $k = n$. Set extra variables to zero to get the identities for $k > n$ from this. (Fun. Can you come up with a proof when $1 \leq k \leq n$?)

Solution.

1. $p_1 = e_1$

$$p_1 = x_1^1 + \cdots x_n^1 = e_1$$

2. $p_2 = e_1 p_1 - 2e_2$

$$\begin{aligned} e_1 p_1 - 2e_2 &= e_1^2 - 2e_2 \\ &= (x_1 + \cdots + x_n)^2 - 2\left(\sum_{1 \leq i < j \leq n} x_i x_j\right) \\ &= \left(x_1^2 + \cdots + x_n^2 + 2\left(\sum_{1 \leq i < j \leq n} x_i x_j\right)\right) - \left(\sum_{1 \leq i < j \leq n} x_i x_j\right) \\ &= x_1^2 + \cdots + x_n^2 \\ &= p_2 \end{aligned}$$

3. $p_3 = e_1 p_2 - e_2 p_1 + 3e_3$

$$
\begin{aligned}
e_1p_2 &= \sum_{1\le i,j\le n} x_ix_j^2 \\
&= \sum_{1\le i,j\le n, i=j} x_i^3 + \sum_{1\le i<j\le n} x_ix_j^2 + \sum_{1\le j<i\le n} x_ix_j^2 \\
&= \sum_{1\le i,j\le n, i=j} x_i^3 + \sum_{1\le i<j\le n} x_ix_j^2 + \sum_{1\le i<j\le n} x_i^2x_j \\
e_2p_1 &= \sum_{\substack{1\le i<j\le n\\1\le k\le n}} x_ix_jx_k \\
&\text{(Using cases: } j<i<k,\ i<k<j,\ i<k<j,\ k=i,\ k=j) \\
&= \sum_{1\le k<i<j\le n} x_ix_jx_k + \sum_{1\le i<k<j\le n} x_ix_jx_k + \sum_{1\le i<j<k\le n} x_ix_jx_k + \sum_{1\le i<j\le n} x_i^2x_j + \sum_{1\le i<j\le n} x_ix_j^2 \\
&\text{(After re-indexing, we get:)} \\
&= 3\sum_{1\le i<j<k\le n} x_ix_jx_k + \sum_{1\le i<j\le n} x_i^2x_j + \sum_{1\le i<j\le n} x_ix_j^2 \\
e_3 &= \sum_{1\le i<j<k\le n} x_ix_jx_k
\end{aligned}
$$

Substituting values in,

$$
\begin{aligned}
e_1p_2 - e_2p_1 + 3e_2 &= \sum_{1\le i,j\le n, i=j} x_i^3 + \sum_{1\le i<j\le n} x_ix_j^2 + \sum_{1\le i<j\le n} x_i^2x_j \\
&\quad - \left(3\sum_{1\le i<j<k\le n} x_ix_jx_k + \sum_{1\le i<j\le n} x_i^2x_j + \sum_{1\le i<j\le n} x_ix_j^2\right) + 3\sum_{1\le i<j<k\le n} x_ix_jx_k \\
&= \sum_{1\le i,j\le n, i=j} x_i^3 \\
&= p_3
\end{aligned}
$$

4. Newton's identities for $k \ge n$.

   Let $f = (x-x_1)\cdots(x-x_n)$. For each x_i, $(x-x_i)$ is a factor of f. Thus, $f(x_i) = 0$ for all i. Summing these from 1 to n and using part (c) we get,

$$\begin{aligned}
0 &= \sum_{i=1}^{n} f(x_i) \\
&= \sum_{i=1}^{n} \left(\sum_{j=0}^{n} (-1)^{n-j} e_{n-j} x^j \right) \\
&= \sum_{i=1}^{n} \left(x_i^n - e_1 x_i^{n-1} + e_2 x^{n-1} + \cdots + e_{n-1} x (-1)^n e_n \right) \\
&= \sum_{i=1}^{n} x_i^n - \sum_{i=1}^{n} e_1 x_i^{n-1} + \sum_{i=1}^{n} e_2 x_i^{n-2} + \cdots + (-1)^{n-1} \sum_{i=1}^{n} e_{n-1} x_i + (-1)^n e_n \\
&= \sum_{i=1}^{n} x_i^n - e_1 \sum_{i=1}^{n} x_i^{n-1} + e_2 \sum_{i=1}^{n} x_i^{n-2} + \cdots + (-1)^{n-1} e_{n-1} \sum_{i=1}^{n} x_i + (-1)^n e_n \\
&= p_n - e_1 p_{n-1} + e_2 p_{n-2} + \cdots + (-1)^{n-1} e_{n-1} p_1 + (-1)^n e_n
\end{aligned}$$

Consider the ring $R[x_1, \cdots, x_n, \cdots x_k]$. Here the equation,

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} + \cdots + (-1)^{k-n} e_n p_{k-n} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k e_k = 0$$

holds. Since, $\forall i > n, e_i = 0$,

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} + \cdots + (-1)^{k-n} e_n p_{k-n} = 0$$

Problem 6. *Use the force, my Newton!*

1. If x, y, z are complex numbers satisfying

$$x + y + z = 1, \qquad x^2 + y^2 + z^2 = 6, \qquad x^3 + y^3 + z^3 = 7,$$

   then prove that $x^n + y^n + z^n$ is rational for any positive integer n.

2. Calculate $x^4 + y^4 + z^4$.

3. Prove that each of x, y, z are not rational numbers.

Solution.

1. Base Case: for $n = 1, 2, 3$ we know that $x^n + y^n + z^n$ is rational.
   Induction Hypothesis: Assume $x^k + y^k + z^k$ is rational, $\forall k < n$.
   Inductive Step: For n, as $n > 3$, the following holds (from 5(d)), also $e_i = 0, \forall i > 3$:

$$p_n - e_1 p_{n-1} + e_2 p_{n-2} - e_3 p_{n-3} = 0$$

   And,

$$p_n = e_1 p_{n-1} - e_2 p_{n-2} + e_3 p_{n-3}$$

   In $R[x, y, z]$, $p_n := x^n + y^n + z^n$.
   Note: $e_1 = p_1, e_2 = (e_1 p_1 - p_2)/2, e_3 = (p_3 - e_1 p_2 + e_2 p_1)/3, e_4 = 0$ as $4 > 3$.
   So, $e_1 = 1, e_2 = -5/2, e_3 = -1/2, e_4 = 0$, which are all rational. And since, $p_{n-1}, p_{n-2}, p_{n-3}$ are also rational (by IH) we have a sum of products of rationals, which is rational as $\mathbb{Q}$ is a field.

2. In $R[x, y, z]$, $p_n := x^n + y^n + z^n$

$$\begin{aligned} 0 &= p_4 - e_1 p_3 + e_2 p_2 + (-1)^3 e_3 p_1 + (-1)^4 e_4 \\ &= p_4 - e_1(7) + e_2(6) - e_3(1) + e_4 \\ p_4 &= e_1 7 - e_2(6) + e_3(1) - e_4 \end{aligned}$$

   Note: $e_1 = p_1, e_2 = (e_1 p_1 - p_2)/2, e_3 = (p_3 - e_1 p_2 + e_2 p_1)/3, e_4 = 0$ as $4 > 3$.
   So, we get $e_1 = 1, e_2 = -5/2, e_3 = -1/2, e_4 = 0$.
   Substituting the values:

$$\begin{aligned} p_4 &= 7 - (-5/2)(6) + (-1/2) - 0 \\ &= 7 + 15 - 1/2 \\ &= 21.5 = 43/2 \end{aligned}$$

3. Let $f(a) = (a-x)(a-y)(a-z)$ in $\mathbb{Q}[n]$.

$$\begin{aligned} f(a) &= a^3 - a^2(x+y+z) + a(xy+yz+zx) - xyz \\ &= a^3 - a^2(e_1) + a(e_2) - e_3 \\ &= a^3 - a^2 - (5/2)a + 1/2 \end{aligned}$$

If $2f(a)$ has a root, then $f(a)$ also has the same root. Rationalizing denominators,

$$2f(a) = 2a^3 - 5a + 1$$

By rational root test, any root of $2f(a)$, say $\left(\frac{p}{q}\right)$ must be such that $p|2$ and $q|1$. Since two is prime, there is only one root we need to check, namely $\left(\frac{p}{q}\right) = 2$. Checking,

$$2f(2) = 2(2)^3 - 5(2) + 1 = 16 - 10 + 1 = 7 \neq 0$$

Hence, 2 is not a root, $f(a)$ doesn't have any rational roots and x, y, z must be irrational.