Cosc 30: Discrete Mathematics

Prishita Dharampal

Problem 1. Provide a good upper bound for the following recurrence. For all problems below assume $T(0) = T(1) = 1$.

1. $T(n) \leq 4 \cdot T(n-2) + 1 \quad \text{if } n > 1.$
2. $T(n) \leq T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \quad \text{if } n > 1.$
3. Let C be a positive constant.

$$T(n) \leq 3 \cdot T\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) + C \quad \text{if } n > 1.$$

Solution.

1. $T(n) \leq 4 \cdot T(n-2) + 1$

   We prove that $T(n) \leq \dfrac{4^{\lfloor \frac{n}{2} \rfloor + 1} - 1}{3}$ for all $n \geq 1$ using induction. Let n be an arbitrary positive integer. Assume $T(k) \leq \dfrac{4^{\lfloor \frac{k}{2} \rfloor + 1} - 1}{3}$ for all k less than n. We check the following cases:

   (a) If $n \geq 1$, we can upper bound $T(n)$ by $4 \cdot T(n-2) + 1$. Since $n - 2 < n$, we substitute,

$$\begin{aligned} T(n) &\leq 4 \cdot T(n-2) + 1 \\ &= 4 \cdot \frac{4^{\lfloor \frac{n-2}{2} \rfloor + 1} - 1}{3} + 1 \\ &= \frac{4^{\lfloor \frac{n-2}{2} \rfloor + 2} - 4 + 3}{3} \\ &= \frac{4^{\lfloor \frac{n}{2} \rfloor + 1} - 1}{3} \end{aligned}$$

Hence, $T(n) \leq \dfrac{4^{\lfloor \frac{n-2}{2} \rfloor + 1} - 1}{3}$ holds for $n > 1$.

(b) If $n = 1$, we get

$$T(1) = \frac{4^{\lfloor \frac{1}{2} \rfloor + 1} - 1}{3} = \frac{4^1 - 1}{3} = \frac{3}{3} = 1$$

So the base case holds.

Therefore the statement is proved.

Note: We use the floor function $k = \lfloor \frac{n}{2} \rfloor$ so that $n - 2k \in \{0, 1\}$, and odd for n we get there one step quicker.

2. $T(n) \leq T\left(\lfloor \frac{n}{3} \rfloor\right) + n$

We prove that $T(n) \leq \dfrac{3n-1}{2}$ by induction for all $n \geq 1$. Let n be an arbitrary positive integer. Assume $T(k) \leq \dfrac{3n-1}{2}$ for all k less than n. We check the following cases:

(a) If $n \geq 1$, we can upper bound $T(n)$ by $T(\lfloor \frac{n}{3} \rfloor) + n$. Since $\lfloor \frac{n}{3} \rfloor < n$, we substitute,

$$\begin{aligned}
T(n) &\leq T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \\
&= \frac{3(\lfloor \frac{n}{3} \rfloor) - 1}{2} + n \\
&= \frac{n-1}{2} + n \\
&= \frac{3n-1}{2}
\end{aligned}$$

Hence, $T(n) \leq \dfrac{3n-1}{2}$ holds for $n > 1$.

(b) If $n = 1$, we get

$$T(1) = \frac{3-1}{2} = 1$$

So the base case holds.

Therefore the statement is proved.

3. $T(n) \leq 3 \cdot T\left(\lfloor \frac{2n}{3} \rfloor\right) + C \quad$ if $n > 1$, for some positive constant C

We prove that $T(n) \leq 3^{(log_{3/2} n)} + C \cdot \dfrac{3^{(log_{3/2} n)} - 1}{2}$ by induction for all $n \geq 1$. Let n be an arbitrary positive integer. Assume $T(k) \leq 3^{(log_{3/2} k)} + C \cdot \dfrac{3^{(log_{3/2} k)} - 1}{2}$ for all k less than n. We check the following cases:

(a) If $n \geq 1$, we can upper bound $T(n)$ by $3 \cdot T(\lfloor \frac{2n}{3} \rfloor) + C$. Since $\lfloor \frac{2n}{3} \rfloor < n$, we substitute,

$$
\begin{aligned}
T(n) &\leq 3 \cdot T\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) + C \\
&= 3 \cdot \left(3^{\left(log_{3/2} \lfloor \frac{2n}{3} \rfloor\right)} + C \cdot \frac{3^{\left(log_{3/2} \lfloor \frac{2n}{3} \rfloor\right)} - 1}{2} \right) + C
\end{aligned}
$$

But,

$$
log_{3/2} \left\lfloor \frac{2n}{3} \right\rfloor \leq log_{3/2} \left(\frac{2n}{3} \right)
$$

Using the upper bound,

$$
\begin{aligned}
T(n) &\leq 3 \cdot T\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) + C \\
&= 3 \cdot \left(3^{\left(log_{3/2} \frac{2n}{3}\right)} + C \cdot \frac{3^{\left(log_{3/2} \frac{2n}{3}\right)} - 1}{2} \right) + C \\
&= 3 \cdot \left(3^{\left(log_{3/2} n\right) - 1} + C \cdot \frac{3^{\left(log_{3/2} n\right) - 1} - 1}{2} \right) + C \\
&= \left(3^{\left(log_{3/2} n\right)} + C \cdot \frac{3^{\left(log_{3/2} n\right)} - 3}{2} \right) + C \\
&= 3^{\left(log_{3/2} n\right)} + C \cdot \left(\frac{3^{\left(log_{3/2} n\right)} - 1}{2} \right)
\end{aligned}
$$

Hence, $T(n) \leq 3^{\left(log_{3/2} n\right)} + C \cdot \left(\frac{3^{\left(log_{3/2} n\right)} - 1}{2} \right)$ holds for $n > 1$.

(b) If $n = 1$, we get

$$
T(1) = 3^0 + C \cdot \left(\frac{3^0 - 1}{2} \right) = 1
$$

So the base case holds.

Therefore the statement is proved.