

Math 63: Real Analysis

Prishita Dharampal

Credit Statement: Talked to Sair Shaikh'26, and Math Stack Exchange. **TODO:**

Problem 1. If A, B, C are sets show that:

$$A - (B - C) = (A - B) \cup (A \cap B \cap C)$$

Solution.

To show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ we first show that

$$A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$$

and then

$$A - (B - C) \supset (A - B) \cup (A \cap B \cap C).$$

1. $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$

$$\forall x \in A - (B - C),$$

$$\begin{aligned} &\implies x \in A \text{ and } x \notin (B - C) \\ &\implies x \in A \text{ and } (x \notin B \text{ or } x \in B \cap C) \\ &\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in B \cap C) \\ &\implies (x \in A \cap \bar{B}) \cup (x \in A \cap B \cap C) \\ &\implies (x \in A - B) \cup (x \in A \cap B \cap C) \\ &\implies x \in (A - B) \cup (A \cap B \cap C) \\ &\implies A - (B - C) \subset (A - B) \cup (A \cap B \cap C) \end{aligned}$$

$$2. A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$$

$$\forall x \in (A - B) \cup (A \cap B \cap C),$$

$$\begin{aligned} &\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \cap B \cap C) \\ &\implies (x \in A) \text{ and } (x \notin B \text{ or } x \in B \cap C) \\ &\implies x \in A \cap (\bar{B} \cup (B \cap C)) \\ &\implies x \in A \cap ((\bar{B} \cup B) \cap (\bar{B} \cup C)) \\ &\implies x \in A \cap (1 \cap (\bar{B} \cup C)) \\ &\implies x \in A \cap (\bar{B} \cup C) \\ &\implies x \in A - (\overline{\bar{B} \cup C}) \\ &\implies x \in A - (B \cap \bar{C}) \\ &\implies x \in A - (B - C) \end{aligned}$$

Since, $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$ and $A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$ we can say that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

Problem 2. Let I be a set and for each $i \in I$ let X_i , be a set. Prove that for any set B we have:

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$$

Solution.

To show that $B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$ we first show that $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$ and then $B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$.

$$1. B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$$

If $x \in B \cap \bigcup_{i \in I} X_i$ then $x \in B$ and $x \in \bigcup_{i \in I} X_i$.

I.e. x is at least in one X_j for some $j \in I \implies x \in B \cap X_j$.

Thus, $x \in \bigcup_{i \in I} (B \cap X_i) \implies B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$.

$$2. B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$$

If $x \in \bigcup_{i \in I} (B \cap X_i)$, then x is at least in one $B \cap X_j$ for some $j \in I$

$$\begin{aligned} &\implies x \in B \text{ and } x \in X_j \\ &\implies x \in B \text{ and } x \in \bigcup_{i \in I} X_i \\ &\implies x \in B \cap \bigcup_{i \in I} X_i \end{aligned}$$

$$\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i).$$

Since, $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$ and $\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$, we can say that
 $\implies B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$.

Problem 3. Let $f : X \rightarrow Y$ be a function, let A and B be subsets of X , and let C and D be subsets of Y . Prove that:

1. $f(A \cap B) \subset f(A) \cap f(B)$
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Solution.

1. Let $x \in f(A \cap B), x = f(y)$

$$\begin{aligned} &\implies y \in A, B \\ &\implies x \in f(A), x \in f(B) \\ &\implies x \in f(A) \cap f(B) \\ &\implies f(A \cap B) \subset f(A) \cap f(B) \end{aligned}$$

For an arbitrary f the reverse containment isn't true. To show this, consider distinct elements $a \in A, a \notin B, b \in B, b \notin A$ such that for some $c, f(a) = c, f(b) = c$. Then $c \in f(A) \cap f(B)$ but $c \notin f(A \cap B)$. I.e. equality won't hold unless f is injective.

2. Let $x \in f^{-1}(C \cap D), y = f(x)$

(\implies)

$$\begin{aligned} &\implies y \in C, y \in D \\ &\implies x \in f^{-1}(C), x \in f^{-1}(D) \\ &\implies x \in f^{-1}(C) \cap f^{-1}(D) \\ &\implies f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

(\Leftarrow)

- Let $x \in f^{-1}(C) \cap f^{-1}(D), y = f(x)$

$$\begin{aligned} &\implies x \in f^{-1}(C), x \in f^{-1}(D) \\ &\implies y \in C, D \\ &\implies y \in C \cap D \\ &\implies x \in f^{-1}(C \cap D) \end{aligned}$$

Problem 4.

1. How many functions are there from a nonempty set S into the \emptyset ?
2. Show that the notation $\{X_i\}_{i \in I}$ implicitly involves the notion of function.

Solution.

1. There are no functions from a nonempty set to the empty set. A function needs to assign a definite output to every input. Since there are no elements in \emptyset , that is impossible.
2. The notation $\{X_i\}_{i \in I}$ describes a rule from elements of I to corresponding objects X_i . This by definition is a function from I to the set consisting objects X_i .

Problem 5. Prove in detail that for any $a, b \in \mathbb{R}$:

$$-(a - b) = b - a$$

Solution.

Because R is a field, we can say that,

$$\begin{aligned} -(a - b) &= -(a) - (-b) \text{ (Field Property 8)} \\ &= -a + b \text{ (Field Property 6)} \\ &= b - a \text{ (Commutativity)} \end{aligned}$$

Hence, $-(a - b) = b - a$ is true for any $a, b \in \mathbb{R}$.

Problem 6. Show that if $a, b, x, y \in \mathbb{R}$ and $a < x < b, a < y < b$, then $|y - x| < b - a$.

Solution.

$b > a \implies b - a > a - a \implies b - a > 0$. Hence, $b - a$ is always positive.

There are 3 cases:

1. $x = y$

Then $y - x < b - a$, is trivially true since $0 < b - a$.

2. $x > y$

$$\begin{aligned} y &< x \\ y - x &< x - x \\ y - x &< 0 \\ y - x &< 0 < b - a \end{aligned}$$

3. $x < y$

$$\begin{aligned} y &< b \\ a &< x \\ \implies y - x &< b - a \end{aligned}$$

Problem 7. Find the g.l.b. and l.u.b. of $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, giving reasons if you can.

Solution.

Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$.

The set $S \subset \mathbb{R}$ is nonempty, and bounded from above since $\forall \frac{1}{n}, \frac{1}{n+1} \in S, \frac{1}{n} > \frac{1}{n+1}$. Hence $y = l.u.b. S$ exists. From the inequality above, we can see that the larger n is, the smaller the resulting fraction is. I.e. the largest value occurs at $n = 1$. Because the maximum element of the set is 1, **the least upper bound is 1**.

$\frac{1}{n} > 0, \forall n > 0$ (7th consequence of the order property). Hence, the set $S \subset \mathbb{R}$ is nonempty, and bounded from below by 0. I.e. $y = g.l.b. S$ exists. To show that 0 is the greatest lower bound, let $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$. Then $\epsilon > \frac{1}{n}$ (L.U.B. 2).

$$\implies 0 < \frac{1}{n} < \epsilon$$

Hence, no $\epsilon > 0$ can be a lower bound for the set S . Therefore, **the greatest lower bound is 0**. **TODO:**

Problem 8. Prove that if $a \in \mathbb{R}, a > 1$, then the set $\{a, a^2, a^3, \dots\}$ is not bounded from above. (Hint: First find a positive integer n such that $a > 1 + \frac{1}{n}$ and prove that $a^n > (1 + \frac{1}{n})^n \geq 2$).

Solution.

Problem 9. If S_1, S_2 are nonempty subsets of \mathbb{R} that are bounded from above, prove that

$$\text{l.u.b. } \{x + y : x \in S_1, y \in S_2\} = \text{l.u.b. } S_1 + \text{l.u.b. } S_2$$

Solution.

If S_1, S_2 are nonempty subsets of \mathbb{R} that are bounded from above, we know that $\text{l.u.b. } S_1$, and $\text{l.u.b. } S_2$ exist.

Let $S = \{x + y : x \in S_1, y \in S_2\}$, $\text{l.u.b. } S_1 = l_1$, $\text{l.u.b. } S_2 = l_2$.

By definition $\forall x \in S_1, x \leq l_1$ and $\forall y \in S_2, y \leq l_2$. Adding the inequalities, we get $x + y \leq l_1 + l_2$. Hence, $l_1 + l_2$ is an upper bound of S .

Let ϵ be an arbitrarily small positive number. Since $l_1 = \text{l.u.b. } S_1$, $\exists x_0 \in S_1$ such that $l_1 - \epsilon/2 < x_0$.

$$l_1 - \epsilon/2 < x_0 \leq l_1$$

Similarly, since $l_2 = \text{l.u.b. } S_2$, $\exists y_0 \in S_2$ such that $l_2 - \epsilon/2 < y_0$.

$$l_2 - \epsilon/2 < y_0 \leq l_2$$

Adding the two inequalities gives,

$$l_1 + l_2 - \epsilon < x_0 + y_0 \leq l_1 + l_2$$

Hence, $l_1 + l_2$ is the least upper bound for the set S .