Cosc 30: Discrete Mathematics

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Problem 1. For any integer n, prove that n^3 is even if and only if n is even.

Solution.

$(\implies)$

If n^3 is even, then by definition $n * n * n$ is even. Since,

$$\begin{aligned} even * even &= even \\ even * odd &= even \\ odd * even &= even \\ odd * odd &= odd \end{aligned}$$

for a product to be even at least one of the terms must be even. Hence, n must be even.

$(\impliedby)$

We know that for any even integer a, $a * a$ is even. I.e. if n is even, then $n * n$ is even. Again, since $n * n$ is even, and n is even, $(n * n) * n$ must be even. Hence, if n is even n^3 is even.

Problem 2. $\sqrt[3]{4}$ is not a fraction; in other words, $\sqrt[3]{4}$ cannot be written as a ratio of two integers.
Hint: Does your proof remain unchanged if we replace $\sqrt[3]{4}$ with $\sqrt[3]{8}$? We are in trouble

Solution.

We will prove that $\sqrt[3]{4}$ isn't rational by contradiction.

Let $\sqrt[3]{4} = \frac{a}{b}$, such that $a, b \in \mathbb{Z}$, $b \neq 0$, $gcd(a, b) = 1$. Then,

$$\begin{aligned}\sqrt[3]{4} &= \frac{a}{b} \\ 4 &= \left(\frac{a}{b}\right)^3 \\ 2^2 b^3 &= a^3\end{aligned}$$

Hence, a^3 is divisible by 2. From (Problem 1) we know that if a^3 is even, then a must be even. I.e. a must be divisible by 2. Substitute $a = 2k$.

$$\begin{aligned}2^2 b^3 &= (2k)^3 \\ 2^2 b^3 &= 2^3 k^3 \\ b^3 &= 2k^3\end{aligned}$$

Similarly, b^3 is divisible by 2. From (Problem 1) we know that if b^3 is even, then b must be even. I.e. b must be divisible by 2. But we defined $gcd(a, b) = 1$. Hence this is a contradiction! $\sqrt[3]{4}$ cannot be written as a ratio of two integers.

Checking the proof with $\sqrt[3]{8}$. Let $\sqrt[3]{8} = \frac{a}{b}$, such that $a, b \in \mathbb{Z}$, $b \neq 0$, $gcd(a, b) = 1$. Then,

$$\begin{aligned}\sqrt[3]{8} &= \frac{a}{b} \\ 8 &= \left(\frac{a}{b}\right)^3 \\ 2^3 b^3 &= a^3\end{aligned}$$

Hence, a^3 is divisible by 2 $\implies$ a must be divisible by 2. Substitute $a = 2k$.

$$\begin{aligned}2^3 b^3 &= (2k)^3 \\ 2^3 b^3 &= k^3 \\ b^3 &= k^3\end{aligned}$$

We can't move ahead from here, and the proof breaks.

Problem 3. Given an array $A[1 \ldots n]$, a value x is called *abundant* if there are more than one-hundredth of the elements in array A with value equal to x.
Prove that every array has at most 100 distinct abundant values.

Solution.

Let $a \in \mathbb{Z}$ be the number of distinct abundant values in the array. By definition, an abundant value occurs in the array more than $n/100$ times.

Define an array $B[c_1, c_2, \cdots, c_k]$ to denote the number of times each abundant value appears in the array. Then we can say that,

$$\sum_{i=1}^{k} c_i > k \cdot \frac{n}{100}$$

But also since there are n elements in the array,

$$\sum_{i=1}^{k} c_i \leq n$$

$$\begin{aligned} &\implies k \cdot \frac{n}{100} \leq n \\ &\implies k \leq 100 \end{aligned}$$

Therefore, there are at most 100 distinct abundant values.

Problem 4*. We call numbers like $\sqrt{2}$ which cannot be written as a fraction *irrational*. Are there two irrational numbers x and y such that x^y is rational? Why?

Solution.

Yes.

We know that square roots of prime numbers are irrational. Let $x = \sqrt{5}$ and $y = \sqrt{2}$. Then, x^y is either rational, in which case we are done, or x^y is irrational. If x^y is irrational, redefine $x = \sqrt{5}^{\sqrt{2}}, y = \sqrt{2}$. Then

$$x^y = (\sqrt{5}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{5}^{\sqrt{2} * \sqrt{2}} = \sqrt{5}^2 = 5 = \frac{5}{1}$$

which is rational. However, I couldn't come up with a general proof for all irrational numbers.