Cosc 30: Discrete Mathematics

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Problem 1. Prove that any positive integer n greater than 1 can be written as a product of primes.

Solution.

Let $P(n) = \forall n \in \mathbb{Z}_{>1}$: n is a product of primes.

Assume that $P(k)$ holds where $1 < k < n$. I.e. the inductive hypothesis is that any integer greater than 1 and less than n can be written as a product of primes.

Then we have two cases:

1. n is prime:

   Then n is a product of itself, and we are done.

2. n is not prime:

   Then we can write n as a product of two integers a, b. Then,

   (a) a, b are prime.
   By definition, $n = a * b$. Hence, n can be written as a product of primes.

   (b) Either one or both of a, b is not prime.
   Since $b = \frac{n}{a}, a = \frac{n}{b}$ we can see that $a, b < n$. Then by the induction hypothesis, both a and b can be written as products of primes. Therefore, their product $n = ab$ can also be written as a product of primes.

Problem 2. Prove that for every real number $r \neq 1$ and every nonnegative integer n,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Solution.

Let $P(x)$ be the statement:

$$\sum_{i=0}^{x} r^i = \frac{r^{x+1} - 1}{r - 1},$$

for a fixed real number $r \neq 1$.

Assume $P(x)$ holds upto $x = n - 1$. Then we have two cases: $x = 0$ or $x > 0$

1. Base Case: $x = 0$
   $\sum_{i=0}^{0} r^i = r^0 = 1$ and $\frac{r^1-1}{r-1} = \frac{r-1}{r-1} = 1$. Hence, $P(0)$ holds.
2. $x = n$, then $P(n)$:

$$\begin{aligned}
\sum_{i=0}^{n} r^i &= \sum_{i=0}^{n-1} r^i + r^n \\
&= \frac{r^n - 1}{r - 1} + r^n \\
&= \frac{r^n - 1}{r - 1} + \frac{r^n(r - 1)}{r - 1} \\
&= \frac{r^n - 1 + r^{n+1} - r^n}{r - 1} \\
&= \frac{r^{n+1} - 1}{r - 1}
\end{aligned}$$

Hence the proposition holds.

Problem 3. Prove that, given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make change for any amount that is at least 30 cents.
[Hint: You need many cases.]

Solution.

Let $P(n) = \forall n \geq 30 : n$ cents can be made using 6, 10, and 15-cent coins.

Assume that for any integer x such that $30 \leq x < n$, we can make x cents using $6, 10, 15$-cent coins. Then there are seven cases to consider: $x = 30, x = 31, x = 32, x = 33, x = 34, x = 35, x > 35$

1. $30 = 10 + 10 + 10$
2. $31 = 10 + 15 + 6$
3. $32 = 10 + 10 + 6 + 6$
4. $33 = 15 + 6 + 6 + 6$
5. $34 = 10 + 6 + 6 + 6 + 6$
6. $35 = 10 + 10 + 15$
7. $x > 35$

   If $n > 35$, then $n - 6 \geq 30$. By the induction hypothesis, $n - 6$ cents can be made. Adding one 6-cent coin gives us a representation for n cents.

Therefore for any $n \geq 30$, we can make n cents using $6, 10, 15$-cent coins.