Math 63: Real Analysis

Prishita Dharampal

Credit Statement: Talked to Sair Shaikh'26, and Math Stack Exchange. **TODO:**

Problem 1. If A, B, C are sets show that:

$$A - (B - C) = (A - B) \cup (A \cap B \cap C)$$

Solution.

To show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ we first show that

$$A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$$

and then

$$A - (B - C) \supset (A - B) \cup (A \cap B \cap C).$$

1. $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$
   $\forall x \in A - (B - C)$,

$$\begin{aligned}
&\implies x \in A \text{ and } x \notin (B - C) \\
&\implies x \in A \text{ and } (x \notin B \text{ or } x \in B \cap C) \\
&\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in B \cap C) \\
&\implies (x \in A \cap \bar{B}) \cup (x \in A \cap B \cap C) \\
&\implies (x \in A - B) \cup (x \in A \cap B \cap C) \\
&\implies x \in (A - B) \cup (A \cap B \cap C) \\
&\implies A - (B - C) \subset (A - B) \cup (A \cap B \cap C)
\end{aligned}$$

2. $A-(B-C) \supset (A-B) \cup (A \cap B \cap C)$
$\forall x \in (A-B) \cup (A \cap B \cap C)$,

$$
\begin{aligned}
&\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \cap B \cap C)\\
&\implies (x \in A) \text{ and } (x \notin B \text{ or } x \in B \cap C)\\
&\implies x \in A \cap (\bar{B} \cup (B \cap C))\\
&\implies x \in A \cap ((\bar{B} \cup B) \cap (\bar{B} \cup C))\\
&\implies x \in A \cap (1 \cap (\bar{B} \cup C))\\
&\implies x \in A \cap (\bar{B} \cup C)\\
&\implies x \in A - (\overline{\bar{B} \cup C})\\
&\implies x \in A - (B \cap \bar{C})\\
&\implies x \in A - (B - C)
\end{aligned}
$$

Since, $A-(B-C) \subset (A-B) \cup (A \cap B \cap C)$ and $A-(B-C) \supset (A-B) \cup (A \cap B \cap C)$ we can say that $A-(B-C) = (A-B) \cup (A \cap B \cap C)$.

Problem 2. Let I be a set and for each $i \in I$ let X_i , be a set. Prove that for any set B we have:

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$$

Solution. To show that $B \cap \bigcup\limits_{i \in I} X_i = \bigcup\limits_{i \in I} (B \cap X_i)$ we first show that $B \cap \bigcup\limits_{i \in I} X_i \subset \bigcup\limits_{i \in I} (B \cap X_i)$ and then $B \cap \bigcup\limits_{i \in I} X_i \supset \bigcup\limits_{i \in I} (B \cap X_i)$.

1. $B \cap \bigcup\limits_{i \in I} X_i \subset \bigcup\limits_{i \in I} (B \cap X_i)$

   If $x \in B \cap \bigcup\limits_{i \in I} X_i$ then $x \in B$ and $x \in \bigcup\limits_{i \in I} X_i$.

   I.e. x is at least in one X_j for some $j \in I \implies x \in B \cap X_j$.

   Thus, $x \in \bigcup\limits_{i \in I} (B \cap X_i) \implies B \cap \bigcup\limits_{i \in I} X_i \subset \bigcup\limits_{i \in I} (B \cap X_i)$.

2. $B \cap \bigcup\limits_{i \in I} X_i \supset \bigcup\limits_{i \in I} (B \cap X_i)$

   If $x \in \bigcup\limits_{i \in I} (B \cap X_i)$, then x is at least in one $B \cap X_j$ for some $j \in I$

$$\begin{aligned} &\implies x \in B \text{ and } x \in X_j \\ &\implies x \in B \text{ and } x \in \bigcup_{i \in I} X_i \\ &\implies x \in B \cap \bigcup_{i \in I} X_i \end{aligned}$$

   $\implies B \cap \bigcup\limits_{i \in I} X_i \supset \bigcup\limits_{i \in I} (B \cap X_i)$.

   Since, $B \cap \bigcup\limits_{i \in I} X_i \subset \bigcup\limits_{i \in I} (B \cap X_i)$ and $\implies B \cap \bigcup\limits_{i \in I} X_i \supset \bigcup\limits_{i \in I} (B \cap X_i)$, we can say that $\implies B \cap \bigcup\limits_{i \in I} X_i = \bigcup\limits_{i \in I} (B \cap X_i)$.

Problem 3. Let $f : X \to Y$ be a function, let A and B be subsets of X, and let C and D be subsets of Y. Prove that:

1. $f(A \cap B) \subset f(A) \cap f(B)$
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Solution.

1. Let $x \in f(A \cap B), x = f(y)$

$$\begin{aligned} &\implies y \in A, B \\ &\implies x \in f(A), x \in f(B) \\ &\implies x \in f(A) \cap f(B) \\ &\implies f(A \cap B) \subset f(A) \cap f(B) \end{aligned}$$

   For an arbitrary f the reverse containment isn't true. To show this, consider distinct elements $a \in A, a \notin B, b \in B, b \notin A$ such that for some c, $f(a) = c, f(b) = c$. Then $c \in f(A) \cap f(B)$ but $c \notin f(A \cap B)$. I.e. equality won't hold unless f is injective.

2. Let $x \in f^{-1}(C \cap D), y = f(x)$

   $(\implies)$

$$\begin{aligned} &\implies y \in C, y \in D \\ &\implies x \in f^{-1}(C), f^{-1}(D) \\ &\implies x \in f^{-1}(C) \cap f^{-1}(D) \\ &\implies f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

   $(\impliedby)$

   Let $x \in f^{-1}(C) \cap f^{-1}(D), y = f(x)$

$$\begin{aligned} &\implies x \in f^{-1}(C), x \in f^{-1}(D) \\ &\implies y \in C, D \\ &\implies y \in C \cap D \\ &\implies x \in f^{-1}(C \cap D) \end{aligned}$$

Problem 4.

1. How many functions are there from a nonempty set S into the $\emptyset$?
2. Show that the notation $\{X_i\}_{i\in I}$ implicitly involves the notion of function.

Solution.

1. There are no functions from a nonempty set to the empty set. A function needs to assign a definite output to every input. Since there are no elements in $\emptyset$, that is impossible.
2. The notation $\{X_i\}_{i\in I}$ describes a rule from elements of I to corresponding objects X_i. This by definition is a function from I to the set consisting objects X_i.

Problem 5. Prove in detail that for any $a, b \in \mathbb{R}$:

$$-(a-b) = b-a$$

Solution.

Because R is a field, we can say that,

$$\begin{aligned}-(a-b) &= -(a)-(-b) \text{ (Field Property 8)}\\ &= -a+b \text{ (Field Property 6)}\\ &= b-a \text{ (Commutativity)}\end{aligned}$$

Problem 6. Show that if $a, b, x, y \in \mathbb{R}$ and $a < x < b, a < y < b$, then $\mid y - x \mid < b - a$.

Solution.

There are two cases,

1. $x = y$

   Here $\mid y - x \mid < b - a$ trivially true as $0 < b - a, \forall\, b > a$.

2. $x \neq y$

   WLOG, assume $y > x$. Since, $y < b$ and $x > a$ then $max(y - x) < b - a$.

Problem 7. Find the g.l.b. and l.u.b. of $\left\{1, \frac{1}{2}, \frac{1}{4}, \cdots\right\}$, giving reasons if you can.

Solution.

g.l.b $= 0$, l.u.b $= 1$.

Assume

Problem 8. Prove that if $a \in \mathbb{R}, a > 1$, then the set $\{a, a^2, a^3, ...\}$ is not bounded from above. (Hint: First find a positive integer n such that $a > 1 + \frac{1}{n}$ and prove that $a^n > \left(1 + \frac{1}{n}\right)^n \geq 2$).

Problem 8. If S_1, S_2 are nonempty subsets of $\mathbb{R}$ that are bounded from above, prove that

$$\text{l.u.b}\ \{x + y : x \in S_1, y \in S_2\} = \text{l.u.b.}\ S_1 + \text{l.u.b.}\ S_2$$