Math 63: Real Analysis

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Problem 1. If A, B, C are sets show that:

$$A - (B - C) = (A - B) \cup (A \cap B \cap C)$$

Solution.

To show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ we first show that

$$A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$$

and then

$$A - (B - C) \supset (A - B) \cup (A \cap B \cap C).$$

1. $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$
   $\forall x \in A - (B - C)$,

$$\begin{aligned}
&\implies x \in A \text{ and } x \notin (B - C) \\
&\implies x \in A \text{ and } (x \notin B \text{ or } x \in B \cap C) \\
&\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in B \cap C) \\
&\implies (x \in A \cap \bar{B}) \cup (x \in A \cap B \cap C) \\
&\implies (x \in A - B) \cup (x \in A \cap B \cap C) \\
&\implies x \in (A - B) \cup (A \cap B \cap C) \\
&\implies A - (B - C) \subset (A - B) \cup (A \cap B \cap C)
\end{aligned}$$

2. $A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$

$\forall x \in (A - B) \cup (A \cap B \cap C)$,

$$
\begin{aligned}
&\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \cap B \cap C) \\
&\implies (x \in A) \text{ and } (x \notin B \text{ or } x \in B \cap C) \\
&\implies x \in A \cap (\bar{B} \cup (B \cap C)) \\
&\implies x \in A \cap ((\bar{B} \cup B) \cap (\bar{B} \cup C)) \\
&\implies x \in A \cap (1 \cap (\bar{B} \cup C)) \\
&\implies x \in A \cap (\bar{B} \cup C) \\
&\implies x \in A - (\overline{\bar{B} \cup C}) \\
&\implies x \in A - (B \cap \bar{C}) \\
&\implies x \in A - (B - C)
\end{aligned}
$$

Since, $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$ and $A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$ we can say that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

Problem 2. Let I be a set and for each $i \in I$ let X_i , be a set. Prove that for any set B we have:

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I}(B \cap X_i)$$

Solution.

To show that $B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I}(B \cap X_i)$ we first show that $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I}(B \cap X_i)$ and then $B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I}(B \cap X_i)$.

1. $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I}(B \cap X_i)$

   If $x \in B \cap \bigcup_{i \in I} X_i$ then $x \in B$ and $x \in \bigcup_{i \in I} X_i$.

   I.e. x is at least in one X_j for some $j \in I \implies x \in B \cap X_j$.

   Thus, $x \in \bigcup_{i \in I}(B \cap X_i) \implies B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I}(B \cap X_i)$.

2. $B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I}(B \cap X_i)$

   If $x \in \bigcup_{i \in I}(B \cap X_i)$, then x is at least in one $B \cap X_j$ for some $j \in I$

   $$\begin{aligned} &\implies x \in B \text{ and } x \in X_j \\ &\implies x \in B \text{ and } x \in \bigcup_{i \in I} X_i \\ &\implies x \in B \cap \bigcup_{i \in I} X_i \end{aligned}$$

   $\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I}(B \cap X_i)$.

   Since, $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I}(B \cap X_i)$ and $\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I}(B \cap X_i)$, we can say that $\implies B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I}(B \cap X_i)$.

Problem 3. Let $f : X \to Y$ be a function, let A and B be subsets of X, and let C and D be subsets of Y. Prove that:

1. $f(A \cap B) \subset f(A) \cap f(B)$
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Solution.

1. Let $x \in f(A \cap B), x = f(y)$

$$\begin{aligned} &\implies y \in A, B \\ &\implies x \in f(A), x \in f(B) \\ &\implies x \in f(A) \cap f(B) \\ &\implies f(A \cap B) \subset f(A) \cap f(B) \end{aligned}$$

   For an arbitrary f the reverse containment isn't true. To show this, consider distinct elements $a \in A, a \notin B, b \in B, b \notin A$ such that for some c, $f(a) = c, f(b) = c$. Then $c \in f(A) \cap f(B)$ but $c \notin f(A \cap B)$. I.e. equality won't hold unless f is injective.

2. Let $x \in f^{-1}(C \cap D), y = f(x)$

   $(\implies)$

$$\begin{aligned} &\implies y \in C, y \in D \\ &\implies x \in f^{-1}(C), f^{-1}(D) \\ &\implies x \in f^{-1}(C) \cap f^{-1}(D) \\ &\implies f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

   $(\impliedby)$

   Let $x \in f^{-1}(C) \cap f^{-1}(D), y = f(x)$

$$\begin{aligned} &\implies x \in f^{-1}(C), x \in f^{-1}(D) \\ &\implies y \in C, D \\ &\implies y \in C \cap D \\ &\implies x \in f^{-1}(C \cap D) \end{aligned}$$

Problem 4.

1. How many functions are there from a nonempty set S into the $\emptyset$?
2. Show that the notation $\{X_i\}_{i \in I}$ implicitly involves the notion of function.

Solution.

1. There are no functions from a nonempty set to the empty set. A function needs to assign a definite output to every input. Since there are no elements in $\emptyset$, that is impossible.
2. The notation $\{X_i\}_{i \in I}$ describes a rule from elements of I to corresponding objects X_i. This by definition is a function from I to the set consisting objects X_i.

Problem 5. Prove in detail that for any $a, b \in \mathbb{R}$:

$$-(a - b) = b - a$$

Solution.

Because R is a field, we can say that,

$$\begin{aligned} -(a-b) &= -(a) - (-b) \text{ (Field Property 8)} \\ &= -a + b \text{ (Field Property 6)} \\ &= b - a \text{ (Commutativity)} \end{aligned}$$

Hence, $-(a - b) = b - a$ is true for any $a, b \in \mathbb{R}$.

Problem 6. Show that if $a, b, x, y \in \mathbb{R}$ and $a < x < b, a < y < b$, then $\mid y - x \mid < b - a$.

Solution.

$b > a \implies b - a > a - a \implies b - a > 0$. Hence, $b - a$ is always positive.

There are 3 cases:

1. $x = y$

   Then $y - x < b - a$, is trivially true since $0 < b - a$.

2. $x > y$

$$\begin{aligned} y &< x \\ y - x &< x - x \\ y - x &< 0 \\ y - x &< 0 < b - a \end{aligned}$$

3. $x < y$

$$\begin{aligned} y &< b \\ a &< x \\ \implies y - x &< b - a \end{aligned}$$

Problem 7. Find the g.l.b. and l.u.b. of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$, giving reasons if you can.

Solution.

Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$. We know that, $\forall \frac{1}{n}$, $\frac{1}{n+1} \in S$

$$\frac{1}{n} > \frac{1}{n+1}$$

From the inequality above, we can see that the larger n is, the smaller the resulting fraction is. I.e. the largest value occurs at $n = 1$. Hence, the set is bounded from above. Because the set $S \subset \mathbb{R}$ is nonempty, and bounded from above $y = l.u.b.\, S$ exists. As the maximum element of the set is 1, **the least upper bound is** 1.

$\frac{1}{n} > 0, \forall n > 0$ (7th consequence of the order property). Hence, the set $S \subset \mathbb{R}$ is nonempty, and bounded from below by 0. I.e. $y = g.l.b.\, S$ exists. To show that 0 is the greatest lower bound, let $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$. Then $\epsilon > \frac{1}{n}$ (L.U.B. 2).

$$\implies 0 < \frac{1}{n} < \epsilon$$

Hence, no $\epsilon > 0$ can be a lower bound for the set S. Therefore, **the greatest lower bound is** 0.

Problem 8. Prove that if $a \in \mathbb{R}, a > 1$, then the set $\{a, a^2, a^3, ...\}$ is not bounded from above. (**Hint:** First find a positive integer n such that $a > 1 + \frac{1}{n}$ and prove that $a^n > \left(1 + \frac{1}{n}\right)^n \geq 2$).

Solution.

We will prove this in two parts,

1. $a^n > \left(1 + \frac{1}{n}\right)^n$

   From (L.U.B. 1) we know that $\exists n \in \mathbb{N}$ such that $n > x$ for any given $x \in \mathbb{R}$. Also,

   $$a > 1 \implies a - 1 > 0$$

   Then, we can say that, for some $n \in \mathbb{N}$,

   $$n(a-1) > 1$$

   Rearranging the inequality gives us,

   $$\begin{aligned} n(a-1) &> 1 \\ a - 1 &> \frac{1}{n} \\ a &> 1 + \frac{1}{n} \end{aligned}$$

   Since both a and $1 + \frac{1}{n}$ are positive numbers, $a > 1 + \frac{1}{n}$, and $n \in \mathbb{N}$, we know,

   $$a^n > \left(1 + \frac{1}{n}\right)^n$$

2. $\left(1 + \frac{1}{n}\right)^n \geq 2$

   $\left(1 + \frac{1}{n}\right)^n = \underbrace{\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) \cdots \left(1 + \frac{1}{n}\right)}_{n-times}$

   We only need to partially expand this to prove the inequality. First we multiply the first term (1) in all of the factors.

   $$\underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{n-times} = 1^n = 1$$

   Then, we multiply $\frac{1}{n}$ from the first factor and multiply it with 1 from all of the other factors.

   $$\frac{1}{n}\underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{(n-1)-times} = \frac{1}{n}$$

Then, we multiply $\frac{1}{n}$ from the second factor and multiply it with 1 from all of the other factors.

$$1 \cdot \frac{1}{n} \cdot \underbrace{1 \cdot 1 \cdot \dots \cdot 1}_{(n-2)-times} = \frac{1}{n}$$

$$\vdots$$

We do this for a total of n times. As a result of the partial expansion we have:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1 + \left(\frac{1}{n} + \dots + \frac{1}{n}\right) + D \\ &= 1 + \frac{n}{n} + D \\ &= 1 + 1 + D \\ &= 2 + D\end{aligned}$$

where D is the difference between $\left(1+\frac{1}{n}\right)^n$ and 2. Also, $D \geq 0$ because $1, \frac{1}{n} > 0$ and summation and product of positive numbers is a positive number. Hence, $\left(1+\frac{1}{n}\right)^n \geq 2$.

Now, we have the inequality,

$$a^n > \left(1+\frac{1}{n}\right)^n \geq 2$$

Because raiseing positive numbers to positive powers, we maintain the inequality. Let $k \in \mathbb{N}$, then

$$a^{nk} = (a^n)^k \geq 2^k$$

Now, to prove that $\{2^k\}$ is unbounded by contradiction.

Assume there exists an least upper bound y of 2^k.

By definition, $\forall k, y > 2^k$. By $(L.U.B.1)$ $\exists m > y$, and $2^m > m \implies 2^m > m > y$. Hence contradicting our assumption! Hence, no upper limits exist. Therefore, by our inequality, the set $\{a^n, n \in \mathbb{N}\}$ is also not bounded from above.

Problem 9. If S_1, S_2 are nonempty subsets of $\mathbb{R}$ that are bounded from above, prove that

$$\text{l.u.b } \{x + y : x \in S_1, y \in S_2\} = \text{l.u.b. } S_1 + \text{l.u.b. } S_2$$

Solution.

If S_1, S_2 are nonempty subsets of $\mathbb{R}$ that are bounded from above, we know that $l.u.b.\, S_1$, and $l.u.b.\, S_2$ exist.

Let $S = \{x + y, x \in S_1, y \in S_2\}$, $l.u.b.\, S_1 = l_1$, $l.u.b.\, S_2 = l_2$.

By definition $\forall x \in S_1, x \leq l_1$ and $\forall y \in S_2, y \leq l_2$. Adding the inequalities, we get $x + y \leq l_1 + l_2$. Hence, $l_1 + l_2$ is an upper bound of S.

Let ϵ be an arbitrarily small positive number. Since $l_1 = l.u.b.\, S_1$, $\exists x_0 \in S_1$ such that $l_1 - \epsilon/2 < x_0$.

$$l_1 - \epsilon/2 < x_0 \leq l_1$$

Similarly, since $l_2 = l.u.b.\, S_2$, $\exists y_0 \in S_2$ such that $l_2 - \epsilon/2 < y_0$.

$$l_2 - \epsilon/2 < y_0 \leq l_2$$

Adding the two inequalities gives,

$$l_1 + l_2 - \epsilon < x_0 + y_0 \leq l_1 + l_2$$

Hence, $l_1 + l_2$ is the least upper bound for the set S.