Math 81: Abstract Algebra

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Credit Statement: Talked to Sair Shaikh'26, and Math Stack Exchange.

Problem 1. For $f(x) = x^4 - 1$ and $g(x) = 3x^2 + 3x$ find: the quotient and remainder after dividing f by g; the gcd of f and g; and the expression of this gcd in the form $af + bg$ for some $a, b \in \mathbb{Q}[x]$. For the last two, you'll need to recall the Euclidean Algorithm and the Bezout Identity.

Problem 2. Prove that two polynomials $f, g \in \mathbb{Z}[x]$ are relatively prime in $\mathbb{Q}[x]$ (i.e., they share no common nonconstant factor) if and only if the ideal $(f, g) \subset \mathbb{Z}[x]$ contains a nonzero integer.

Problem 3. Decide whether each of the following polynomials is irreducible, and if not, then find the factorization into monic irreducibles.

1. $x^4 + 1 \in \mathbb{R}[x]$
2. $x^4 + 1 \in \mathbb{Q}[x]$
3. $x^7 + 66x^6 - 77x + 737 \in \mathbb{Q}[x]$
4. $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$
5. $x^3 + 5x^2 - 9x + 3 \in \mathbb{Q}[x]$

Problem 4. *Irreducible polynomials over finite fields.* Let $\mathbb{F}_3$ be the field with three elements.

1. Determine all the monic irreducible polynomials of degree ≤ 3 in $\mathbb{F}_3[x]$.

2. Determine the number of monic irreducible polynomials of degree 4 in $\mathbb{F}_3[x]$. **Hint.** This is easier than determining all of them.

Problem 5(a).*Symmetric polynomials.* Let R be a commutative ring with 1 and $R[x_1, \dots, x_n]$ the ring of polynomials in the variables $x_1, \dots, x_n$ with coefficients in R. Consider the symmetric group S_n acting on the set $\{x_1, \dots, x_n\}$ by permutations. Extend this action linearly to $R[x_1, x_2, \dots, x_n]$; for example, if $\sigma = (123) \in S_3$, then

$$\sigma \cdot (x_1x_2 - 6x_3^2 + 7x_2x_3^2) = x_2x_3 - 6x_1^2 + 7x_3x_1^2.$$

Then this action satisfies $\sigma \cdot (f + g) = \sigma \cdot f + \sigma \cdot g$ and $\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$ for all $\sigma \in S_n$ and all $f, g \in R[x_1, \dots, x_n]$.
Let $S \subset R[x_1, \dots, x_n]$ be the subset fixed under the action of S_n. Prove that S is a subring with 1. This is called the **ring of symmetric polynomials**.

Problem 5(b). For each $n \geq 0$, define polynomials $e_i \in R[x_1, \dots, x_n]$ by $e_0 = 1$ and

$$e_1 = x_1 + \cdots + x_n, \quad e_2 = \sum_{1 \leq i < j \leq n} x_ix_j, \quad \dots, \quad e_n = x_1 \cdots x_n$$

and $e_k = 0$ for $k > n$. In words, e_k is the sum of all distinct products of subsets of k distinct variables. Prove that each e_k is a symmetric polynomial. These are called the **elementary symmetric polynomials**.

Problem 5(c).The **generic polynomial** of degree n is the polynomial

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

in the ring $R[x_1, \ldots, x_n][x]$ of polynomials in x with coefficients in $R[x_1, \ldots, x_n]$. Prove (by induction) that

$$\begin{aligned} f(x) = (x - x_1)(x - x_2) \cdots (x - x_n) &= x^n - e_1 x^{n-1} + e_2 x^{n-2} + \cdots + (-1)^n e_n \\ &= \sum_{j=0}^{n} (-1)^{n-j} e_{n-j} x^j. \end{aligned}$$

Problem 5(d).For each $k \geq 1$, define the **power sums** $p_k = x_1^k + \cdots + x_n^k$ in $R[x_1, \ldots, x_n]$. Clearly, the power sums are symmetric. Verify the following identities by hand:

$$p_1 = e_1, \quad p_2 = e_1 p_1 - 2e_2, \quad p_3 = e_1 p_2 - e_2 p_1 + 3e_3$$

In general **Newton's identities** in $R[x_1, \ldots, x_n]$ are (recall that $e_k = 0$ for $k > n$):

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0.$$

Prove Newton's identities whenever $k \geq n$.

Hint. For each i, consider the equation in part (c) for $f(x_i)$ and sum all these equations together. This gives Newton's identity for $k = n$. Set extra variables to zero to get the identities for $k > n$ from this. (Fun. Can you come up with a proof when $1 \leq k \leq n$?)

Problem 6.*Use the force, my Newton!*

1. If x, y, z are complex numbers satisfying

   $$x + y + z = 1, \qquad x^2 + y^2 + z^2 = 6, \qquad x^3 + y^3 + z^3 = 7,$$

   then prove that $x^n + y^n + z^n$ is rational for any positive integer n.

2. Calculate $x^4 + y^4 + z^4$.

3. Prove that each of x, y, z are not rational numbers.