Math 63: Real Analysis

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Problem 1. Prove that if the points of a convergent sequence of points in a metric space are reordered, then the new sequence converges to the same limit.

Solution.

Let (p_n) be a convergent sequence, $f : \mathbb{N} \to \mathbb{N}$ be a bijective function that reorders points of (p_n). Define the reordered sequence (p'_n) by

$$p'_n = p_{f(n)}$$

Since (p_n) is convergent, we know that there exists a p such that for any $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$, such that $d(p, p_i) < \epsilon$ whenever $i > N(\epsilon)$.

Fix $\epsilon > 0$. Since $N(\epsilon)$ is a positive intger, the set $\{f(1), f(2), \cdots, f(N-1)\}$ is finite. Hence, there exists $N'(\epsilon) \in \mathbb{N}$ such that

$$max\{f(1), f(2), \cdots, f(N-1)\} < N'(\epsilon)$$

Therefore, for every $n > N'(\epsilon)$,

$$d(p, p'_n) < \epsilon$$

This shows that (p'_n) converges to p. Hence, any reordering of a convergent sequence converges to the same limit.

Problem 2. Show that if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers that converges to a, then,

$$\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = a.$$

Solution.

Let (a_n) be the sequence $(a_n) = a_1, a_2, a_3, \ldots$, and let (a'_n) be the sequence $(a'_n) = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)$. Fix $\epsilon/2$. Because (a_n) converges, we know there exists N_1 such that for all $n > N_1$, $d(a, a_n) < \epsilon/2$. Define $K = \sum_{i=1}^{N_1}(a_i - a)$, and define $N_2 \geq 2K/\epsilon \implies K/N_2 < \epsilon/2$. Pick $N = \max\{N_1, N_2\}$. Then for any $n > N$, by distributing a, we can write,

$$\sum_{i=1}^{n} \frac{a_i}{n} - a = \sum_{i=1}^{N_1} \frac{a_i - a}{n} + \sum_{i=N_1+1}^{n} \frac{a_i - a}{n}$$

But for all $n > N \geq N_2$, we note that the first term is $\leq \epsilon/2$, because $K/n < K/N_2 < \epsilon/2$. Moreover, each of the $a_i - a$ in the second term is less than $\epsilon/2$, so we get

$$\sum_{i=N_1+1}^{n} \frac{a_i - a}{n} < \frac{n \cdot \epsilon/2}{n} = \epsilon/2$$

Overall, we get,

$$\sum_{i=1}^{N_1} \frac{a_i - a}{n} + \sum_{i=N_1+1}^{n} \frac{a_i - a}{n} < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus the series converges to a.

Problem 3. Prove that any sequence in $\mathbb{R}$ has a monotonic subsequence.
(Hint: This is easy if there exists a subsequence with no least term, hence we may suppose that each subsequence has a least term.) (Note that this result and the theorem on the convergence of bounded monotonic sequences gives another proof that $\mathbb{R}$ is complete.)

Solution.

Let (x_n) be a sequence in $\mathbb{R}$. We consider two cases.

1. There exists a subsequence with no least term.

   If there exists a subsequence with no least term, pick that subsequence. Let this subsequence be (p_n). Fix an index i.

   Claim: There exist some $j > i$ such that $p_j < p_i$.

   We prove this by contradiction. Assume there exist no such j. Since (p_n) has no least term, then all $p_k < p_i$ must have indices k less than i. But there are only a finite number of indices less than i.

   $$\{p_1, p_2, \cdots, p_i\}$$

   But a finite set has a least element which would be the least element of the subsequence (p_n). This contradicts our original assumption that (p_n) has no least term. Hence there must exist some $j > i$ such that $p_j < p_i$ for all i.

   Using the claim, we can build a monotonic strictly descreasing sequence by induction. Choose $i_1 = 1$, and having chosen i_k, choose $i_{k+1} > i_k$ such that

   $$p_{i_{k+1}} < p_{i_k}.$$

2. All subsequences have least terms.

   If all subsequences have a least term than we can build an increasing monotonic sequence by picking the least terms of every consequent subsequence.

   Let x_{n_1} be the least term of the original sequence. Consider the subsequence,

   $$(x_n)_{n>n_1}.$$

   By assumption, it has a least term; call it x_{n_2}. Continue inductively: having chosen x_{n_k}, let $x_{n_{k+1}}$ be the least term of the subsequence with indices greater than n_k.

   By construction,

   $$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots,$$

   so this is a monotone increasing subsequence.

Hence, we can always build a monotonic subsequence for any sequence in $\mathbb{R}$.

Problem 4. Let S be a subset of the metric space E . Define the closure of S, denoted $\bar{S}$, to be the intersection of all closed subsets of E that contain S. Show that

1. $\bar{S} \supset S$, and S is closed if and only if $\bar{S} = S$.
2. $\bar{S}$ is the set of all limits of sequences of points of S that converge in E.
3. a point $p \in E$ is in $\bar{S}$ if and only if any ball in E of center p contains points of S, which is true if and only if p is not an interior point of S' (cf. Prob. 15).

Solution.

1. Let $\{X_i\}_{i \in I}$, where I is the indexing set, be the set of all closed subsets of E containing S. By definition,
$$\bar{S} = \bigcap_{i \in I} X_i$$
Again, by definition, $S \subset X_i$, for all $i \in I$, then $S \subset \bigcap_{i \in I} X_i = \bar{S}$.

   To show: S is closed if and only if $\bar{S} = S$.

   ($\implies$)

   Assume S is closed. Then, S is the minimal closed subset of E containing S. Since $\bar{S}$ is the intersection of all closed sets containing S, and S itself is closed and contains S, it must be one of the sets in the intersection. That is, $\bar{S} \subset S$. Combining this with the containment above, we get $\bar{S} = S$.

   ($\impliedby$)

   Assume $\bar{S} = S$. Since $\bar{S}$ is an intersection of closed sets, it is closed. Therefore S is closed.

2. We know that an intersection of any collection of closed subsets of E is a closed subset (Proposition on page 41). Hence, $\bar{S}$ is a closed subset. We claim that a subset $S \in E$ is closed if and only if every convergent sequence (p_n) of points, converges to a point $p \in S$.

   Claim: A subset $S \in E$ is closed if and only if every convergent sequence (p_n) of points,
$$\lim_{n \to \infty} p_n = p$$
converges to a point $p \in S$.

   ($\implies$)

Assume S is closed. Let (p_n) be a sequence with $\forall n,\ p_n \in S$ and $\lim_{n\to\infty} p_n = p$ exists. Assume $p \notin S$. Then, $\exists r > 0 : B_r(p) \cap S = \emptyset$ But by convergence, $\exists N$ such that for all $n > N, p_n \in B_r(p)$. This contradicts the assumption that $\forall n,\ p_n \in S$. Hence, p must be in S.

$(\Longleftarrow)$

We will prove this by contrapositive. Assume S is not closed. Then, we need to construct a sequence (p_n) with $\forall n,\ p_n \in S$ such that $\lim_{n\to\infty} p_n = p$ exists, and $p \notin S$. Since, S is not closed, there exists a $p \notin S$ such that $B_r(p) \cap S \neq \emptyset$ for every $r > 0$. For each n, choose a point

$$p_n \in B_{1/n}(p) \cap S$$

This is possible by the assumption above. Then $\forall n,\ p_n \in S$, and,

$$d(p_n, p) < \frac{1}{n}$$

Hence,

$$\lim_{n\to\infty} p_n = p.$$

Since $p \notin S$, we have constructed a convergent sequence (p_n) in S whose limit does not lie in S.

Thus proving the claim that a subset S is closed if and only if eveyr convergent sequence (p_n) of points converges to a point $p \in S$.

Moreover, from part (1), we know that $S \subset \bar{S}$. So $\bar{S}$ is the set of all limits of sequences of points of S.

3. **To show:** Point $p \in \bar{S} \implies$ any ball in E of center p contains points of S.

   Assume $p \in \bar{S}$. We will prove this by contradiction. Assume that there exists an open ball $B_r(p)$, for some $r > 0$, such that $B_r(p) \cap S = \emptyset$. Then the complement of $B_r(p)$, is a closed set $(B_r(p))' \subset \bar{S}$. That is, $p \notin \bar{S}$. A contradiction! Hence, for all $r > 0$, $B_r(p) \cap S \neq \emptyset$.

   Note: It suffices to consider open balls, since if a closed ball were disjoint from S the open ball contained inside it would also be disjoint from S.

   To show: Any ball in E of center p contains points of $S \implies p \in \bar{S}$.

   We prove this by contradiction.

   Assume all balls of center p contains points of S. That is, $B_r(p) \cap S \neq \emptyset$, for all $r > 0$. For the sake of contradiction, assume there exists a closed set X that contains S, but does not contain p. Then, the complement X' of X is an open set, and contains p. Since by definition, every open set has an open ball around any given point, we can define an open ball $B'_{r'}(p) \in X'$ centered at p with $r' > 0$. But then, $B'_{r'}(p) \cap S = \emptyset$. A contradiction! Hence, there exist no closed sets X that contain S but not p. And since, $\bar{S}$ is the intersection of all closed subsets containing S, p must be in $\bar{S}$.

Hence, any point $p \in E$ is in $\bar{S}$ if and only if any ball in E of center p contains points of S.

To show: If any ball in E of center p contains points of S then p is not an interior point of S'.

We prove this by contradiction.

Assume for any ball $B_r(p) \in E$, $r > 0$, $B_r(p) \cap S \neq \emptyset$. For the sake of contradiction assume p is an interior point of S'. Then there exists an open ball in E of center p which is entirely contained in S'. But there there exists an ball in E of center p that contains no points of S. A contradiction! Therefore, p cannot be an interior point of S'.

To show: If p is not an interior point of S' then any ball in E of center p contains points of S.

If p is not an interior point of S' then there exist no open balls $B_r(p)$, for some $r > 0$ in E centered at p contained entirely in S'. Therefore, $B_r(p) \cap S \neq \emptyset$, i.e. any open ball in E of center p contains points of S.

Hence, any ball in E of center p contains points of S if and only if p is not an interior point of S'.

Note: It suffices to consider open balls, since if a closed ball were disjoint from S the open ball contained inside it would also be disjoint from S.

Problem 5. Show that a complete subspace of a metric space is a closed subset.

Solution.

Let S be a complete subspace of a metric space (E, d). By the proposition on page 51, we know that every convergent sequence of points in a metric space is a Cauchy sequence, and moreover by defition of a complete space we know that every cauchy sequence of points $p \in S$ converges to a point of S. From the claim in Problem (4.2) we know that if every convergenet sequence of points in S converges to a point in S, S is a closed set. Hence, a complete subspace of a metric space is a closed subset.