Cosc 30: Discrete Mathematics

Prishita Dharampal

Problem 1. *Negation.* First write down a proposition using a predicate with variables and quantifiers for the following statement. Then write down its negation. Make sure to push the "not" all the way inside; in particular, there should be no "not" before any quantifier. (Assume `isBlack(x)` is the predicate that "cow x is black", and `isWhite(x)` is the predicate that "cow x is white".)

(a) Every cow is either white or black. [To think about later: Why is it ambiguous if we say "all cows are either white or black?"]

(b) If one of the cows in the group is black, and every (strict) subset of cows has the same color, then all cows are black. [Hint: Try to express "cows in a chosen subset all have the same color" using a statement with two quantifiers. Assume all cows are either black or white.]

Solution.

(a) **Proposition:** $\forall \, cow \, c \, (\texttt{isWhite(c)} \text{ OR } \texttt{isBlack(c)})$
Negation: $\exists \, cow \, c((\text{NOT } \texttt{isWhite(c)}) \text{ AND } (\text{NOT}\texttt{isBlack(c)}))$

(b) Let C be the set of cows, and let $P_1, P_2,$ and Q be defined as follows:

$P_1 : \exists \, b \in C \, (\texttt{isBlack(b)})$

$P_2 : \forall S \subsetneq C \text{ AND } \forall c, x \in S(\texttt{isBlack(c)} \implies \texttt{isBlack(x)} \text{ OR } (\texttt{isWhite(c)} \implies \texttt{isWhite(x)}))$

$Q : \forall a \in C \, (\texttt{isBlack(a)})$

Proposition: $P_1 \text{ AND } P_2 \implies Q$

Negation: $\text{NOT } ((P_1 \text{AND} P_2) \implies Q)$

$$\begin{aligned} & \text{NOT } ((P_1 \text{AND} P_2) \implies Q) \\ \implies & \text{NOT } (\text{ NOT } (P_1 \text{AND} P_2) \text{ OR } Q) \\ \implies & (P_1 \text{AND} P_2) \text{ AND } (\text{NOT } Q) \end{aligned}$$

$\text{NOT } Q : \exists a \in C \,(\text{NOT } \texttt{isBlack(a)})$

So, negation of the statement would be:

If one of the cows in the group is black and every (strict) subset of cows has the same color, then there exists at least one cow that is not black.

Problem 2. *Contrapositives.* Write down the contrapositive of the following propositions.

(a) If there is a cow that is not black, then no cows can be black.

(b) If one of the cows in the group is black, and every (strict) subset of cows has the same color, then all cows are black.

Solution.

(a) **Proposition:** $\exists\, cow\, c\,(\text{ NOT } (\texttt{isBlack(c)})) \implies \forall\, cow\, a\,(\text{ NOT } (isBlack(a)))$

Contrapositive: $\exists\, cow\, a(\texttt{isBlack(a)}) \implies \forall\, cow\, c(\texttt{isBlack(c)})$

(b) **Proposition:** (Same as Problem 1(b))

Contrapositive:

$$\begin{aligned} &\text{NOT } Q \implies \text{NOT } (P_1 \text{ AND } P_2) \\ \implies &\text{NOT } Q \implies \text{NOT } P_1 \text{ OR NOT } P_2 \end{aligned}$$

NOT $P_1 : \forall b \in C\,(\text{NOT } (\texttt{isBlack(b)}))$

$P_2 : \exists S \subsetneq C \text{ AND } \exists c, x \in S(\texttt{isBlack(c)} \implies \text{NOT } (\texttt{isBlack(x)}) \text{ AND } (\texttt{isWhite(c)} \implies \text{NOT } (\texttt{isWhite(x)})))$

Assuming all cows are either black or white, P_2 simplifies to

$P_2 : \exists S \subsetneq C \text{ AND } \exists c, x \in S(\texttt{isBlack(c)} \text{ AND } \texttt{isWhite(x)})$

So the contrapositive of the statement would be:

If not all cows are black, then either no cow is black or there exists a (strict) subset of cows that don't have the same color.