

Math 63: Real Analysis

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Credit Statement: Talked to Sair Shaikh'26, and Math Stack Exchange. **TODO:**

Problem 1. If A, B, C are sets show that:

$$A - (B - C) = (A - B) \cup (A \cap B \cap C)$$

Solution.

To show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ we first show that

$$A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$$

and then

$$A - (B - C) \supset (A - B) \cup (A \cap B \cap C).$$

1. $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$

$$\forall x \in A - (B - C),$$

$$\begin{aligned} &\implies x \in A \text{ and } x \notin (B - C) \\ &\implies x \in A \text{ and } (x \notin B \text{ or } x \in B \cap C) \\ &\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in B \cap C) \\ &\implies (x \in A \cap \bar{B}) \cup (x \in A \cap B \cap C) \\ &\implies (x \in A - B) \cup (x \in A \cap B \cap C) \\ &\implies x \in (A - B) \cup (A \cap B \cap C) \\ &\implies A - (B - C) \subset (A - B) \cup (A \cap B \cap C) \end{aligned}$$

$$2. A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$$

$$\forall x \in (A - B) \cup (A \cap B \cap C),$$

$$\begin{aligned} &\implies (x \in A \text{ and } x \notin B) \text{ or } (x \in A \cap B \cap C) \\ &\implies (x \in A) \text{ and } (x \notin B \text{ or } x \in B \cap C) \\ &\implies x \in A \cap (\bar{B} \cup (B \cap C)) \\ &\implies x \in A \cap ((\bar{B} \cup B) \cap (\bar{B} \cup C)) \\ &\implies x \in A \cap (1 \cap (\bar{B} \cup C)) \\ &\implies x \in A \cap (\bar{B} \cup C) \\ &\implies x \in A - (\overline{\bar{B} \cup C}) \\ &\implies x \in A - (B \cap \bar{C}) \\ &\implies x \in A - (B - C) \end{aligned}$$

Since, $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$ and $A - (B - C) \supset (A - B) \cup (A \cap B \cap C)$ we can say that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

Problem 2. Let I be a set and for each $i \in I$ let X_i , be a set. Prove that for any set B we have:

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$$

Solution. To show that $B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$ we first show that $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$ and then $B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$.

1. $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$

If $x \in B \cap \bigcup_{i \in I} X_i$ then $x \in B$ and $x \in \bigcup_{i \in I} X_i$.

I.e. x is at least in one X_j for some $j \in I \implies x \in B \cap X_j$.

Thus, $x \in \bigcup_{i \in I} (B \cap X_i) \implies B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$.

2. $B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$

If $x \in \bigcup_{i \in I} (B \cap X_i)$, then x is at least in one $B \cap X_j$ for some $j \in I$

$$\begin{aligned} &\implies x \in B \text{ and } x \in X_j \\ &\implies x \in B \text{ and } x \in \bigcup_{i \in I} X_i \\ &\implies x \in B \cap \bigcup_{i \in I} X_i \end{aligned}$$

$$\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i).$$

Since, $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$ and $\implies B \cap \bigcup_{i \in I} X_i \supset \bigcup_{i \in I} (B \cap X_i)$, we can say that
 $\implies B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i)$.

Problem 3. Let $f : X \rightarrow Y$ be a function, let A and B be subsets of X , and let C and D be subsets of Y . Prove that:

1. $f(A \cap B) \subset f(A) \cap f(B)$
2. $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$

Solution.

1. If $x \in f(A \cap B)$

Problem 4.

1. How many functions are there from a nonempty set S into the \emptyset ?
2. Show that the notation $\{X_i\}_{i \in I}$ implicitly involves the notion of function.

Solution.

1. There are no functions from a nonempty set to the empty set because a function needs a pre-image and an image and the co-domain is empty.
- 2.

Problem 5. Prove in detail that for any $a, b \in \mathbb{R}$:

$$-(a - b) = b - a$$

Problem 6. Show that if $a, b, x, y \in \mathbb{R}$ and $a < x < b, a < y < b$, then $|y - x| < b - a$.

Problem 7. Find the g.l.b. and l.u.b. of $\{1, \frac{1}{2}, \frac{1}{4}, \dots\}$, giving reasons if you can.

Problem 8. Prove that if $a \in \mathbb{R}, a > 1$, then the set $\{a, a^2, a^3, \dots\}$ is not bounded from above. (Hint: First find a positive integer n such that $a > 1 + \frac{1}{n}$ and prove that $a^n > (1 + \frac{1}{n})^n \geq 2$).

Problem 8. If S_1, S_2 are nonempty subsets of \mathbb{R} that are bounded from above, prove that

$$\text{l.u.b. } \{x + y : x \in S_1, y \in S_2\} = \text{l.u.b. } S_1 + \text{l.u.b. } S_2$$