

Math 63: Real Analysis

Prishita Dharampal

Problem 1. Verify that the following are metric spaces:

1. all n-tuples of real numbers, with

$$d((x_1, \dots, x_n)(y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$$

2. all bounded infinite sequences $x = (x_1, x_2, x_3, \dots)$ of elements of \mathbb{R} with

$$d(x, y) = \text{l.u.b.}\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \dots\}$$

Solution.

1. Let $E = \{\text{all n-tuples of real numbers}\}$, with the distance between two points x and y $d(x, y) = d((x_1, \dots, x_n)(y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$. To see if (E, d) defines a metric space, we check the following:

(a) $d((x_1, \dots, x_n)(y_1, \dots, y_n)) \geq 0$

Since, $d((x_1, \dots, x_n)(y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$, and each term $|x_i - y_i| \geq 0$, the sum of these terms is also non-negative. Hence, we can say that $\forall x, y \in E$, $d((x_1, \dots, x_n)(y_1, \dots, y_n)) \geq 0$.

(b) $d((x_1, \dots, x_n)(y_1, \dots, y_n)) = 0 \iff (x_1, \dots, x_n) = (y_1, \dots, y_n)$

(\implies)

Assume $d((x_1, \dots, x_n)(y_1, \dots, y_n)) = 0$. Since every term $|x_i - y_i|$ is non-negative, the sum being zero implies that every term is also zero. I.e. $|x_i - y_i| = 0 \implies x_i = y_i$, for all x_i, y_i . Hence, $(x_1, \dots, x_n) = (y_1, \dots, y_n)$.

(\impliedby)

Assume $(x_1, \dots, x_n) = (y_1, \dots, y_n)$. Then,

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n |x_i - x_i| = \sum_{i=1}^n 0 = 0$$

(c) $d(x, y) = d(y, x), \forall x, y \in E$

For every term $|x_i - y_i|$ in the summation, we know that by definition, $|x_i - y_i| = |y_i - x_i|$. Hence, $d((x_1, \dots, x_n)(y_1, \dots, y_n)) = d((y_1, \dots, y_n), (x_1, \dots, x_n))$ or $d(x, y) = d(y, x)$.

(d) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$

$$\begin{aligned} d(x, y) + d(y, z) &= d((x_1, \dots, x_n), (y_1, \dots, y_n)) + d((y_1, \dots, y_n), (z_1, \dots, z_n)) \\ &= \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i| \\ &= \sum_{i=1}^n |x_i - y_i| + |y_i - z_i| \end{aligned}$$

We know that,

$$\begin{aligned} |x_i - y_i| + |y_i - z_i| &\geq |x_i - y_i + y_i - z_i| \\ \implies |x_i - y_i| + |y_i - z_i| &\geq |x_i - z_i| \end{aligned}$$

I.e.,

$$\begin{aligned} \sum_{i=1}^n |x_i - y_i| + |y_i - z_i| &\geq \sum_{i=1}^n |x_i - z_i| \\ d((x_1, \dots, x_n), (y_1, \dots, y_n)) + d((y_1, \dots, y_n), (z_1, \dots, z_n)) &\geq d((x_1, \dots, x_n), (z_1, \dots, z_n)) \\ d(x, y) + d(y, z) &\geq d(x, z) \end{aligned}$$

Hence, (E, d) defines a metric space.

2. Let $E = \{\text{all bounded infinite sequences of real numbers}\}$ with the distance function d defined as $d(x, y) = l.u.b.\{|x_1 - y_1|, |x_2 - y_2|, \dots\} = \{|x_i - y_i|\}_{i \in \mathbb{N}}$. To see if (E, d) defines a metric space, we check the following:

(a) $d(x, y) \geq 0$

$$d(x, y) = l.u.b.\{|x_i - y_i|\}_{i \in \mathbb{N}}$$

with each element in the set $|x_i - y_i| \geq 0, \forall i \in \mathbb{N}$ and $l.u.b.\{|x_i - y_i|\}_{i \in \mathbb{N}} \geq |x_j - y_j|$ for any $j \in \mathbb{N}$. On combining the inequalities we get,

$$\begin{aligned} l.u.b.\{|x_i - y_i|\}_{i \in \mathbb{N}} &\geq |x_j - y_j| \geq 0 \\ l.u.b.\{|x_i - y_i|\}_{i \in \mathbb{N}} &\geq 0 \end{aligned}$$

$$(b) d(x, y) = 0 \iff x = y$$

(\implies)

Assume $d(x, y) = 0$. Then $\text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} = 0 \implies |x_i - y_i| \leq 0, \forall i \in \mathbb{N}$. But by definition, the absolute value is positive, i.e. $|x_i - y_i| \geq 0, \forall i \in \mathbb{N}$. On combining the inequalities, we get: $|x_i - y_i| = 0, \forall i \in \mathbb{N}$.

I.e. $\forall i \in \mathbb{N}, x_i = y_i \implies x = y$.

(\Leftarrow)

Assume $x = y$.

Then $\forall i \in \mathbb{N}, x_i = y_i \implies x_i - y_i = 0 \implies \text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} = 0 = d(x, y)$.

$$(c) d(x, y) = d(y, x)$$

By definition, $|x_i - y_i| = |y_i - x_i|$, so $\text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} = \text{l.u.b.}\{|y_i - x_i|\}_{i \in \mathbb{N}}$. Hence, $d(x, y) = d(y, x)$.

$$(d) d(x, z) \leq d(x, y) + d(y, z)$$

TODO:

WTS $\text{l.u.b.}\{|x_i - z_i|\}_{i \in \mathbb{N}} \leq \text{l.u.b.}\{|x_i - y_i|\}_{i \in \mathbb{N}} + \text{l.u.b.}\{|y_i - z_i|\}_{i \in \mathbb{N}}$

Problem 2. What are the open and closed balls in the metric space of example (4), § 1? Show that two balls of different centers and radii may be equal. What are the open sets in this metric space?

Example (4), § 1: Let E be an arbitrary set and, for $p, q \in E$, define $d(p, q) = 0$ if $p = q$, $d(p, q) = 1$ if $p \neq q$.

Solution.

$$1. (a) 0 < r < 1$$

i. Open Balls:

$$B_r(p) = \{d(p, q) < r; q \in E\} = \{p\}$$

Since $d(p, q) < 1$ implies $p = q$.

ii. Closed Balls:

$$B_r(p) = \{d(p, q) \leq r; q \in E\} = \{p\}$$

Since by definition $d(p, q) < 1$ implies $p = q$.

$$(b) r = 1$$

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ii. Closed Balls:

$$B_r(p) = \{d(p, q) \leq r; q \in E\} = E$$

Since $d(p, q) \leq 1$ for all $q \in E$.

(c) $r > 1$

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2. To see that two balls with different centers and radii may be equal, consider two open balls $B_{r_1}(p), B_{r_2}(q)$ with $p \neq q, r_1 \neq r_2$, and $r_1, r_2 > 0$. Then, as we saw above,

$$B_{r_1}(p) = E, \quad B_{r_2}(q) = E$$

I.e. two balls with different centers and radii may be equal.

3. Every singleton set $\{p\} \in E$ is open, as we can construct an open ball $B_r(p)$ with $0 < r < 1$ centered at p . Since unions of any collection of open sets are open, any subset of E is open.

Problem 3. Show that the subset of E^2 given by $\{(x_1, x_2) \in E^2 : x_1 > x_2\}$ is open.

Solution.

Let $S = \{(x_1, x_2) \in E^2 : x_1 > x_2\}$, with distance function defined as $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for any two points $(x_1, x_2), (y_1, y_2) \in E^2$. Let $B_{r/4}((x_1, x_2))$ be an open ball centered around an arbitrary point $(x_1, x_2) \in S$, with $r = x_1 - x_2$ (obviously, $r > 0$).

Let $(y_1, y_2) \in B_{r/4}((x_1, x_2))$. Then,

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &< r/4 \\ \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &< r/4 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 &< r^2/16 \\ \implies (x_1 - y_1)^2 &\leq (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2/16 \\ |x_1 - y_1| &< r/4 \end{aligned}$$

Similarly, we get

$$|x_2 - y_2| < r/4$$

We can then write: $y_1 - y_2 = x_1 - x_2 - (x_1 - y_1) + (x_2 - y_2)$.

We know that, $x_1 - x_2 = r, |x_1 - y_1| < r/4, |x_2 - y_2| < r/4$. Then,

$$y_1 - y_2 > r/2 \implies y_1 - y_2 > 0 \implies y_1 > y_2 \implies (y_1, y_2) \in S$$

I.e. $B_{r/4}((x_1, x_2)) \subset S$. Since, (x_1, x_2) was an arbitrary point, this works for any element of S . S is thus an open subset because of every point of S we can construct such a ball.

Problem 4. Prove that any bounded open subset of \mathbb{R} is the union of disjoint open intervals.

Solution.