Public goods, voting, and growth

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Abstract

We study a parametric politico-economic model of economic growth with productive public goods and public consumption goods. The provision of public goods is funded by a proportional tax. Agents are heterogeneous in their initial capital endowments, discount factors and the relative weights of public consumption in overall private utility. They vote on the shares of public goods in GDP. We propose a definition of voting equilibrium, prove the existence and provide a characterization of voting equilibria, and obtain a closed-form solution for the voting outcomes. Also we introduce a "fictitious" representative agent and interpret the outcome of voting as a choice made by a central planner for his benefit. Finally, we undertake comparative static analysis of the shares of public goods in GDP and of the rate of balanced growth with respect to the discount factors and the preferences for public consumption. The results of this analysis suggest that the representative-agent version of our model is capable of capturing the interaction between many voting heterogeneous agents only if the heterogeneity is one-dimensional.

Keywords: Intertemporal choice, Growth, Public goods, Voting

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1 Introduction

Public intervention is one of the key factors explaining differences in economic growth patterns among countries in the world. The role of governments and government expenditures in the growth process has been at the heart of many theoretical and empirical contributions.

There are two alternative approaches to deal with public intervention in the economy: either a social planner is assumed to look for an optimal solution and policy instruments are then designed to decentralize this solution or policy decisions are assumed to be the outcome of a political process inside which policy makers and/or voters interact. As far as economic growth is concerned, both efficiency issues and equity issues can be dealt with in both approaches.

In this paper, we propose a parametric politico-economic model of economic growth with productive public goods that increase private production possibilities and public consumption goods that contribute to private utility. The government levies a proportional tax on the consumers' income, which funds the provision of public goods. We assume logarithmic preferences, the Cobb-Douglas production function and full depreciation of capital. Unlike the vast majority of the literature, in our model agents are heterogeneous not only in their initial endowments of private capital, but also in their discount factors and preferences for public consumption goods.

At each time period, agents vote on two shares and hence the policy space is 2-dimensional. Therefore, generically a Condorcet winner fails to exist. To overcome this difficulty, following Kramer (1972) and Shepsle (1979), we assume that agents vote separately on the two shares within the same period. They cast their vote on each share assuming that the other share has been settled. A solution is consistent if the pair of shares obtained through that procedure is self-supporting in a Nash-like manner.

We propose a definition of voting equilibrium, prove the existence of voting equilibria, provide their characterization, obtain a closed-form solution for the voting outcomes, which do not depend on the initial distribution of private capital and expectations. It follows that if at each time the shares of public goods in GDP are determined by voting, then they are constant over time.

Also we introduce a "fictitious" representative agent and interpret the outcome of voting in our model as a choice made by a central planner for his benefit. Finally, we undertake comparative static analysis of the shares of public goods in GDP and of the rate of balanced growth with respect to the discount factors and the preferences for public consumption. Some outcomes of this analysis in the general case of many agents are somewhat different from the outcomes in the case of a representative agent. They show that the representative-agent version of our model is a reasonable approximation of the general case with many heterogeneous voting agents only if the heterogeneity is one-dimensional.

At first glance it would seem impossible to propose a consistent voting procedure if the uniqueness of a competitive equilibrium cannot be guaranteed for some policies. It turns out that this is not the case. We show that under some additional assumption about agents' beliefs it is possible to generalize the voting procedure in a consistent way to the case where the uniqueness of competitive equilibria is not ensured and that the outcome of this generalized voting procedure is the same as in the case of uniqueness.

The structure of the paper is as follows. Section 2 provides a short literature review. Section 3 presents the main building blocks describing the production technology, the government spending and the agents' preferences. Section 4 provides a preliminary analysis of competitive equilibria assuming that the shares of public goods in GDP are given. It studies the existence and uniqueness of an intertemporal equilibrium and the key characteristics of a balanced growth equilibrium for given shares. Section 5 endogenizes the shares of public goods through a voting procedure and describes the outcome of voting under the assumption that for any public policy the competitive equilibrium is unique. Finally Section 6 extends the analysis to the case where the uniqueness is not ensured. Section 7 concludes.

2 Literature review

Government expenditures on goods and services are traditionally classified as productivity-enhancing or utility-enhancing. Barro (1990), Barro and Sala-i-Martin (1992), Glomm and Ravikumar (1994a) and many others study the optimal level of government expenditures when they take the form of public production services (see de Haan and Romp (2007) for a recent survey of empirical literature and Irmen and Kuehnel (2009) for a survey of theoretical literature). Other constributions study the role of public investment to alleviate fixed costs associated with production and responsible of poverty traps: Dechert and Nishimura (1983) for the study of nonconvexities due to fixed costs and recently Le Van *et al* (2016) for an analysis of lock-ins to underdevelopment caused by the lack of core infrastructure (road, rail, power supply,...). Bianconi and Turnovsky (1997), Devereux and Wen (1998) and others study the more conventional case in which government expenditures take the form of utility-enhancing public services that provide direct utility to households. Some analyses have included both aspects of public spending (see, e.g., Baier and Glomm (2001), Baxter and King (1993), Chang (1999), Chen (2006), Marrero (2010), Economides *et al* (2011)). In the vast majority of papers on economic growth with public intervention the shares of government expenditures are either exogenous or chosen by a benevolent planner.

The political-economy literature describes collective choice mechanisms and explains how fundamentals (preferences and technologies) together with political institutions determine political outcomes. Glomm and Ravikumar (1994b) and Koulovatianos and Mirman (2004, 2005) consider infinite-horizon economies where public sector investments and public consumption goods are financed by income taxes. They endogenize the provision of public goods through majority voting. In their models, households can differ only with respect to their initial endowments of private capital. However, this heterogeneity does not lead to any disagreement in voting. All agents vote unanimously. In Park and Philippopoulos (2003) households also can differ only with respect to initial capital endowments.

In our paper, in addition to heterogeneity in initial endowments, we introduce heterogeneity in discount factors and preferences for public consumption goods. Heterogeneity in discount factors exists for a number of reasons: differences in life expectancy, health, family background, etc. and is well documented (for a survey, see Frederick *et al* (2002)). Recent results of Hübner and Vannoorenberghe (2015) and Dohmen *et al* (2015) show that average patience explains a considerable fraction of the between-country variation in growth and income. Patience varies not only between countries, but also within countries. According to Dohmen *et al* (2015), between-country variance accounts for about 13,5 % and within-country variation for about 86,5% of total variation. Another source of heterogeneity in our model stems from the weight attached to public consumption. Competing preferences for consumption public goods are a key element of the political debate in many countries [1].

There is a rich literature on models of economic growth with consumers having different rates of impatience. This literature started from Becker (1980) (for a survey, see Becker (2006)). However, most existing papers on this topic ignore public sector. Few exceptions include papers by Sarte (1997), Sorger (2002), Li and Sarte (2004) and Bosi and Seegmuller (2010), who study the impact of a progressive income tax on the long-run growth and distribution.

One difficulty when applying the tools of politico-economic analysis in neoclassical growth models with rational optimizing agents is to account for voters' expectations of current and future equilibrium prices and of future equilibrium policies (see e.g., Krusell

[1] More generally, in models dealing with voting on public expenditures, heterogeneity in preferences can come from many other sources. For instance, in models which study the accumulation of human or physical capital, individuals may be heterogenous in the weight they place on a relative indicator of social status like relative consumption (Tournemaine and Tsoukis (2015)), relative human capital (Fershtman *et al*) or relative wealth (Pham (2005)).

et al (1997) and Krusell and Ríos-Rull (1999)). Alesina and Rodrik (1994) ignore this difficulty. Kaas (2003), Creedy *et al* (2011) and Burlon (2017) overcome it in OLG frameworks.

We overcome this difficulty by assuming logarithmic preferences, the Cobb-Douglas production function and full depreciation of capital. These simplifications also eliminate a possible strategic motive to influence the outcomes of future votes, hence they can be taken as given. Models where voters can ignore the effect of their choices on future political environment due to logarithmic preferences and the Cobb-Douglas technology are not untypical in the literature (see e.g., Glomm and Ravikumar (1994a,b), Zhang et al. (2003), Gradstein and Kaganovich (2004), Koulovatianos and Mirman (2004), Creedy *et al* (2011)).

Several recent papers are devoted to collective decisions by individuals heterogeneous in their time preferences (see e.g., Gollier and Zeckhauser (2005), Zuber (2011), Heal and Millner (2014)). The results of these papers indicate that in a collective choice context there is a conflict between two attractive properties of intertemporal preferences – time consistency and time invariance – though these properties are indistinguishable for individuals[2].

Jackson and Yariv (2015) consider, among other things, voting as a method of aggregating of preferences over common consumption. Their results suggest that it cannot lead to an unambiguous outcome if agents have different time preferences. Their analysis assumes, however, that voting takes place only once, at time zero. In this sense their model is atemporal. Our framework is essentially intertemporal[3] so that in each period agents are only voting over current values of the shares of public goods in GDP.

3 Main building blocks of the model

3.1 Production sector

The aggregate output (GDP) at each time t, Y_t, is given by a production function

$$Y_t = q(g_t)F(K_t, L),$$

where K_t is the capital stock, g_t is the time t per capita quantity of productive public goods and L is the input of labor, which is assumed to be equal to the number of agents

[2] In our model, the shares of public goods in GDP determined by voting are constant over time and hence voting is time consistent and time invariant.

[3] Similar approach is applied by Borissov and Pakhnin (2018) to a growth model with exhaustible natural resources and by Borissov *et al* (2016) to a non-parametric growth model with public consumption.

(each agent supplies one unit of labor) and constant over time. Capital fully depreciates during one time period. We assume that the production function is Cobb-Douglas:

$$F(K, L) = K^{\alpha} L^{1-\alpha}, \qquad 0 < \alpha < 1,$$

and that

$$q(g) = g^{1-\alpha}.$$

Thus,

$$Y_t = g_t^{1-\alpha} K_t^{\alpha} L^{1-\alpha}.$$

Per capita GDP, $y_t = Y_t/L$, can be written as

$$y_t = q(g_t) f(k_t) = g_t^{1-\alpha} k_t^{\alpha},$$

where $k_t = K_t/L$ and $f(k_t) = k_t^{\alpha}$. The wage rate w_t and the gross interest rate $1 + r_r$ are the marginal products of labor and capital respectively:

$$w_t = q(g_t)(f(k_t) - f'(k_t)k_t) \ (= (1-\alpha)(g_t)^{1-\alpha}(k_t)^{\alpha}),$$

$$1 + r_t = q(g_t) f'(k_t) \ (= \alpha(g_t/k_t)^{1-\alpha}).$$

For simplicity, we assume constant returns to scale to the reproducible factors. It is noteworthy that most of our results concerning voting equilibria have their counterparts in a model with decreasing returns to scale (with somewhat more tedious formulas).

3.2 Government

The government provides utility-enhancing public consumption goods and productive public goods. They are financed by the use of a proportional income tax. The government runs a balanced budget. Therefore, if the per capita level of public consumption goods provision at time t is h_t and the per capita level of productive public goods provision at time $t+1$ is g_{t+1}, then the total tax rate is equal to $\theta_t + \lambda_t$, where

$$\theta_t := \frac{h_t L}{Y_t} = \frac{h_t}{y_t} \text{ and } \lambda_t := \frac{g_{t+1} L}{Y_t} = \frac{g_{t+1}}{y_t}$$

are the shares consumptive and productive public goods in GDP respectively. We assume that the part of taxes collected in period t and used for the productive public goods is spent in period $t+1$.

3.3 Consumers

The agents indexed by $j = 1, \ldots, L$. Their number, L, is assumed to be odd. Each agent $j = 1, , L$ is endowed with one unit of labor, derives utility from personal and public consumption, and discounts future utilities by the factor β_j. Her instantaneous utility at time t is given by $\ln c_t^j + \delta_j \ln h_t$, where c_t^j is her personal consumption and $\delta_j \geq 0$ is the weight she gives to public consumption relative to private consumption. We assume that $1 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_L > 0$.

By J we denote the set of agents with the highest discount factor:

$$J := \{j \mid \beta_j = \beta_1\}.$$

The agents take the sequences $(w_t)_{t=0}^{\infty}$, $(r_t)_{t=0}^{\infty}$, $(h_t)_{t=0}^{\infty}$, $(\lambda_t)_{t=0}^{\infty}$ and $(\theta_t)_{t=0}^{\infty}$ as given and have perfect foresight of all these sequences. The time t the budget constraint of agent j is given by

$$c_t^j + s_t^j = (1 - \theta_t - \lambda_t)[(1 + r_t)s_{t-1}^j + w_t],$$

where s_t^j denotes the time t savings of this agent j. They are assumed to be non-negative, which implies that future wage income cannot be discounted to the present.

Given $s_{\tau-1}^j$, at time τ agent j maximizes her intertemporal utility under the budget constraints, i.e. solves the following problem:

$$\max \sum_{t=\tau}^{\infty} \beta_j^{t-\tau} (\ln c_t^j + \delta_j \ln h_t), \tag{1}$$

s.t.

$$c_t^j + s_t^j = (1 - \theta_t - \lambda_t)[(1 + r_t)s_{t-1}^j + w_t], \; t = \tau, \tau + 1, ...,$$

$$s_t^j \geq 0, \; t = \tau, \tau + 1, ...$$

Since the sequence $(h_t)_{t=\tau}^{\infty}$ is taken as given by consumer j, maximizing (1) is equivalent to maximizing the utility obtained from personal consumption only, which is equal to

$$\sum_{t=\tau}^{\infty} \beta_j^{t-\tau} \ln c_t^j.$$

3.4 Central planner first-best optimum

Prior to consider the model with heterogenous agents, suppose that there is a central planner acting for the benefit of a representative agent. The existence of a representative agent for a given population is an ubiquitous assumption in macroeconomic theory, though it is very controversial from a microeconomic perspective. In the title of his highly cited paper, Alan Kirman (1992) poses the following question: Whom or what does the representative individual represent? His answer is nobody and nothing. According to

Kirman, the reduction of the behavior of a group of heterogeneous agents to the behavior of one representative agent "is not simply an analytical convenience as often explained, but is both unjustified and leads to conclusions which are usually misleading and often wrong" (p. 117).

This seems to be especially true of models with heterogeneous discounting, in which the existence of a social welfare function satisfying some natural assumptions is highly questionable (see e.g. Zuber, 2011). However, even in such models the fiction of a representative agent may be useful. For example, it is convenient to interpret outcomes of votes in our model as choices made by a central planner for the benefit of a "fictitious" representative agent[4]. This interpretation will help us to understand the structure of voting equilibria and their dependence on agents' preferences .

The first-best central planner's optimum is obtained by solving

$$\max \sum_{t=0}^{\infty} \beta_R^t (\ln c_t + \delta_R \ln h_t),$$

s.t.

$$\text{s.t. } (c_t + h_t) + (k_{t+1} + g_{t+1}) = k_t^{\alpha} g_t^{1-\alpha},\ t = 0, 1, ...$$

Here c_t and h_t are the representative agent's private and public consumption, β_R is her discount factor and δ_R is the weight she gives to public consumption relative to private consumption.

The solution to this problem is determined by the following relationships:

$$c_t + h_t = (1 - \beta_R) k_t^{\alpha} g_t^{1-\alpha},\ c_t = \frac{1}{1 + \delta_R}(c_t + h_t),\ h_t = \frac{\delta_R}{1 + \delta_R}(c_t + h_t),$$

$$k_{t+1} + g_{t+1} = \beta_R k_t^{\alpha} g_t^{1-\alpha},\ k_{t+1} = \alpha(k_{t+1} + g_{t+1}),\ g_{t+1} = (1 - \alpha)(k_{t+1} + g_{t+1}).$$

Hence, the optimal shares of consumptive and productive public goods in GDP, $\theta_t = h_t/y_t$ and $\lambda_t = g_{t+1}/y_t$, are constant over time and given by

$$\theta_t = \frac{\delta_R(1 - \beta_R)}{1 + \delta_R},\ \lambda_t = (1 - \alpha)\beta_R. \tag{2}$$

It is interesting to note that the optimal share of public productive goods in GDP, $\lambda_t = (1 - \alpha)\beta_R$, does not depend on the weight the representative agent gives to public consumption relative to private consumption.

[4]In what follows we call this agent *"fictitious" representative agent* (F) in order to distinguish between the case with many agents and the case with a single agent (i.e. $L = 1$). In the latter case the single agent is called simply *representative agent* (R).

In subsection 5.3 we will show that in the second-best optimum, where the central planner maximizes the representative agent welfare in equilibrium taking the decision rule of the representative agent as given, shares of the two public goods are the same [5].

It is tempting to conjecture from (2) that if there are many agents with diverse preferences and the shares of the two public goods are determined by voting, this vote will lead to λ_t equal to $(1-\alpha)\beta_{med}$[6], where β_{med} is the median value of $\beta_j,\ j=1,\ldots,L$, and θ_t equal to either the median value of $\frac{\delta_j(1-\beta_j)}{1+\delta_j},\ j=1,...,L$, or $\frac{\delta_{med}(1-\beta_{med})}{1+\delta_{med}}$, where δ_{med} is the median value of $\delta_j,\ j=1,\ldots,L$. Is this conjecture correct?

We subdivide this question into three questions. From the least to the most general:

1) If agents differ only in their tastes for the public consumption good (sharing the same discount factor), is it true that the agent with the median taste for the consumption public goods determines both shares of public goods?

2) If agents differ only in their discount factors (having the same taste for the consumption public good), can we say that the agent with the median discount factor determines both shares of public goods?

3) And finally, if agents are heterogeneous in their discount factors and in their tastes for the consumption public good, what is the outcome of voting? Is it fully determined by the median value of $\beta_j,\ j=1,\ldots,L$, and the median value of either $\delta_j,\ j=1,\ldots,L$, or $\frac{\delta_j(1-\beta_j)}{1+\delta_j},\ j=1,...,L$?

As was noted above, the results of Jackson and Yariv (2015) suggest that if agents have different discount factors, voting cannot lead to an unambiguous outcome and hence the expected answers to questions 2) and 3) should be negative.

4 Equilibria at given shares of public goods in GDP

As said above, we first analyze competitive equilibria under given public policies and then endogenize the public policies by letting the agents vote at the beginning of each period. In this section we make the first step of this procedure.

[5]The share of the productive public goods, $\lambda_t=(1-\alpha)\beta_R$, is the same as in Glomm and Ravikumar (1994a). It differs from one in Barro (1990) by the factor β_R because the tax revenues in their model are converted into the productive public goods with a one-period lag, whereas in Barro's continuous time model tax revenues are converted into the productive public goods instantaneously.

[6]Indeed, the second formula in (2) suggests that the agents with discount factor lower than or equal to β_{med} would prefer $\lambda_\tau=(1-\alpha)\beta_{med}$ to any λ_τ higher than $(1-\alpha)\beta_{med}$ while the agents with discount factor higher than or equal to β_{med} would prefer $\lambda_\tau=(1-\alpha)\beta_{med}$ to any λ_τ lower than $(1-\alpha)\beta_{med}$. This seems to imply that $\lambda_\tau=(1-\alpha)\beta_{med}$ must be a Condorcet winner in voting on λ_τ. Moreover this argument applies *irrespective of* θ_t.

4.1 Competitive equilibria at given shares of public goods in GDP

Suppose that the economy at time τ is in a non-generate state $\mathcal{I}^*_\tau = ((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$[7] and that the shares of utility-enhancing and productive public goods in GDP, θ_t and λ_t, $t = \tau, \tau+1, \dots,$ are given. It follows that the tax rate at time $t = \tau, \tau+1, \dots$ is $\theta_t + \lambda_t$. We assume that

$$\liminf_{t\to\infty} \theta_t > 0,\ \liminf_{t\to\infty} \lambda_t > 0 \text{ and } \limsup_{t\to\infty}(\theta_t + \lambda_t) < 1. \tag{3}$$

Let $\Theta_\tau = (\theta_t)^\infty_{t=\tau}$ and $\Lambda_\tau = (\lambda_t)^\infty_{t=\tau}$. Further, let $1 + r^*_\tau = q(g^*_\tau)f'(k^*_\tau)$ and $w^*_\tau = q(g^*_\tau)(f(k^*_\tau) - f'(k^*_\tau)k^*_\tau)$ be the pre-tax gross interest and wage rates at time τ.

Definition. *A sequence*

$$\mathcal{E}^* = ((c^{j*}_t)^L_{j=1},\ (s^{j*}_t)^L_{j=1},\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t)^\infty_{t=\tau}$$

is called a competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$ if

1) *for each $j = 1, \dots, L$, the sequence $(c^{j*}_t, s^{j*}_t)^\infty_{t=\tau}$ is a solution to*

$$\begin{cases} \max \sum_{t=\tau}^\infty \beta_j^{t-\tau} \ln c^j_t \\ c^j_t + s^j_t = (1-\theta_t-\lambda_t)[(1+r^*_t)s^j_{t-1} + w^*_t],\ t = \tau, \tau+1, ..., \\ s^j_t \geq 0,\ t = \tau, \tau+1, ...,\ (\text{where } s^j_{\tau-1} = s^{j*}_{\tau-1}); \end{cases} \tag{4}$$

2) $k^*_{t+1}L = \sum_{j=1}^L s^{j*}_t,\ t = \tau, \tau+1, ...;$

3) $1 + r^*_t = q(g^*_t)f'(k^*_t)\ (= \alpha(g^*_t/k^*_t)^{1-\alpha}),\ t = \tau+1, \tau+2, ...;$

4) $w^*_t = q(g^*_t)(f(k^*_t) - f'(k^*_t)k^*_t)\ (= (1-\alpha)(g^*_t)^{1-\alpha}(k^*_t)^\alpha),\ t = \tau+1, \tau+2, ...;$

5) $g^*_{t+1} = \lambda_t q(g^*_t)f(k^*_t)(= \lambda_t(g^*_t)^{1-\alpha}(k^*_t)^\alpha)),\ t = \tau, \tau+1, ...;$

6) $h^*_t = \theta_t q(g^*_t)f(k^*_t)(= \theta_t(g^*_t)^{1-\alpha}(k^*_t)^\alpha)),\ t = \tau, \tau+1, ...$

It is easy to check that in the case of one representative consumer ($L = 1$) for any non-degenerate state of the economy at time τ, $\mathcal{I}^*_\tau = (s^*_{\tau-1},\ k^*_\tau,\ g^*_\tau)$, there exists a unique competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$, $\mathcal{E}^* = (c^*_t,\ s^*_t,\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t)^\infty_{t=\tau}$ (index j is suppressed here because there is only one consumer). It is fully determined by the following relationships:

$$k^*_{t+1} = s^*_t = \alpha\beta_R(1-\theta_t-\lambda_t)q(g^*_t)f(k^*_t),\ c^*_t = (1-\alpha\beta_R)(1-\theta_t-\lambda_t)q(g^*_t)f(k^*_t). \tag{5}$$

The following existence proposition is a modification of the extended existence theorem for the Ramsey model with heterogeneous agents proved in Becker *et al* (1991) and can be proved in a similar way[8].

[7] A tuple $((s^j)^L_{j=1}, k, g)$ is called a *non-degenerate state* of the economy if $s^j \geq 0,\ j = 1, \dots, L,$ $\sum_{j=1}^L s^j = kL > 0$ and $g > 0$.

[8] See Becker *et al* (2015), Borissov and Dubey (2015) and Le Van and Pham (2016) for simpler proofs of the existence of equilibria in Ramsey-type models with heterogeneous agents and borrowing constraints.

Proposition 1. *For any non-degenerate state of the economy at time* τ*,* $\mathcal{I}_\tau^* = ((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$*, there exists a competitive* Θ_τ*-*Λ_τ*-equilibrium starting from* $\mathcal{I}_\tau^*$*.*

The question arises of whether the competitive Θ-Λ-equilibrium is unique. Because of the Cobb-Douglas parametric specification of our model, it is reasonable to conjecture that the answer to this question is yes. However, we have no proof of this conjecture. At the same time, the following proposition says that if the initial state is such that all capital is owned by the most patient consumers, then the answer to the above question is positive. It also says that if initially all capital belongs to the most patient agents, then in a Θ-Λ-equilibrium, all post-tax wage income is spent on consumption. This implies in particular that all agents whose discount factor is lower than β_1 make no savings at all. As for the patient agents, the share of the post-tax gross capital income they spend on savings is equal to β_1 irrespective of their wealth, the tax rates and the levels of public goods provision. It follows that the time t savings rate in the economy as a whole is $\beta_1(1-\theta_t-\lambda_t)\alpha$.

Proposition 2. *For any non-degenerate state* $\mathcal{I}_\tau^* = ((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$ *satisfying*

$$k_\tau^* L = \sum_{j\in J} s_{\tau-1}^{j*}\ (i.e.\ s_{\tau-1}^{j*}=0,\ j\notin J),$$

the Θ_τ*-*Λ_τ*-equilibrium*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=\tau}^\infty$$

starting from $\mathcal{I}_\tau^*$ *is unique and determined as follows* $(t=\tau,\tau+1,\ldots)$:

$$k_{t+1}^* = \beta_1(1-\theta_t-\lambda_t)\alpha q(g_t^*)f(k_t^*),$$

$$s_t^{j*} = \beta_1(1-\theta_t-\lambda_t)(1+r_t^*)s_{t-1}^{j*},\ c_t^{j*} = (1-\theta_t-\lambda_t)[(1-\beta_1)(1+r_t^*)s_{t-1}^{j*}+w_t^*],\ j\in J,$$

$$s_t^{j*}=0,\ c_t^{j*}=(1-\theta_t-\lambda_t)w_t^*,\ j\notin J.$$

$$1+r_{t+1}^* = q(g_{t+1}^*)f'(k_{t+1}^*),\ w_{t+1}^* = q(g_{t+1}^*)(f(k_{t+1}^*)-f'(k_{t+1}^*)k_{t+1}^*),$$

$$g_{t+1}^* = \lambda_t q(g_t^*)f(k_t^*),\ h_t^* = \theta_t q(g_t^*)f(k_t^*).$$

Proof. See Appendix 1.

In his seminal article on optimal capital accumulation, Ramsey (1928) conjectured that in a model with households differentiated by their rates of time preference eventually the most patient households own the entire capital stock of the economy. The literature on Ramsey's conjecture is comprehensively surveyed in Becker (2006). The following proposition maintains that in our model this conjecture is true.

Proposition 3. *For any competitive* Θ_τ*-*Λ_τ*-equilibrium*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=\tau}^\infty$$

there is T *such that*

$$k_t^* L = \sum_{j\in J} s_{t-1}^{j*}\ (i.e.\ s_{t-1}^{j*} = 0,\ j \notin J),\ t = T, T+1, \ldots .$$

Proof. See Appendix 1.

In what follows, if some sequences $\Theta = (\theta_t)_{t=0}^\infty$ and $\Lambda = (\lambda_t)_{t=0}^\infty$ are taken as given, then for all $\tau = 0, 1, \ldots$, we denote the tails of these sequences starting at time τ as Θ_τ and Λ_τ: $\Theta_\tau := (\theta_t)_{t=\tau}^\infty$ and $\Lambda_\tau := (\lambda_t)_{t=\tau}^\infty$. In particular, $\Theta_0 = \Theta$ and $\Lambda_0 = \Lambda$.

It should be noted that if

$$((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

is a competitive Θ_0-Λ_0-equilibrium starting from some initial state $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$, then for each $\tau = 1, 2, \ldots$, its tail

$$((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=\tau}^\infty$$

is a competitive Θ_τ-Λ_τ-equilibrium starting from $((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$. In other terms, any Θ_0-Λ_0-equilibrium is time consistent.

4.2 Balanced-growth equilibria at given shares of public goods

Suppose that the shares of consumption and productive public goods are constant over time and are equal to, respectively, $\theta > 0$ and $\lambda > 0$, i.e. $\Theta = (\theta, \theta, \theta, \ldots)$ and $\Lambda = (\lambda, \lambda, \lambda, \ldots)$, and that $\theta + \lambda < 1$. A competitive Θ-Λ-equilibrium

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

starting from $\mathcal{I}_0^* = ((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$ is called a *balanced growth* θ-λ*-equilibrium* if there is an equilibrium rate of balanced growth, γ^*, such that for $t = 0, 1, \ldots$, and $j = 1, \ldots, L$,

$$\frac{k_{t+1}^*}{k_t^*} = \frac{g_{t+1}^*}{g_{t+1}^*} = \frac{w_{t+1}^*}{w_{t+1}^*} = \frac{h_{t+1}^*}{h_{t+1}^*} = \frac{c_{t+1}^{j*}}{c_t^{j*}} = \frac{s_t^{j*}}{s_{t-1}^{j*}} = 1 + \gamma^*. \tag{6}$$

It is clear that in a balanced growth θ-λ-equilibrium

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

starting from $\mathcal{I}_0^*$, the interest rate r_t^* is constant over time:

$$1+r_t^* = \alpha\left(\frac{k_t^*}{g_t^*}\right)^{\alpha-1} = \alpha\left(\frac{k_0^*}{g_0^*}\right)^{\alpha-1} = 1+r_0^*,\ t=0,1,\ldots. \tag{7}$$

The following proposition is an adaptation of a well-known result by Becker (1980) to our model. It maintains that in a balanced growth θ-λ-equilibrium all capital is owned by the most patient consumers. It also says that the rate of balanced growth, γ^*, is completely determined by the parameters of the production function, the two shares of public goods in GDP, θ and λ, and the discount factor of the most patient consumer, β_1.

Proposition 4. 1) *Let*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

be a balanced growth θ-λ-equilibrium starting from $\mathcal{I}_0^*$ and γ^* be the corresponding equilibrium rate of balanced growth. Then

$$1+\gamma^* = \beta_1(1-\theta-\lambda)(1+r_0^*) = \lambda^{1-\alpha}(1-\theta-\lambda)^\alpha(\alpha\beta_1)^\alpha, \tag{8}$$

and for $t=0,1,\ldots,$

$$\frac{k_t^*}{g_t^*} = \frac{\alpha\beta_1(1-\theta-\lambda)}{\lambda}, \tag{9}$$

$$k_t^* L = \sum_{j\in J} s_{t-1}^{j*}\ (i.e.\ s_{t-1}^{j*}=0,\ j\notin J), \tag{10}$$

2) *Let $k_0^*>0$, $g_0^*>0$ and $(s_{-1}^{j*})_{j=1}^L$ $(s_{-1}^{j*}\geq 0,\ j=1,\ldots,L)$ be such that*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*}\ (i.e.\ s_{\tau-1}^{j*}=0,\ j\notin J) \tag{11}$$

and

$$\frac{k_0^*}{g_0^*} = \frac{\alpha\beta_1(1-\theta-\lambda)}{\lambda}. \tag{12}$$

Then there is a balanced growth θ-λ-equilibrium

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

starting from $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$, which is completely determined by (6)-(7) with γ^* given by (8).

It follows from this proposition that the rate of balanced growth, γ^*, is increasing in β_1 and does not depend on $\beta_j \notin J$. It is clearly decreasing in the share of public consumption goods in GDP, θ. As for the dependence of the rate of balanced growth on the share of productive public goods, λ, it has an inverted U-shaped form, but we shall see that voting leads to shares lying in a range where this dependence is increasing.

The following proposition maintains that if the shares of public goods are constant over time ($\Theta = (\theta, \theta, \theta, \ldots)$ and $\Lambda = (\lambda, \lambda, \lambda, \ldots)$) and at the initial state all capital is owned by the most patient consumers, then the unique competitive Θ-Λ-equilibrium settles on a balanced growth equilibrium in the first time period.

Proposition 5. *Suppose that $\Theta = (\theta, \theta, \theta, \ldots)$ and $\Lambda = (\lambda, \lambda, \lambda, \ldots)$. Then for any competitive Θ-Λ-equilibrium*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^{\infty},$$

there exists T such that $\mathcal{E}^*$ satisfies (6)-(7) and (9)-(10) for $t > T$, where γ^* is given by (8).

Moreover, if the initial state $\mathcal{I}_0^* = ((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$ is such that $k_\tau^* L = \sum_{j\in J} s_{\tau-1}^{j*}$ (i.e. $s_{\tau-1}^{j*} = 0,\ j \notin J$), then the unique competitive Θ-Λ-equilibrium

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^{\infty}$$

starting from $\mathcal{I}_0^*$ satisfies (6)-(7) and (9)-(10) for $t = 1, 2, \ldots$.

Proof. It follows from Proposition 2 and Proposition 3. □

5 Voting equilibria

5.1 Definitions

5.1.1 Time τ voting equilibrium

Suppose that at time τ the economy is in a non-degenerate state $\mathcal{I}_\tau^* = ((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$ and the agents are asked to vote on the time τ shares of public goods in GDP, θ_τ and λ_τ. We assume that when voting on θ_τ and λ_τ, they have some expectations about future shares of public goods in GDP, $\Theta_{\tau+1}^e = (\theta_t^e)_{t=\tau+1}^{\infty}$ and $\Lambda_{\tau+1}^e = (\lambda_t^e)_{t=\tau+1}^{\infty}$, such that $\liminf_{t\to\infty} \theta_t^e > 0$, $\liminf_{t\to\infty} \lambda_t^e > 0$ and $\limsup_{t\to\infty}(\theta_t^e + \lambda_t^e) < 1$.

To describe the voting outcome, it is necessary to specify the indirect utility functions of the agents that represent their policy preferences when they vote. This can easily be done *if for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences of tax rates Θ_τ and Λ_τ given by*

$$\Theta_\tau = (\theta_\tau, \theta_{\tau+1}^e, \theta_{\tau+2}^e, \ldots),\ \Lambda_\tau = (\lambda_\tau, \lambda_{\tau+1}^e, \lambda_{\tau+2}^e, \ldots), \tag{13}$$

there is a unique competitive Θ_τ-Λ_τ-equilibrium

$$\mathcal{E}^*(\theta_\tau, \lambda_\tau) = ((c_t^{j*}(\theta_\tau, \lambda_\tau))_{j=1}^L,\ (s_t^{j*}(\theta_\tau, \lambda_\tau))_{j=1}^L,\ k_{t+1}^*(\theta_\tau, \lambda_\tau),$$
$$r_{t+1}^*(\theta_\tau, \lambda_\tau),\ w_{t+1}^*(\theta_\tau, \lambda_\tau),\ g_{t+1}^*(\theta_\tau, \lambda_\tau),\ h_t^*(\theta_\tau, \lambda_\tau))_{t=\tau}^{\infty}$$

starting from $\mathcal{I}^*_\tau$.[9] In this case, the indirect utility function representing the policy preferences of agent j over θ_τ and λ_τ is given by the utility of this agent in $\mathcal{E}^*(\theta_\tau, \lambda_\tau)$:

$$V^j(\theta_\tau, \lambda_\tau) := \sum_{t=\tau}^{\infty} \beta_j^{t-\tau} (\ln c_t^{j*}(\theta_\tau, \lambda_\tau) + \delta_j \ln h_t^*(\theta_\tau, \lambda_\tau)).$$

Since voting on θ_τ and λ_τ is 2-dimensional, there can be no Condorcet winner if the agents vote on θ_τ and λ_τ together. Because of this, following Kramer (1972) and Shepsle (1979), we assume that the agents vote on the time τ shares of public goods in GDP, θ_τ and λ_τ, separately: when voting on θ_τ, each agent j takes λ_τ as given and seeks to maximize $V^j(\theta_\tau, \lambda_\tau)$ over $\theta_\tau \in (0, 1)$, and when voting on λ_τ, each agent j takes θ_τ as given and seeks to maximize $V^j(\theta_\tau, \lambda_\tau)$ over $\lambda_\tau \in (0, 1)$.

Definition. *Suppose that for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and the sequences of shares of public goods in GDP, Θ_τ and Λ_τ, given by (13), there is a unique Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$. Suppose also that $\theta^*_\tau > 0$ and $\lambda^*_\tau > 0$ are the Condorcet winners in voting on θ_τ at $\lambda_\tau = \lambda^*_\tau$ and voting on λ_τ at $\theta_\tau = \theta^*_\tau$ and that $\theta^*_\tau + \lambda^*_\tau < 1$. Then we say that the couple $(\theta^*_\tau, \lambda^*_\tau)$ is a time τ (temporary) voting equilibrium.*

It should be noticed that in this definition it is not presupposed that the expected shares of public goods in GDP, θ^e_t and λ^e_t, are time t voting equilibria for $t > \tau$. Moreover, the expectations are not assumed to be correct. By contrast, in the next definition, the shares of public goods in GDP are time t voting equilibria for each time t and the agents have perfect foresight about future shares of public goods.

5.1.2 Intertemporal voting equilibrium

Let

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t)_{t=0}^\infty$$

be a competitive Θ^*-Λ^*-equilibrium starting from a non-degenerate initial state $\mathcal{I}^*_0 = ((s_{-1}^{j*})_{j=1}^L,\ k^*_0,\ g^*_0\}$ for some sequences of shares of public goods in GDP, $\Theta^* = (\theta^*_t)_{t=0}^\infty$ and $\Lambda^* = (\lambda^*_t)_{t=0}^\infty$.

Definition. *If, for each $\tau = 0, 1, ...$, the couple $(\theta^*_\tau, \lambda^*_\tau)$ is a time τ voting equilibrium at $\theta^e_t = \theta^*_t,\ t = \tau+1, \tau+2, \ldots,$ and $\lambda^e_t = \lambda^*_t,\ t = \tau+1, \tau+2, \ldots,$ then we say that the triple $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ is an intertemporal voting equilibrium starting from $\mathcal{I}^*_0$.*

In this definition, it is presupposed that for each $\tau = 0, 1, ...$, for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences of shares of public goods in GDP, Θ_τ and Λ_τ,

[9] It should be noted that here the notation is well-defined only because the competitive equilibrium is unique for any θ_τ and λ_τ.

given by

$$\Theta_\tau = (\theta_\tau, \theta^*_{\tau+1}, \theta^*_{\tau+2}, \ldots),\ \Lambda_\tau = (\lambda_\tau, \lambda^*_{\tau+1}, \lambda^*_{\tau+2}, \ldots),$$

there is a unique competitive Θ_τ-Λ_τ-equilibrium starting from $((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$.

5.2 Indirect utility functions

In this subsection we make important comparative dynamics for competitive equilibria, which will help us to describe the functions $V^j(\theta_\tau, \lambda_\tau),\ j = 1, ..., L$, and voting equilibria. For given $\Theta^e_{\tau+1} = (\theta^e_t)^\infty_{t=\tau+1}$ and $\Lambda^e_{\tau+1} = (\lambda^e_t)^\infty_{t=\tau+1}$, take arbitrary $\theta^\circ_\tau > 0$ and $\lambda^\circ_\tau > 0$ such that $\theta^\circ_\tau + \lambda^\circ_\tau < 1$. Let

$$\Theta^\circ_\tau = (\theta^\circ_\tau, \theta^e_{\tau+1}, \theta^e_{\tau+2}, \ldots),\ \Lambda^\circ_\tau = (\lambda^\circ_\tau, \lambda^e_{\tau+1}, \lambda^e_{\tau+2}, \ldots), \tag{14}$$

and let

$$\mathcal{E}^*_\tau = ((c^{j*}_t)^L_{j=1},\ (s^{j*}_t)^L_{j=1},\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t)^\infty_{t=\tau}$$

be a competitive Θ°_τ-Λ°_τ-equilibrium starting from an initial state $((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$.

Suppose that we change the shares of utility-enhancing and productive public goods at time τ, θ°_τ to $\theta_\tau > 0$ and λ°_τ to $\lambda_\tau > 0$ ($\theta_\tau + \lambda_\tau < 1$), leaving these shares intact for $t > \tau$. What is the impact of this change on the competitive equilibrium? Proposition 2 suggests that if in the initial state $((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$ all capital is owned by the most patient agents, then these agents will keep the savings rates[10] unchanged and, after the policy change, will use the savings rates that they applied in equilibrium $\mathcal{E}^*_\tau$. As for the savings rate of all other agents, we know from Proposition 2 that their savings must be equal to zero forever.

Indeed, since the tax is proportional, at time τ, the share of the accumulated savings in the wealth of a patient agent $j \in J$ is equal to $\frac{(1+r^*_\tau)s^{j*}_{\tau-1}}{(1+r^*_\tau)s^{j*}_{\tau-1}+w^*_\tau}$ and hence will not change after the policy change. At the same time Proposition 2 tells us that the share of the post-tax gross capital income this agent spends on savings is equal to β_1 irrespective of her wealth, the tax rate and the levels of public goods provision. Therefore, it seems reasonable to conjecture that the savings rate of the patient agents will not change after the policy change.

It turns out that this conjecture is correct. Moreover, in a sense, it is correct not only in the case where all capital belongs to the most patient agent in the initial state. This seems to be natural, because the policy change leads to a rescaling of the economy without any change of proportions between real variables.

[10]By the savings rate of agent j at time t we mean the proportion $\frac{s^j_t}{(1-\theta_t-\lambda_t)[(1+r_t)s^j_{t-1}+w_t]}$.

Formally, suppose that all agents keep the savings rates unchanged and, after the policy change, use the savings rates that they applied in equilibrium $\mathcal{E}^*_\tau$. Then the economy will switch to a new path

$$\mathbf{E}(\theta_\tau, \lambda_\tau, \theta^\circ_\tau, \lambda^\circ_\tau, \mathcal{E}^*_\tau) = ((\tilde{c}^j_t)^L_{j=1},\ (\tilde{s}^j_t)^L_{j=1},\ \tilde{k}_{t+1}, \tilde{r}_{t+1},\ \tilde{w}_{t+1},\ \tilde{g}_{t+1},\ \tilde{h}_t)^\infty_{t=\tau}.$$

Let us describe its structure[11] and show that it is a post-change equilibrium[12]. The replacement of θ°_τ by θ_τ and λ°_τ by λ_τ leads to a change in the share of the private sector of the economy at time τ by a factor of

$$\frac{1-\lambda_\tau-\theta_\tau}{1-\lambda^\circ_\tau-\theta^\circ_\tau},$$

and to a proportional change in the after-tax wealth of each agent. Since the agents do not change their savings rates, the short-run impact of the policy change on the economy is as follows: the per capita provision of public consumption goods at time τ will become

$$\tilde{h}_\tau = (\theta_\tau/\theta^\circ_\tau)h^*_\tau \tag{15}$$

and the per capita provision of productive public goods at time $\tau+1$ will become

$$\tilde{g}_{\tau+1} = (\lambda_\tau/\lambda^\circ_\tau)g^*_{\tau+1};$$

the time τ consumption and savings of each agent j will become

$$\tilde{c}^j_\tau = \frac{1-\lambda_\tau-\theta_\tau}{1-\lambda^\circ_\tau-\theta^\circ_\tau}c^{j*}_\tau \tag{16}$$

and

$$\tilde{s}^j_\tau = \frac{1-\lambda_\tau-\theta_\tau}{1-\lambda^\circ_\tau-\theta^\circ_\tau}s^{j*}_\tau$$

respectively. Hence the time $\tau+1$ per capita stock of private capital will become

$$\tilde{k}_{\tau+1} = \frac{1-\lambda_\tau-\theta_\tau}{1-\lambda^\circ_\tau-\theta^\circ_\tau}k^*_{\tau+1}.$$

It follows, that the time $\tau+1$ per capita output, the wage rate and the pre-tax income of each agent will change by a factor of

$$\left(\frac{1-\lambda_\tau-\theta_\tau}{1-\lambda^\circ_\tau-\theta^\circ_\tau}\right)^\alpha \left(\frac{\lambda_\tau}{\lambda^\circ_\tau}\right)^{1-\alpha}.$$

[11] A complete set of formulas for calculating $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta^\circ_\tau, \lambda^\circ_\tau, \mathcal{E}^*_\tau)$) can be found in Appendix 2. To be more precise, these formulas define the map $\mathbf{E}_\tau(\cdot,\cdot,\cdot,\cdot,\cdot)$ that takes each tuple $(\theta_\tau, \lambda_\tau, \theta^\circ_\tau, \lambda^\circ_\tau, \mathcal{E}^*_\tau)$ that consists of numbers $\theta_\tau > 0$, $\lambda_\tau > 0$, $\theta^\circ_\tau > 0$ and $\lambda^\circ_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and $\theta^\circ_\tau + \lambda^\circ_\tau < 1$ and a competitive Θ°_τ-Λ°_τ-equilibrium starting from an initial state $((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$, $\mathcal{E}^*_\tau$, to the sequence $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta^\circ_\tau, \lambda^\circ_\tau, \mathcal{E}^*_\tau)$ (here Θ°_τ and Λ°_τ are given by (14)). Clearly, $\mathbf{E}(\theta^\circ_\tau, \lambda^\circ_\tau, \theta^\circ_\tau, \lambda^\circ_\tau, \mathcal{E}^*_\tau) = \mathcal{E}^*_\tau$.

[12] In the case of a representative agent, this follows from (5).

Since we do not change the shares of utility-enhancing and productive public goods in GDP at time $\tau+1$, the time $\tau+1$ per capita provision of public consumption goods and the time $\tau+2$ per capita productive public expenditures will change by the same factor as the time $\tau+1$ per capita output. Therewith, the share of each agent in national income will not change. Therefore, the time $\tau+1$ consumption and savings of each agent j will change by the same factor as the time $\tau+1$ per capita output. From time $\tau+2$ onward the same is true of all variables except the interest rate. Thus,

$$\tilde{h}_t = \left(\frac{1-\lambda_\tau-\theta_\tau}{1-\lambda_\tau^\circ-\theta_\tau^\circ}\right)^\alpha \left(\frac{\lambda_\tau}{\lambda_\tau^\circ}\right)^{1-\alpha} h_t^*,\ t=\tau+1,\tau+2,..., \tag{17}$$

$$\tilde{c}_t^j = \left(\frac{1-\lambda_\tau-\theta_\tau}{1-\lambda_\tau^\circ-\theta_\tau^\circ}\right)^\alpha \left(\frac{\lambda_\tau}{\lambda_\tau^\circ}\right)^{1-\alpha} c_t^{j*},\ t=\tau+1,\tau+2,...,\ j=1,...,L. \tag{18}$$

Lemma 1. *$\mathbf{E}(\theta_\tau,\lambda_\tau,\theta_\tau^\circ,\lambda_\tau^\circ,\mathcal{E}_\tau^*)$ is a competitive Θ_τ-Λ_τ-equilibrium starting from $((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$ for*

$$\Lambda_\tau=(\lambda_\tau,\lambda_{\tau+1}^e,\lambda_{\tau+2}^e,\ldots),\ \Theta_\tau=(\theta_\tau,\theta_{\tau+1}^e,\theta_{\tau+1}^e,\ldots).$$

Proof. See Appendix 2.

It should be emphasized that *it is not assumed here that $\mathbf{E}(\theta_\tau,\lambda_\tau,\theta_\tau^\circ,\lambda_\tau^\circ,\mathcal{E}_\tau^*)$ is the unique* competitive Θ_τ-Λ_τ-equilibrium starting from $((s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*)$

Let us now *assume that for any $\theta_\tau>0$ and $\lambda_\tau>0$ such that $\theta_\tau+\lambda_\tau<1$ and for the sequences of tax rates Θ_τ and Λ_τ given by (13) the competitive Θ_τ-Λ_τ-equilibrium is unique* and hence we can denote the utility of agent j in $\mathcal{E}_\tau^*$ and in $\mathbf{E}(\theta_\tau,\lambda_\tau,\theta_\tau^\circ,\lambda_\tau^\circ,\mathcal{E}_\tau^*)$ as $V^j(\theta_\tau^\circ,\lambda_\tau^\circ)$ and $V^j(\theta_\tau,\lambda_\tau)$ respectively:

$$V^j(\theta_\tau,\lambda_\tau)=\sum_{t=\tau}^{\infty}\beta_j^{t-\tau}(\ln\tilde{c}_t^j+\delta_j\ln\tilde{h}_t),\quad V^j(\theta_\tau^\circ,\lambda_\tau^\circ)=\sum_{t=\tau}^{\infty}\beta_j^{t-\tau}(\ln c_t^{j*}+\delta_j\ln h_t^*).$$

We want to calculate the impact of the policy change on the utility of agent j, $V^j(\theta_\tau,\lambda_\tau)-V^j(\theta_\tau^\circ,\lambda_\tau^\circ)$. By (15), $\delta_j\ln\tilde{h}_\tau-\delta_j\ln h_\tau^*=\delta_j(\ln\theta_\tau-\ln\theta_\tau^\circ)$ and by (16), $\ln\tilde{c}_\tau^j-\ln c_\tau^{j*}=\ln(1-\lambda_\tau-\theta_\tau)-\ln(1-\lambda_\tau^\circ-\theta_\tau^\circ)$. Therefore,

$$(\ln\tilde{c}_\tau^j+\delta_j\ln\tilde{h}_\tau)-(\ln c_\tau^{j*}+\delta_j\ln h_\tau^*)$$
$$=\ln(1-\lambda_\tau-\theta_\tau)+\delta_j\ln\theta_\tau-(\ln(1-\lambda_\tau^\circ-\theta_\tau^\circ)+\delta_j\ln\theta_\tau^\circ).$$

This is the *short-run effect* of the change in policy.

To obtain the *long-run effect* (beyond time τ), note that, by (17) and (18), for $t=\tau+1,\tau+2,...$ we have

$$\ln\tilde{c}_t^j+\delta_j\ln\tilde{h}_t^j-(\ln c_t^{j*}+\delta_j\ln h_t^{j*})$$
$$=(1+\delta_j)\ln[(1-\lambda_\tau-\theta_\tau)^\alpha\lambda_\tau^{1-\alpha}]-(1+\delta_j)\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha(\lambda_\tau^\circ)^{1-\alpha}].$$

Hence,

$$\begin{aligned}V^j(\theta_\tau,\lambda_\tau)-V^j(\theta_\tau^\circ,\lambda_\tau^\circ)&=\ln(1-\lambda_\tau-\theta_\tau)+\delta_j\ln\theta_\tau-(\ln(1-\lambda_\tau^\circ-\theta_\tau^\circ)+\delta_j\ln\theta_\tau^\circ)\\&+\beta_j(1+\delta_j)\ln[(1-\lambda_\tau-\theta_\tau)^\alpha\lambda_\tau^{1-\alpha}]-\beta_j(1+\delta_j)\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha(\lambda_\tau^\circ)^{1-\alpha}]\\&+\beta_j^2(1+\delta_j)\ln[(1-\lambda_\tau-\theta_\tau)^\alpha\lambda_\tau^{1-\alpha}]-\beta_j^2(1+\delta_j)\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha(\lambda_\tau^\circ)^{1-\alpha}]+\dots\\&=\underbrace{\ln(1-\lambda_\tau-\theta_\tau)+\delta_j\ln\theta_\tau\;-\;(\ln(1-\lambda_\tau^\circ-\theta_\tau^\circ)+\delta_j\ln\theta_\tau^\circ)}_{\textit{Short run effect of the policy change}}\\&+\underbrace{\frac{\beta_j}{1-\beta_j}(1+\delta_j)\left[\ln((1-\lambda_\tau-\theta_\tau)^\alpha\lambda_\tau^{1-\alpha})\;-\;\ln((1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha(\lambda_\tau^\circ)^{1-\alpha})\right]}_{\textit{Long run discounted effect of the policy change}}.\end{aligned}$$

Thus,

$$V^j(\theta_\tau,\lambda_\tau)-V^j(\theta_\tau^\circ,\lambda_\tau^\circ)=v^j(\theta_\tau,\lambda_\tau)-v^j(\theta_\tau^\circ,\lambda_\tau^\circ),$$

where

$$v^j(\theta,\lambda):=\delta_j\ln\theta+\frac{1-\beta_j+(1+\delta_j)\alpha\beta_j}{1-\beta_j}\ln(1-\lambda-\theta)\;+\frac{(1+\delta_j)(1-\alpha)\beta_j}{1-\beta_j}\ln\lambda.$$

This proves the following lemma.

Lemma 2. *Assume that for any $\theta_\tau>0$ and $\lambda_\tau>0$ such that $\theta_\tau+\lambda_\tau<1$ and for the sequences of tax rates Θ_τ and Λ_τ given by (13) there is a unique competitive Θ_τ-Λ_τ-equilibrium. Then*

$$V^j(\theta_\tau,\lambda_\tau)=v^j(\theta_\tau,\lambda_\tau)+(V^j(\theta_\tau^\circ,\lambda_\tau^\circ)-v^j(\theta_\tau^\circ,\lambda_\tau^\circ)),\;\; j=1,...,L.\;\square$$

It follows from this lemma that, in the definition of a time τ voting equilibrium, the indirect utility functions $V^j(\theta_\tau,\lambda_\tau),\; j=1,\dots,L$, that represent the policy preferences of the agents can be replaced by the functions $v^j(\theta_\tau,\lambda_\tau),\; j=1,\dots,L$.

5.3 Representative agent case: second-best optimal policy

Before we describe voting equilibria, consider the representative agent case of our model and suppose that at each time τ the government chooses the shares of public goods to maximize the representative agent welfare taking into account the competitive equilibrium, where the latter includes the optimal reaction of the representative agent to policy instruments. Since to finance the provision of public goods the government employs distorting taxation, this is a second-best policy problem.

It follows from Lemma 2 that to find the second-best shares, it is necessary to maximize $v^R(\theta_\tau,\lambda_\tau)$, where

$$v^R(\theta,\lambda):=\delta_R\ln\theta+\frac{1-\beta_R+(1+\delta_R)\alpha\beta_R}{1-\beta_R}\ln(1-\lambda-\theta)\;+\frac{(1+\delta_R)(1-\alpha)\beta_R}{1-\beta_R}\ln\lambda$$

To do this, it is sufficient to solve the following system of two equations:

$$\frac{\partial v^R(\theta_\tau, \lambda_\tau)}{\partial \theta_\tau} = 0, \; \frac{\partial v^R(\theta_\tau, \lambda_\tau)}{\partial \lambda_\tau} = 0.$$

The solutions to these equations are given by

$$\theta_t = \kappa(\beta_R, \delta_R)(1 - \lambda_t) \tag{19}$$

and

$$\lambda_t = \chi(\beta_R, \delta_R)(1 - \theta_t), \tag{20}$$

where

$$\kappa(\beta, \delta) := \frac{\delta(1-\beta)}{(1+\delta)(1-\beta+\alpha\beta)},$$

and

$$\chi(\beta, \delta) := \frac{(1+\delta)(1-\alpha)\beta}{1+\delta\beta}.$$

It is natural to call $\kappa(\beta, \delta)$ the *conditional propensity to spend on public consumption goods* and $\chi(\beta, \delta)$ the *conditional propensity to spend on productive public goods.* The conditional propensity to spend on public consumption goods, $\kappa(\beta, \delta)$, is decreasing in β and increasing in δ while the conditional propensity to spend on productive public goods, $\chi(\beta, \delta)$, is increasing in both β and δ.

It is easy to infer that the solution to the problem of maximizing $v(\theta_\tau, \lambda_\tau)$, $(\theta^{R*}, \lambda^{R*})$, is given by

$$\theta^{R*} = \frac{\delta_R(1-\beta_R)}{1+\delta_R}, \; \lambda^{R*} = (1-\alpha)\beta_R. \tag{21}$$

In other words, the second-best solution to the problem of the choice of shares of public goods coincides with the first-best one given by (2).This does not mean, however, that the second-best optimum coincides with the first-best optimum. Due to distortive taxation, the second-best rate of growth is lower than the first-best one.

It should be noticed that θ_τ chosen by (19) solves the problem of maximizing $v^R(\theta_\tau, \lambda_\tau)$ over $\theta_\tau \geq 0$ at any given $\lambda_\tau \in (0, 1)$. Respectively, λ_τ chosen by (20) solves the problem of maximizing $v^R(\theta_\tau, \lambda_\tau)$ over λ_τ at any given $\theta_\tau \in (0, 1)$. Suppose that the choice of θ_τ and λ_τ is made by two different government agencies. Then (i) the optimal choice of the agency responsible for θ_τ is given by (19) irrespective of whether λ_τ is chosen optimally or not; (ii) the optimal choice of the agency responsible for λ_τ is given by (20) irrespective of whether θ_τ is chosen optimally or not[13]. Thus, the role of the two agencies is just to"compute" the conditional propensities to spend on consumptive and productive public goods and after that to coordinate their decisions.

[13]It is possible to show that the same is true of the first-best optimum.

As noted above, it seems reasonable to conjecture, that if the shares of the two public goods are obtained by voting, this voting will lead to the outcome fully determined by the median value of $\beta_j,\ j=1,\dots,L$, and the median value of either $\delta_j,\ j=1,\dots,L$, or $\frac{\delta_j(1-\beta_j)}{1+\delta_j},\ j=1,...,L$. In the next subsection we will discuss this conjecture in detail.

5.4 Time τ voting equilibria

Suppose that a non-degenerate state of the economy at time τ, $\mathcal{I}^*_\tau = ((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$, and expectations about future shares of public goods in GDP, $\Theta^e_{\tau+1} = (\theta^e_t)^\infty_{t=\tau+1}$ and $\Lambda^e_{\tau+1} = (\lambda^e_t)^\infty_{t=\tau+1}$, are such that a time τ voting equilibrium exists.

Taking into account Lemma 1 and Lemma 2, we can describe the outcomes of the votes on θ_τ and λ_τ. When voting on θ_τ, agent j seeks to maximize $v^j(\theta_\tau,\lambda_\tau)$ over $\theta_\tau \in (0,1)$. To find the maximum, it is sufficient to solve the equation

$$\frac{\partial v^j(\theta_\tau,\lambda_\tau)}{\partial\theta_\tau} = 0.$$

The solution to this equation, which is the most-preferred value of θ_τ for agent j, is equal to

$$\theta_\tau = \kappa(\beta_j,\delta_j)(1-\lambda_\tau).$$

In other terms, *given λ_τ, the most-preferred value of θ_τ for agent j coincides with the value the central planner would choose (see (19)) if the latter considered agent j as the representative agent.*

The median voter theorem applies to voting on θ_τ at a given λ_τ and the outcome of voting is

$$\theta_\tau = \kappa_{med}(1-\lambda_\tau),$$

where κ_{med} is the median value of $\kappa(\beta_j,\delta_j),\ j=1,\dots,L$.

When voting on λ_τ, agent j seeks to maximize $v^j(\theta_\tau,\lambda_\tau)$ over $\lambda_\tau \in (0,1)$. To find the maximum, it is sufficient to solve the equation

$$\frac{\partial v^j(\theta_\tau,\lambda_\tau)}{\partial\lambda_\tau} = 0.$$

The solution to this equation is

$$\lambda_\tau = \chi(\beta_j,\delta_j)(1-\theta_\tau).$$

In other terms, *given θ_τ, the most-preferred value of λ_τ for agent j coincides with the value the central planner would choose (see (20)) if the latter considered agent j as the representative agent.*

The median voter theorem applies to voting on λ_τ at a given θ_τ and the outcome of voting is

$$\lambda_\tau = \chi_{med}(1-\theta_\tau),$$

where χ_{med} is the median value of $\chi(\beta_j, \delta_j),\ j = 1, \ldots, L$.

Therefore, the pair $(\theta^*_\tau, \lambda^*_\tau)$ is a time τ voting equilibrium if and only if it is a solution to the following system of two equations:

$$\theta = \kappa_{med}(1-\lambda),\ \lambda = \chi_{med}(1-\theta) \tag{22}$$

or, equivalently, if and only if $\theta^*_\tau = \theta^*,\ \lambda^*_\tau = \lambda^*$, where

$$\theta^* := \frac{\kappa_{med} - \kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}},\ \lambda^* := \frac{\chi_{med} - \kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}}.$$

Thus, we have proved the following theorem.

Theorem 1. *Suppose that a non-degenerate state of the economy at time τ, $\mathcal{I}^*_\tau = ((s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau)$, and sequences of expected shares of public goods in GDP, $\Theta^e_{\tau+1} = (\theta^e_t)^\infty_{t=\tau+1}$ and $\Lambda^e_{\tau+1} = (\lambda^e_t)^\infty_{t=\tau+1}$, are given. Suppose also that, for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences Θ_τ and Λ_τ given by (13), there is a unique competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$. Then there is a unique time τ voting equilibrium $(\theta^*_\tau, \lambda^*_\tau)$, which is given by*

$$\theta^*_\tau = \theta^*,\ \lambda^*_\tau = \lambda^*. \ \square$$

To interpret this theorem, note that agents do not implement the public policy themselves. The government is the political institution to do this. Suppose again that the government consists of two agencies the role of the first agency being to determine the share of public consumption goods in GDP, θ_t, and the role of the second one being to determine the share of productive public goods in GDP, λ_t. Then, to perform their roles, the two agencies adopt the following two-stage procedure. First, they organize votes to find the median values of the conditional propensity to spend on public consumption goods, κ_{med}, and the conditional propensity to spend on productive public goods, χ_{med}. Second, they coordinate their decisions and find equilibrium levels of θ_t and λ_t, θ^* and λ^*, by solving the system of two equations (22). These two equations can be interpreted as the best-response functions of the two agencies (see Fig.1). Theorem 1 claims that the outcome of the described procedure will be exactly the time τ voting equilibrium.

5.5 Intertemporal voting equilibria

The following theorem maintains that in any intertemporal voting equilibrium, the shares of public goods in GDP are constant over time. It follows directly from Theorem 1.

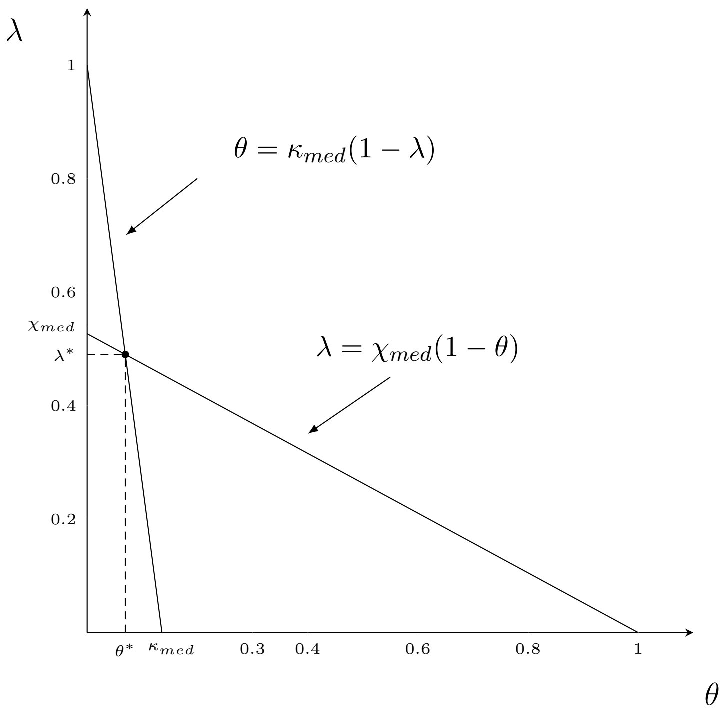

Figure 1: *Determining θ^* and λ^**

Theorem 2. *For any intertemporal voting equilibrium $(\Theta^*, \Lambda^*, \mathcal{E}^*)$, the sequences of the shares of public goods in GDP, Θ^* and Λ^*, are constant over time and determined as follows:*

$$\Theta^* = (\theta^*, \theta^*, \ldots), \ \Lambda^* = (\lambda^*, \lambda^*, \ldots). \tag{23}$$

Theorem 2 describes the structure of intertemporal voting equilibria, but leaves the question about their existence unanswered. However, if we take into account Proposition 2, we can formulate the following theorem, which maintains that a unique intertemporal voting equilibrium exists if in the initial state all capital is owned by the most patient consumers.

Theorem 3. *Suppose that the initial state $\mathcal{I}_0^* = ((s_{-1}^{j*})_{j=1}^L, \ k_0^*, \ g_0^*)$ is such that*

$$k_0^* L = \sum_{j \in J} s_{-1}^{j*}, \ (i.e. \ s_{-1}^{j*} = 0, \ j \notin J).$$

Then there is a unique intertemporal voting equilibrium $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ starting from $\mathcal{I}_0^*$, which is constructed as follows: Θ^* and Λ^* are determined by (23) and

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L, \ (s_t^{j*})_{j=1}^L, \ k_{t+1}^*, \ r_{t+1}^*, \ w_{t+1}^*, \ g_{t+1}^*, \ h_t^*)_{t=0}^\infty$$

is determined by the following relationships $(t = 0, 1, \ldots)$:

$$k_{t+1}^* = \beta_1 (1 - \theta^* - \lambda^*) \alpha q(g_t^*) f(k_t^*),$$

$$s_t^{j*} = \beta_1(1-\theta^*-\lambda^*)(1+r_t^*)s_{t-1}^{j*},$$
$$c_t^{j*} = (1-\theta^*-\lambda^*)[(1-\beta_1)(1+r_t^*)s_{t-1}^{j*} + w_t^*],\ j \in J,$$
$$s_t^{j*} = 0,\ c_t^{j*} = (1-\theta^*-\lambda^*)w_t^*, j \notin J,$$
$$1 + r_{t+1}^* = q(g_{t+1}^*)f'(k_{t+1}^*),\ w_{t+1}^* = q(g_{t+1}^*)(f(k_{t+1}^*) - f'(k_{t+1}^*)k_{t+1}^*),$$
$$g_{t+1}^* = \lambda^* q(g_t^*)f(k_t^*),\ h_t^* = \theta^* q(g_t^*)f(k_t^*),$$
where $1 + r_0^* = q(g_0^*)f'(k_0^*)$ *and* $w_0^* = q(g_0^*)(f(k_0^*) - f'(k_0^*)k_0^*)$.

Corollary. *If the discount factors of all consumers are the same, i.e. if* $\beta_1 = \ldots = \beta_L$*, then for any non-degenerate initial state there exists a unique itertemporal voting equilibrium starting from this initial state.*

We can now answer the questions put forward in Subsection 3.4 of whether it is possible to generalize (2) to the case where the shares of public goods are determined by voting. To do this, let us formulate the following proposition.

Proposition 6. *If*

$$\kappa_{med} = \kappa(\beta_{med}, \delta_{med}),\ \chi_{med} = \chi(\beta_{med}, \delta_{med}), \tag{24}$$

then

$$\theta^* = \frac{\delta_{med}(1-\beta_{med})}{1+\delta_{med}},\ \lambda^* = (1-\alpha)\beta_{med}.$$

Proof is in Appendix 3.

Thus, we can summarise as follows:

1) Suppose agents differ only in their taste for the public consumption goods, sharing the same discount factor ($\beta_1 = \ldots = \beta_L =: \beta$). Then, by Proposition 6, the voting equilibrium shares of the consumption and the production public goods are fully determined by δ_{med} and β:
$$\theta^* = \frac{\delta_{med}(1-\beta)}{1+\delta_{med}},\ \lambda^* = (1-\alpha)\beta.$$

2) Suppose agents differ only in their discount factor, having the same taste for the consumption public goods ($\delta_1 = \ldots = \delta_L =: \delta$). Then, by Proposition 6, the voting equilibrium shares of the consumption and the production public goods are fully determined by δ and β_{med}:
$$\theta^* = \frac{\delta(1-\beta_{med})}{1+\delta},\ \lambda^* = (1-\alpha)\beta_{med}.$$

3) Finally, suppose agents are heterogeneous in their discount factor and in their taste for the consumption public good. In this most general case the votes on θ_τ and λ_τ do

not boil down to searching the median $\beta_j,\ j = 1, \ldots, L$, and either $\delta_j,\ j = 1, \ldots, L$, or $\frac{\delta_j(1-\beta_j)}{1+\delta_j},\ j = 1, ..., L$, but boil down to searching the median $\kappa(\beta_j, \delta_j),\ j = 1, \ldots, L$, and $\chi(\beta_j, \delta_j),\ j = 1, \ldots, L$. It well may be that $\chi_{med} \neq \chi(\beta_j, \delta_j),\ j = 1, \ldots, L$, and $\kappa_{med} \neq \kappa(\beta_j, \delta_j),\ j = 1, \ldots, L$, and hence $\chi_{med} \neq \chi(\beta_{med}, \delta_{med})$ and $\kappa_{med} \neq \kappa(\beta_{med}, \delta_{med})$. In such a configuration, illustrated by Fig. 2[14] , where $L = 3$, $\kappa_{med} = \kappa(\beta_1, \delta_1)$ and $\chi_{med} = \chi(\beta_2, \delta_2)$, the outcomes of the two votes are not predetermined by β_{med} and either δ_{med} or the median value of $\frac{\delta_j(1-\beta_j)}{1+\delta_j},\ j = 1, ..., L$.

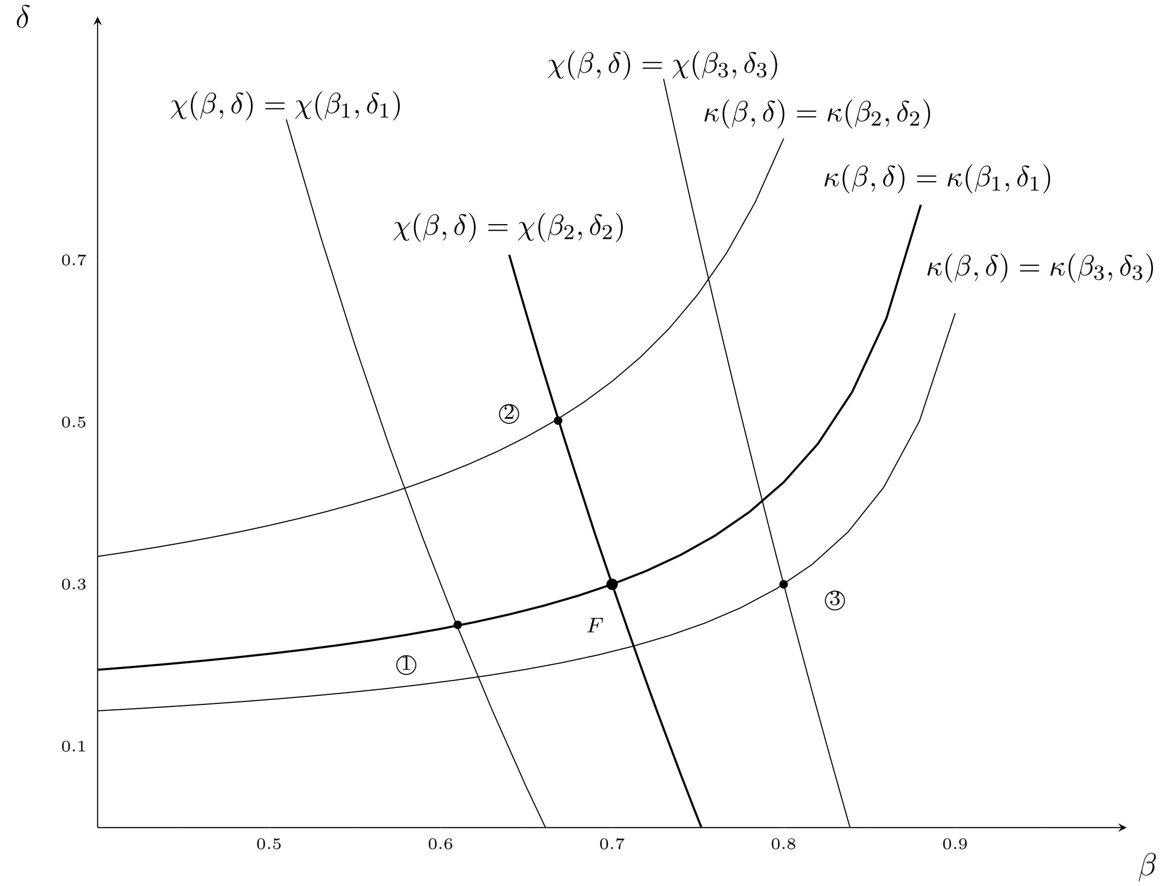

Figure 2: *Voting equilibrium and "fictitious" representative agent*

As noted above, it may be instructive to represent the outcome of voting as a choice made by a "fictitious" representative agent (labeled by F in Fig. 2). His preference parameters, β_F and δ_F are found as the solution to the following system of two equations in β and δ:

$$\kappa(\beta, \delta) = \kappa_{med},\ \chi(\beta, \delta) = \chi_{med}.$$

[14]Recall that the function $\kappa(\beta, \delta)$ is decreasing in β and increasing in δ while the function $\chi(\beta, \delta)$ is increasing in both β and δ. Therefore, the level lines of $\kappa(\beta, \delta)$ are upward sloping while the level lines of $\chi(\beta, \delta)$ are downward sloping.

It is easy to note that the outcome of voting on the two shares coincides with the first- and second-best optimum for the "fictitious" representative agent:

$$\theta^* = \theta^{F*} := \frac{\delta_F(1-\beta_F)}{1+\delta_F}, \ \lambda^* = \lambda^{F*} := (1-\alpha)\beta_F$$

(θ^{F*} and λ^{F*} are complete analogues of θ^{R*} and λ^{R*} given by (21)).

5.6 Balanced growth voting equilibria

Let us move to balanced growth voting equilibria.

Definition. *An intertemporal voting equilibrium* $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ *starting from* $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$ *is called a balanced growth voting equilibrium if*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

is a balanced growth θ^**-*λ^**-equilibrium starting from* $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$*. The equilibrium rate of growth corresponding to* $\mathcal{E}^*$ *is called the voting-equilibrium rate of balanced growth.*

The following theorem describes the structure of balanced growth voting equilibria.

Theorem 4. 1) *Let* $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ *be a balanced growth voting equilibrium starting from* $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$*,*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty,$$

and let γ^* *be the corresponding voting-equilibrium rate of balanced growth. Then*

$$1+\gamma^* = (\lambda^*)^{1-\alpha}(1-\theta^*-\lambda^*)^\alpha(\alpha\beta_1)^\alpha, \tag{25}$$

and for $t = 0, 1, \dots,$

$$\frac{k_t^*}{g_t^*} = \frac{\alpha\beta_1(1-\theta^*-\lambda^*)}{\lambda^*}, \tag{26}$$

$$k_t^* L = \sum_{j\in J} s_{t-1}^{j*} \ (i.e.\ s_{t-1}^{j*} = 0,\ j \notin J). \tag{27}$$

2) *Let* $k_0^* > 0$*,* $g_0^* > 0$ *and* $(s_{-1}^{j*})_{j=1}^L$ $(s_{-1}^{j*} \geq 0,\ j = 1, \dots, L)$ *be such that*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*} \ (i.e.\ s_{\tau-1}^{j*} = 0,\ j \notin J)$$

and

$$\frac{k_0^*}{g_0^*} = \frac{\alpha\beta_1(1-\theta^*-\lambda^*)}{\lambda^*}.$$

Then there is a balanced growth voting equilibrium $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ *starting from* $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$*, which is constructed as follows:* Θ^* *and* Λ^* *are determined by (23) and*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

is determined by (6)-(7) with γ^* given by (25).

Proof. It follows from Proposition 4, Theorem 2 and Theorem 3. $\square$

The following proposition maintains that if at the initial state all capital is owned by the most patient consumers, then the unique intertemporal voting equilibrium settles on a balanced growth voting equilibrium at the first time period.

Proposition 7. *Suppose that the initial state $\mathcal{I}_0^* = ((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*)$ is such that $k_0^* L = \sum_{j\in J} s_{-1}^{j*}$ (i.e. $s_{-1}^{j*} = 0,\ j \notin J$). Then the unique intertemporal voting equilibrium $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ starting from $\mathcal{I}_0^*$ is such that*

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

satisfies (6)-(7) and (26)-(27) for $t = 1, 2, \ldots,$ with γ^* given by (25).

Proof. It follows from Theorem 2 and Proposition 5. $\square$

5.7 Comparative statics

In this section we study the dependence of the shares of productive and consumptive publics goods in GDP, the total tax rate and the rate of balanced growth on the agents' preference parameters. It should be noted that comparative statics for the voting equilibrium is not a mere generalization of comparative statics for the second-best optimal policy.

5.7.1 Comparative statics of the shares of public goods in GDP

Prior to making comparative statics note that the share of public consumption goods in voting equilibrium, θ^*, is increasing in the median conditional propensity to spend on public consumption goods, κ_{med}, and decreasing in the median conditional propensity to spend on productive public goods, χ_{med}, while the share of productive public goods in voting equilibrium, λ^*, is decreasing in κ_{med} and increasing in χ_{med}. As for the total tax rate $\theta^* + \lambda^*$, it is increasing in both κ_{med} and χ_{med}.

Recall also that the conditional propensity to spend on public consumption goods, $\kappa(\beta, \delta)$, is decreasing in β and increasing in δ while the conditional propensity to spend on productive public goods, $\chi(\beta, \delta)$, is increasing in both β and δ. That the conditional propensity to spend on *productive* public goods, $\chi(\beta, \delta)$, is increasing in δ, which is used to weight public *consumption* relative to private consumption, deserves special mention. To explain this, suppose that the share of public consumption good in GDP is given and ask an agent maximizing at time τ her intertemporal utility $\sum_{t=\tau}^\infty \beta^{t-\tau}(\ln c_t + \delta \ln h_t)$ to choose the share of productive public goods in GDP. This agent can determine her today's and future individual consumption. Also her choice of the share of productive public goods in GDP can impact future levels of public consumption. At the same time her choice has no effect on today's level of public consumption, which is predetermined. Therefore her objective function can be written as $\ln c_\tau + \sum_{t=\tau+1}^{\infty} \beta^{t-\tau}(\ln c_t + \delta \ln h_t)$. We can see that an increase in her preferences for public consumption will increase the overall weight of the future in her objective function and therefore her conditional propensity to spend on productive public goods, $\chi(\beta, \delta)$, will go up.

Let us start the discussion by considering again the case of the *representative agent* R. From (19)-(20) and (21) we obtain the following:

A. Suppose that the discount factor of the representative agent, β_R, increases. Then

1) *the share of productive public goods in GDP increases (λ^* goes up);*

2) *the share of public consumption goods in GDP decreases (θ^* goes down);*

3) *The total tax rate $\theta^* + \lambda^*$ increases if δ_R is sufficiently small ($\delta_R < (1-\alpha)/\alpha$) and decreases if δ_R is sufficiently large ($\delta_R < (1-\alpha)/\alpha$).*

B. Suppose that the weight the representative gives to public consumption relative to private consumption, δ_R, increases. Then

1) *the share of productive public goods in GDP stays the same (λ^* does not change);*

2) *the share of public consumption goods in GDP increases (θ^* goes up);*

3) *The total tax rate $\theta^* + \lambda^*$ increases.*

Let us turn to the general case with many agents and assume for simplicity that they are all different in their preference parameters:

$$\beta_i \neq \beta_j \text{ and } \delta_i \neq \delta_j \text{ for } i \neq j.$$

If there is i such that $\kappa_{med} = \kappa(\beta_i, \delta_i)$ and $\chi_{med} = \chi(\beta_i, \delta_i)$[15], then, clearly, agent i becomes the "fictitious" representative agent and for $j \neq i$, a small variation of β_j or δ_j will lead to no variation in θ^* and λ^*. As for the dependence of θ^*, λ^* and $\theta^* + \lambda^*$ on $\beta_i (= \beta_F)$ or $\delta_i (= \delta_F)$, it is just as in the case of a representative agent.

Now consider the generic case where $\kappa_{med} = \kappa(\beta_i, \delta_i)$ and $\chi_{med} = \chi(\beta_j, \delta_j)$ hold for $i \neq j$[16] (agent i determines the median conditional propensity to spend on public consumption goods, κ_{med}, and agent j determines the median conditional propensity to spend on productive public goods, χ_{med}). Clearly, for $k \neq i, j$, a small variation of β_k or δ_k will lead to no variation in θ^* and λ^*.

[15]This is a non-generic case.

[16]In Figure 2, $i = 1$, $j = 2$.

A small increase in β_i will not change χ_{med}, but will decrease κ_{med}[17], while an increase in β_j will not change κ_{med}, but will increase χ_{med}[18]. In both cases, λ^* will increase and θ^* will decrease because λ^* is increasing in χ_{med} and decreasing in κ_{med} while θ^* is increasing in κ_{med} and decreasing in χ_{med}. As for the total tax rate $\theta^* + \lambda^*$, since it is increasing in both κ_{med} and χ_{med}, an increase in β_i leads to its decrease and an increase in β_j to its increase.

In terms of the "fictitious" representative agent, a higher patience of either agent i or agent j implies a higher patience of the "fictitious" representative agent.

Thus, using a continuity argument, we infer that *if all agents become more patient, then*

1. *the "fictitious" representative agent becomes more patient (β_F goes up) and either more or less inclined to consume public consumption goods (δ_F goes either up or down);*

2. *the share of productive public goods in GDP, λ^*, increases;*

3. *the share of public consumption goods in GDP, θ^*, decreases;*

4. *and the total tax rate $\theta^* + \lambda^*$ either increases or decreases.*

Let us pass to the dependence of the shares of public goods in GDP and the total tax rate on the preferences for public consumption of the critical voters, i and j. Figures 3 and 4 illustrate the effect of a rise in the preferences for public consumption of the critical voters. In these figures, where we label post-change values of relevant variables and the position of the "fictitious" representative agent by prime, they are respectively $i = 1$ and $j = 2$.

First consider a small increase in the preferences for public consumption of agent i, who determines the median conditional propensity to spend on public consumption goods, κ_{med}. This leads to an increase in κ_{med}. Since $\kappa(\beta, \delta)$ is increasing in δ and decreasing in β, the level line $\kappa(\beta, \delta) = \kappa_{med}$ will move to the left in Figure 3 (its after-change position is represented by the dashed curve). Therefore, the "fictitious" representative agent will become less patient (β_F will go down) and more inclined to consume public consumption goods (δ_F will go up).This will lead to a lower share of productive public goods in GDP ($\lambda^{*\prime} < \lambda^*$) and a higher share of public consumption goods in GDP ($\theta^{*\prime} > \theta^*$), which is not surprising. As for the total tax rate $\theta^* + \lambda^*$, it will go up, because it is increasing in κ_{med}.

[17] In Figure 2, the level line $\kappa(\beta, \delta) = \kappa(\beta_1, \delta_1)$ will move to the right.

[18] In Figure 2, the level line $\chi(\beta, \delta) = \chi(\beta_2, \delta_2)$ will move to the right.

Now consider a small increase in the preferences for public consumption of agent j, who determines the median conditional propensity to spend on productive public goods, χ_{med}. This leads to an increase in χ_{med}. Since $\chi(\beta, \delta)$ is increasing both in δ and β, the level line $\chi(\beta, \delta) = \chi_{med}$ will move to the right in Figure 4 (its after-change position is represented by the dashed curve). The "fictitious" representative agent will become more patient (β_F will go up) and hence the share of productive public goods in GDP will increase ($\lambda^{*\prime} > \lambda^*$). As for the relative preference of the "fictitious" representative agent for public consumption, it will also increase. However, the share of public consumption goods in GDP will decrease ($\theta^{*\prime} < \theta^*$), because this share is decreasing in χ_{med}. The total tax rate $\theta^* + \lambda^*$ it will go up because it is increasing in χ_{med}.

Using a continuity argument, it follows that *if the relative preferences of all agents for public consumption increase by small amounts then*

1. *the relative preference of the "fictitious" representative agent for public consumption unambiguously increases (δ_F goes up);*

2. *the discount factor of the "fictitious" representative agent either decreases or increases (β_F goes either up or down);*

3. *the share of productive public goods in GDP either increases or decreases (λ^* goes either up or down);*

4. *the share of public consumption goods in GDP either increases or decreases (θ^* goes either up or down);*

5. *the total tax rate $\theta^* + \lambda^*$ unambiguously increases.*

In particular, it is possible that *an increase in preferences for public consumption can lead to a lower share of public consumption goods in GDP and a higher share of productive public goods in GDP.* This contrasts with the intuition gained from the representative-agent version of our model.

In the representative version of our model a rise in δ_R does not change β_R while in the many voting-agents case a rise in all deltas leads to a rise of δ_F and a change in β_F (rise or fall). This is why the pattern of outcomes is so diverse in the many voting-agents case. The following subsection performs similar comparative statics on the rate of balanced growth.

5.7.2 Comparative statics of the rate of balanced growth

To describe the dependence of the voting-equilibrium rate of balanced growth, γ^*, on the the discount factors and the weights the agents give to public consumption relative

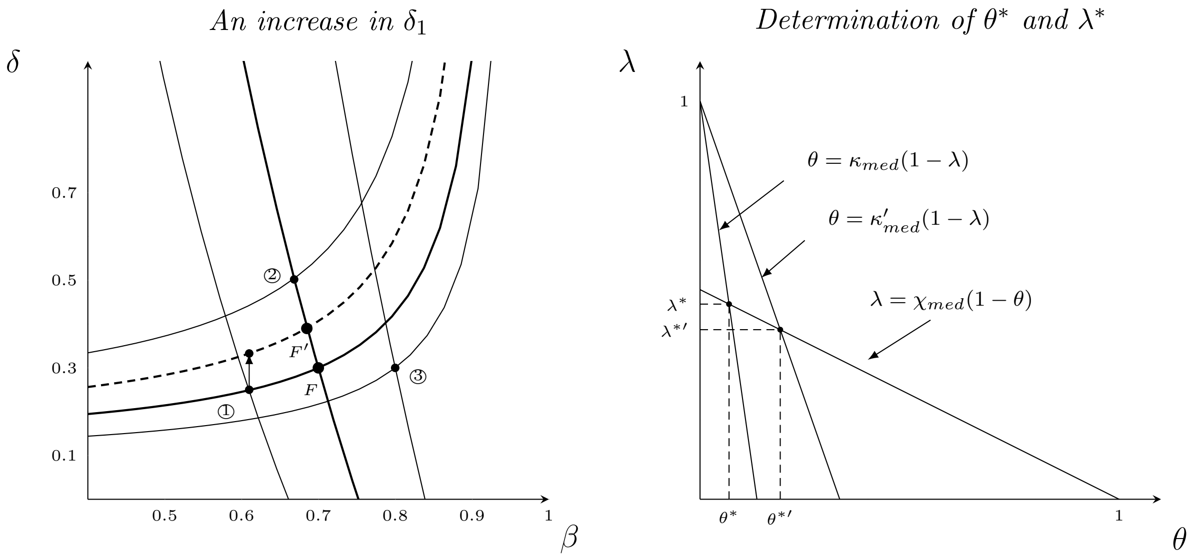

Figure 3: *Effects of an increase in δ_1 on θ^* and λ^**

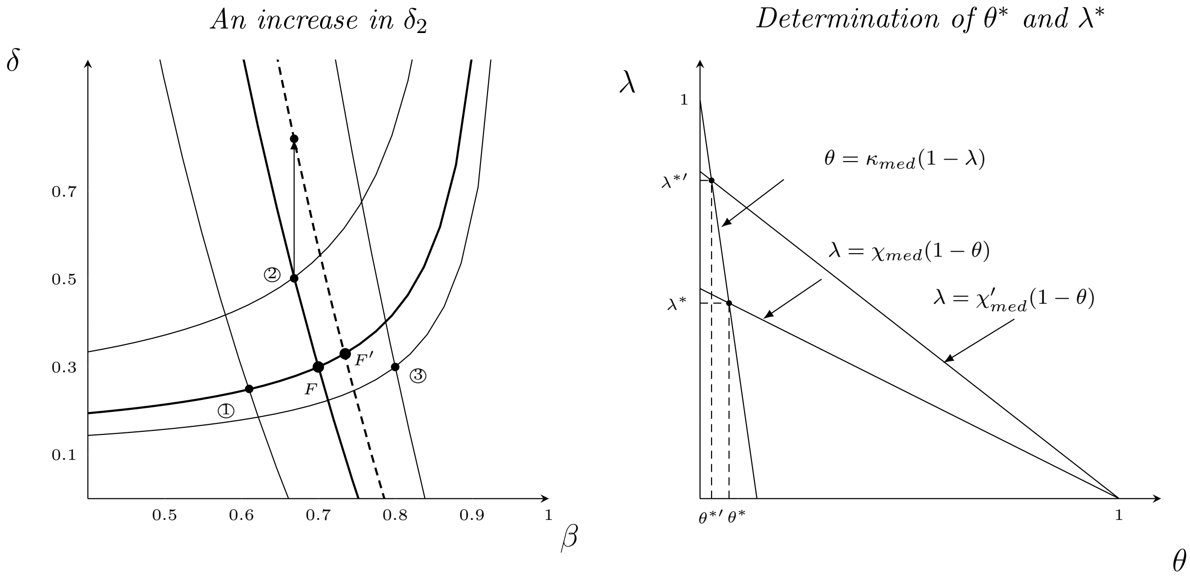

Figure 4: *Effects of an increase in δ_2 on θ^* and λ^**

to private consumption, note that, by (25), γ^* is decreasing in θ^* for $0 < \theta^* < 1 - \lambda^*$ and increasing in λ^* for $0 < \lambda^* < (1-\alpha)(1-\theta^*)$. Therefore, γ^* is decreasing in κ_{med} for $0 \leq \kappa_{med} < 1$ and increasing in χ_{med} for $0 < \chi_{med} < 1-\alpha$. At the same time, $0 \leq \kappa_{med} < 1$ and $0 < \chi_{med} < 1 - \alpha$ for any $\beta_j \in (0,1),\ j = 1,\ldots,L$, and any $\delta_j \geq 0,\ j = 1,\ldots,L$.

It follows that *in the case of a representative agent the rate of equilibrium balanced growth, γ^*, is increasing in the discount factor of the representative agent, β_R, and decreasing in the weight the representative agent gives to public consumption, δ_R.* This result is not surprising.

In the case of many agent, the picture is somewhat ambiguous. On the one hand, we know that if all agents become more patient, λ^* increases and θ^* decreases. Therefore, *if all agents become more patient, then the voting equilibrium rate of balanced growth increases.*

On the other hand, as we noted above, increases in all $\delta_j,\ j = 1, \ldots, L$, can make the "fictitious" representative agent either more or less patient and can lead to an outcome where θ^* goes down and λ^* goes up or to an outcome where θ^* goes up and λ^* goes down. Since γ^* is increasing in λ^* and decreasing in θ^*, it follows that *the dependence of the voting-equilibrium rate of balanced growth on the weights given to public consumption relative to private consumption is ambiguous.* In particular, it is possible that *increases in preferences for public consumption can lead to a higher voting-equilibrium rate of balanced growth*, while in the case of a representative agent an increase in the relative preference for public consumption unambiguously lead to a lower rate of growth.

At the same time, *if all $\delta_j,\ j = 1, \ldots, L$, increase and the equalities in (24) hold both before and after the changes (i.e. in the case where β_{med} and δ_{med} determine the equilibrium), the voting equilibrium rate of balanced growth, γ^*, will decrease.* This follows from Proposition 6 and from the fact that γ^* is decreasing in θ^*. Thus, if all agents have the same discount factor or they give the same weight to public consumption relative to private consumption, then increasing in the weight(s) they give to public consumption relative to private consumption will unambiguously lead to a decrease in the rate of growth.

6 Generalized intertemporal voting equilibria

We proved the existence of intertemporal voting equilibria only in the case where initially all capital is owned by the most patient consumers. A natural question arises: do intertemporal voting equilibria exist if less patient consumers also own some fraction of capital in the initial state? To give a positive answer to this question, it is necessary to prove the uniqueness of a competitive Θ-Λ-equilibrium starting from arbitrary given initial states, which is a difficult task. However, we can go around this difficulty. To do this, we need to modify the notion of intertemporal voting equilibrium. Lemma 1 will help us.

Let policy sequences $\Theta^* = (\theta_t^*)_{t=0}^{\infty}$ and $\Lambda^* = (\lambda_t^*)_{t=0}^{\infty}$ be given and let

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^{L},\ (s_t^{j*})_{j=1}^{L},\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^{\infty}$$

be a competitive Θ^*-Λ^*-equilibrium starting from a non-degenerate initial state

$((s_{-1}^{j*})_{j=1}^{L},\ k_0^*,\ g_0^*)$ and hence, for each τ,

$$\mathcal{E}_\tau^* = ((c_t^{j*})_{j=1}^{L},\ (s_t^{j*})_{j=1}^{L},\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=\tau}^{\infty}$$

is a Θ_τ^*-Λ_τ^*-equilibrium starting from $\mathcal{I}_\tau^* = ((s_{\tau-1}^{j*})_{j=1}^{L},\ k_\tau^*,\ g_\tau^*)$, where $\Theta_\tau^* = (\theta_t^*)_{t=\tau}^{\infty}$ and $\Lambda_\tau^* = (\lambda_t^*)_{t=\tau}^{\infty}$.

Suppose that the economy has settled on the competitive Θ^*-Λ^*-equilibrium $\mathcal{E}^*$ and at each time τ, when the economy is at the state $\mathcal{I}_\tau^* = ((s_{\tau-1}^{j*})_{j=1}^{L},\ k_\tau^*,\ g_\tau^*)$, the agents are asked to vote if they want to change the time τ policy given by the couple $(\theta_\tau^*, \lambda_\tau^*)$. To answer this question, the agents need to have indirect utility functions.

If for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences Θ_τ and Λ_τ given by

$$\Theta_\tau = (\theta_\tau, \theta_{\tau+1}^*, \theta_{\tau+2}^*, \ldots),\ \Lambda_\tau = (\lambda_\tau, \lambda_{\tau+1}^*, \lambda_{\tau+2}^*, \ldots) \tag{28}$$

there is a unique competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}_\tau^*$, we can specify the indirect utilities of agents in this vote unambiguously as described in subsection 5.1.1. Otherwise, the choice of indirect utility functions representing the policy preferences of agents is ambiguous.

To overcome this ambiguity, assume that, *when voting at time τ, all agents are aware of Lemma 1, but ignore possible multiplicity of competitive equilibria and believe that for any θ_τ and λ_τ, there is a unique Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}_\tau^*$.*

Recall that in Lemma 1 the uniqueness of competitive equilibrium is not presupposed. This lemma says that if $\mathcal{E}_\tau^*$ is a Θ_τ^*-Λ_τ^*-equilibrium, then $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta_\tau^*, \lambda_\tau^*, \mathcal{E}_\tau^*)$[19] is *a* Θ_τ-Λ_τ-equilibrium for Θ_τ and Λ_τ given by (28). Under the uniqueness assumption $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta_\tau^*, \lambda_\tau^*, \mathcal{E}_\tau^*)$ is *the* Θ_τ-Λ_τ-equilibrium. Therefore, if the agents *believe* that for any θ_τ and λ_τ, there is a unique Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}_\tau^*$, then they are sure that replacing the policy $(\theta_\tau^*, \lambda_\tau^*)$ by another policy $(\theta_\tau, \lambda_\tau)$ will result in switching the economy to $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta_\tau^*, \lambda_\tau^*, \mathcal{E}_\tau^*)$. In this case, the indirect utility function that represents the policy preferences of agent j over θ_τ and λ_τ when voting is defined unambiguously as the utility of this agent in $\mathbf{E}(\theta_\tau, \lambda_\tau, \theta_\tau^*, \lambda_\tau^*, \mathcal{E}_\tau^*)$. It follows from Lemma 2 that the function $v_\tau^j(\theta_\tau, \lambda_\tau)$ also represents the policy preferences of agent j.

As above, we assume that the agents vote on θ_τ and λ_τ separately: when voting on θ_τ, agent j takes λ_τ as given and seeks to maximize $v_\tau^j(\theta_\tau, \lambda_\tau)$ over $\theta_\tau \in (0, 1)$, and when voting on λ_τ she takes θ_τ as given and seeks to maximize $v_\tau^j(\theta_\tau, \lambda_\tau)$ over $\lambda_\tau \in (0, 1)$.

Definition. *If for each $\tau = 0, 1, \ldots,$ there are Condorcet winners in the votes on θ_τ and λ_τ described above and they coincide with θ_τ^* and λ_τ^* respectively, the triple* $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ *is called a generalized intertemporal voting equilibrium starting from $\mathcal{I}_0^*$.*

[19]Here it is assumed that the expectations about future shares of public goods in GDP, $\Theta_{\tau+1}^e = (\theta_t^e)_{t=\tau+1}^{\infty}$ and $\Lambda_{\tau+1}^e = (\lambda_t^e)_{t=\tau+1}^{\infty}$ are given for $t = \tau+1, \tau+2, ...$ by $\theta_t^e = \theta_t^*$ and $\lambda_t^e = \lambda_t^*$.

It is clear that any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium and any generalized intertemporal voting equilibrium starting from the initial state where all capital is owned by the most patient consumers is an intertemporal voting equilibrium.

The following theorem maintains that the shares of public goods in GDP in generalized voting intertemporal equilibria are the same as in intertemporal voting equilibria. At the same time, unlike intertemporal voting equilibria, the existence of generalized intertemporal voting equilibria is assured for any non-degenerate initial state.

Theorem 5. *For any generalized intertemporal voting equilibrium* $(\Theta^*, \Lambda^*, \mathcal{E}^*)$*, the sequences of the shares of public goods in GDP,* Θ^* *and* Λ^**, are constant over time and given by (23).*

Moreover, for any non-degenerate initial state there exists a generalized intertemporal voting equilibrium starting from that state.

Proof. To prove the theorem, it is sufficient to repeat the argument used in the proof of Theorem 1 and to refer to Proposition 1. □

The long-run behavior of generalized intertemporal voting equilibria is described by the following theorem. It says that any generalized intertemporal voting equilibrium at some time settles on a balanced growth voting equilibrium[20].

Theorem 6. *For any generalized intertemporal voting equilibrium* $(\Theta^*, \Lambda^*, \mathcal{E}^*)$ *there is* T *such that for* $t > T$*,* $\mathcal{E}^*$ *satisfies (6)-(7) and (26)-(27) with* γ^* *given by (25).*

Proof. It follows from Theorem 2 and Proposition 5. □

7 Conclusion

Since the publication of the seminal paper by Barro (1990), growth effects of public spending have been one of the popular topics in economic literature, which usually analyzes the size of government and composition of different types of government expenditure. In this literature the shares of government expenditure are either exogenous or chosen optimally by a social planner. These assumptions do not reflect the fundamental characteristics of collective decision-making and the distribution of preferences, and models based on these assumptions cannot explain substantial variations among countries in their tax and expenditure policies, even among developed democracies sharing similar economic and political regimes.

In this paper we try to answer the question of why tax and expenditure policies

[20]Of course, in a model with decreasing returns to scale sustained growth is impossible and the economy does not settle in a balanced growth path, but converges to a stationary equilibrium.

differ among democratic countries. In order to understand how fundamentals (preferences and technologies) together with political institutions determine the tax and expenditure policies, this paper examines the determination of the shares of productive public goods and utility-enhancing public consumption goods in GDP under majority voting in the context of a parametric model with infinitely-lived agents that are heterogeneous in their discount factors and preferences for public consumption goods.

In our model, the shares of public goods in GDP are obtained as an outcome of a dynamic voting process. We have introduced the notions of temporary and (generalized) intertemporal voting equilibrium, proved their existence and shown that any (generalized) intertemporal voting equilibrium at some time settles on a balanced growth voting equilibrium.

Due to log preferences and Cobb-Douglas technology with complete depreciation of capital we have obtained a closed-form solution for the voting outcomes. These outcomes are constant over time, they are fully determined by the the technology and preference parameters: the elasticity of output with respect to physical capital, the discount factors and the parameters that measure how much the agents value public consumption. It is noteworthy that it is not necessarily the median values of the discount factors and the weights given to public consumption relative to private consumption that determine the voting outcome. At the same time it is possible introduce a "fictitious" representative agent and to interpret the outcome of voting as the second-best optimum of this "fictitious" representative agent.

We have also undertaken some comparative static analysis of the shares of public goods in GDP and of the long-run rate of growth. This analysis shows that if *all* agents become more patient, then the share of productive public goods will increase, the share of public consumption goods will decrease, and the long-run rate of growth will increase. As for the dependence of the two shares and the long-run rate of growth on preferences for public goods, it is ambiguous: if *all* agents increase the weights they give to public consumption relative to private consumption, then the shares of productive and utility-enhancing public goods in GDP and the long-run rate of growth can either increase or decrease, while in the case of a representative agent an increase in the relative preference for public consumption unambiguously leads to a higher share of public consumption goods in GDP and a lower rate of growth. Our comparative static analysis suggests that a representative-agent model is a reasonable approximation of an economy with many voting heterogeneous agents only if the agents' heterogeneity is one-dimensional.

Much of the recent literature on optimal public expenditures in growth models assumes specific functional forms and computes optimal public policies [21]. We also assumes specific

[21] See, however, Glomm and Ravikumar (1999).

functional forms, which is not entirely satisfactory. Existence and characterization of voting equilibria for reasonably general preferences and technologies could be a topic of future research.

A Appendix 1. Proofs of Proposition 2, and Proposition 3 and Proposition 4

With no loss of generality we prove Proposition 2 and Proposition 3 for $\tau = 0$. Suppose we are given sequences $\Theta = (\theta_t)_{t=0}^{\infty}$ and $\Lambda = (\lambda_t)_{t=0}^{\infty}$ satisfying (3) and denote

$$\psi_t := 1 - \theta_t - \lambda_t, \ t = 0, 1, \ldots .$$

It follows from (3) that

$$\liminf_{t\to\infty} \psi_t > 0. \tag{29}$$

Consider a competitive Θ-Λ-equilibrium

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L, \ (s_t^{j*})_{j=1}^L, \ k_{t+1}^*, \ r_{t+1}^*, \ w_{t+1}^*, \ g_{t+1}^*, \ h_t^*)_{t=0}^{\infty}$$

starting from $((s_{-1}^{j*})_{j=1}^L, \ k_0^*, \ g_0^*\}$.

For each $j = 1, \ldots, L$, the sequence $(c_t^{j*}, s_t^{j*})_{t=0,1,2,\ldots}$ is a solution to

$$\max \ \textstyle\sum_{t=0}^{\infty} \beta_j^t c_t^j,$$
$$c_t^j + s_t^j = \psi_t[(1 + r_t^*)s_{t-1}^j + w_t^*], \ t = 0, 1, 2, ...,$$
$$s_t^j \geq 0, \ t = 0, 1, 2, ..., \ (\text{where } s_{-1}^j = s_{-1}^{j*}).$$

Therefore, it satisfies the following first-order conditions:

$$\beta_j \psi_{t+1}(1 + r_{t+1}^*)c_t^{j*} \leq c_{t+1}^{j*} \ (= \ \text{if } s_t^{j*} > 0), \ t = 0, 1, 2, ...$$

Lemma A1. *Let $\beta > 0$ be such that for some t',*

$$k_{t+1}^* > \beta\psi_t(1 + r_t^*)k_t^* = \beta\psi_t\alpha q(g_t^*)f(k_t^*), \ t > t'.$$

If $\beta_j < \beta$, then $s_t^{j*} = 0$ for all sufficiently large t.

Proof. Let us take j such that $\beta_j < \beta$ and show that $s_t^{j*} = 0$ for some $t \geq t'$. To do this, assume the converse. Then, by the first-order conditions, for all $t \geq t'$,

$$\beta_j\psi_t(1 + r_t^*)c_{t-1}^{j*} = c_t^{j*}$$

and hence

$$\frac{c_t^{j*}}{k_{t+1}^*} \leq \frac{\beta_j\psi_t(1 + r_t^*)c_{t-1}^{j*}}{\beta\psi_t(1 + r_t^*)k_t^*} \leq \frac{\beta_j}{\beta}\frac{c_{t-1}^{j*}}{k_t^*}.$$

Since $\beta_j/\beta < 1$, $c_t^{j*}/k_{t+1}^* \to 0$ as $t \to \infty$. Taking account of the evident inequality $k_{t+1}^* \leq \psi_t q(g_t^*) f(k_t^*)$, $t = 0, 1, \ldots$, and (29), we get

$$\frac{c_t^{j*}}{\psi_t w_t^*} = \frac{c_t^{j*}}{\psi_t(1-\alpha)q(g_t^*)f(k_t^*)} \leq \frac{c_t^{j*}}{\psi_t(1-\alpha)k_{t+1}^*} \xrightarrow[t\to\infty]{} 0.$$

This means that $c_t^{j*} < \psi_t w_t^*$ for all sufficiently large t, which is clearly not optimal for consumer j.

Let us now show that if $s_{t_1}^{j*} = 0$ for $t_1 > t'$, then $s_t^{j*} = 0$ for all $t > t_1$. Assume the converse. Then there are $t_2 \geq t_1$ and $t_3 > t_2 + 1$ such that

$$s_{t_2}^{j*} = 0,\ s_{t_3}^{j*} = 0,\ s_t^{j*} > 0,\ t_2 < t < t_3.$$

Therefore,

$$\begin{aligned} c_{t_2+1}^{j*} &< \psi_{t_2+1} w_{t_2+1}^*, \\ c_{t_3}^{j*} &> \psi_{t_3} w_{t_3}^*, \end{aligned} \tag{30}$$

which is impossible. Indeed, for $t > t'$, we have

$$\frac{\alpha}{1-\alpha}\frac{w_{t+1}^*}{1+r_{t+1}^*} = k_{t+1}^* > \beta\psi_t(1+r_t^*)k_t^* = \frac{\alpha}{1-\alpha}\beta\psi_t w_t^*$$

and hence $\beta(1+r_{t+1}^*)\psi_t w_t^* < w_{t+1}^*$. Therefore, taking account of the first-order conditions, we get

$$\begin{aligned} c_{t_2+2}^{j*} = \beta_j\psi_{t_2+2}(1+r_{t_2+2}^*)c_{t_2+1}^{j*} &< \beta_j\psi_{t_2+2}(1+r_{t_2+2}^*)\psi_{t_2+1}w_{t_2+1}^* \\ &\leq \beta_1\psi_{t_2+2}(1+r_{t_2+2}^*)\psi_{t_2+1}w_{t_2+1}^* < \psi_{t_2+2}w_{t_2+2}^* \end{aligned}$$

and, repeating the argument,

$$c_{t+1}^{j*} < \psi_{t+1}w_{t+1}^*,\ t_2 < t < t_3.$$

This implies $c_{t_3}^{j*} < \psi_{t_3}w_{t_3}^*$, which contradicts (30). $\Box$

Lemma A2. $k_{t+1}^* \leq \beta_1\psi_t(1+r_t^*)k_t^* = \beta_1\psi_t\alpha q(g_t^*)f(k_t^*)$, $t = 0, 1, \ldots$.

Proof. Assume the converse. Then there are t' and $\zeta > 1$ such that

$$k_{t'+1}^* \geq \zeta\beta_1\psi_{t'}(1+r_{t'}^*)k_{t'}^* = \zeta\beta_1\psi_{t'}\alpha q(g_{t'}^*)f(k_{t'}^*).$$

Let us show that for $t > t'$,

$$k_{t+1}^* \geq \zeta\beta_1\psi_t\alpha q(g_t^*)f(k_t^*). \tag{31}$$

Denote

$$J(t') := \{j \in J \mid s_{t'}^{j*} > 0\}.$$

Since

$$\frac{w_t^*}{(1+r_t^*)k_t^*} = \frac{1-\alpha}{\alpha},\ t = 0,1,\dots,$$

we have

$$\sum_{j\in J(t')} (s_{t'+1}^{j*} + c_{t'+1}^{j*}) = \sum_{j\in J(t')} [\psi_{t'+1}(1+r_{t'+1}^*)s_{t'}^{j*} + \psi_{t'+1}w_{t'+1}^*] =$$

$$\psi_{t'+1}(1+r_{t'+1}^*)k_{t'+1}^*L + |J(t')|\psi_{t'+1}w_{t'+1}^*$$

$$= (L + \frac{1-\alpha}{\alpha}|J(t')|)\psi_{t'+1}(1+r_{t'+1}^*)k_{t'+1}^*$$

$$\geq (L + \frac{1-\alpha}{\alpha}|J(t')|)\psi_{t'+1}(1+r_{t'+1}^*)\zeta\beta_1\psi_{t'}(1+r_{t'}^*)k_{t'}^*$$

$$= \zeta\psi_{t'+1}\beta_1(1+r_{t'+1}^*)[\psi_{t'}(1+r_{t'}^*)k_{t'}^*L + \frac{1-\alpha}{\alpha}|J(t')|\psi_{t'}(1+r_{t'}^*)k_{t'}^*]$$

$$= \zeta\psi_{t'+1}\beta_1(1+r_{t'+1}^*)[\psi_{t'}(1+r_{t'}^*)\sum_{j=1}^{L} s_{t'-1}^{j*} + |J(t')|\psi_{t'}w_{t'}^*]$$

$$\geq \zeta\psi_{t'+1}\beta_1(1+r_{t'+1}^*)\sum_{j\in J(t')} [\psi_{t'}(1+r_{t'}^*)s_{t'-1}^{j*} + \psi_{t'}w_{t'}^*]$$

$$= \zeta\psi_{t'+1}\beta_1(1+r_{t'+1}^*)\sum_{j\in J(t')} (s_{t'}^{j*} + c_{t'}^{j*}).$$

At the same time, by the first-order conditions,

$$c_{t'+1}^{j*} = \beta_j\psi_{t'+1}(1+r_{t'+1}^*)c_{t'}^{j*},\ j \in J(t'),$$

and hence

$$\sum_{j\in J(t')} c_{t'+1}^{j*} = \sum_{j\in J(t')} \beta_j\psi_{t'+1}(1+r_{t'+1}^*)c_{t'}^{j*}$$

$$\leq \sum_{j\in J(t')} \beta_1\psi_{t'+1}(1+r_{t'+1}^*)c_{t'}^{j*} < \zeta\beta_1\psi_{t'+1}(1+r_{t'+1}^*)\sum_{j\in J(t')} c_{t'}^{j*}.$$

It follows that

$$k_{t'+2}^*L = \sum_{j=1}^{L} s_{t'+1}^{j*} \geq \sum_{j\in J(t')} s_{t'+1}^{j*} \geq \zeta\beta_1\psi_{t'+1}(1+r_{t'+1}^*)\sum_{j\in J(t')} s_{t'}^{j*}$$

$$= \zeta\beta_1\psi_{t'+1}(1+r_{t'+1}^*)k_{t'+1}^*L = \zeta\beta_1\psi_{t'+1}\alpha q(g_{t'+1}^*)f(k_{t'+1}^*)L.$$

Therefore, $k_{t'+2}^* \geq \zeta\beta_1\psi_{t'+1}\alpha q(g_{t'+1}^*)f(k_{t'+1}^*)$.

Repeating the argument we infer that (31) holds for all $t > t'$.

By Lemma A1, $s_t^{j*} = 0$ for each j and for all sufficiently large t, which contradicts the evident positivity of k_t^* for all $t = 0,1,\dots$. This contradiction proves Lemma A2. $\square$

Lemma A3. $\frac{w^*_{t+1}}{1+r^*_{t+1}} \le \beta_1 \psi_t w^*_t,\ t = 0, 1, \ldots.$

Proof. By Lemma A2, for all $t = 0, 1, \ldots,$

$$\frac{w^*_{t+1}}{1+r^*_{t+1}} = \frac{(1-\alpha)q(g^*_{t+1})f(k^*_{t+1})}{1+r^*_{t+1}} = \frac{(1-\alpha)(1+r^*_{t+1})k^*_{t+1}}{\alpha(1+r^*_{t+1})}$$
$$\le \frac{(1-\alpha)\beta_1\psi_t(1+r^*_t)k^*_t}{\alpha} = \beta_1\psi_t(1-\alpha)q(g^*_t)f(k^*_t) = \beta_1\psi_t w^*_t. \ \square$$

Lemma A4. $s^{j*}_{t+1} \ge \beta_1\psi_{t+1}(1+r^*_{t+1})s^{j*}_t,\ j \in J,\ t = -1, 0, 1, \ldots.$

Proof. Let $j \in J$. By the first-order conditions,

$$\beta_1^t c^{j*}_0 \le \frac{c^{j*}_t}{\psi_1(1+r^*_1)\ldots\psi_t(1+r^*_t)},\ t = 1, 2, \ldots,$$

and hence

$$c^{j*}_0(1+\beta_1+\beta_1^2+\ldots) \le c^{j*}_0 + \frac{c^{j*}_1}{\psi_1(1+r^*_1)} + \frac{c^{j*}_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)} + \ldots. \tag{32}$$

Also we have

$$c^{j*}_0 + s^{j*}_0 = \psi_0(1+r^*_0)s^{j*}_{-1} + \psi_0 w^*_0,$$
$$\frac{c^{j*}_1 + s^{j*}_1}{\psi_1(1+r^*_1)} = s^{j*}_0 + \frac{\psi_1 w^*_1}{\psi_1(1+r^*_1)},$$
$$\frac{c^{j*}_2 + s^{j*}_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)} = \frac{s^{j*}_1}{\psi_1(1+r^*_1)} + \frac{\psi_2 w^*_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)},$$
$$\ldots.$$

Summing these equalities over t, we find that

$$c^{j*}_0 + \frac{c^{j*}_1}{\psi_1(1+r^*_1)} + \frac{c^{j*}_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)} + \ldots$$
$$= \psi_0(1+r^*_0)s^{j*}_{-1} + \psi_0 w^*_0 + \frac{\psi_1 w^*_1}{\psi_1(1+r^*_1)} + \frac{\psi_2 w^*_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)} + \ldots. \tag{33}$$

Moreover, taking account of Lemma A3, we obtain

$$\frac{\psi_{t+1} w^*_{t+1}}{\psi_1(1+r^*_1)\ldots\psi_{t+1}(1+r^*_{t+1})} \le \frac{\beta_1\psi_t w^*_t}{\psi_1(1+r^*_1)\ldots\psi_t(1+r^*_t)} \le \ldots \le \beta_1^{t+1}\psi_0 w_0,$$

which implies

$$\psi_0(1+r^*_0)s^{j*}_{-1} + \psi_0 w^*_0 + \frac{\psi_1 w^*_1}{\psi_1(1+r^*_1)} + \frac{\psi_2 w^*_2}{\psi_1(1+r^*_1)\psi_2(1+r^*_2)} + \ldots$$
$$\le \psi_0(1+r^*_0)s^{j*}_{-1} + \psi_0 w^*_0(1+\beta_1+\beta_1^2+\ldots). \tag{34}$$

Combining (32), (33) and (34), we get

$$c^{j*}_0(1+\beta_1+\beta_1^2+\ldots) \le \psi_0(1+r^*_0)s^{j*}_{-1} + \psi_0 w^*_0(1+\beta_1+\beta_1^2+\ldots)$$

and therefore

$$c_0^{j*} \le \psi_0(1+r_0^*)(1-\beta_1)s_{-1}^{j*} + \psi_0 w_0^*.$$

Thus,

$$\begin{aligned} s_0^{j*} &= \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* - c_0^{j*} \\ &\ge \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* - \psi_0(1+r_0^*)(1-\beta_1)s_{-1}^{j*} - \psi_0 w_0^* \;= \beta_1\psi_0(1+r_0^*)s_{-1}^{j*}. \end{aligned}$$

This proves the inequality $s_{t+1}^{j*} \ge \beta_1\psi_{t+1}(1+r_{t+1}^*)s_t^{j*}$ for $t=-1$. To prove it for $t = 0,1,\dots,$ it is sufficient to repeat the argument. □

Lemma A5. *If*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*} \ (i.e.\ s_{-1}^{j*} = 0,\ j \notin J),$$

then for all $t = 0,1,\dots,$

$$k_{t+1}^* = \beta_1\psi_t\alpha q(g_t^*)f(k_t^*), \tag{35}$$

$$s_t^{j*} = \beta_1\psi_t(1+r_t^*)s_{t-1}^{j*},\ c_t^{j*} = (1-\beta_1)\psi_t(1+r_t^*)s_{t-1}^{j*} + \psi_t w_t^*,\ j\in J, \tag{36}$$

$$s_t^{j*} = 0,\ c_t^{j*} = \psi_t w_t^*,\ j\notin J. \tag{37}$$

Proof. By Lemma A4, $\beta_1\psi_0(1+r_0^*)k_0^*L = \beta_1\psi_0(1+r_0^*)\sum_{j\in J} s_{-1}^{j*} \le \sum_{j\in J} s_0^{j*} \le k_1^*L$. At the same time, by Lemma A2, $k_1^* \le \beta_1\psi_0(1+r_0^*)k_0^*$. Therefore, $k_1^* = \beta_1\psi_0(1+r_0^*)k_0^*$ and hence $s_0^{j*} = \beta_1\psi_0(1+r_0^*)s_{-1}^{j*},\ j\in J$, and $s_0^{j*}=0,\ j\notin J$. We have proved (35)-(37) for $t=0$. To prove (35)-(37) for $t=1,2,\dots,$ it is sufficient to repeat the argument. □

Proof of Proposition 2. It follows from Lemma A5. □

Proof of Proposition 3. It follow from the first-order conditions and Lemma A3 that for $j\in J$,

$$\frac{c_{t+1}^{j*}}{w_{t+1}^*} \ge \frac{c_t^{j*}}{w_t^*},\ t = 0,1,\dots,$$

and hence the sequence $(c_t^{j*}/w_t^*)_{t=0}^\infty$ is non-decreasing. This sequence is bounded, because

$$c_t^{j*} \le L\frac{w_t^*}{1-\alpha},\ t=0,1,\dots.$$

Hence, it converges. It follows that the sequence $\left(\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}}\right)_{t=0}^\infty$ converges to 1. Since

$$\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}} = \frac{w_{t+1}^*}{\beta_1\psi_{t+1}(1+r_{t+1}^*)w_t^*},\ t=0,1,\dots,$$

the sequence $\left(\frac{w_{t+1}^*}{\beta_1\psi_{t+1}(1+r_{t+1}^*)w_t^*}\right)_{t=0}^\infty$ converges to 1 as well. It follows that if $\beta < \beta_1$, then

$$k_{t+1}^* = \frac{\alpha}{1-\alpha}\frac{w_{t+1}^*}{1+r_{t+1}^*} > \frac{\alpha}{1-\alpha}\beta\psi_t w_t^* = \beta\psi_t(1+r_t^*)k_t^*,$$

for all sufficiently large t. To complete the proof, it is sufficient to take $\beta < \beta_1$ such that $\beta > \max_{j \notin J} \beta_j$ and to refer to Lemma A1. $\square$

Proof of Proposition 4. 1) Let

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

be a balanced growth θ-λ-equilibrium starting from $\mathcal{I}_0^*$ and γ^* be the corresponding equilibrium rate of balanced growth. Repeating a well-known argument (Becker (1980, 2006)), we find that

$$1 + \gamma^* = \beta_1(1 - \theta - \lambda)(1 + r_0^*), \tag{38}$$

and that $\mathcal{E}^*$ satisfies (10).

At the same time,

$$(1 + \gamma^*)g_t^* = \lambda(g_t^*)^{1-\alpha}(k_t^*)^\alpha,\ t = 0, 1, \ldots,$$

and hence

$$1 + \gamma^* = \lambda \left(\frac{k_t^*}{g_t^*}\right)^\alpha,\ t = 0, 1, \ldots.$$

Taking account of (7) and (38), we get

$$\lambda \left(\frac{k_t^*}{g_t^*}\right)^\alpha = \beta_1(1 - \theta - \lambda)\alpha \left(\frac{k_t^*}{g_t^*}\right)^{\alpha-1},$$

which implies (9).

2) Suppose that $k_0^* > 0$, $g_0^* > 0$ and $(s_{-1}^{j*})_{j=1}^L$ ($s_{-1}^{j*} \geq 0,\ \ j = 1, \ldots, L$) satisfy (11) and (12) and that

$$\mathcal{E}^* = ((c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*)_{t=0}^\infty$$

is determined by (6)-(7). Taking into account the above argument, it is not difficult to check that $\mathcal{E}^*$ is a competitive Θ-Λ-equilibrium starting from $((s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ with $\Theta = (\theta, \theta, \theta, \ldots)$ and $\Lambda = (\lambda, \lambda, \lambda, \ldots)$. $\square$

B Appendix 2. Proof of Lemma 1

Proof of Lemma 1. Denote

$$\nu_\tau = \frac{1 - \lambda_\tau - \theta_\tau}{1 - \lambda_\tau^\circ - \theta_\tau^\circ}$$

and write out the formulas that define the sequence

$$\mathbf{E}(\theta_\tau, \lambda_\tau, \theta_\tau^\circ, \lambda_\tau^\circ, \mathcal{E}_\tau^*) = ((\tilde{c}_t^j)_{j=1}^L,\ (\tilde{s}_t^j)_{j=1}^L,\ \tilde{k}_{t+1}, \tilde{r}_{t+1},\ \tilde{w}_{t+1},\ \tilde{g}_{t+1},\ \tilde{h}_t)_{t=\tau}^\infty :$$

$$\tilde{h}_\tau = (\theta_\tau/\theta_\tau^\circ)h_\tau^*,\ \tilde{g}_{\tau+1} = (\lambda_\tau/\lambda_\tau^\circ)g_{\tau+1}^*,$$

$$\tilde{c}^j_\tau = \nu_\tau c^{j*}_\tau,\ \tilde{s}^j_\tau = \nu_\tau s^{j*}_\tau,\ \tilde{k}_{\tau+1} = \nu_\tau k^*_{\tau+1},$$

$$\tilde{w}_{\tau+1} = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} w^*_{\tau+1},\ 1+\tilde{r}_{\tau+1} = \nu^{\alpha-1}_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1+r^*_{\tau+1}),$$

$$\tilde{h}_t = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} h^*_t,\ t = \tau+1, \tau+2, ...,$$

$$\tilde{g}_{t+1} = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} g^*_{t+1},\ t = \tau+1, \tau+2, ...,$$

$$\tilde{c}^j_t = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} c^{j*}_t,\ t = \tau+1, \tau+2, ...,\ j = 1, ..., L,$$

$$\tilde{s}^j_t = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} s^{j*}_t,\ t = \tau+1, \tau+2, ...,\ j = 1, ..., L,$$

$$\tilde{k}_{t+1} = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} k^*_{t+1},\ t = \tau+1, \tau+2, ...,$$

$$\tilde{w}_{t+1} = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} w^*_{t+1},\ t = \tau+1, \tau+1, ...,$$

$$1+\tilde{r}_{t+1} = 1 + r^*_{t+1},\ t = \tau+1, \tau+2, ...$$

Direct calculations show that

$$\tilde{k}_{t+1} L = \sum_{j=1}^{L} \tilde{s}^j_t,\ t = \tau, \tau+1, ...$$

and

$$q(\tilde{g}_{t+1}) f(\tilde{k}_{t+1}) = \nu^\alpha_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} q(g^*_{t+1}) f(k^*_{t+1}),\ t = \tau, \tau+1, \dots.$$

It follows that

$$\tilde{w}_{t+1} = (1-\alpha) q(\tilde{g}_{t+1}) f(\tilde{k}_{t+1}) = q(\tilde{g}_{t+1})(f(\tilde{k}_{t+1}) - f'(\tilde{k}_{t+1})\tilde{k}_{t+1}),\ t = \tau, \tau+1, \dots,$$

$$\tilde{g}_{t+1} = \lambda^e_t q(\tilde{g}_t) f(\tilde{k}_t) \text{ and } \tilde{h}_t = \theta^e_t q(\tilde{g}_t) f(\tilde{k}_t),\ t = \tau+1, \tau+2, \dots.$$

Also it is clear that $\tilde{g}_{\tau+1} = \lambda_\tau q(\tilde{g}_\tau) f(\tilde{k}_\tau)$ and $\tilde{h}_\tau = \theta_\tau q(\tilde{g}_\tau) f(\tilde{k}_\tau)$.

We have

$$\begin{aligned} 1+\tilde{r}_{\tau+1} &= \nu^{\alpha-1}_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1+r^*_{\tau+1}) = \nu^{\alpha-1}_\tau (\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha} q(g^*_{\tau+1}) f'(k^*_{\tau+1}) \\ &= ((\lambda_\tau/\lambda^\circ_\tau) g^*_{\tau+1})^{1-\alpha} (\nu_\tau k^*_{\tau+1})^{\alpha-1} = (\tilde{g}_{\tau+1})^{1-\alpha} (\tilde{k}_{\tau+1})^{\alpha-1} = q(\tilde{g}_{\tau+1}) f'(\tilde{k}_{\tau+1}). \end{aligned}$$

Also we have $\tilde{g}_{\tau+1}/\tilde{k}_{\tau+1} = g^*_{\tau+1}/k^*_{\tau+1},\ t = \tau+1, \tau+2, \dots$, and hence

$$1+\tilde{r}_{t+1} = q(\tilde{g}_{t+1}) f'(\tilde{k}_{t+1}),\ t = \tau+1, \tau+2, \dots.$$

To complete the proof of the lemma, we need to show that the sequence $\{(\tilde{c}^j_t)^L_{j=1},\ (\tilde{s}^j_t)^L_{j=1})^\infty_{t=\tau}$ is a solution to the problem

$$\begin{gathered} \max\ \textstyle\sum_{t=\tau}^\infty \beta_j^{t-\tau} c^j_t, \\ c^j_\tau + s^j_\tau = (1-\theta_\tau - \lambda_\tau)[(1+r^*_\tau) s^{j*}_{\tau-1} + w^*_\tau], \\ c^j_t + s^j_t = (1-\theta^e_t - \lambda^e_t)[(1+\tilde{r}_t) s^j_{t-1} + \tilde{w}_t],\ t = \tau+1, \tau+2, ..., \\ s^j_t \geq 0,\ t = \tau, \tau+1, ... \end{gathered}$$

To do this, it is sufficient to show that the following conditions are satisfied:

$$\tilde{c}^j_\tau + \tilde{s}^j_\tau = (1 - \theta_\tau - \lambda_\tau)[(1 + r^*_\tau)s^{j*}_{\tau-1} + w^*_\tau], \tag{39}$$

$$\tilde{c}^j_t + \tilde{s}^j_t = (1 - \theta^e_t - \lambda^e_t)[(1 + \tilde{r}_t)\tilde{s}^j_{t-1} + \tilde{w}_t],\ t = \tau + 1, \tau + 2, ..., \tag{40}$$

$$\beta_j(1 - \theta^e_{t+1} - \lambda^e_{t+1})(1 + \tilde{r}_{t+1})\tilde{c}^j_t \le \tilde{c}^j_{t+1}\ (= \text{ if } \tilde{s}^j_t > 0),\ t = \tau, \tau + 1, ..., \tag{41}$$

$$\frac{\beta^t_j \tilde{s}^j_{t-1}}{\tilde{c}^j_t} \xrightarrow[t\to\infty]{} 0. \tag{42}$$

These relationships follow from the fact that $(c^{j*}_t, s^{j*}_t)^\infty_{t=\tau}$ is a solution to (4) at $\theta_\tau = \theta^\circ_\tau,\ \theta_t = \theta^e_t,\ t = \tau+1, \tau+2, \ldots,\ \lambda_\tau = \lambda^\circ_\tau,\ \lambda_t = \lambda^e_t,\ t = \tau+1, \tau+2, \ldots$, and hence satisfies the budget constraints

$$c^{j*}_\tau + s^{j*}_\tau = (1 - \theta_\tau - \lambda_\tau)[(1 + r^*_\tau)s^{j*}_{\tau-1} + w^*_\tau], \tag{43}$$

$$c^{j*}_t + s^{j*}_t = (1 - \theta^e_t - \lambda^e_t)[(1 + r^*_t)s^{j*}_{t-1} + w^*_t],\ t = \tau, \tau + 1, ..., \tag{44}$$

the first-order conditions

$$\beta_j(1 - \theta^e_{t+1} - \lambda^e_{t+1})(1 + r^*_{t+1})c^{j*}_t \le c^{j*}_{t+1}\ (= \text{ if } s^{j*}_t > 0),\ t = \tau, \tau + 1, ..., \tag{45}$$

and the transversality condition

$$\frac{\beta^t_j s^*_{t-1}}{c^*_t} \xrightarrow[t\to\infty]{} 0. \tag{46}$$

Indeed, the validity of (39) follows directly from (43) and the choice of $\tilde{c}^j_\tau$ and $\tilde{s}^j_\tau$. The validity of (40) for $t = \tau + 2, \tau + 3, \ldots$ follows from (44) and the choice of $\tilde{c}^j_t$, $\tilde{s}^j_t$, $\tilde{s}^j_{t-1}$, $\tilde{r}_t$ and $\tilde{w}_t$ for $t = \tau + 2, \tau + 3, \ldots$. As for the validity of (40) for $t = \tau + 1$, it is also readily checked. Indeed, since

$$1 + r^*_{\tau+1} = \nu^{1-\alpha}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}(1 + \tilde{r}_{\tau+1}),\ s^{j*}_\tau = \nu^{-1}_\tau \tilde{s}^j_\tau,\ w^{j*}_{\tau+1} = \nu^{-\alpha}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}\tilde{w}_{\tau+1},$$

by the choice of $\tilde{c}^j_{\tau+1}$ and $\tilde{s}^j_{\tau+1}$, we have

$$\begin{aligned}
\tilde{c}^j_{\tau+1} + \tilde{s}^j_{\tau+1} &= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(c^{j*}_{\tau+1} + s^{j*}_{\tau+1}) \\
&= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})[(1 + r^*_{\tau+1})s^{j*}_\tau + w^*_{\tau+1}] \\
&= (1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})[\nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}\nu^{1-\alpha}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}(1 + \tilde{r}_{\tau+1})\nu^{-1}_\tau \tilde{s}^j_\tau \\
&\quad + \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}\nu^{-\alpha}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}\tilde{w}^j_{\tau+1}] \\
&= (1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})[(1 + \tilde{r}_{\tau+1})\tilde{s}^j_\tau + \tilde{w}_{\tau+1}].
\end{aligned}$$

The validity of (41) for $t = \tau + 1, \tau + 2, \ldots$ follows directly from (45) and the choice of $\tilde{c}^j_t$ and $1 + \tilde{r}_{t+1}$ for $t = \tau + 1, \tau + 2, \ldots$. The validity of (41) for $t = \tau$ is also verified with ease. Indeed, by the choice of $\tilde{c}^j_\tau$, $\tilde{c}^j_{\tau+1}$ and $\tilde{r}_{\tau+1}$ and by (45),

$$\begin{aligned}
(1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})(1 + \tilde{r}_{\tau+1})\tilde{c}^j_\tau &= (1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})\nu^{\alpha-1}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1 + r^{j*}_{\tau+1})\nu_\tau c^{j*}_\tau \\
&= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1 - \theta^e_{\tau+1} - \lambda^e_{\tau+1})(1 + r^{j*}_{\tau+1})c^{j*}_\tau \le \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}c^{j*}_{\tau+1} = \tilde{c}^j_{\tau+1}.
\end{aligned}$$

Moreover, if $\tilde{s}^j_\tau > 0$, which is equivalent to $s^{j*}_\tau > 0$, then

$$\begin{aligned}(1-\theta^e_{\tau+1}-\lambda^e_{\tau+1})(1+\tilde{r}_{\tau+1})\tilde{c}^j_\tau &= (1-\theta^e_{\tau+1}-\lambda^e_{\tau+1})\nu^{\alpha-1}_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1+r^{j*}_{\tau+1})\nu_\tau c^{j*}_\tau \\ &= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1-\theta^e_{\tau+1}-\lambda^e_{\tau+1})(1+r^{j*}_{\tau+1})c^{j*}_\tau = \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}c^{j*}_{\tau+1} = \tilde{c}^j_{\tau+1}.\end{aligned}$$

Finally, (42) follows directly from (46). □

C Appendix 3. Proof of Proposition 6.

Proof of Proposition 6. Suppose the equalities in (24) are satisfied. Then we have

$$\kappa_{med} = \frac{\delta_{med}(1-\beta_{med})}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})},\ \chi_{med} = \frac{(1+\delta_{med})(1-\alpha)\beta_{med}}{1+\delta_{med}\beta_{med}}.$$

Therefore,

$$\begin{aligned}1-\kappa_{med} &= \frac{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})-\delta_{med}(1-\beta_{med})}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})} \\ &= \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})},\end{aligned}$$

$$1-\chi_{med} = \frac{1+\delta_{med}\beta_{med}-(1+\delta_{med})(1-\alpha)\beta_{med}}{1+\delta_{med}\beta_{med}} = \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{1+\delta_{med}\beta_{med}},$$

$$\kappa_{med}\chi_{med} = \frac{\delta_{med}(1-\beta_{med})(1-\alpha)\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})},$$

$$\begin{aligned}1-\kappa_{med}\chi_{med} &= \frac{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})-\delta_{med}(1-\beta_{med})(1-\alpha)\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})} \\ &= \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})}.\end{aligned}$$

It follows that

$$\theta^* = \frac{\kappa_{med}-\kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}} = \frac{\delta_{med}(1-\beta_{med})}{1+\delta_{med}},\ \lambda^* = \frac{\chi_{med}-\kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}} = (1-\alpha)\beta_{med}.\ \square$$

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