PROOF OF AN ASYMPTOTIC EXPANSION

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In this short note we prove the asymptotic expansion given in Bornemann and März (2006, Footnote 6), namely

$$c^1|_{\theta=\pi/2} = \sqrt{\frac{2}{\pi}}\log(\mu)\mu^{-1} + \frac{\gamma+\log 2}{\sqrt{2\pi}}\mu^{-1} + O(\mu^{-2}), \qquad \mu \to \infty. \tag{1}$$

To this end we split the defining integral at the angle $\pi/2 - 2/\sqrt{\mu}$,

$$c^1|_{\theta=\pi/2} = \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \frac{1-\exp\left(-\frac{\mu^2}{2}\cos^2\phi\right)}{\mu\cos\phi}\,d\phi$$

$$= \underbrace{c^1|_{\theta=\pi/2-2/\sqrt{\mu}}}_{=I_1} + \underbrace{\sqrt{\frac{2}{\pi}} \int_{\pi/2-2/\sqrt{\mu}}^{\pi/2} \frac{1-\exp\left(-\frac{\mu^2}{2}\cos^2\phi\right)}{\mu\cos\phi}\,d\phi}_{=I_2}.$$

As we have already shown in the proof of (2006, Thm. 2), there holds the asymptotic expansion

$$I_1 = \sqrt{\frac{2}{\pi}}\log\left(\cot\left(\frac{1}{\sqrt{\mu}}\right)\right)\mu^{-1} + O(\mu^{-2}) = \frac{1}{\sqrt{2\pi}}\log(\mu)\,\mu^{-1} + O(\mu^{-2}).$$

On the other hand we obtain by simple substitutions and the mean value theorem

$$I_2 = \sqrt{\frac{2}{\pi}} \int_0^{2/\sqrt{\mu}} \frac{1-\exp\left(-\frac{\mu^2}{2}\sin^2\phi\right)}{\mu\sin\phi}\,d\phi = \sqrt{\frac{2}{\pi}} \int_0^{\mu\sin(2/\sqrt{\mu})} \frac{1-\exp(-\xi^2/2)}{\xi\sqrt{\mu^2-\xi^2}}\,d\xi$$

$$= \underbrace{\sqrt{\frac{2}{\pi}} \int_0^{\mu\sin(2/\sqrt{\mu})} \frac{1-\exp(-\xi^2/2)}{\xi}\,d\xi}_{=I_3} \left(\mu^{-1} + O(\mu^{-5/2})\right).$$

A further substitution of $t = \xi^2/2$ gives, using the exponential integral E_1 (see Abramowitz and Stegun 1964, Eq. (5.1.39)),

$$I_3 = \frac{1}{\sqrt{2\pi}} \int_0^{\mu^2\sin^2(2/\sqrt{\mu})/2} \frac{1-e^{-t}}{t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left(E_1\left(\frac{\mu^2}{2}\sin^2\left(\frac{2}{\sqrt{\mu}}\right)\right) + \log\left(\frac{\mu^2}{2}\sin^2\left(\frac{2}{\sqrt{\mu}}\right)\right) + \gamma\right),$$

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where γ denotes Euler's constant. Since

$$\frac{\mu^2}{2}\sin^2\left(\frac{2}{\sqrt{\mu}}\right) = 2\mu + O(1),$$

we get by the asymptotic expansion $E_1(z) = e^{-z}(z^{-1} + O(z^{-2}))$ of the exponential integral for large arguments (see Abramowitz and Stegun 1964, Eq. (5.1.51))

$$I_3 = \frac{1}{\sqrt{2\pi}}\Big(O(\mu^{-1}) + \log(2) + \log(\mu) + O(\mu^{-1}) + \gamma\Big).$$

Collecting all the terms finally yields

$$\begin{aligned}
c^1|_{\theta=\pi/2} &= I_1 + I_3\left(\mu^{-1} + O(\mu^{-5/2})\right) \\
&= \frac{1}{\sqrt{2\pi}}\log(\mu)\,\mu^{-1} + \frac{\mu^{-1} + O(\mu^{-5/2})}{\sqrt{2\pi}}\Big(\log(\mu) + \gamma + \log(2) + O(\mu^{-1})\Big) + O(\mu^{-2}) \\
&= \sqrt{\frac{2}{\pi}}\log(\mu)\mu^{-1} + \frac{\gamma + \log 2}{\sqrt{2\pi}}\mu^{-1} + O(\mu^{-2}),
\end{aligned}$$

which proves the asserted expansion (1).

References.

Abramowitz, M. and Stegun, I. A.: 1964, *Handbook of Mathematical Functions*, Dover Publications, New York.

Bornemann, F. and März, T.: 2006, Fast image inpainting based on coherence transport, *Technical report*, Technische Universität München.