Supplementary material to Estimating Lower Limb Kinematics using a Lie Group Constrained EKF and a Reduced Wearable IMU Count

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1 Additional details for Section II-C.1 Predication update

Below is the explicit definition of the motion model $\Omega(\mathbf{X}k)$ and $\mathscr{C}_k$.

$$\begin{aligned}\Omega(\mathbf{X}_k) = \big[&(\Delta t\, \mathbf{v}_k^{mp} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{p})^T\, \breve{\mathbf{R}}_k^{p} \;\; \mathbf{0}_{1\times 3} \;\; (\Delta t\, \mathbf{v}_k^{la} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{ls})^T\, \breve{\mathbf{R}}_k^{ls} \;\; \mathbf{0}_{1\times 3} \\ &(\Delta t\, \mathbf{v}_k^{ra} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{rs})^T\, \breve{\mathbf{R}}_k^{rs} \;\; \mathbf{0}_{1\times 3} \;\; \Delta t(\breve{\mathbf{a}}_k^{mp})^T \;\; \Delta t(\breve{\mathbf{a}}_k^{la})^T \;\; \Delta t(\breve{\mathbf{a}}_k^{ra})^T\big]^T\end{aligned} \tag{1}$$

$$\mathscr{C}_k = \frac{\partial}{\partial \boldsymbol{\epsilon}}\Omega\left(\boldsymbol{\mu}_k^{\epsilon}\right)|_{\boldsymbol{\epsilon}=0} = \left[\begin{array}{c|ccc} \mathbf{0}_{18\times 18} & \begin{matrix} \Delta t(\breve{\mathbf{R}}_k^{p})^T & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \Delta t(\breve{\mathbf{R}}_k^{ls})^T & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \Delta t(\breve{\mathbf{R}}_k^{rs})^T \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{matrix} \\ \hline \multicolumn{2}{c}{\mathbf{0}_{9\times 27}} \end{array}\right] \tag{2}$$

2 Additional details for Section II-C.2 Measurement update

Only the derivation for $\mathcal{H}_{mp}$ will be shown below, as the derivation for the other measurements are either trivial or can be solved similarly. The derivation for $\mathcal{H}_{ori}$ and $\mathcal{H}_{lim}$ are trivial as solving for $[\log(h_a\left(\hat{\boldsymbol{\mu}}_k^-\right)^{-1} h_a\left(\boldsymbol{\mu}_k^{\epsilon}\right))]^{\vee}_{G_a}$ for measurement type a simply gives us the exponential coordinates of the corresponding perturbations, $\boldsymbol{\epsilon}$. The zero velocity part of $\mathcal{H}_{ls}$ and $\mathcal{H}_{rs}$ can also be calculated trivially, while the flat floor assumption can be calculated similarly as $\mathcal{H}_{mp}$ but the Z position set to floor height, z_f, instead of the pelvis standing height, z_p.

Since the measurement function $h_{mp}(\mathbf{X}_k) \in \mathbb{R})$ instead of $\in SE(3)$, then $\mathbf{X}_1^{-1}\mathbf{X}_2 = \mathbf{X}_2 - \mathbf{X}_1$. It then follows that $\boldsymbol{\delta}\mathbf{h}_{mp} = [\log(h_{mp}(\hat{\boldsymbol{\mu}}_k^-)^{-1} h_{mp}(\boldsymbol{\mu}_k^{\epsilon}))]^{\vee}_{G_{mp}} = h(\boldsymbol{\mu}_k^{\epsilon}) - h(\hat{\boldsymbol{\mu}}_k^-)$; and that $\frac{\partial}{\partial \boldsymbol{\epsilon}}\boldsymbol{\delta}\mathbf{h}_{mp}|_{\boldsymbol{\epsilon}=0} = \frac{\partial}{\partial \boldsymbol{\epsilon}} h\left(\boldsymbol{\mu}_k^{\epsilon}\right)|_{\boldsymbol{\epsilon}=0}$. Also note of a useful property (Eq. (3)) for $\mathbf{a}, \mathbf{b} \in \mathfrak{se}(3)$ [1, Eq. (72)].

$$[\mathbf{a}]^{\wedge}_{SE(3)}\, \mathbf{b} = \mathbf{a}\, [\mathbf{b}]^{\odot}_{SE(3)}\,, \qquad \begin{bmatrix} \epsilon \\ \eta \end{bmatrix}^{\odot} = \begin{bmatrix} \eta \mathbf{I}_{3\times 3} & -[\epsilon]^{\wedge}_{SO(3)} \\ \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} \end{bmatrix} \tag{3}$$

$$\mathbf{i}_z = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{i}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad h_{mp}\left(\mathbf{X}_k\right) = \mathbf{i}_z^T\, \mathbf{T}^p \mathbf{i}_0 \tag{4}$$

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$$h_{mp}(\boldsymbol{\mu}_k^{\epsilon}) = \mathbf{i}_z^T\,\bar{\mathbf{T}}^p \exp\left([\boldsymbol{\epsilon}^p]^{\wedge}_{SE(3)}\right)\mathbf{i}_0 \qquad \text{Linearize } \exp(\boldsymbol{\phi}) \approx \mathbf{I} + [\boldsymbol{\phi}]^{\wedge} \tag{5}$$

$$= \mathbf{i}_z^T\,\bar{\mathbf{T}}^p\,[\boldsymbol{\epsilon}^p]^{\wedge}\,\mathbf{i}_0 = \mathbf{i}_z^T\,\bar{\mathbf{T}}^p\,[\mathbf{i}_0]^{\odot}\,\boldsymbol{\epsilon}^p \qquad \text{Use Eq. (3) to swap } \epsilon \text{ to the right} \tag{6}$$

$$\mathcal{H}_{mp} = \frac{\partial}{\partial \boldsymbol{\epsilon}} h_{mp}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0} = \left[\ \mathbf{i}_z^T\,\bar{\mathbf{T}}^p\,[\mathbf{i}_0]^{\odot}\ \ \mathbf{0}_{1\times 6}\ \ \mathbf{0}_{1\times 6} \,\vdots\, \mathbf{0}_{1\times 9}\ \right] \tag{7}$$

3 Additional details for Section II-C.3 Constraint update

3.1 Thigh length

Below is the derivation of $\mathcal{C}_{ltl,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{ltl}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0}$ obtained from the thigh length constraint (Eq. (10)) where $\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+})$ is the thigh vector (Eq. (9)). $\mathcal{C}_{rtl,k}$ is derived similarly.

$$\mathbf{E} = \begin{bmatrix}\mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1}\end{bmatrix} \qquad {}^{p}\mathbf{p}^{lh} = \begin{bmatrix}0 & \frac{d^p}{2} & 0 & 1\end{bmatrix}^T \qquad {}^{ls}\mathbf{p}^{lk} = \begin{bmatrix}0 & 0 & d^{ls} & 1\end{bmatrix}^T \tag{8}$$

$$\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+}) = \overbrace{\mathbf{E}\,\mathbf{T}^p\,{}^{p}\mathbf{p}^{lh}}^{\text{hip joint pos.}} - \overbrace{\mathbf{E}\,\mathbf{T}^{ls}\,{}^{ls}\mathbf{p}^{lk}}^{\text{knee joint pos.}} \tag{9}$$

$$c_{ltl}(\tilde{\boldsymbol{\mu}}_k^{+}) = \boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+})^T\,\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+}) - (d^{lt})^2 = 0 = \mathbf{D}_{ltl} \tag{10}$$

$$\text{For simplicity let us first define } \boldsymbol{\tau}_z^{lt}(\boldsymbol{\mu}_k^{\epsilon}) \text{ and linearize } \exp(\boldsymbol{\phi}) \approx \mathbf{I} + [\boldsymbol{\phi}]^{\wedge} \tag{11}$$

$$\boldsymbol{\tau}_z^{lt}(\boldsymbol{\mu}_k^{\epsilon}) = \mathbf{E}\Big(\,\bar{\mathbf{T}}^p \exp\left([\boldsymbol{\epsilon}^p]^{\wedge}\right){}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\exp\left(\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\right){}^{ls}\mathbf{p}^{lk}\Big) \tag{12}$$

$$= \mathbf{E}\left(\bar{\mathbf{T}}^p\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\,{}^{ls}\mathbf{p}^{lk} + \bar{\mathbf{T}}^p\,[\boldsymbol{\epsilon}^p]^{\wedge}\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\,{}^{ls}\mathbf{p}^{lk}\right) \tag{13}$$

$$= \mathbf{E}\big(\overbrace{\bar{\mathbf{T}}^p\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\,{}^{ls}\mathbf{p}^{lk}}^{\mathbf{A}} + \overbrace{\bar{\mathbf{T}}^p\left[{}^{p}\mathbf{p}^{lh}\right]^{\odot}\boldsymbol{\epsilon}^p - \bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot}\boldsymbol{\epsilon}^{ls}}^{\mathbf{B}}\big) \tag{14}$$

Calculating for $c_{ltl}(\boldsymbol{\mu}_k^{\epsilon})$ and noting that $\mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{B} = \mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{A}$ since it is scalar

$$c_{ltl}(\boldsymbol{\mu}_k^{\epsilon}) = (\mathbf{A}+\mathbf{B})^T\,\mathbf{E}^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) - (d^{lt})^2 \tag{15}$$

$$= \mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{A} + 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{B} + \mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{B} - (d^{lt})^2 \tag{16}$$

Assume second order error $\mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{B} \approx 0$

$$= \mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{A} + 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\big(\,\bar{\mathbf{T}}^p\left[{}^{p}\mathbf{p}^{lh}\right]^{\odot}\boldsymbol{\epsilon}^p - \bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot}\boldsymbol{\epsilon}^{ls}\big) - (d^{lt})^2 \tag{17}$$

$$\mathcal{C}_{ltl,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{ltl}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0} = \Big[\ 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^p\left[{}^{p}\mathbf{p}^{lh}\right]^{\odot}\ \ -2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot}\ \ \mathbf{0}_{1\times 6} \,\vdots\, \mathbf{0}_{1\times 9}\ \Big] \tag{18}$$

3.2 Hinge knee joint

Below is the derivation of $\mathcal{C}_{lkh,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{lkh}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0}$) obtained from the constraint for the hinge knee joint that enforces the long (z) axis of the thigh to be perpendicular to the mediolateral axis (y) of the shank, as shown in Eq. (19). $\mathcal{C}_{rkh,k}$ is derived similarly.

$$c_{lkh}(\boldsymbol{\mu}_k) = (\mathbf{r}_y^{ls})^T\,\boldsymbol{\tau}_z^{lt} = \left(\mathbf{E}\,\mathbf{T}^{ls}\mathbf{i}_y\right)^T\boldsymbol{\tau}_z^{lt} = 0 = \mathbf{D}_{lkh} \tag{19}$$

Linearize $\exp(\boldsymbol{\phi}) \approx \mathbf{I} + [\boldsymbol{\phi}]^{\wedge}$

$$c_{lkh}\left(\boldsymbol{\mu}_k^{\epsilon}\right) = \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\exp\left(\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\right)\mathbf{i}_y\right)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) = \left(\mathbf{E}\left(\bar{\mathbf{T}}^{ls} + \bar{\mathbf{T}}^{ls}\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\right)\mathbf{i}_y\right)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) \tag{20}$$

$$= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\right)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) + \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\mathbf{i}_y\right)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) \tag{21}$$

Assume second order error ≈ 0 and scalar so transposable

$$= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\right)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[\mathbf{i}_y]^{\odot}\,\boldsymbol{\epsilon}^{ls} \tag{22}$$

$$= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\right)^T \mathbf{E}\left(\mathbf{A} + \bar{\mathbf{T}}^p \left[{}^p\mathbf{p}^{lh}\right]^{\odot} \boldsymbol{\epsilon}^p - \bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} \boldsymbol{\epsilon}^{ls}\right) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\mathbf{i}_y\right]^{\odot}\boldsymbol{\epsilon}^{ls} \tag{23}$$

$$\mathcal{C}_{lkh,k} = \left[\; \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\right)^T \mathbf{E}\,\bar{\mathbf{T}}^p \left[{}^p\mathbf{p}^{lh}\right]^{\odot} \;\middle|\; -\left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\right)^T \mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\mathbf{i}_y\right]^{\odot} \;\middle|\; \mathbf{0}_{1\times 6} \;\middle|\; \mathbf{0}_{1\times 9}\;\right] \tag{24}$$

3.3 Knee range of motion

Below is the derivation of $\mathcal{C}_{lkr,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{lkr}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0})$ obtained from the constraint for the knee range of motion (ROM) which is enforced if the knee angle is outside the allowed ROM. $\mathcal{C}_{rkr,k}$ is derived similarly.

$$\begin{aligned} c_{lkr}(\tilde{\boldsymbol{\mu}}_k^{+}) &= ((\mathbf{r}_z^{ls})^T \cos(\alpha'_{lk} - \tfrac{\pi}{2}) - (\mathbf{r}_x^{ls})^T \sin(\alpha'_{lk} - \tfrac{\pi}{2}))\,\mathbf{r}_z^{lt} \\ &= (\mathbf{E}\,\mathbf{T}^{ls}\overbrace{(\mathbf{i}_z\cos(\alpha'_{lk} - \tfrac{\pi}{2})\mathbf{i}_x\sin(\alpha'_{lk} - \tfrac{\pi}{2}))}^{\boldsymbol{\psi}})^T\,\boldsymbol{\tau}_z^{lt} = 0 = \mathbf{D}_{lkr} \end{aligned} \tag{25}$$

$$\begin{aligned} c_{lkr}\left(\boldsymbol{\mu}_k^{\epsilon}\right) &= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\exp\left(\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\right)\boldsymbol{\psi}\right)^T \mathbf{E}\left(\mathbf{A}+\mathbf{B}\right) \qquad \text{Linearize } \exp(\boldsymbol{\phi}) \approx \mathbf{I} + [\boldsymbol{\phi}]^{\wedge} \\ &= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\boldsymbol{\psi}\right)^T \mathbf{E}\left(\mathbf{A}+\mathbf{B}\right) + \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge}\boldsymbol{\psi}\right)^T \mathbf{E}\left(\mathbf{A}+\mathbf{B}\right) & (26) \\ &\quad \text{Assume second order error } \approx 0 \text{ and scalar therefore transposable} \\ &= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\boldsymbol{\psi}\right)^T \mathbf{E}\left(\mathbf{A}+\mathbf{B}\right) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\boldsymbol{\psi}\right]^{\odot}\boldsymbol{\epsilon}^{ls} & (27) \\ &= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\boldsymbol{\psi}\right)^T \mathbf{E}\left(\mathbf{A} + \bar{\mathbf{T}}^p \left[{}^p\mathbf{p}^{lh}\right]^{\odot} \boldsymbol{\epsilon}^p - \bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} \boldsymbol{\epsilon}^{ls}\right) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\boldsymbol{\psi}\right]^{\odot}\boldsymbol{\epsilon}^{ls} & (28) \end{aligned}$$

$$\mathcal{C}_{lkr,k} = \left[\; \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\boldsymbol{\psi}\right)^T \mathbf{E}\,\bar{\mathbf{T}}^p \left[{}^p\mathbf{p}^{lh}\right]^{\odot} \;\middle|\; -\left(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\boldsymbol{\psi}\right)^T \mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\left[\boldsymbol{\psi}\right]^{\odot} \;\middle|\; \mathbf{0}_{1\times 6} \;\middle|\; \mathbf{0}_{1\times 9}\;\right] \tag{29}$$

References

[1] T. D. Barfoot, *State Estimation for Robotics*. Cambridge University Press, 2017.