Supplementary material to Estimating Lower Limb Kinematics using a Lie Group Constrained EKF and a Reduced Wearable IMU Count

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1 Additional details for Section II-C.1 Predication update

Below is the explicit definition of the motion model $\Omega(\mathbf{X}k)$ and $\mathscr{C}_k$.

$$\begin{aligned}\Omega(\mathbf{X}_k) = \big[&(\Delta t\, \mathbf{v}_k^{mp} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{p})^T\, \breve{\mathbf{R}}_k^{p} \;\; \mathbf{0}_{1\times3} \;\; (\Delta t\, \mathbf{v}_k^{la} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{ls})^T\, \breve{\mathbf{R}}_k^{ls} \;\; \mathbf{0}_{1\times3} \\ &(\Delta t\, \mathbf{v}_k^{ra} + \tfrac{\Delta t^2}{2}\, \breve{\mathbf{a}}_k^{rs})^T\, \breve{\mathbf{R}}_k^{rs} \;\; \mathbf{0}_{1\times3} \;\; \Delta t(\breve{\mathbf{a}}_k^{mp})^T \;\; \Delta t(\breve{\mathbf{a}}_k^{la})^T \;\; \Delta t(\breve{\mathbf{a}}_k^{ra})^T\big]^T\end{aligned} \tag{1}$$

$$\mathscr{C}_k = \tfrac{\partial}{\partial \boldsymbol{\epsilon}}\Omega\left(\boldsymbol{\mu}_k^{\epsilon}\right)|_{\boldsymbol{\epsilon}=0} = \left[\begin{array}{c:ccc} & \Delta t(\breve{\mathbf{R}}_k^{p})^T & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{18\times18} & \mathbf{0}_{3\times3} & \Delta t(\breve{\mathbf{R}}_k^{ls})^T & \mathbf{0}_{3\times3} \\ & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \Delta t(\breve{\mathbf{R}}_k^{rs})^T \\ & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \hdashline \multicolumn{4}{c}{\mathbf{0}_{9\times27}} \end{array}\right] \tag{2}$$

2 Additional details for Section II-C.2 Measurement update

Only the derivation for $\mathcal{H}_{mp}$ will be shown below. The derivation for the other measurements are either trivial or can be solved similarly. The derivation for $\mathcal{H}_{ori}$ and $\mathcal{H}_{lim}$ are trivial as solving for $[\log(h_a\left(\hat{\boldsymbol{\mu}}_k^-\right)^{-1} h_a\left(\boldsymbol{\mu}_k^{\epsilon}\right))]_{G_a}^{\vee}$ where $a \in \{ori, lim\}$ simply gives us the exponential coordinates of the corresponding perturbations, $\boldsymbol{\epsilon}$. The zero velocity part of $\mathcal{H}_{ls}$ and $\mathcal{H}_{rs}$ can also be calculated trivially, while the flat floor assumption can be calculated similarly as $\mathcal{H}_{mp}$ but the Z position set to floor height, z_f, instead of the pelvis standing height, z_p.

Since the measurement function $h_{mp}(\mathbf{X}_k) \in \mathbb{R}$, $\mathbf{X}_1^{-1}\mathbf{X}_2 = \mathbf{X}_2 - \mathbf{X}_1$. It then follows that $\boldsymbol{\delta}\mathbf{h}_{mp} = [\log(h_{mp}(\hat{\boldsymbol{\mu}}_k^-)^{-1} h_{mp}(\boldsymbol{\mu}_k^{\epsilon}))]_{G_{mp}}^{\vee} = h(\boldsymbol{\mu}_k^{\epsilon}) - h(\hat{\boldsymbol{\mu}}_k^-)$; and that $\frac{\partial}{\partial \boldsymbol{\epsilon}}\boldsymbol{\delta}\mathbf{h}_{mp}|_{\boldsymbol{\epsilon}=0} = \frac{\partial}{\partial \boldsymbol{\epsilon}} h\left(\boldsymbol{\mu}_k^{\epsilon}\right)|_{\boldsymbol{\epsilon}=0}$. Also note of a useful property (Eq. (3)) for $\mathbf{a}, \mathbf{b} \in \mathfrak{se}(3)$ [1, Eq. (72)].

$$[\mathbf{a}]_{SE(3)}^{\wedge}\, \mathbf{b} = \mathbf{a}\, [\mathbf{b}]_{SE(3)}^{\odot}\,, \quad \begin{bmatrix}\epsilon \\ \eta\end{bmatrix}^{\odot} = \begin{bmatrix}\eta \mathbf{I}_{3\times3} & -[\epsilon]_{SO(3)}^{\wedge} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3}\end{bmatrix} \tag{3}$$

$$\mathbf{i}_z = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^T, \quad \mathbf{i}_0 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T, \quad h_{mp}\left(\mathbf{X}_k\right) = \mathbf{i}_z^T\, \mathbf{T}^p \mathbf{i}_0 \tag{4}$$

$$h_{mp}(\boldsymbol{\mu}_k^{\epsilon}) = \mathbf{i}_z^T\, \bar{\mathbf{T}}^p \exp\left([\boldsymbol{\epsilon}^p]_{SE(3)}^{\wedge}\right) \mathbf{i}_0 \qquad \text{Linearize } \exp(\boldsymbol{\epsilon}) \approx \mathbf{I} + [\boldsymbol{\epsilon}]^{\wedge} \text{ where } \epsilon \approx 0 \text{ (very small).} \tag{5}$$

$$= \mathbf{i}_z^T\, \bar{\mathbf{T}}^p\, [\boldsymbol{\epsilon}^p]^{\wedge}\, \mathbf{i}_0 = \mathbf{i}_z^T\, \bar{\mathbf{T}}^p\, [\mathbf{i}_0]^{\odot}\, \boldsymbol{\epsilon}^p \qquad \text{Use Eq. (3) to swap } \epsilon \text{ to the right} \tag{6}$$

$$\mathcal{H}_{mp} = \tfrac{\partial}{\partial \boldsymbol{\epsilon}} h_{mp}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0} = \left[\begin{array}{ccc:c} \mathbf{i}_z^T\, \bar{\mathbf{T}}^p\, [\mathbf{i}_0]^{\odot} & \mathbf{0}_{1\times6} & \mathbf{0}_{1\times6} & \mathbf{0}_{1\times9}\end{array}\right] \tag{7}$$

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3 Additional details for Section II-C.3 Constraint update

3.1 Thigh length

Below is the derivation of $\mathcal{C}_{ltl,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{ltl}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0}$ obtained from the thigh length constraint (Eq. (10)) where $\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+})$ is the thigh vector (Eq. (9)). $\mathcal{C}_{rtl,k}$ is derived similarly.

$$\mathbf{E} = \begin{bmatrix}\mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1}\end{bmatrix} \quad {}^{p}\mathbf{p}^{lh} = \begin{bmatrix}0 & \frac{d^p}{2} & 0 & 1\end{bmatrix}^T \quad {}^{ls}\mathbf{p}^{lk} = \begin{bmatrix}0 & 0 & d^{ls} & 1\end{bmatrix}^T \tag{8}$$

$$\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+}) = \overbrace{\mathbf{E}\,\mathbf{T}^{p}\,{}^{p}\mathbf{p}^{lh}}^{\text{hip joint pos.}} - \overbrace{\mathbf{E}\,\mathbf{T}^{ls}\,{}^{ls}\mathbf{p}^{lk}}^{\text{knee joint pos.}} \tag{9}$$

$$c_{ltl}(\tilde{\boldsymbol{\mu}}_k^{+}) = \boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+})^T\,\boldsymbol{\tau}_z^{lt}(\tilde{\boldsymbol{\mu}}_k^{+}) - (d^{lt})^2 = 0 = \mathbf{D}_{ltl} \tag{10}$$

$$\text{For simplicity let us first define } \boldsymbol{\tau}_z^{lt}(\boldsymbol{\mu}_k^{\epsilon}) \text{ and linearize } \exp(\boldsymbol{\epsilon}) \approx \mathbf{I} + [\boldsymbol{\epsilon}]^{\wedge} \tag{11}$$

$$\boldsymbol{\tau}_z^{lt}(\boldsymbol{\mu}_k^{\epsilon}) = \mathbf{E}(\bar{\mathbf{T}}^{p}\exp([\boldsymbol{\epsilon}^{p}]^{\wedge})\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\exp([\boldsymbol{\epsilon}^{ls}]^{\wedge})\,{}^{ls}\mathbf{p}^{lk}) \tag{12}$$

$$= \mathbf{E}(\bar{\mathbf{T}}^{p}\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\,{}^{ls}\mathbf{p}^{lk} + \bar{\mathbf{T}}^{p}\,[\boldsymbol{\epsilon}^{p}]^{\wedge}\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\,[\boldsymbol{\epsilon}^{ls}]^{\wedge}\,{}^{ls}\mathbf{p}^{lk}) \tag{13}$$

$$= \mathbf{E}\big(\overbrace{\bar{\mathbf{T}}^{p}\,{}^{p}\mathbf{p}^{lh} - \bar{\mathbf{T}}^{ls}\,{}^{ls}\mathbf{p}^{lk}}^{\mathbf{A}} + \overbrace{\bar{\mathbf{T}}^{p}\,[{}^{p}\mathbf{p}^{lh}]^{\odot}\,\boldsymbol{\epsilon}^{p} - \bar{\mathbf{T}}^{ls}\,[{}^{ls}\mathbf{p}^{lk}]^{\odot}\,\boldsymbol{\epsilon}^{ls}}^{\mathbf{B}}\big) \tag{14}$$

$$\text{Calculating for } c_{ltl}(\boldsymbol{\mu}_k^{\epsilon}) \text{ and noting that } \mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{B} = \mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{A} \text{ since it is scalar}$$

$$c_{ltl}(\boldsymbol{\mu}_k^{\epsilon}) = (\mathbf{A}+\mathbf{B})^T\,\mathbf{E}^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) - (d^{lt})^2 \tag{15}$$

$$= \mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{A} + 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{B} + \mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{B} - (d^{lt})^2 \tag{16}$$

$$\text{Assume second order error } \mathbf{B}^T\mathbf{E}^T\mathbf{E}\mathbf{B} \approx 0$$

$$= \mathbf{A}^T\mathbf{E}^T\mathbf{E}\mathbf{A} + 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\big(\,\bar{\mathbf{T}}^{p}\,[{}^{p}\mathbf{p}^{lh}]^{\odot}\,\boldsymbol{\epsilon}^{p} - \bar{\mathbf{T}}^{ls}\,[{}^{ls}\mathbf{p}^{lk}]^{\odot}\,\boldsymbol{\epsilon}^{ls}\big) - (d^{lt})^2 \tag{17}$$

$$\mathcal{C}_{ltl,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{ltl}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0} = \begin{bmatrix} 2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{p}\,[{}^{p}\mathbf{p}^{lh}]^{\odot} & -2\mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[{}^{ls}\mathbf{p}^{lk}]^{\odot} & \mathbf{0}_{1\times 6} & \mathbf{0}_{1\times 9}\end{bmatrix} \tag{18}$$

3.2 Hinge knee joint

Below is the derivation of $\mathcal{C}_{lkh,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{lkh}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0}$ obtained from the constraint for the hinge knee joint (Eq. (19)). $\mathcal{C}_{rkh,k}$ is derived similarly.

$$\mathbf{i}_y = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T, \quad c_{lkh}(\boldsymbol{\mu}_k) = (\mathbf{r}_y^{ls})^T\,\boldsymbol{\tau}_z^{lt} = \big(\mathbf{E}\,\mathbf{T}^{ls}\mathbf{i}_y\big)^T\,\boldsymbol{\tau}_z^{lt} = 0 = \mathbf{D}_{lkh} \tag{19}$$

$$\text{Linearize } \exp(\boldsymbol{\epsilon}) \approx \mathbf{I} + [\boldsymbol{\epsilon}]^{\wedge}$$

$$c_{lkh}\,(\boldsymbol{\mu}_k^{\epsilon}) = (\mathbf{E}\,\bar{\mathbf{T}}^{ls}\exp([\boldsymbol{\epsilon}^{ls}]^{\wedge})\mathbf{i}_y)^T\mathbf{E}(\mathbf{A}+\mathbf{B}) = (\mathbf{E}(\bar{\mathbf{T}}^{ls} + \bar{\mathbf{T}}^{ls}\,[\boldsymbol{\epsilon}^{ls}]^{\wedge})\mathbf{i}_y)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) \tag{20}$$

$$= \big(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\big)^T\,\mathbf{E}\,(\mathbf{A}+\mathbf{B}) + (\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[\boldsymbol{\epsilon}^{ls}]^{\wedge}\,\mathbf{i}_y)^T\mathbf{E}\,(\mathbf{A}+\mathbf{B}) \tag{21}$$

$$\text{Assume second order error } \approx 0\text{, scalar so transposable, and using Eq. (3)}$$

$$= \big(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\big)^T\,\mathbf{E}\,(\mathbf{A}+\mathbf{B}) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[\mathbf{i}_y]^{\odot}\,\boldsymbol{\epsilon}^{ls} \tag{22}$$

$$= \big(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\big)^T\,\mathbf{E}(\mathbf{A} + \bar{\mathbf{T}}^{p}\,[{}^{p}\mathbf{p}^{lh}]^{\odot}\,\boldsymbol{\epsilon}^{p} - \bar{\mathbf{T}}^{ls}\,[{}^{ls}\mathbf{p}^{lk}]^{\odot}\,\boldsymbol{\epsilon}^{ls}) + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[\mathbf{i}_y]^{\odot}\,\boldsymbol{\epsilon}^{ls} \tag{23}$$

$$\mathcal{C}_{lkh,k} = \begin{bmatrix} \big(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\big)^T\,\mathbf{E}\,\bar{\mathbf{T}}^{p}\,[{}^{p}\mathbf{p}^{lh}]^{\odot} & -\big(\mathbf{E}\,\bar{\mathbf{T}}^{ls}\mathbf{i}_y\big)^T\,\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[{}^{ls}\mathbf{p}^{lk}]^{\odot} + \mathbf{A}^T\mathbf{E}^T\mathbf{E}\,\bar{\mathbf{T}}^{ls}\,[\mathbf{i}_y]^{\odot} & \mathbf{0}_{1\times 6} & \mathbf{0}_{1\times 9}\end{bmatrix} \tag{24}$$

3.3 Knee range of motion

Below is the derivation of $\mathcal{C}_{lkr,k} = \frac{\partial}{\partial \boldsymbol{\epsilon}} c_{lkr}(\boldsymbol{\mu}_k^{\epsilon})|_{\boldsymbol{\epsilon}=0}$ obtained from the constraint for the knee range of motion (ROM) which is enforced if the knee angle is outside the allowed ROM (Eq. (25)). $\mathcal{C}_{rkr,k}$ is derived similarly.

$$
\begin{aligned}
c_{lkr}(\tilde{\boldsymbol{\mu}}_k^{+}) &= ((\mathbf{r}_z^{ls})^T \cos(\alpha'_{lk} - \tfrac{\pi}{2}) - (\mathbf{r}_x^{ls})^T \sin(\alpha'_{lk} - \tfrac{\pi}{2}))\, \mathbf{r}_z^{lt} \\
&= (\mathbf{E}\, \mathbf{T}^{ls} \overbrace{\left(\mathbf{i}_z \cos(\alpha'_{lk} - \tfrac{\pi}{2}) \mathbf{i}_x \sin(\alpha'_{lk} - \tfrac{\pi}{2})\right)}^{\boldsymbol{\psi}})^T\, \boldsymbol{\tau}_z^{lt} = 0 = \mathbf{D}_{lkr}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
c_{lkr}\,(\boldsymbol{\mu}_k^{\epsilon}) &= (\mathbf{E}\,\bar{\mathbf{T}}^{ls} \exp([\boldsymbol{\epsilon}^{ls}]^{\wedge})\boldsymbol{\psi})^T \mathbf{E}\,(\mathbf{A} + \mathbf{B}) \qquad \text{Linearize } \exp(\boldsymbol{\epsilon}) \approx \mathbf{I} + [\boldsymbol{\epsilon}]^{\wedge} \\
&= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls} \boldsymbol{\psi}\right)^T \mathbf{E}\,(\mathbf{A} + \mathbf{B}) + (\mathbf{E}\,\bar{\mathbf{T}}^{ls} \left[\boldsymbol{\epsilon}^{ls}\right]^{\wedge} \boldsymbol{\psi})^T \mathbf{E}\,(\mathbf{A} + \mathbf{B}) && (26) \\
&\ \text{Assume second order error } \approx 0\text{, scalar so transposable, and using Eq. (3)} \\
&= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls} \boldsymbol{\psi}\right)^T \mathbf{E}\,(\mathbf{A} + \mathbf{B}) + \mathbf{A}^T \mathbf{E}^T \mathbf{E}\,\bar{\mathbf{T}}^{ls} \left[\boldsymbol{\psi}\right]^{\odot} \boldsymbol{\epsilon}^{ls} && (27) \\
&= \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls} \boldsymbol{\psi}\right)^T \mathbf{E}(\mathbf{A} + \bar{\mathbf{T}}^{p} \left[{}^{p}\mathbf{p}^{lh}\right]^{\odot} \boldsymbol{\epsilon}^{p} - \bar{\mathbf{T}}^{ls} \left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} \boldsymbol{\epsilon}^{ls}) + \mathbf{A}^T \mathbf{E}^T \mathbf{E}\,\bar{\mathbf{T}}^{ls} \left[\boldsymbol{\psi}\right]^{\odot} \boldsymbol{\epsilon}^{ls} && (28) \\
\mathcal{C}_{lkr,k} &= \left[\ \left(\mathbf{E}\,\bar{\mathbf{T}}^{ls} \boldsymbol{\psi}\right)^T \mathbf{E}\,\bar{\mathbf{T}}^{p} \left[{}^{p}\mathbf{p}^{lh}\right]^{\odot} \;\middle|\; -\left(\mathbf{E}\,\bar{\mathbf{T}}^{ls} \boldsymbol{\psi}\right)^T \mathbf{E}\,\bar{\mathbf{T}}^{ls} \left[{}^{ls}\mathbf{p}^{lk}\right]^{\odot} + \mathbf{A}^T \mathbf{E}^T \mathbf{E}\,\bar{\mathbf{T}}^{ls} \left[\boldsymbol{\psi}\right]^{\odot} \;\middle|\; \mathbf{0}_{1\times 6} \;\middle|\; \mathbf{0}_{1\times 9}\ \right] && (29)
\end{aligned}
$$

References

[1] T. D. Barfoot, *State Estimation for Robotics.* Cambridge University Press, 2017.