Direct Proof of the Uncountability of the Transcendental Numbers

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We usually prove that the set of the real transcendental numbers $\mathbb{R} \setminus \mathbb{A}$ is uncountable *indirectly* by proving that the set of algebraic numbers $\mathbb{A}$ is countable. Here, we present a *direct* proof that $\mathbb{R} \setminus \mathbb{A}$ is uncountable.

Theorem. *The set* $\mathbb{R} \setminus \mathbb{A}$ *is uncountable.*

Proof. The function $f\colon [0, +\infty) \to \mathbb{R} \setminus \mathbb{A}$ defined by

$$f(x) = \begin{cases} \pi + x & \text{if} \pi + x \notin \mathbb{A} \\ \pi - x & \text{if} \pi + x \in \mathbb{A} \end{cases} \tag{1}$$

is: (1) *well-defined*, because if $\pi + x \in \mathbb{A}$

$$\pi = \frac{(\pi + x) + (\pi - x)}{2} \in \mathbb{A} \tag{2}$$

which is false; (2) *injective*, because if $f(x) = f(y)$, then $x = \|f(x) - \pi\| = \|f(y) - \pi\| = y$. (The function f is inspired by the folklore theorem "$\pi + e \notin \mathbb{A}$ or $\pi - e \notin \mathbb{A}$", and its proof "otherwise $\pi = \frac{(\pi+e)+(\pi-e)}{2} \in \mathbb{A}$ which is false".)

$\square$

Note. *This works because* $\mathbb{A}$ *is a field so we are certain that*

$$\{a \in \mathbb{A} : b \in \mathbb{A} : a + b \in \mathbb{A}\} \tag{3}$$

is true.