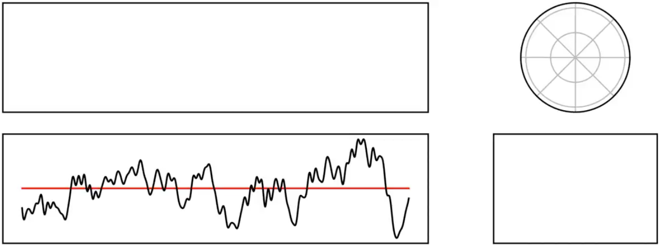
COMS20011 – Data-Driven Computer Science

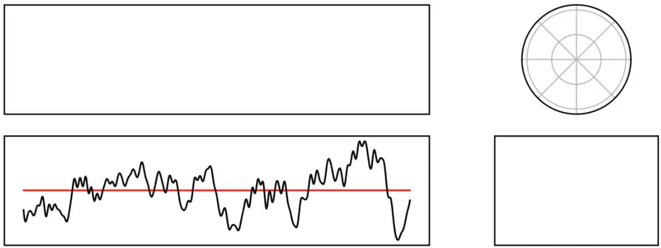


March 2024
Signals & Frequencies

Majid Mirmehdi

Lecture MM-04

Next in DDCS



Feature Selection and Extraction

- **Signal basics**
- 1D Fourier Transform
- Characteristics of features
- PCA for dimensionality reduction
- Convolutions

Summary: Typical Data Analysis Problem (Reminder)

Steps:

1. Pre-processing [Unit - Part 1] → Majid Mirmehdi (~10%)
2. Feature Selection [Unit - Part 3] → Majid Mirmehdi (~40%)
3. Modelling & Classification [Unit - Part 2] → Charles Kind (~50%)

Supervised Training

Pre-processed data

→ Image Features → Classifier Training → Trained Classifier

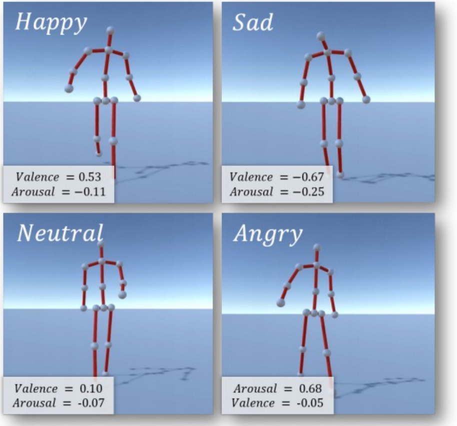
Training Labels → Classifier Training

Testing

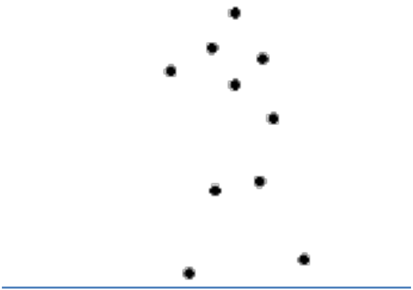
Test Data

→ Image Features → Trained Classifier → Prediction *Outdoor*

Features help simplify the problem



Patient with mild Parkinson's Disease



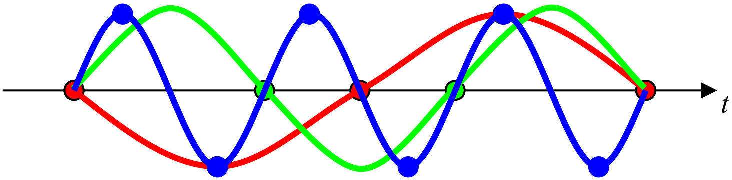
- Even "impoverished" motion data can evoke a strong perception

Image from https://medium.com/artificial-intelligence-network/

Nyquist-Shannon Sampling Theory - Reminder

*"An analogue signal containing components up to some maximum frequency **u** (Hz) may be completely reconstructed by regularly spread samples, provided the sampling rate is at least **$2u$** samples per second"*

Also referred to as the Nyquist-Shannon criterion: sampling rate s should be at least twice the highest spatial frequency u.



sampling period $T \leq \dfrac{1}{2u}$

equivalent to *sampling rate* $s \geq 2u$

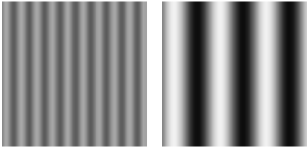
Basic Signals

AAAAATAAAAA

0000001000000

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

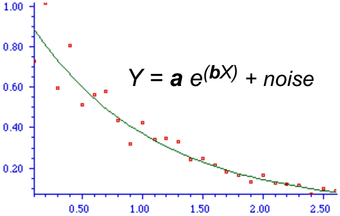
$x = sin(t) = sin(t+2\pi)$

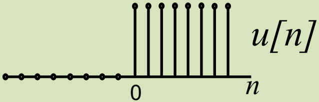


Some basic signals:

- **Unit impulse signal**
- **Unit step signal**
- **Exponential signal**
- **Periodic signal**

All signals can be represented by these basic signals!



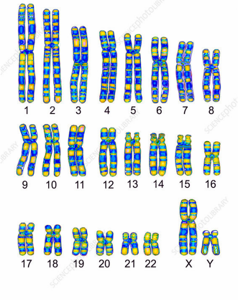


$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

Signals as Functions

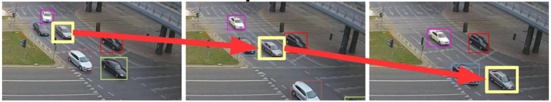
A signal is a physical quantity that is a function of one or more independent variable(s), such as space and/or time.

Data from a Gene pool



https://www.sciencephoto.com/media/458364/view/human-karyotype

Position of a car in a video sequence

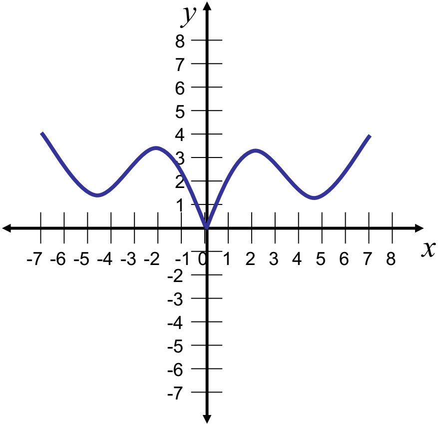


https://fayyen.com/miris/

Example signals:

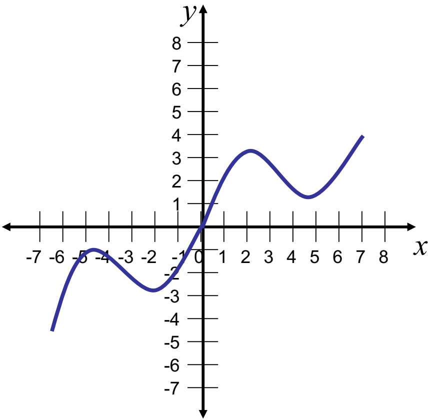
1D signal: $f(t)$
2D signal: $f(x,y)$
3D signal: $f(x,y,t)$ etc.

Reminder: Even Functions have y-axis Symmetry



So a function is **even** when $y = f(x) = f(-x)$

Reminder: Odd Functions have origin Symmetry



So a function is **odd** when $y = f(x) = -f(-x)$

Examples: odd or even?

$$f(x) = 3x^4 - 7x^2 + 1$$

$$f(-x) = 3(-x)^4 - 7(-x)^2 + 1 = 3x^4 - 7x^2 + 1$$

Even

$$f(x) = 4x^3 - x$$

$$f(-x) = 4(-x)^3 - (-x) = -4x^3 + x$$

Odd

$$f(x) = 3x^5 - x + 1$$

$$f(-x) = 3(-x)^5 - (-x) + 1 = -3x^5 + x + 1$$

Neither Odd, nor Even!

Function Decomposition

Any function $f(x) \neq 0$ can be decomposed as the sum of an even function $f_e(x)$ and an odd function $f_o(x)$.

$$f(x) = g(x) + h(x)$$

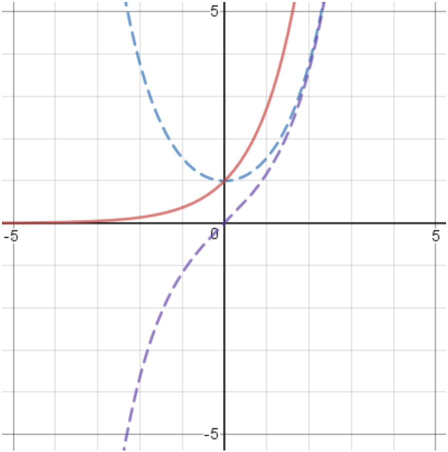
$$g(x) = \frac{1}{2}[f(x) + f(-x)]$$ Even Part

$$h(x) = \frac{1}{2}[f(x) - f(-x)]$$ Odd Part

Example

$$f(x) = e^{-x}$$

$$f(x) = g(x) + h(x)$$



Even Part

$$g(x) = \frac{1}{2}(e^{-x} + e^{x})$$

Odd Part

$$h(x) = \frac{1}{2}(e^{-x} - e^{x})$$

Signals as Functions

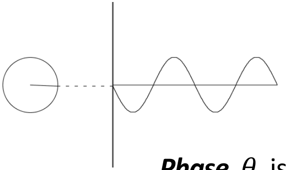
period is the time T it takes to finish one oscillation.

frequency $u = \frac{1}{T}$ is the number of periods per second, measured in Hz.

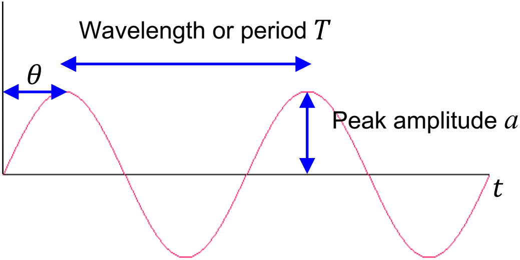
amplitude a is a measure of how much it changes over a single period.

Example: sine function

$$f(t) = a \sin 2\pi ut$$



From giphy.com



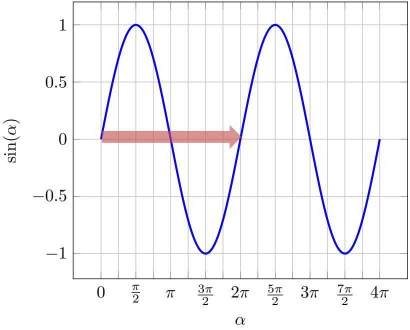
Phase θ is where the signal is in the oscillation (or the angular position).

Periodic Signals

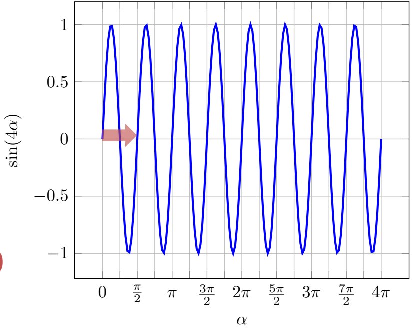
A continuous-time signal $f(t)$ is ***periodic*** if $\quad f(t+T) = f(t)$

Fundamental period T_0 of $f(t)$ is the smallest T that satisfies the above.

Fundamental frequency is then $\quad \nu_0 = \dfrac{1}{T_0}$



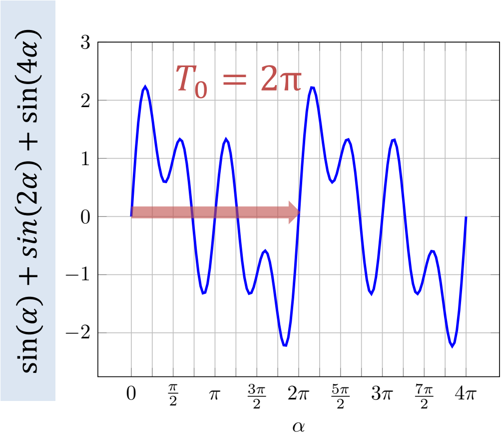
T_0



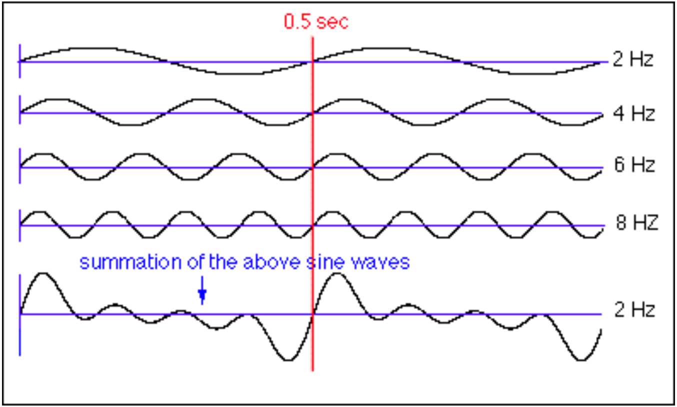
Periodic Signals

$$f(t + T) = f(t)$$

What is T_0 for this example?



Example I: a simple signal

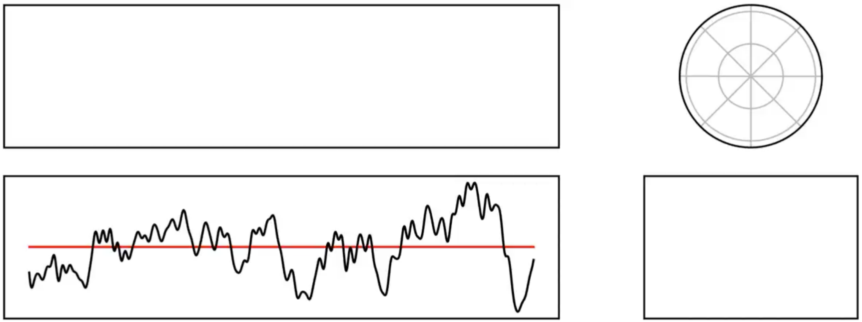


The fundamental period or combined frequency is the highest common factor.

Example I: a simple signal

$$f(\alpha) = \sum_{n=0}^{7} \sin(2\pi n\alpha)$$

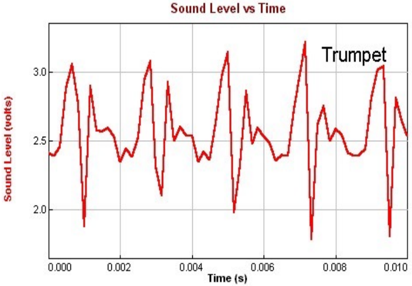
Example II: another signal

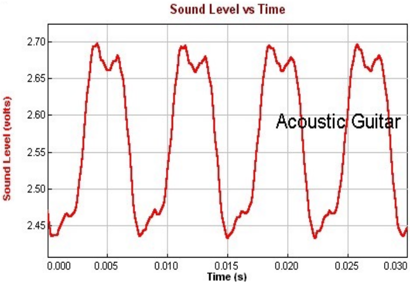


Example III: Visible light

How should we interpret these musical instrument signals?

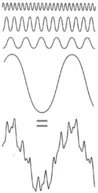
Characteristics of sound in audio signals:
High pitch - rapidly varying signal
Low pitch - slowly varying signal





Fourier Series

Trigonometric Fourier Series: Any *periodic* function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier* (1822).



$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

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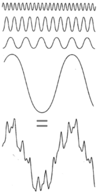
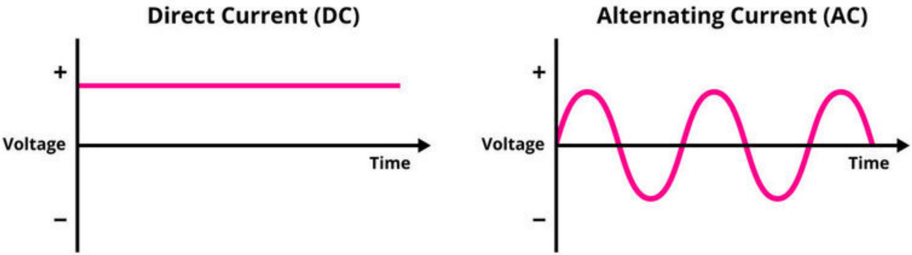


$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) \; + \; b_n \sin\left(\frac{2\pi nx}{T}\right)$$

A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.
- **Note the function has an even part and an odd part!**
- **a_0 is often referred to as the DC term or the average of the signal.**


Direct Current (DC)
+
Voltage
Time
−
Alternating Current (AC)
+
Voltage
Time
−

Fourier Series Solution

A *Fourier series* provides an equivalent representation of the function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) \ + \ b_n \sin\left(\frac{2\pi n x}{T}\right)$$

The coefficients are:

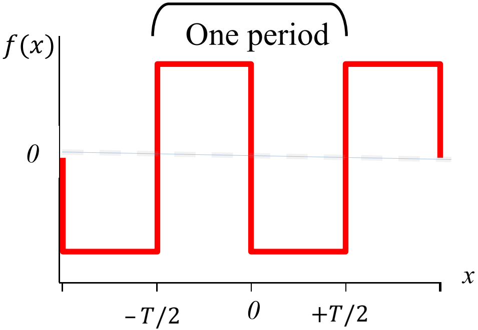
$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \cos\left(\frac{2\pi n x}{T}\right) dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \sin\left(\frac{2\pi n x}{T}\right) dx$$

Fourier Series Example: Square Wave

$f(x)$ → a square wave

$$f(x) = \begin{cases} +1 & \frac{-T}{2} \leq x < 0 \\ -1 & 0 \leq x < \frac{T}{2} \end{cases}$$



Example periodic function on $-T/2, +T/2$

Fourier Series Example: Square Wave

$f(x)$ → a square wave

$$f(x) = \begin{cases} +1 & \frac{-T}{2} \leq x < 0 \\ -1 & 0 \leq x < \frac{T}{2} \end{cases}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \cos(2\pi nx/T) dx$$

$$= \frac{2}{T} \int_{-T/2}^{0} \cos(2\pi nx/T) dx - \frac{2}{T} \int_{0}^{+T/2} \cos(2\pi nx/T) dx = \mathbf{0}$$

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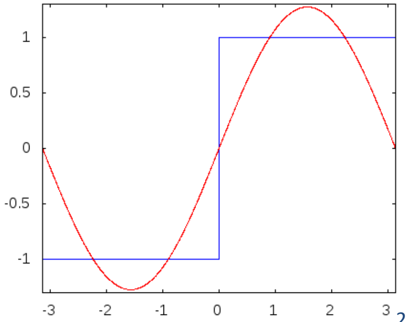
$$= \frac{2}{T}\int_{-T/2}^{0} \cos(2\pi nx/T)dx - \frac{2}{T}\int_{0}^{+T/2} \cos(2\pi nx/T)dx = \mathbf{0}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\sin(2\pi nx/T)dx$$

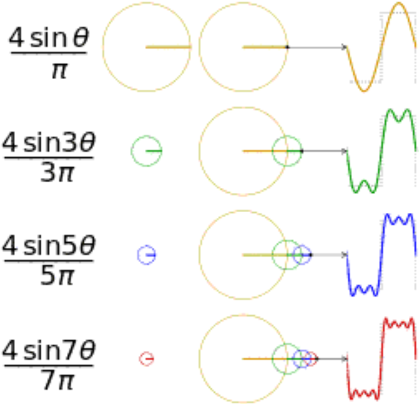
$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$n = 1,3,5,7,\ldots$



$$f(x) = \frac{4}{\pi}\cdot\sin\frac{2\pi x}{T} + \frac{4}{3\pi}\cdot\sin 3\cdot\frac{2\pi x}{T} + \frac{4}{5\pi}\cdot\sin 5\cdot\frac{2\pi x}{T} + \ldots$$

Approximating the Square Wave

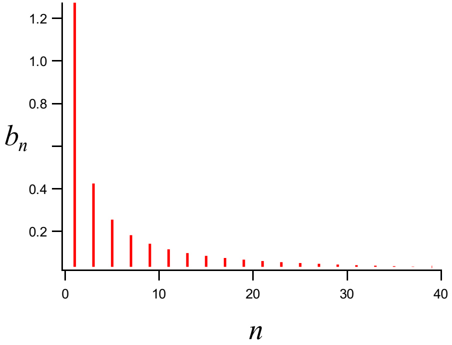


$$\theta = \frac{2\pi x}{T}$$

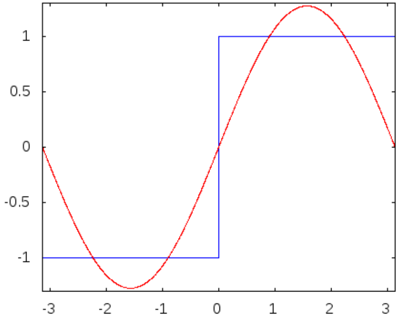
$$f(x) = \frac{4}{\pi} \cdot \sin \frac{2\pi x}{T} + \frac{4}{3\pi} \cdot \sin 3 \cdot \frac{2\pi x}{T} + \frac{4}{5\pi} \cdot \sin 5 \cdot \frac{2\pi x}{T} + \frac{4}{7\pi} \cdot \sin 7 \cdot \frac{2\pi x}{T} + \ldots$$

Fourier Space/Domain for the Square Wave

- The set of *Fourier Space* coefficients b_n contain complete information about the function
- Although *f(x)* is periodic to infinity, b_n is negligible beyond a finite range
- Sometimes the Fourier representation is more convenient to use, or just view



Next in DDCS



Feature Selection and Extraction

- Signal basics
- **1D Fourier Transform**
- Another look at features
- PCA for dimensionality reduction
- Convolutions