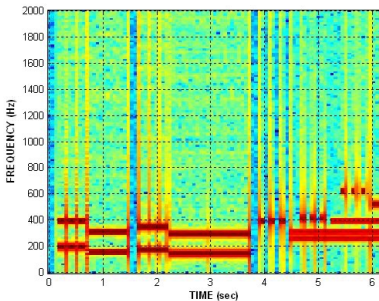# COMS20011 – Data-Driven Computer Science
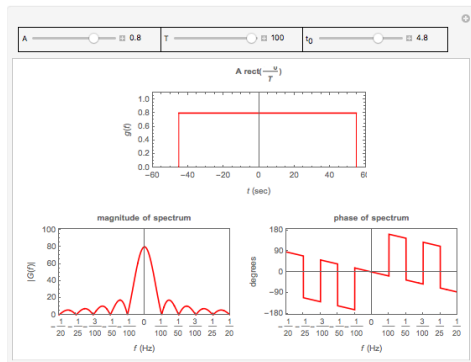
**March 2023**
**1D Fourier Transform**

**Majid Mirmehdi**

**Lecture MM-05**

# Next in DDCS

## Feature Selection and Extraction

➢ Signal basics

➢ **1D Fourier Transform**

➢ Another look at features

➢ PCA for dimensionality reduction

➢ Convolutions

2

# Frequency Analysis for Feature Extraction & more...

➤ The aim of processing a signal using Fourier analysis is to *manipulate the spectrum of a signal* rather than manipulating the signal itself. Example: *simple compression*



512 bytes      50 bytes      50 bytes      512 bytes

➤ Functions that are ***not periodic*** can also be expressed as the integral of sines and/or cosines weighted by a coefficient, using the Fourier transform.

# Visualising the outcome of the FT
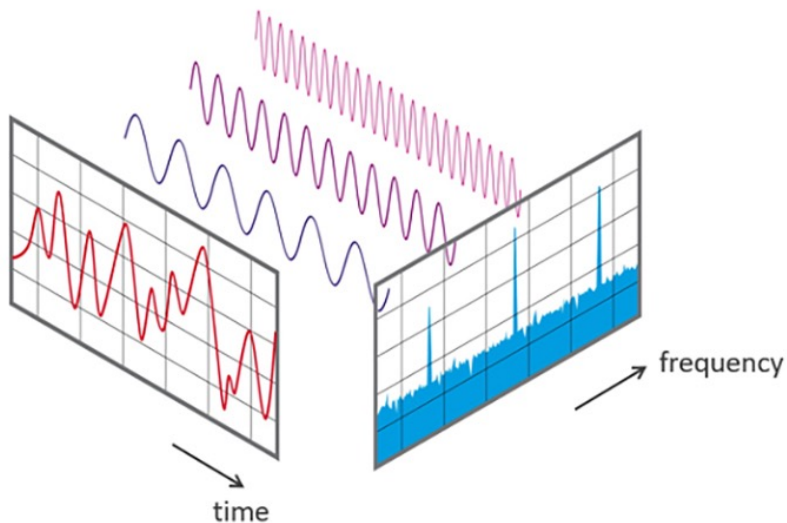
# 1D Fourier Transform

The Fourier Transform of a single variable continuous function $f(x)$ is:

$$F(u) = \int\limits_{-\infty}^{\infty} f(x)\ e^{-j2\pi ux} dx$$

Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \int\limits_{-\infty}^{\infty} F(u)\ e^{j2\pi ux} du$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

## 1D Fourier Transform: discrete form

The Fourier Transform of a discrete function of one variable, $f(x)$, $x=0,1,2...,N-1$ is:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-j2\pi ux}{N}} \qquad \text{for } u = 0,1,2,...,N-1.$$

Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \sum_{u=0}^{N-1} F(u)\, e^{\frac{j2\pi ux}{N}} \qquad \text{for } x = 0,1,2,...,N-1.$$

These two equations are also known as the Fourier Transform Pair.
Note, they constitute a lossless representation of data.

# 1D Fourier Transform

The concept of the frequency domain follows from *Euler's Formula*:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Thus, each term of the Fourier Transform is composed of the sum of *all* values of the function *f(x)* multiplied by sines and cosines of various frequencies:
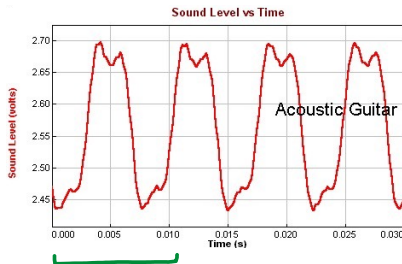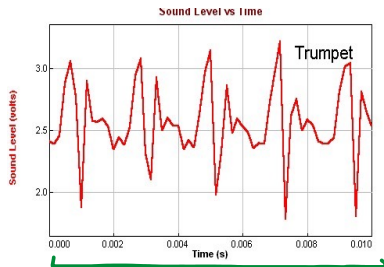
$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - j\sin\left(\frac{2\pi ux}{N}\right)\right]$$

for $u = 0, 1, 2, \ldots, N-1$.

We have transformed from a time domain to a frequency domain representation.
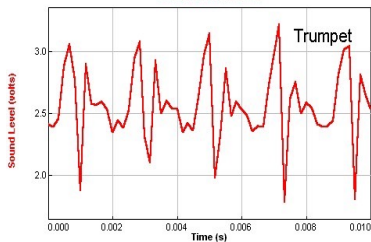
# Example: low and high frequencies
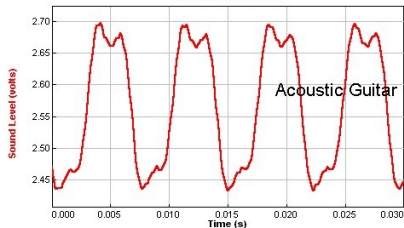
Characteristics of sound in audio signals:

# Example: low and high frequencies
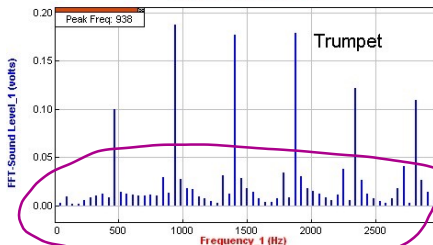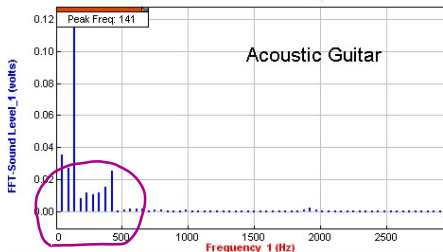
Characteristics of sound in audio signals:

# Example: Acoustic Data Analysis

**Spectrogram**
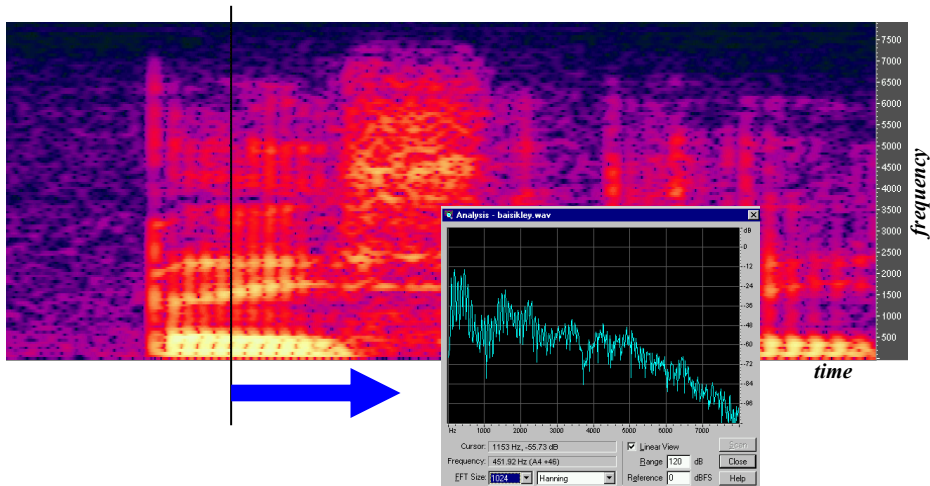
# Can you match the sound to the frequencies?



Beethovens FIFTH   (Robby GRIFFIN)
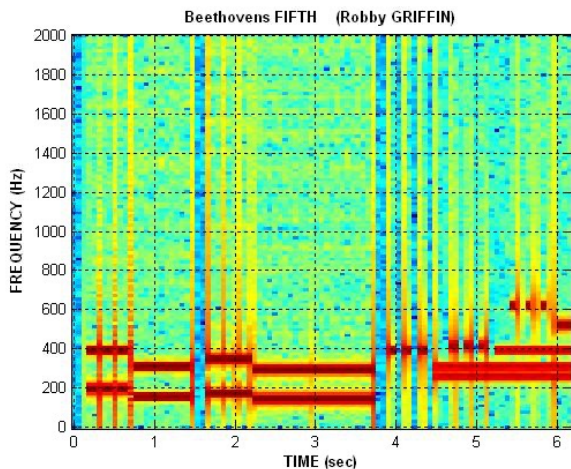
# 1D Fourier Transform

$F(u)$ is a complex number & has real and imaginary parts:

$$F(u) = R(u) + jI(u)$$

*Magnitude* or *spectrum* of the FT:

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

Phase angle or phase spectrum:

$$\varphi(u) = \tan^{-1}\frac{I(u)}{R(u)}$$

Expressing $F(u)$ in polar coordinates:

$$F(u) = |F(u)|e^{j\varphi(u)}$$

# Very Simple Application Example

Automatic speech recognition between two speech utterances $x(n)$ and $y(n)$:
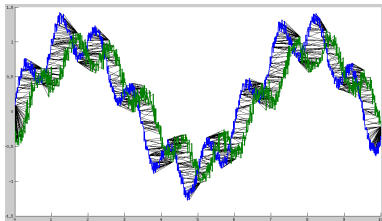
Naïve approach: 🥱

$$E = \sum_{\forall n} (x(n) - y(n))^2$$

Problems with this approach? $\Bigg\{$

$x(n) = K\, y(n)$, yet $E \neq 0$
*$K$ being a scaling parameter*

$x(n) = y(n - m)$, yet $E \neq 0$
*$m$ causing a delay shift*

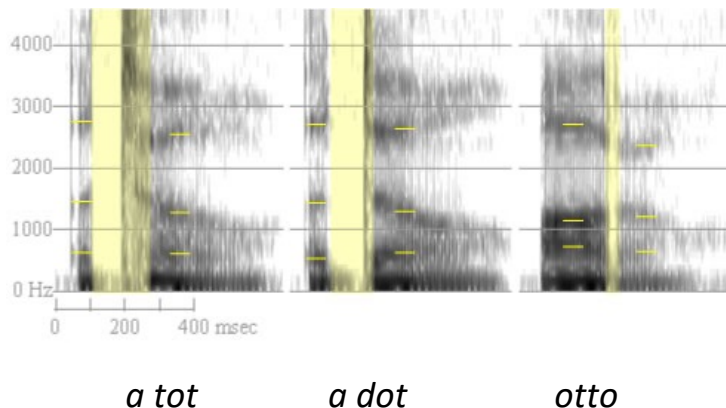One solution could be Dynamic Time Warping? (recall from earlier lecture)

# Use Frequency Domain Features!

> Take the Fourier transform of both utterances to get $X(u)$ and $Y(u)$.

> Then consider the Euclidean distance between their magnitude spectrums: $|X(u)|$ and $|Y(u)|$:

$$d_E = \sum_{\forall u} (|X(u)| - |Y(u)|)^2$$

# Use Frequency Domain Features!

Still a difficult task even in the frequency domain.



*a tot*        *a dot*        *otto*

# DNA Sequence Example

➢ The analysis of correlations in DNA sequences is used to identify protein coding genes in genomic DNA.

➢ Locating and characterizing repeats and periodic clusters provides certain information about the structural and functional characteristics of the molecule.

➢ DNA sequences are represented by letters, A, C, G or T, and **–**

➢ e.g. ACAATG-GCCATAAT-ATGTGAAC--GCTCA…
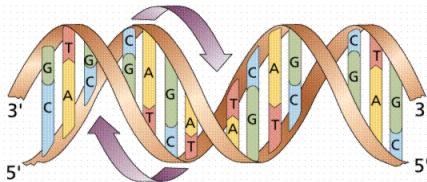
# DNA Sequence Example
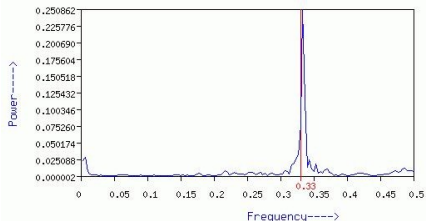
Consider the periodic sequence A--A--A--A--.....
where blanks can be filled randomly by A, C, G
or T. This shows a periodicity of 3.

The spectral density of such a sequence is
significantly non-zero only at one frequency
(0.33) which corresponds to the perfect
periodicity of base A ($1/0.333 = 3.0$).
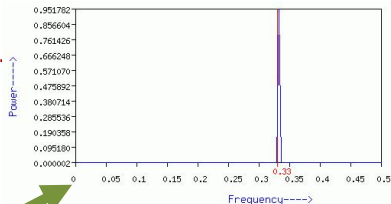
Destroy the perfect repetition by randomly
replacing the A's with any of the letters…

# DNA Sequence Analysis

The computation of Fourier & other linear transforms of *symbolic data* is a big problem.

**The simplest solution is to map each symbol to a number**.

Consider, for example, the following symbolic periodic sequence:
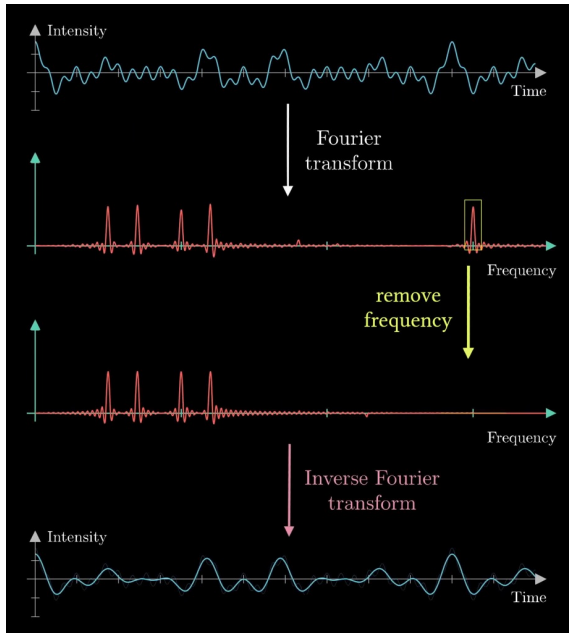s = (ATAGACATAGACATAGAC . . .).

The mapping:
A → 1,
T → 0,        Period = 2
G → 0,
C → 0,

The mapping:
A → 1,
T → 2,        Period = 6
G → 3,
C → 4,

This clearly shows that some of the relevant harmonic structure can be exposed by the symbolic-to-numeric labelling.

# Another Example: Noise Removal

https://www.3blue1brown.com/lessons/fourier-transforms

# The 2D Fourier Transform (NOT IN EXAM)
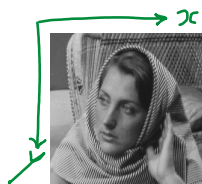


FT → straightforward extension to 2D:

> Images are functions of two variables → e.g. $f(x, y)$

> Defined in terms of *spatial frequency* → 2D frequency.

> Fourier Transform is particularly useful for characterising intensity variations across an image.

> FT identifies the *Rate of change of intensity* along each dimension.

# Examples: 2D Spatial Frequency (NOT IN EXAM)
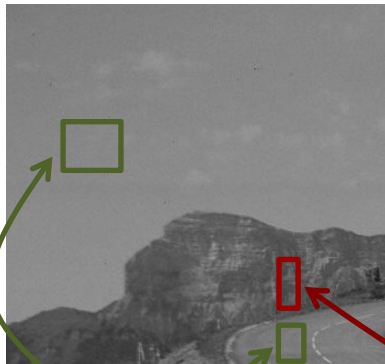


Slowly changing → low frequency

Rapidly changing → high frequency

More on the 2D FT next year in the *Image processing and Computer Vision* unit
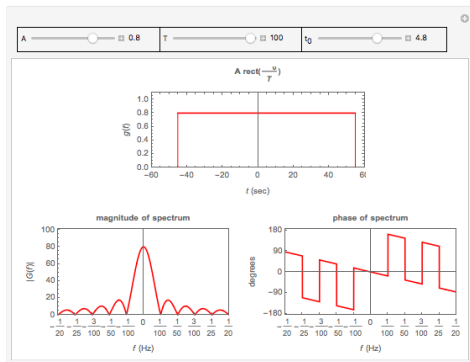
# Frequency Spectrum – Summary

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

> Distribution of $|F(u)| \rightarrow$ frequency spectrum/space of signal.

> Features often extracted from the *Power Spectrum*:

$$\omega(f) = |F(u)|^2$$

> Slowly changing signals $\rightarrow$ spectrum concentrated around low frequencies.

> Rapidly changing signals $\rightarrow$ spectrum concentrated around high frequencies.

> Also bandlimited signals $\rightarrow$ frequency content confined within some frequency band.

# Next in DDCS



## Feature Selection and Extraction

- ➤ Signal basics
- ➤ 1D Fourier Transform
- ➤ **Another look at features**
- ➤ PCA for dimensionality reduction
- ➤ Convolutions