Algorithms Design
Chap01-Introduction

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1.2 Five Representative Problems

1.1 Matching

Scenarios

Matching med-school students to hospitals

- **Input:** med-school students, hospitals, a set of preferences among hospitals and med-school students
- **Output**: a self-reinforcing admissions results.

Matching high school students to colleges

- **Input:** students, colleges, a set of preferences among colleges and students
- **Output**: an admissions results.

1.1.1 Problem Formulation

Input: n men and n women

- Participants rate members of opposite gender.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

Output: a "suitable" matching

1.1.1 Problem Formulation

- Given a set $M = \{m_1, \cdots, m_n\}$, a set $W = \{w_1, \cdots, w_n\}$.
- Let $M \times W$ denote the set of all possible ordered pairs of the form (m, w),where $m \in M$ and $w \in W$.

Def.(1-1) A **matching** S is a set of ordered pairs, each from $M \times W$ s.t.

- Each man of M appears in at most one pair of S.
- Each woman of W appears in at most one pair of S.

1.1.1 Problem Formulation

Def.(1-2). A **perfect matching** S' is a matching with the property that each member of M and each member of W appears in ***exactly*** one pair of S'.

Def.(1-3). Given a perfect matching S, man m and woman w' form an **unstable pair w.r.t.** $\boldsymbol{S}$ if both:

- m prefers w' s to matched woman w in S.
- w prefers m' to matched man m in S.

Note. An unstable pair $(m, w) \notin S$, but each of m and w prefers the other to their partner in S.

Quiz 1

Which pair is unstable w.r.t matching { A–X, B–Z, C–Y } ?

A. A-Y.

B. B-X.

C. B-Z.

D. None of the above.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

1.1.1 Problem Formulation

Def.(1-4). A **stable matching** is a perfect matching with no existence of unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

1.1.1 Problem Formulation

Q: Does stable matchings always exist?

A: No.

Q: Given a set of preference lists, can we efficiently construct a stable matching if there is one?

A:Yes

1.1.1 Problem Formulation

Stable roommate problem

- $2n$ people; each person ranks others from 1 to $2n-1$.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

no perfect matching is stable

A–B, C–D $\Rightarrow$ B–C unstable

A–C, B–D $\Rightarrow$ A–B unstable

A–D, B–C $\Rightarrow$ A–C unstable

1.1.2 Examples

$n = 2$,two men $\{m, m'\}$,two women $\{w, w'\}$

- S1=$\{(m, w),\ (m', w')\}$
  - Stable matching?
- S2=$\{(m, w'), (m', w)\}$
  - Stable matching?

preference	1st	2nd
m	w	w'
m'	w	w'

preference	1st	2nd
w	m	m'
w	m	m'

1.1.2 Examples

$n = 3, M = \{\text{Atlanta, Boston, Chicago}\},$
$\text{W} = \{\text{Xavier, Yalanda, Zeus}\}$

	favorite 1st	2nd	least favorite 3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

hospitals' preference lists

	favorite 1st	2nd	least favorite 3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

students' preference lists

$S1$ ={A-Z, B-Y,C-X} stable maching?

$S2$ ={A-X, B-Y,C-Z} stable matching?

1.1.3 Gale-Shapley Algorithms

Propose-and-reject algorithm. [Gale-Shapley 1962]

- Intuitive method that guarantees to find a stable matching

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

1.1.3 Gale-Shapley Algorithms

$n = 2$,two men $\{m, m'\}$,two women $\{w, w'\}$

preference	1st	2nd
m	w	w'
m'	w	w'

preference	1st	2nd
w	m	m'
w'	m	m'

Algorithms:

(1) m proposes to w

(1) (m, w)

(2) m' proposes to w

(2) w rejects m'

(3) m' proposes to w'

(3) (m', w')

1.1.3 Gale-Shapley Algorithms

$$n = 3, M = \{\text{Atlanta}, \text{Boston}, \text{Chicago}\},$$
$$\text{W} = \{\text{Xavier}, \text{Yalanda}, \text{Zeus}\}$$

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

hospitals' preference lists

(1) (A-X) (2) (B-Y) (3) (C-Z)

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

students' preference lists

(1) (B-X) (2) (A-Y) (3) (C-Z)

1.1.3 Gale-Shapley Algorithms

Algorithms:

(1) A proposed to X

(1) (A,X)

(2) B proposed to Y

(2) (B-Y)

(3) C proposed to X

(3) X reject C

	1st (favorite)	2nd	3rd (least favorite)
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

hospitals' preference lists

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

students' preference lists

1.1.3 Gale-Shapley Algorithms

Algorithms:(cont.)

(4) C proposes to Y

(4) Y rejects C

(5) C proposes to Z

(5) (Z,C)

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

hospitals' preference lists

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

students' preference lists

1.1.3 Gale-Shapley Algorithms

n=4, M={A,B,C,D},W={W,X,Y,Z}

	1st	2nd	3rd	4th
A	W	Z	Y	X
B	W	X	Y	Z
C	X	Z	Y	W
D	Y	W	X	Z

	1st	2nd	3rd	4th
W	C	A	B	D
X	A	D	C	B
Y	B	D	C	A
Z	D	B	A	C

1.1.4 Analysis

Proof of correctness: termination

Observation 1: Men propose to women in decreasing order of preference.

Observation 2 : Once a woman is matched, she never becomes unmatched; she only "trades up.“

1.1.4 Analysis

Theorem (1-1). G-S Algorithm terminates after at most n^2 iterations of **while** *loop*.

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

- Let $P(t)$ denote the set of pairs (m, w) such that m has proposed to w by the end of iteration t.
- $|P(t + 1)| \geq |P(t)|$

1.1.4 Analysis

Proof of Correctness: Perfection

Theorem (1-2). All men and women get matched.

Pf. (by contradiction)

Suppose, for sake of contradiction, that **Zeus** is not matched upon termination of algorithm.

Then some woman, say **Amy**, is not matched upon termination.

By Observation 2, **Amy** was never proposed to.

But, **Zeus** proposes to everyone, since he ends up unmatched.

1.1.4 Analysis

Proof of Correctness: Stability

Theorem(1-3). In Gale-Shapley matching S^*, there are no unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

Case 1: Z never proposed to A. ⇒ Z prefers his GS partner to A. ⇒ A-Z is not unstable.

Case 2: Z proposed to A. ⇒ A rejected Z (right away or later) ⇒ A prefers her GS partner to Z. ⇒ A-Z is not unstable.

In either case, A-Z is not unstable ⇒ a contradiction.

1.1.4 Analysis

Stable matching problem.

- Given n hospitals and n students, and their preference lists, find a stable matching if one exists.

Theorem.

- The Gale–Shapley algorithm guarantees to find a stable matching for any problem instance. [Gale–Shapley 1962]

Quiz 2

Do all executions of Gale–Shapley lead to the same stable matching?

A. No, because the algorithm is nondeterministic.
B. No, because an instance can have several stable matchings.
C. Yes, because each instance has a unique stable matching.
D. Yes, even though an instance can have several stable matchings and the algorithm is nondeterministic.

1.1.4 Analysis

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

1.1.4 Analysis

Def.(1-5) A women w is a **valid partner** of a man m if there exists a stable matching that contains the pair (m, w).

E.g. Both X and Y are valid partners for A.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

1.1.4 Analysis

E.g.

Both X and Y are valid partners of B.

Z is the only valid partner of C.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

1.1.4 Analysis

Def.(1-6) A woman w is a **best valid partner** of a man m if w is a valid partner of m, and no woman whom m ranks higher than w is a valid partner of his.

Q:Who is the best valid partner of A (or B)?

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

Quiz 3

Who is the best valid partner of W in the following instance?

	1st	2nd	3rd	4th
A	Y	Z	X	W
B	Z	Y	W	X
C	W	Y	X	Z
D	X	Z	W	Y

	1st	2nd	3rd	4th
W	D	A	B	C
X	C	B	A	D
Y	C	B	A	D
Z	D	A	B	C

6 stable matchings
{ A–W, B–X, C–Y, D–Z }
{ A–X, B–W, C–Y, D–Z }
{ A–X, B–Y, C–W, D–Z }
{ A–Z, B–W, C–Y, D–X }
{ A–Z, B–Y, C–W, D–X }
{ A–Y, B–Z, C–W, D–X }

1.1.4 Analysis

Man-optimal assignment. Each man receives best valid partner.

$$S^* = \{(m, best(m)) | m \in M\}$$

Theorem(1-4). GS matching S^* is man-optimal assignment.

Corollary(1-1). Man-optimal assignment is a stable matching.

1.1.4 Analysis

Theorem(1-4). GS matching S^* is man-optimal assignment.

Pf. (by contradiction)

- **Suppose** some man is paired with someone other than best partner. Men propose in decreasing order of preference ⇒ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.(TBC)

1.1.4 Analysis

- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z is not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S, a contradiction.

1.1.4 Analysis

Def.(1-7) Man m is a **worst valid partner** of woman w if m is a valid partner of w, and no man whom w ranks lower than m is a valid partner of hers.

Q:Who is the worst valid partner for X (or Y)?

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

1.1.4 Analysis

Q: Does man-optimality come at the expense of the women?

A: Yes

Woman-pessimal assignment. Each woman receives worst valid partner.

$$S^* = \{(worst(w), w) : w \in W\}$$

Theorem(1-5). GS matching S^* is a woman-pessimal assignment.

Corollary(1-2). Woman-pessimal assignment is a stable matching.

1.1.4 Analysis

Theorem(1-5). GS matching S^* is a woman-pessimal assignment.

Pf.

- Suppose A-Z matched in S^*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B.
- Thus, A-Z is an unstable in S.

1.1.5 Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

Gale-Shapley algorithm : finds a stable matching in $O(n^2)$ time.

Gale-Shapley algorithm finds a man-optimal stable matching S^*

Gale-Shapley algorithm finds a woman-pessimal stable matching S^*.

1.1.6 Extensions

Men $\approx$ hospitals, Women $\approx$ med school students.

Variant 1. Some **participants** declare **others** as unacceptable.

Variant 2. Unequal number of positions and students.

Variant 3. Some hospitals have more than one position.

1.1.6 Extensions

Def.(1-8) Matching S is **unstable** if there is a hospital *h* and student s such that:

- *h* and *s* are acceptable to each other; and
- Either *s* is unmatched, or *s* prefers *h* to the assigned hospital; and
- Either *h* does not have all its places filled, or *h* prefers s to at least one of its assigned students.

1.1.6 Extensions

Theorem(1-6). For instances by Def.(1-7) There exists a stable matching.

Pf.

Straightforward generalization of Gale–Shapley algorithm.

1.1.6 Extensions

Applications

National resident matching program (NRMP).

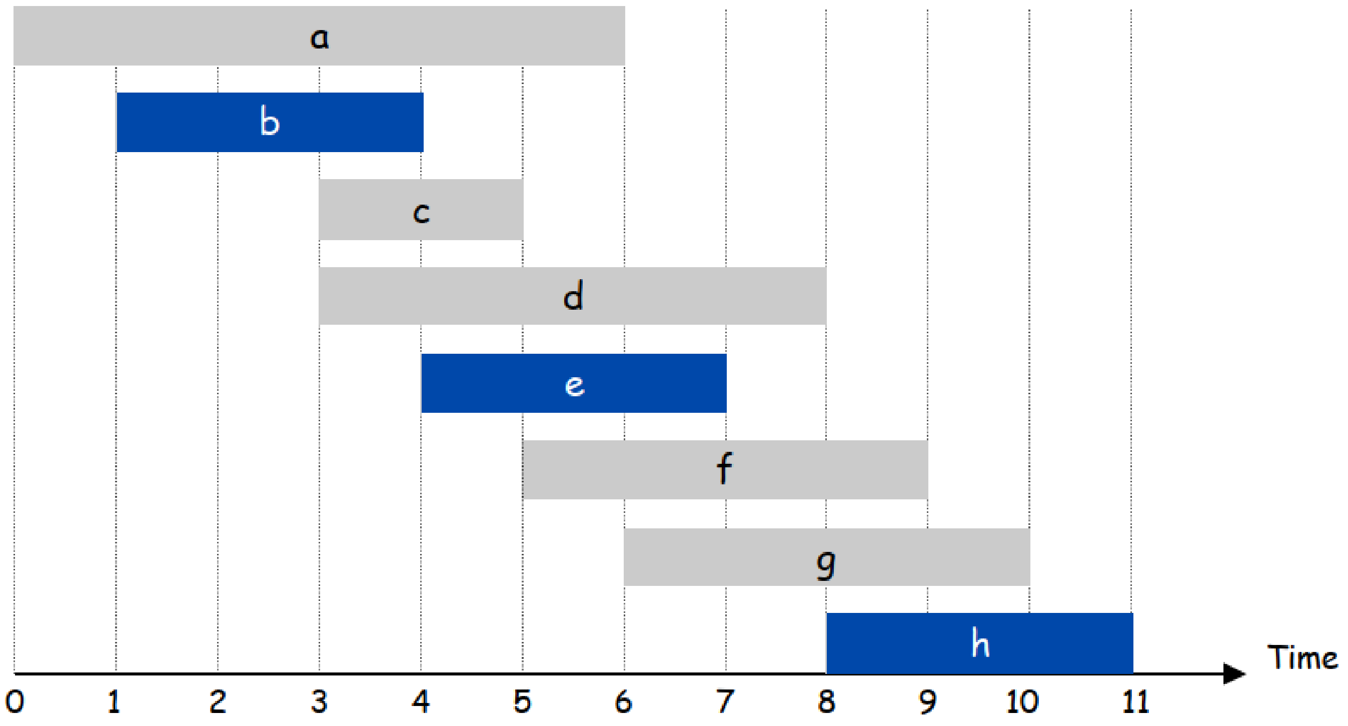
New York City high school match (Match 90K students to 500 high school programs).

National college match(Match students to colleges)

1.2 Five Representative Problems-(1/5)Interval Scheduling

Input. Set of jobs with start times and finish times.

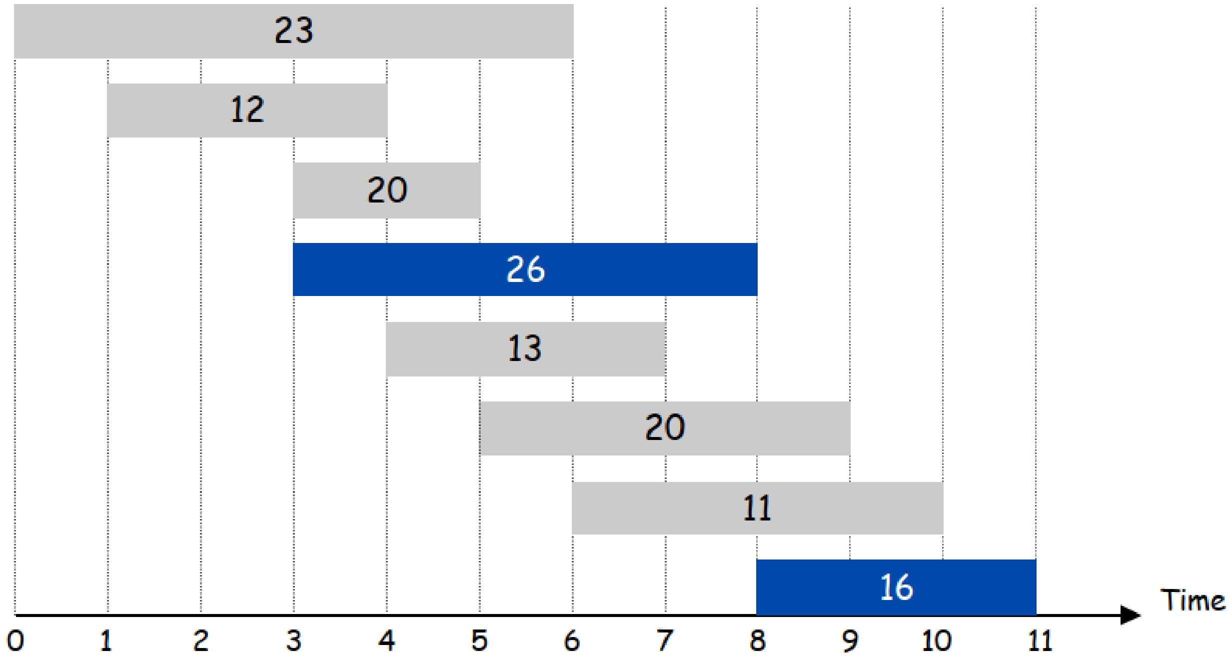
Goal. Select a compatible subset of jobs of maximum possible size.



1.2 Five Representative Problems-(2/5)Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.

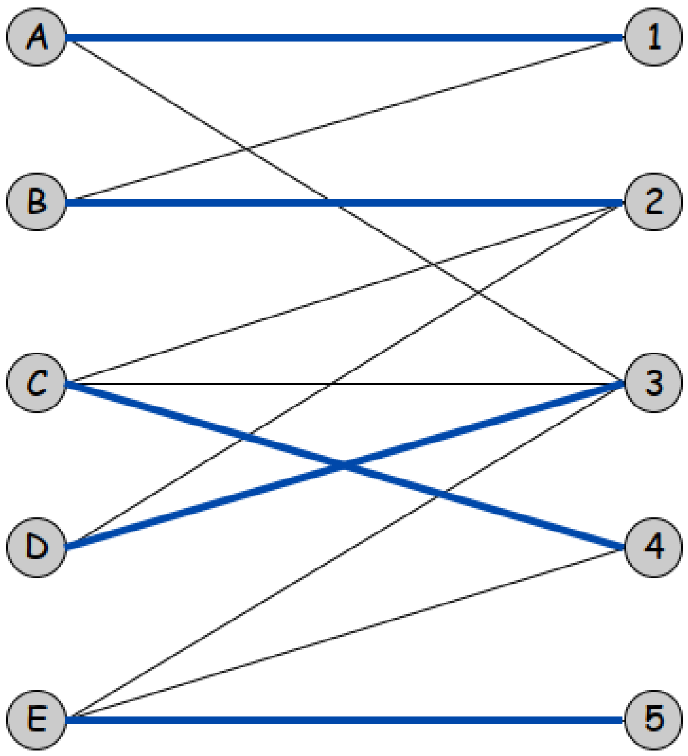
Goal. Find a compatible subset of jobs of maximum total weights.



1.2 Five Representative Problems-(3/5)Bipartite Matching

Input. An arbitrary bipartite graph.

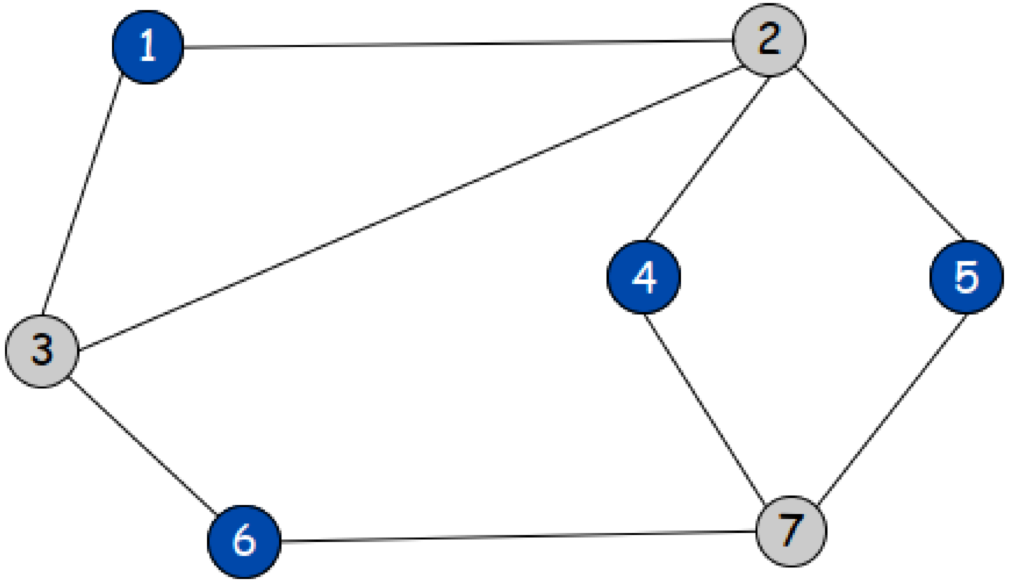
Goal. Find a matching.



1.2 Five Representative Problems-(4/5)Independent Set

Input. Graph.

Goal. Find an independent set with the maximum size.

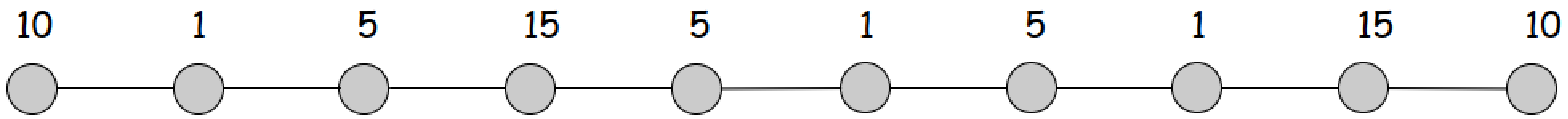


1.2 Five Representative Problems-(5/5)Competitive Facility Location

Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

1.2 Five Representative Problems

(1/5) **Interval scheduling**: $nlog(n)$ greedy algorithm.

(2/5) **Weighted interval scheduling**: $nlog(n)$ dynamic programming algorithm.

(3/5) **Bipartite matching**: n^k max-flow based algorithm.

(4/5) **Independent set**: NP-complete.

(5/5) **Competitive facility location**: PSPACE-complete.