3.1

解：由 $\theta=\int\theta=\int\frac{\frac{L}{r^2}dr}{\sqrt{2m[E-V(r)]-\frac{L^2}{r^2}}}$ 代入 $V=-\frac{\alpha}{r^2}$ $=-L\int\frac{d(\frac{1}{r})}{\sqrt{2mE+(2m\alpha-L^2)\frac{1}{r^2}}}$

令 $u=\frac{1}{r}$ $\therefore\theta=-L\int\frac{du}{\sqrt{(2m\alpha-L^2)\cdot u^2+2mE}}$

(1) 若 $2mE>0$ $2m\alpha<L^2$ $\theta=\frac{-L}{\sqrt{L^2-2m\alpha}}\int\frac{du}{\sqrt{\frac{2mE}{L^2-2m\alpha}-u^2}}=\frac{L}{\sqrt{L^2-2m\alpha}}\arccos\sqrt{\frac{L^2-2m\alpha}{2mE}}u+C$

选适当 $\theta$，使 $C=0$，$u=\frac{1}{r}=\sqrt{\frac{2mE}{L^2-2m\alpha}}\cos\left(\sqrt{1-\frac{2m\alpha}{L^2}}\theta\right)$

(2) $2mE>0$ $2m\alpha>L^2$，$\theta=\frac{-L}{\sqrt{2m\alpha-L^2}}\int\frac{du}{\sqrt{u^2+\frac{2mE}{2m\alpha-L^2}}}=\frac{-L}{\sqrt{2m\alpha-L^2}}\operatorname{arcsh}\left(\sqrt{\frac{2m\alpha-L^2}{2mE}}\cdot u\right)+C$

$\theta$ 若使 $C=0$ $u=\frac{1}{r}=\sqrt{\frac{2mE}{2m\alpha-L^2}}\operatorname{sh}\left(\sqrt{\frac{2m\alpha}{L^2}-1}\,\theta\right)$

(3) $2mE<0$，$2m\alpha>L^2$，$\theta=\frac{-L}{\sqrt{2m\alpha-L^2}}\int\frac{du}{\sqrt{u^2-\frac{2m|E|}{2m\alpha-L^2}}}=\frac{-L}{\sqrt{2m\alpha-L^2}}\operatorname{arcch}\left(\sqrt{\frac{2m\alpha-L^2}{2m|E|}}u\right)+C$

(2) 和 (3) 中若 $r=0$ $t=\int_0^r\frac{dr}{\sqrt{\frac{2E}{m}+\frac{2\alpha}{mr^2}-\frac{L^2}{m^2r^2}}}=\int_0^r\frac{d(r^2)}{\sqrt{\frac{8Er^2}{m}+(\frac{8\alpha}{m}-\frac{4L^2}{m^2})}}$

$=\frac{1}{E}\sqrt{\frac{m}{2}}\left(\sqrt{Er^2+\alpha-\frac{L^2}{2m}}-\sqrt{\alpha-\frac{L^2}{2m}}\right)$

3.2 $V=\frac{1}{2}k(r-l)^2$ $t=\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{2}{m_r}[E-\frac{1}{2}k(r-l)^2-\frac{L^2}{m_r r^2}]}}=\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{1}{m}[E-\frac{1}{2}mv_0^2-\frac{L^2}{mr^2}]}}$

将 $m_r=\frac{m}{2}$，$r_0=\sqrt{\frac{mv_0^2}{k}}+l$ 代入

$\theta=\int_0^r d\theta=\int_{r_0}^{r}\frac{\frac{L}{r^2}dr}{\sqrt{m(E-\frac{1}{2}mv_0^2-\frac{L^2}{mr^2})}}$

3.3 $V(x)=\int_{-\infty}^{x}\left(-\frac{\alpha}{x}dr\right)=-\alpha\ln|x|$

$t=\int\frac{dr}{\sqrt{\frac{2}{m}(E+\alpha\ln|x|)-\frac{L^2}{m^2r^2}}}=\int_{-\alpha}^{0}\frac{dr}{\sqrt{\frac{2}{m}[-\alpha\ln|x|+\alpha\ln|x|]}}$

3.4 $\alpha r>>1$ 时 $V\approx0$ 粒子做自由运动

$\alpha r<<1$ 时，$V\approx-\frac{k}{r}$，粒子近似做在 $-\frac{k}{r}$ 势场的开普勒运动

$\alpha r\approx1$ 时，粒子做在势场 $-\frac{k}{er}$ 的开普勒运动

3.6 求粒子在中心力 $F=-\frac{k}{r^2}+\frac{c}{r^3}$ 作用下的轨道方程

解: $V=-\frac{k}{r}+\frac{c}{2r^2}$

$$\theta=\int d\theta=\int\frac{\frac{L}{r^2}dr}{\sqrt{2m(E+\frac{k}{r}-\frac{c}{2r^2})-\frac{L^2}{r^2}}}=\int\frac{-Ld(\frac{1}{r})}{\sqrt{2mE+\frac{2mk}{r}-\frac{1}{r^2}(mc+L^2)}}$$

令 $u=\frac{1}{r}$, $\theta=\frac{-Ldu}{\sqrt{-(mc+L^2)u^2+2mk\cdot u+2mE}}=\frac{L}{\sqrt{A}}\int\frac{-d(u-\frac{B}{2A})}{\sqrt{\frac{B^2+4AD}{4A^2}-(u-\frac{B}{2A})^2}}$

$$u=\frac{mk}{L^2+mc}+\sqrt{\frac{m^2k^2+2mE(L^2+mc)}{(L^2+mc)^2}}\cos\sqrt{\frac{L^2+mc}{L^2}}\theta$$

$$r=\frac{\frac{L^2+mc}{mk}}{1+\sqrt{1+\frac{2mE(L^2+mc)}{m^2k^2}}\cos\sqrt{\frac{L^2+mc}{L^2}}\theta}=\frac{p}{1+e\cos\alpha\theta},\quad p=\frac{L^2+mc}{mk},\quad \alpha=\sqrt{1+\frac{mc}{L^2}}$$

$$e=\sqrt{1+\frac{2mE(L^2+mc)}{m^2k^2}}$$

3.8 $t=\int\frac{dr}{\sqrt{\frac{2}{m}(E+\frac{\alpha}{r})-\frac{L^2}{m^2r^2}}}=\int\frac{dr}{\sqrt{\frac{2\alpha}{mr}-\frac{L^2}{m^2r^2}}}=\int\frac{mrdr}{\sqrt{2m\alpha r-L^2}}$

$L^2\eta^2=2m\alpha r-L^2$ $\quad r=\frac{L^2\eta^2+L^2}{2m\alpha}$

$$t=\int m\frac{L^2\eta^2+L^2}{2m\alpha}\cdot\frac{L^2\eta}{m\alpha}\cdot\frac{1}{L\eta}d\eta=\int\frac{L^3\eta^2+L^3}{2m\alpha^2}d\eta$$

$$=\frac{L^3}{2m\alpha^2}\int(\eta^2+1)d\eta=\frac{L^3}{2m\alpha^2}(\eta+\frac{\eta^3}{3})$$

若 $p=\frac{L^2}{m\alpha}$, $t=\sqrt{\frac{mp^3}{\alpha}}\cdot\frac{\eta}{2}(1+\frac{\eta^2}{3})$

$$\begin{cases}x=r\cos\theta=p-r=p-\frac{L^2(\eta^2+1)}{2m\alpha}=p-\frac{p(\eta^2+1)}{2}=\frac{p}{2}(1-\eta^2)\\ y=r\sin\theta=\sqrt{r^2-x^2}=\sqrt{\frac{p^2}{4}(\eta^2+1)^2-\frac{p^2}{4}(1-\eta^2)^2}=p\cdot\eta\end{cases}$$

3.11 $r=\frac{p}{1+e\cos\theta}$ $\quad V=-\frac{\alpha}{r}$, 有 $E=-\frac{\alpha}{2a}=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V$

周期 $T=2\pi\sqrt{\frac{\cancel{ma^3}\,ma^3}{\alpha}}$ $\quad$ 即 $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-\frac{\alpha}{r}=\frac{-\alpha}{2a}$

即 $\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}=\frac{\alpha}{r}-\frac{\alpha}{2a}$ $\quad \frac{dr}{dt}=\frac{1}{r}\sqrt{\frac{2\alpha}{m}}\sqrt{r-\frac{r^2}{2a}-\frac{L^2}{2m\alpha}}$

$\frac{dt}{r}=\frac{\sqrt{\frac{ma}{\alpha}}dr}{\sqrt{\alpha^2-\frac{L^2a}{m\alpha}-(r-a)^2}}$ $\quad \bar{V}=\frac{1}{t}\int_0^t dt\left|-\frac{a}{r}\right|=\frac{1}{t}\cdot\frac{4\sqrt{m\alpha a}}{t}\arcsin 1=\frac{\alpha}{a}$

$$\bar{T}=\frac{1}{t}\int_0^t\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2)dt=\frac{1}{t}\int_0^t(\frac{\alpha}{r}-\frac{\alpha}{2a})dt=\frac{\alpha}{a}-\frac{\alpha}{2a}=\frac{\alpha}{2a}$$

卢瑟福公式:

$$V = -\frac{a}{r}$$

取转过角度和距离 $\rho$ 为广义坐标

$$P_\phi = \frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi = J$$

$$E = \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2) + V(\rho)$$

$$V(\rho) = -\frac{a}{\rho} \qquad p = -\frac{J^2}{m|a|}$$

由 $\rho = \frac{p}{1+e\cos(\phi-\theta)}$ ~~将~~ $p = -\frac{J^2}{m|a|}$ 得偏心率 $e = \sqrt{1+\frac{2J^2E}{ma^2}}$

$$\cot\frac{\theta}{2} = \sqrt{e^2-1} = \sqrt{\frac{2J^2E}{ma^2}}$$

将 $J = mrv_0$ $E = \frac{mv_0^2}{2}$ 代入

$$\cot\frac{\theta}{2} = \frac{mrv_0^2}{a}$$

$$d\Omega = 2\pi\sin\theta\, d\theta \qquad d\sigma = 2\pi\, r\,dr$$

$$\frac{d\sigma}{d\Omega} = \frac{r}{\sin\theta}\left|\frac{dr}{d\theta}\right|$$

对 $\cot\frac{\theta}{2} = \frac{mrv_0^2}{a}$ 微分 $-\frac{d\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{mv_0^2\,dr}{a}$ 得 $\left|\frac{dr}{d\theta}\right| = \frac{|a|}{2mv_0^2\sin^2\frac{\theta}{2}}$

$$\frac{d\sigma}{d\Omega} = \frac{\cot\frac{\theta}{2}|a|}{mv_0^2\sin\theta}\cdot\frac{|a|}{2mv_0^2\sin^2\frac{\theta}{2}} = \frac{|a|^2}{4m^2v_0^4\sin^4\frac{\theta}{2}}$$

$$= \frac{a^2}{16E^2\sin^4\frac{\theta}{2}}$$