Exercise 12, Discrete Mathematics for Bioinformatics

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12.1 Inverse Queens Problem

a) Variables

$$x_i \in \{1, ..., n\} \quad \text{for } 1 \leq i \leq n$$

Constraints

$$x_i = x_j \vee |x_i - x_j| = |i - j| \quad \forall i \neq j$$

b) Solve for $n = 4$ and $D_1 = \{2\}$.

Forward checking: $D_1 = D_2 = D_3 = D_4 = \{1, 2, 3, 4\}$

- $x_1 = 2 \Rightarrow D_2 = \{1, 2, 3\}, D_3 = \{2, 4\}, D_4 = \{2\}$
  - $x_2 = 1 \Rightarrow D_3 = \{2\}, D_4 = \{\}$ ... dead end.
  - $x_2 = 2 \Rightarrow D_3 = \{2\}, D_4 = \{2\}$
    - Solution found

Patial lookahead: $D_1 = D_2 = D_3 = D_4 = \{1, 2, 3, 4\}$

- $x_1 = 2 \Rightarrow D_2 = \{2\}$ because values 1 or 3 are not arc consistent with x_4. $D_3 = \{2\}$ because value 4 is not arc consistent with x_4. $D_4 = \{2\}$.
  - Solution found

12.2 Task Scheduling

a) We have variables $A, B, C, D \in \{1, ..., 11\}$ and interpret them as the starting times of the tasks, each of them on a seperate machine (tasks can run in parallel). The upper level of 11 derives from the sum of all durations. An example schedule could look like this:

	1	2	3	4	5	6	7	8	9	10	11
A	X	X	X								
B							X	X			
C					X	X	X	X			
D										X	X

Constraints would be

$$A + 3 \leq B, \quad A + 3 \leq C, \quad B + 2 \leq D, \quad C + 4 \leq D$$

b) We add two additional variables S and E with duration 0, $D_S = \{1\}$ and $D_E = \{1, ..., 11\}$ and derive the new constaints:

$$S \leq X \text{ for } X \in \{A, B, C, D\}, \quad A + 3 \leq E, \quad B + 2 \leq E, \quad C + 4 \leq E, \quad D + 2 \leq E$$

c) Arc consistency: We won't perform the AC-3 algorithm due to its complexity. Instead, we use a rather intuitive iteration:

$$\begin{aligned}
D_A &= \{1,2,3,4,5,6,7,8\} \text{ due to } A + 3 \leq B, \quad A + 3 \leq C \\
D_B &= \{4,5,6,7,8,9\} \text{ due to } A + 3 \leq B, \quad B + 2 \leq D \\
D_C &= \{4,5,6,7\} \text{ due to } A + 3 \leq C, \quad C + 4 \leq D \\
D_D &= \{8,9,10,11\} \text{ due to } C + 4 \leq D \\
D_E &= \{10,11\} \text{ due to } D + 2 \leq E \\
D_A &= \{1,2,3,4\} \text{ due to } A + 3 \leq C
\end{aligned}$$

d) By fixing E to the minimal value 10, arc consistency can restrain the domains further:

$$\begin{aligned}
D_D &= \{8\} \text{ due to } D + 2 \leq E \\
D_C &= \{4\} \text{ due to } C + 4 \leq D \\
D_B &= \{4,5,6\} \text{ due to } B + 2 \leq D \\
D_A &= \{1\} \text{ due to } A + 3 \leq B
\end{aligned}$$

12.3 Bin Packing

12.4 IP

Each constraint of the form $|x_i - x_j| \geq 2$ can be rewritten as

$$x_i - x_j \geq 2 \vee x_j - x_i \geq 2$$

We can express the logical *or* by adding a new variable d_{ij} (decision variable):

$$\begin{array}{rcl}
x_i - x_j \geq 2 & \vee & x_j - x_i \geq 2 \\
\Leftrightarrow x_i - x_j \geq d_{ij}(-2 - m) + 2 & \wedge & x_j - x_i \geq (1 - d_{ij})(-2 - m) + 2 \\
\Leftrightarrow x_i - x_j + (2 + m)d_{ij} \geq 2 & \wedge & x_j - x_i - (2 + m)d_{ij} \geq -m
\end{array}$$

The decision variable decides which constraint must be satisfied:

$$\begin{array}{lrcl}
\text{if } d_{ij} = 0 & x_i - x_j \geq 2 & \wedge & x_j - x_i \geq -m \\
\text{if } d_{ij} = 1 & x_i - x_j \geq -m & \wedge & x_j - x_i + \geq 2
\end{array}$$

Note that these equivalencies only hold since $x_i - x_j \geq 2$ and $x_j - x_i \geq 2$ cannot be true at the same time.

In order to model these inequalities $\forall i \neq j$, we have to apply this technique to $n^2 - n$ constraints, thus introducing $(n^2 - n)$ new binary variables.