Exercise 8, Discrete Mathematics for Bioinformatics

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8.1 Bases and Basic Solutions

a)

$$x_1 + x_2 \leq 4, \tag{1}$$
$$x_1 \leq 4, \tag{2}$$
$$-x_1 \leq 0, \tag{3}$$
$$x_2 \leq 2, \tag{4}$$
$$-x_2 \leq 0. \tag{5}$$

We note that (2) is dependent on the other inequalities and can be left out. What remains is

$$Ax = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq b = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

b) Let $M = \{1, 2, 3, 4\}$. We look for bases, i.e. cardinality 2 subsets $I \subset M$ for which A_{I*} is regular. The bases are

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}.$$

The corresponding basic solutions are $A_{I*}^{-1} b_I$, i.e.

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

c) Feasible means $Ax \leq b$. This is the case for (7)–(10) but not for (6), since there $4 = x_2 > 2$.

d) $I = \{1, 3\}$: $x = (2, 2)$,
$I = \{1, 4\}$: $x = (4, 0)$,
$I = \{2, 3\}$: $x = (0, 2)$,
$I = \{2, 4\}$: $x = (0, 0)$.

8.2 Simplex Algorithm

a) Let x_1 = volume of coke, and x_2 = volume of beer. Then we have

$$\text{profit} = u^T x = x_1 + 2x_2 = \max!,$$

$$\text{weight} = x_1 + 1.5x_2 \le 150,$$

$$\text{beer availability} = x_2 \le 35,$$

$$\text{maximum alcoholic volume} = (x_1 + x_2) \cdot 2/3 \ge x_2 \Leftrightarrow -2x_1 + x_2 \le 0,$$

$$-x_1, -x_2 \le 0.$$

A and b can simply be read off.

b) Simplex algorithm: see handwritten sheet.

8.3 Duality

a) By reducing the inequalities we obtain

$$24x_1 + 36x_2 + 18x_3 \le 360 \Leftrightarrow 4x_1 + 6x_2 + 3x_3 \le 60,$$

$$12 \cdot 6x_1 + 12 \cdot 6x_2 + 12 \cdot 2x_3 \le 48 \Leftrightarrow 3x_1 + 3x_2 + x_3 \le 2,$$

$$3x_1 + 4x_2 + 2x_3 = \max!.$$

(The second inequality holds because there are 12 acres of land.)

b) Dual problem:

$$60u_1 + 2u_2 = \min!,$$

$$u_1, u_2 \ge 0,$$

$$\begin{pmatrix} 4 & 3 \\ 6 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.$$

Economic interpretation: minimize the total production cost, while proceeds on potatoes ≥ 3, proceeds on corn ≥ 4, proceeds on wheat ≥ 2, with non-negative prices.

c) Original problem: profit $= 4$, $(x_1, x_2, x_3) = (0, 0, 2)$. Dual problem: cost $= 4$, $(u_1, u_2) = (0, 2)$. Same optimal values.