Empirical IO Homework
Dynamic Entry/Exit game

Suppose that in each time period, a set of firms producing homogeneous products play a simultaneous move Nash quantity-setting game. The market-level inverse demand curve in market in time t is given by

$$p_t = 10 - Q_t + x_t$$

where Q_t is the total quantity produced by all the firms, and x_t is a demand shifter. Assume that all firms have 1) a marginal cost of 0, and 2) a fixed cost of operation of 5 per-period.

1) Given these cost and demand primitives, derive a equation expressing the Nash-equilibrium single period profits per firm, as depending on the total number of firms N_t and the demand shifter x_t. In other words, derive the formula for $\pi(N_t, x_t)$. This formula will be useful for the computations later.

Next, consider the dynamic entry and exit process. First, assume that the demand shifter x_t takes on 3 possible values $\{-5, 0, 5\}$, and that this evolves over time according to the following first order Markov transition matrix (e.g. given $x_t = 5$, the probability that $x_{t+1} = -5$ is 0.2):

			x_{t+1}	
		-5	0	5
	-5	0.6	0.2	0.2
x_t	0	0.2	0.6	0.2
	5	0.2	0.2	0.6

For the exit process, assume that in each period, each active firm gets a stochastic sell off value:

$$\mu_{it} \sim N(\mu, \sigma_\mu^2)$$

This value μ_{it} is assumed to be private information for the firm, and it is assumed to be i.i.d. across firms and time. If a firm decides to exit at period t, it receives current payoff of just μ_{it} (note that this is slightly different than the model in class) and is then out of the market forever. This implies that the value function for an active firm is

$$V(N_t, x_t, \mu_{it}) = \max_{d_{it} \in \{0,1\}} \left\{ \mu_{it}, \pi(N_t, x_t) + 0.9E\left[V(N_{t+1}, x_{t+1}, \mu_{it+1}) | N_t, x_t, d_{it} = 1\right] \right\}$$

where $d_{it} = 0$ if the firm exits and $d_{it} = 1$ if the firm remains in the market.

For the entry process, assume that in each period, there is one "potential entrant" into the market. This potential entrant is endowed with a stochastic entry cost:

$$\gamma_{it} \sim N(\gamma, \sigma_\gamma^2)$$

that they must pay to enter the market (γ_{it} is also private information to the entrant and assumed iid across time). If the firm decides to enter at t, their single period payoffs at t are $-\gamma_{it}$ - i.e. they only begin to produce and make profits in the next period. If the firm decides not to enter at t, they disappear from the market forever. This implies that the value function for a potential entrant is:

$$V^E(N_t, x_t, \gamma_{it}) = \max_{e_{it} \in \{0,1\}} \left\{ 0, -\gamma_{it} + 0.9E\left[V(N_{t+1}, x_{t+1}, \gamma_{it+1}) | N_t, x_t, e_{it} = 1\right] \right\}$$

where $e_{it} = 1$ if the firm enters and $e_{it} = 0$ if the firm does not enter the market.

We will focus attention on symmetric Markov-Perfect Nash Equilibria of this game that take the following "cutoff" form:

$$\text{For an incumbent Firm:} \qquad d_{it} = \begin{cases} 0 & \text{if } \mu_{it} > \overline{\mu}(N_t, x_t) \\ 1 & \text{if } \mu_{it} \leq \overline{\mu}(N_t, x_t) \end{cases}$$

$$\text{For a potential entrant:} \qquad e_{it} = \begin{cases} 0 & \text{if } \gamma_{it} > \overline{\gamma}(N_t, x_t) \\ 1 & \text{if } \gamma_{it} \leq \overline{\gamma}(N_t, x_t) \end{cases}$$

2) Why does a cutoff equlbrium make intuitive sense in this setup?

Assume that the maximum number of firms in the market is 5 (i.e. when $N_t = 5$, assume that the entry cost γ_{it} is ∞ with probability 1.) This means that there are 18 possible values of the state i.e. $N_t \in \{0, 1, 2, ...5\}$, $x_t \in \{-5, 0, 5\}$. Therefore, equilibrium can be summarized by 36 cutoff numbers, $\overline{\mu}(N_t, x_t)$ at each of the 18 possible states, and $\overline{\gamma}(N_t, x_t)$ at each of the 18 possible states. Actually, note that only 30 of these numbers are relevant, since in states where $N_t = 0$ there are no incumbents (so $\overline{\mu}(0, x_t)$ is irrelevant), and in states where $N_t = 5$ there can be no entry, (so $\overline{\gamma}(5, x_t)$ is irrelevant).

Also note that the 2 value functions can be alternatively written as:

$$V(N_t, x_t, \mu_{it}) = \max_{d_{it} \in \{0,1\}} \left\{ \mu_{it} \ , \ \pi(N_t, x_t) + 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \int V(N_{t+1}, x_{t+1}, \mu_{it+1}) p(d\mu_{it+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_{it}, d_{it} = 1) \right.$$

$$V^E(N_t, x_t, \gamma_{it}) = \max_{e_{it} \in \{0,1\}} \left\{ 0, -\gamma_{it} + 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \int V(N_{t+1}, x_{t+1}, \mu_{it+1}) p(d\mu_{it+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, e_{it} = 1) \right\}$$

Then, defining

$$\overline{V}(N_t, x_t) = \int V(N_t, x_t, \mu_{it}) p(d\mu_{it})$$

we can write the value functions as:

$$V(N_t, x_t, \mu_{it}) = \max_{d_{it} \in \{0,1\}} \left\{ \mu_{it}, \pi(N_t, x_t) + 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \overline{V}(N_{t+1}, x_{t+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, d_{it} = 1) \right\}$$

$$V^E(N_t, x_t, \gamma_{it}) = \max_{e_{it} \in \{0,1\}} \left\{ 0, -\gamma_{it} + 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \overline{V}(N_{t+1}, x_{t+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, e_{it} = 1) \right\}$$

Note that $\overline{V}(N_t, x_t)$ is not defined when $N_t = 0$ (this follows because $V(N_t, x_t, \mu_{it})$ is not defined at $N_t = 0$, i.e. there is no value function for an incument at $N_t = 0$ because there is no incumbent with $N_t = 0$). .

3) Describe (at an intuitive level) what $\overline{V}(N_t, x_t)$ measures. How does this relate to the alternative specific value functions of Rust (1987)?

4) Assume that the parameters of the model take the values: $\gamma = 5, \sigma^2_\gamma = 5, \mu = 5$, and $\sigma^2_\mu = 5$. Use the following iterative process to solve for the equilibrium (lets assume at the moment that it is unique). Note that this process is related to, but slightly different than the procedure I outlined in class:

1. Guess $\overline{\mu}(N_t, x_t)$, $\overline{\gamma}(N_t, x_t)$, and $\overline{V}(N_t, x_t)$ (note: this is just 3 vectors of 15 numbers, since $\overline{\mu}(N_t, x_t)$ and $\overline{V}(N_t, x_t)$ are not defined when $N_t = 0$, and $\overline{\gamma}(N_t, x_t)$ is not defined when $N_t = 5$)

2. Using $\overline{\mu}(N_t, x_t)$ and $\overline{\gamma}(N_t, x_t)$, compute the transition matrices $\Pr(N_{t+1}|N_t, x_t, d_{it} = 1)$ and $\Pr(N_{t+1}|N_t, x_t, e_{it} = 1)$. There should be six of these matrices (for $d_{it} = 1$, there should be 1 for each of the 3 x_t's, and for $e_{it} = 1$ there should be one for each of the 3 x_t's) Why are the two matrices different (for a given x_t)?

3. Compute

$$\begin{aligned}
\Psi_1(N_t, x_t) &= 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \overline{V}(N_{t+1}, x_{t+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, d_{it} = 1) \\
\Psi_2(N_t, x_t) &= 0.9 \sum_{N_{t+1}} \sum_{x_{t+1}} \overline{V}(N_{t+1}, x_{t+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, e_{it} = 1)
\end{aligned}$$

4. Solve for "new" optimal cutoffs at each state using:

$$\begin{aligned}
\overline{\mu}'(N_t, x_t) &= \pi(N_t, x_t) + \Psi_1(N_t, x_t) \\
\overline{\gamma}'(N_t, x_t) &= \Psi_2(N_t, x_t)
\end{aligned}$$

   Why do these equations tell us the new optimal cutoffs (given the computed values of $\Pr(N_{t+1}|N_t, x_t, d_{it} = 1)$ and $\Pr(N_{t+1}|N_t, x_t, e_{it} = 1)$)?

5. Solve for "new" $\overline{V}'(N_t, x_t)$ at each state using:

$$\begin{aligned}
\overline{V}'(N_t, x_t) &= \int V(N_t, x_t, \mu_{it}) p(d\mu_{it}) \\
&= \int \max_{d_{it} \in \{0,1\}} \left\{ \mu_{it}, \pi(N_t, x_t) + .9 \sum_{N_{t+1}} \sum_{x_{t+1}} \overline{V}(N_{t+1}, x_{t+1}) \Pr(x_{t+1}|x_t) \Pr(N_{t+1}|N_t, x_t, d_{it}{=}1) \right\} p(d\mu_{it}) \\
&= \int \max_{d_{it} \in \{0,1\}} \{\mu_{it}, \pi(N_t, x_t) + \Psi_1(N_t, x_t)\} p(d\mu_{it}) \\
&= \left[1 - \Phi\left(\frac{\pi(N_t, x_t) + \Psi_1(N_t, x_t) - \mu}{\sigma_\mu}\right)\right] \left[\mu + \sigma_\mu * \left(\frac{\phi\left(\frac{\pi(N_t,x_t)+\Psi_1(N_t,x_t)-\mu}{\sigma_\mu}\right)}{1 - \Phi\left(\frac{\pi(N_t,x_t)+\Psi_1(N_t,x_t)-\mu}{\sigma_\mu}\right)} \right)\right] \\
&\quad + \Phi\left(\frac{\pi(N_t, x_t) + \Psi_1(N_t, x_t) - \mu}{\sigma_\mu}\right) [\pi(N_t, x_t) + \Psi_1(N_t, x_t)]
\end{aligned}$$

   where ϕ and Φ are the standard normal pdf and cdf. Where is the last equality coming from?

6. With the new $\overline{\mu}'(N_t, x_t)$, $\overline{\gamma}'(N_t, x_t)$, and $\overline{V}'(N_t, x_t)$, go back to step 1.

7. Iterate procedure until convergence.

5) Try resolving for the equilibrium (question 4) starting with 5 different guesses of the initial values of $\overline{\mu}(N_t, x_t)$, $\overline{\gamma}(N_t, x_t)$, and $\overline{V}(N_t, x_t)$. Do you find any evidence of multiple equilibria?

6) At the equilibrium (or one of the equilibria - if there are multiple equilibria, you can pick whichever one you like), tell me the values $\overline{\mu}(3, 0)$, $\overline{\gamma}(3, 0)$, $\overline{V}(3, 0)$, and $V(3, 0, -2)$.

7) Consider a market starting out with 0 firms and $x_t = 0$. Simulate this structure of this market 10000 periods into the future. You can do this using only the equilibrium $\overline{\mu}(N_t, x_t)$ and $\overline{\gamma}(N_t, x_t)$ functions you

have computed (along with $\Pr(x_{t+1}|x_t)$) . Note that you will need to take draws from the μ_{it} and γ_{it} distributions to do this. What is the average number of firms in the market across these 10000 periods?

8) Suppose the government decided to implement a 5 unit entry tax. What happens to the average number of firms in equilibrium? (note: you will need to resolve for a new equilibrium to do this). Can you answer this question if there are multiple equilibria? Why or why not?

9) Using the simulated data from Question 7), use a BBL-like estimator to estimate $\gamma, \sigma_\gamma^2, \mu$, and σ_μ^2. In other words assume you know everything about the model (demand, fixed and marginal costs, discount factor, $p(x_{t+1}|x_t)$ distribution) except for these 4 parameters, and estimate them. More specifically, estimation should proceed as follows:

A) Estimate the reduced form *expected* policy functions, $\widehat{d}(N_t, x_t)$ and $\widehat{e}(N_t, x_t)$, directly from the data. At each state, these policy functions tell us, respectively, 1) the probability that any specific incumbent stays in the market, and 2) the probability that the potential entrant enters. You can estimate these with a frequency simulator, e.g. $\widehat{d}(N_t, x_t)$ is the proportion of all incumbents in the data at state (N_t, x_t) who decided to stay in the market, $\widehat{e}(N_t, x_t)$ is the proportion of times in the data where at state (N_t, x_t), the entrant decided to enter. Note that you do not need to estimate $\widehat{d}(N_t, x_t)$ when $N_t = 0$, nor do you need to estimate $\widehat{e}(N_t, x_t)$ when $N_t = 5$ (if you want, you can just set $\widehat{d}(0, x_t) = \widehat{e}(5, x_t) = 0$). If there are other states (N_t, x_t) that are never reached in the dataset, use the values of $\widehat{d}(N_t, x_t)$ and $\widehat{e}(N_t, x_t)$ from a nearby state that was observed in the data. Alternatively, you can estimate $\widehat{d}(N_t, x_t)$ and $\widehat{e}(N_t, x_t)$ using a probit or logit model with a high order polynomial in N_t and x_t.

B) Guess the parameters $\gamma, \sigma_\gamma^2, \mu$, and σ_μ^2. At each possible value of the state, use $\widehat{d}(N_t, x_t)$, $\widehat{e}(N_t, x_t)$, $\pi(N_t, x_t)$ and $\Pr(x_{t+1}|x_t)$ to "forward" simulate the PDV of *one* of the incumbents (call this firm "Firm 1") *conditional on that firm deciding not to exit in the initial period* (clearly, you can't do this at states where $N_t = 0$). This involves simulating 1) decisions of potential entrants in the current period and the future, 2) decisions of firms other than Firm 1 in the current period and the future, and 3) decisions of Firm 1 in the future (since we are already conditioning on Firm 1 *not exiting* in the current period, we don't need to simulate Firm 1's current decision). To simulate all these decisions, you will need to take draws of μ_{it} and γ_{it} (these draws will obviously depend on the parameters. **Create them by taking draws from $N(0,1)$'s, multiplying by σ_μ (or σ_γ) and adding μ (or γ). Importantly (and like we did in the BLP problem set), hold the underlying $N(0,1)$ draws constant as the parameters change** (also hold the draws on the x_t process the same across simulations). Given such draws, an incumbent stays in the market if

$$\Phi\left(\frac{\mu_{it} - \mu}{\sigma_\mu}\right) \leq \widehat{d}(N_t, x_t)$$

and an entrant enters if

$$\Phi\left(\frac{\gamma_{it} - \gamma}{\sigma_\gamma}\right) \leq \widehat{e}(N_t, x_t)$$

Note that these equations are implied by the cutoff equilibrium, because if, e.g. $\widehat{d}(N_t, x_t) = 0.25$, then firms with μ_{it}'s in the lower quartile of the distribution must be the ones that are staying in the market. Simulate forward in time until either Firm 1 exits the market (or until t gets very large, e.g. 100) and add up Firm 1's total discounted profits. Note that if Firm 1 eventually exits, you will need to add her simulated μ_{it} in that period (appropriately discounted) into total profits to compute the simulated total stream of profits.

C) Do the above forward simulation process 50 times for one of the incuments in each of the 15 states (i.e. all the states except those with $N_t = 0$), and average across these 50 runs to get an expectation of this PDV. Call these values $\Lambda(N_t, x_t \mid \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$.

D) Note that according to the model, a firm will stay in the market if

$$\mu_{it} < \Lambda(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$$

which, given the distribution of μ_{it}, should happen with probability

$$\Pr(\text{incumbent "stays in" at } N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) = \Phi\left(\frac{\Lambda(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \mu}{\sigma_\mu}\right)$$

Therefore, if the simulations of $\Lambda(N_t, x_t \mid \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$ were done at the true parameters $\gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2$, one would expect

$$\Phi\left(\frac{\Lambda(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \mu}{\sigma_\mu}\right) \approx \widehat{d}(N_t, x_t) \qquad \text{at every } N_t, x_t \tag{1}$$

E) Do a similar forward simulation process to compute the expected PDV of future profits for a potential entrant in each of the 15 states (ignore states where $N_t = 5$). More specifically, what you want to simulate here is the expected PDV of a potential entrant entering, *net of the entry cost* γ_{it} (by "net of the entry cost γ_{it}" I mean that you should not add the $-\gamma_{it}$ for the entering period into the total PDV). Denote these simulated values for each of the 15 states by $\Lambda^E(N_t, x_t \mid \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$. Similar to Step 4), according to the model, a potential entrant will enter if

$$\Lambda^E(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) > \gamma_{it}$$

which, given the distribution of γ_{it}, should happen with probability

$$\Pr(\text{entrant enters at } N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) = \Phi\left(\frac{\Lambda^E(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \gamma}{\sigma_\gamma}\right)$$

Therefore, if the simulations of $\Lambda^E(N_t, x_t \mid \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$ were done at the true parameters $\gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2$, one would expect

$$\Phi\left(\frac{\Lambda^E(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \gamma}{\sigma_\gamma}\right) \approx \widehat{e}(N_t, x_t) \qquad \text{at every } N_t, x_t \tag{2}$$

F) Given that equations (1) and (2) should hold at the true parameters, we will base estimation on them. More specifically, you should use a search procedure to find the $\gamma, \sigma_\gamma^2, \mu$, and σ_μ^2 that minimizes the following minimum distance criterion:

$$\begin{aligned} \min \sum_{N_t=1}^{5} \sum_{x_t} & \left(\Phi\left(\frac{\Lambda(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \mu}{\sigma_\mu}\right) - \widehat{d}(N_t, x_t) \right)^2 \\ + \sum_{N_t=0}^{4} \sum_{x_t} & \left(\Phi\left(\frac{\Lambda^E(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2) - \gamma}{\sigma_\gamma}\right) - \widehat{e}(N_t, x_t) \right)^2 \end{aligned}$$

10) You do not need to compute standard errors for your estimate (note that you wouldn't be able to use standard minimum distance variance formulas, since those do not take into account the estimation error in $\Lambda(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$ and $\Lambda^E(N_t, x_t | \gamma, \sigma_\gamma^2, \mu, \sigma_\mu^2)$ (due to the fact that these quantities depend on $\widehat{d}(N_t, x_t)$ and $\widehat{e}(N_t, x_t)$). Does your estimation procedure do a good job of estimating the true parameters?