Appendix: Neural Reasoning for Sure through Constructing Explainable Models

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Abstract

This supplementary document starts with an introduction to the geometry of the Poincaré disk and intuitions behind HSphNN. Then, we describe the technical implementation and prove that HSphNN reasons syllogistic statements *for sure* in one epoch ($M = 1$). At the end, we list 24 valid types of Aristotelian syllogistic reasoning.

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1. An introduction to the geometry of Poincaré spheres

The Poincaré disk is a hyperbolic plane defined on an open Euclidean unit sphere in n-dimensional space [1, 2, 3].

$$\mathcal{H} = \{(x_1, \ldots, x_n) | x_1^2 + \cdots + x_n^2 < 1\}$$

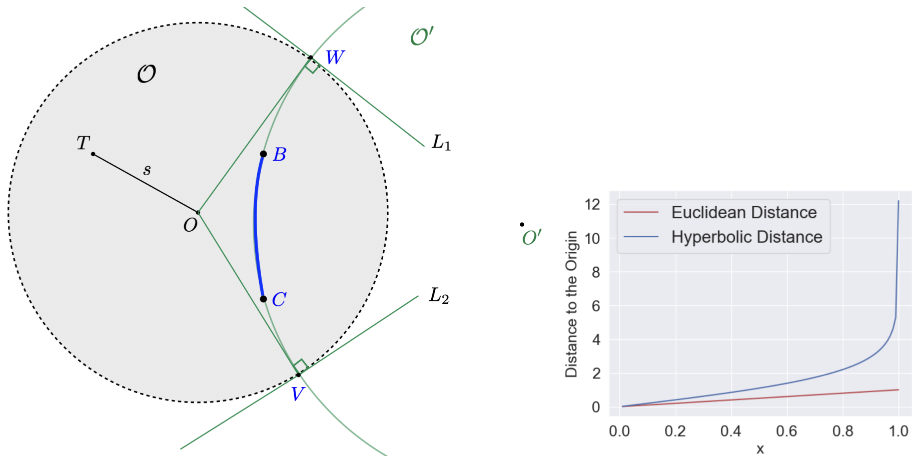
Let O be the centre of the Euclidean unit sphere and T be a point within this sphere, s be the Euclidean distance between O and T, Figure 1(a) shows the case of $n = 2$, the hyperbolic distance between O and A, $d_{\mathbb{D}}(s)$ is defined as follows.

$$d_{\mathbb{D}}(s) \triangleq \log \frac{1+s}{1-s} = 2\tanh^{-1}(s)$$

When $s \to 1$, $d_{\mathbb{D}}(s) \to +\infty$. Thus, the Poincaré disk has an unlimited hyperbolic space. The Poincaré distance metric is defined as

$$d\rho = \frac{2ds}{1-s^2}$$

This formula suggests that the same distance change in Euclidean metric causes different distance changes in hyperbolic metric. The difference is determined by how far away (Euclidean distance) this point is from the centre O. When close to the boundary of the Poincaré disk, a tiny change in Euclidean distance will cause a huge change in hyperbolic distance, as illustrated in Figure 1(b).



(a) (b)

Figure 1: (a) The grey circle is the Poincaré disk; the blue arc is the shortest distance between B and C in hyperbolic metric. This arc is part of the circumference of circle O' that intersects with circle O at W and V. L_1 is a tangential line of O, $|OW|$ is a tangential line of O', L_1 is perpendicular to $|OW|$; L_2 is a tangential line of O, $|OV|$ is a tangential line of O', L_2 is perpendicular to $|OV|$; (b) The relation between Euclidean and Poincaré hyperbolic distances. The picture is copied from [2].

1.1. Shortest Poincaré distances

The shortest hyperbolic distance between B and C in an n-dimensional Poincaré disk is not the straight line segment in Euclidean space, rather an arc (Geodesic) of a 2-dimensional circle $\mathcal{O}'$ [4]. The three points O, B, C determine a 2-dimensional plane. This plane intersects the original disk in a 2-dimensional Poincaré disk $\mathcal{O}$. In this plane, another circle $\mathcal{O}'$ intersects the circumference of $\mathcal{O}$ at W and V, as shown in Figure 2, with the condition that: (1) the line L_1 (L_2) passes W (V) and tangential with the circle O, and (2) the line $|OW|$ ($|OV|$) is tangential with the circle O' and perpendicular with the tangential line L_1 (L_2).

1.2. Euclidean and Poincaré spheres

A Euclidean sphere in the Poincaré disk still has a spherical shape in hyperbolic metric, but the hyperbolic centre is offset towards the boundary of the Poincaré disk. Let $\mathcal{O}_1$ have the Euclidean centre O_1 and the Euclidean radius r_1 and the Euclidean distance between O_1 and the origin of Poincaré disk O be l_1, as illustrated in Figure 2.

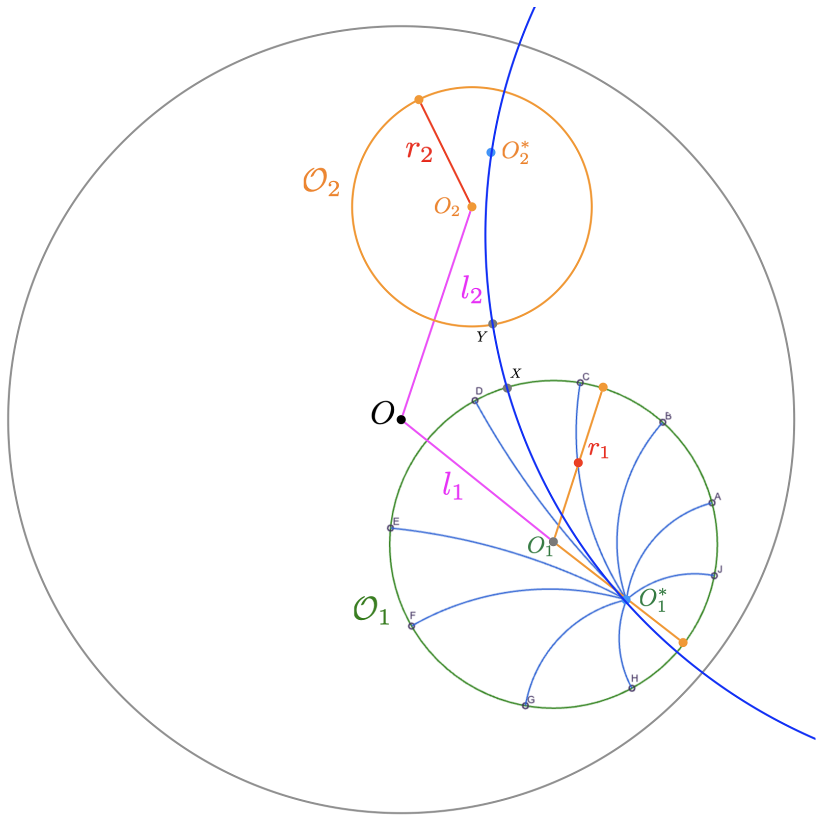


Figure 2: Sphere parameters in Euclidean and hyperbolic metrics: O is the Euclidean centre of Poincaré disk, O_1 and O_1^* are the Euclidean and hyperbolic centres of sphere $\mathcal{O}_1$, respectively. Then, O, O_1, and O_1^* are co-linear (in both Euclidean and hyperbolic metric).

The Poincaré diameter of $\mathcal{O}_1$ is $2\mathbb{D}(r_1)$, where $\mathbb{D}(r_1)$ is its Poincaré radius. We have

$$2\mathbb{D}(r_1) = d_{\mathbb{D}}(l_1 + r_1) - d_{\mathbb{D}}(l_1 - r_1)$$

Thus,

$$\mathbb{D}(r_1) = \tanh^{-1}(l_1 + r_1) - \tanh^{-1}(l_1 - r_1)$$

Let O_1^* be the Poincaré centre of $\mathcal{O}_1$, the Poincaré distance $d_{\mathbb{D}}(|OO_1^*|)$ can be computed as follows.

$$d_{\mathbb{D}}(|OO_1^*|) = d_{\mathbb{D}}(l_1 - r_1) + \mathbb{D}(r_1) = \tanh^{-1}(l_1 + r_1) + \tanh^{-1}(l_1 - r_1)$$

Thus, $|OO_1^*| = \tanh(\frac{d_{\mathbb{D}}(|OO_1^*|)}{2})$. The Euclidean vector of O_1^* is

$$|OO_1^*|\frac{O_1}{\|O_1\|} = \tanh\left(\frac{\tanh^{-1}(l_1 + r_1) + \tanh^{-1}(l_1 - r_1)}{2}\right)\frac{O_1}{\|O_1\|}$$

1.3. Qualitative spatial relations in the hyperbolic metric

In the setting of syllogistic reasoning, we work with configurations of qualitative spatial relations between n-dimensional Poincaré spheres, for example, $\mathbf{D}$, $\mathbf{P}$, $\overline{\mathbf{P}}$, $\mathbf{PO}$. Each is a linear relation of their radii and the distance between their centres; for example, two spheres disconnecting from each other means the distance between their centres is greater than or equal to the sum of their radii. In Euclidean metric, we are familiar with

$$dis(O_1, O_2) \geq r_1 + r_2$$

where $dis(\vec{O}_1, \vec{O}_2) \triangleq \|\vec{O}_1 - \vec{O}_2\|$. In hyperbolic metric, we show the formula will be

$$d_{\mathbb{D}}(O_1^*, O_2^*) \geq \mathbb{D}(r_1) + \mathbb{D}(r_2)$$

As illustrated in Figure 2, the green sphere $\mathcal{O}_1$ disconnects from the orange sphere $\mathcal{O}_2$. In hyperbolic metric, the shortest distance between their centres O_1^* and O_2^* is the blue arc $\widehat{O_1^*O_2^*}$ (Geodesic) that passes O_1^* and O_2^*, perpendicular with the boundary of the Poincaré disk. Let this blue $\widehat{O_1^*O_2^*}$ intersect with the boundary of the orange sphere at Y and with the boundary of the green sphere at X, as shown in Figure 2. As it is a Geodesic, the blue arc $\widehat{O_1^*X}$ ($\widehat{O_2^*Y}$) is the shortest path between O_1^* and X (O_2^* and Y), therefore, it is the hyperbolic radius $\mathbb{D}(r_1)$ ($\mathbb{D}(r_2)$). So, $d_{\mathbb{D}}(O_1^*, O_2^*) \geq \mathbb{D}(r_1) + \mathbb{D}(r_2)$. In our setting of syllogistic reasoning, Euclidean and hyperbolic metrics have the same way of describing qualitative spatial relations between spheres as follows.

Sphere $\mathcal{O}_G$ is part of Sphere $\mathcal{O}_C$, if

$$dis(C, G) + r_G \leq r_C$$

or

$$d_{\mathbb{D}}(C_0, G_0) + \mathbb{D}(r_G) \leq \mathbb{D}(r_C)$$

where r_G and r_C are Euclidean radii of $\mathcal{O}_G$ and $\mathcal{O}_C$, respectively, as illustrated by the **cyan** sphere and the **blue** sphere in Figure 3.

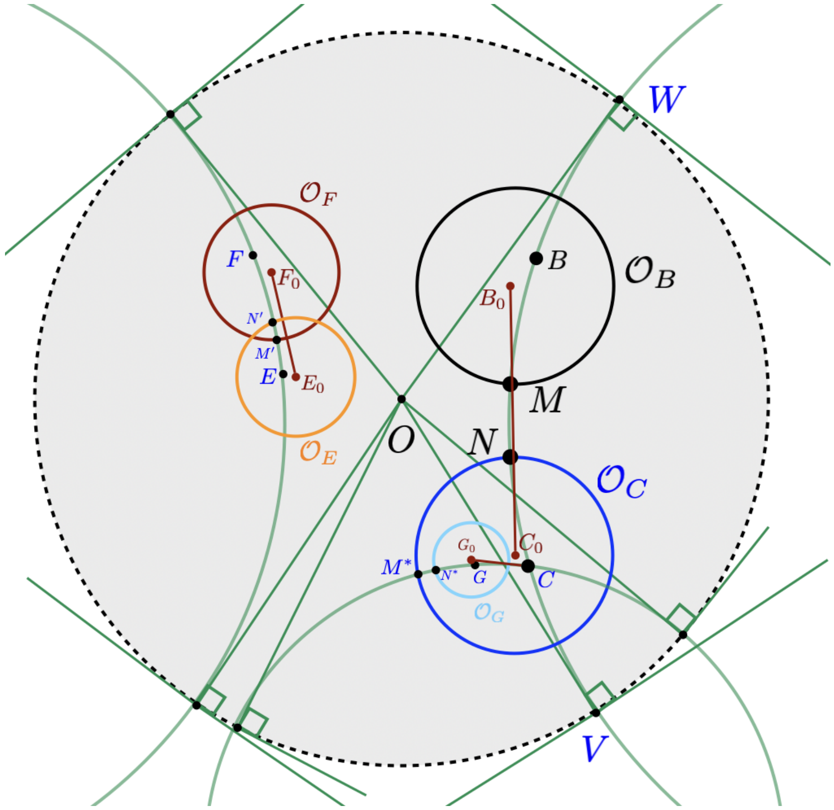


Figure 3: Qualitative spatial relations between two spheres in a Poincaré disk can be being disjoint ($\mathcal{O}_B$ and $\mathcal{O}_C$), being part of ($\mathcal{O}_G$ and $\mathcal{O}_C$), being partially overlapped ($\mathcal{O}_F$ and $\mathcal{O}_E$). Each relation is a linear relation of the centre distance and radii, and can be described in the same way in Euclidean and hyperbolic metric. B, C, F, E, G are hyperbolic centre of Poincaré sphere $\mathcal{O}_B$, $\mathcal{O}_C$, $\mathcal{O}_F$, $\mathcal{O}_E$, $\mathcal{O}_G$, respectively; and B_0, C_0, F_0, E_0, G_0 are their Euclidean centres, respectively.

Sphere $\mathcal{O}_F$ is partially overlapped with Sphere $\mathcal{O}_E$, if

$$|r_G - r_C| < dis(F, E) < r_G + r_C$$

or

$$|\mathbb{D}(r_F) - \mathbb{D}(r_E)| < d_{\mathbb{D}}(F_0, E_0) \leq \mathbb{D}(r_F) + \mathbb{D}(r_E)$$

where r_F and r_E are Euclidean radii of $\mathcal{O}_F$ and $\mathcal{O}_E$, respectively, as illustrated by the **brownish** sphere and the **orange** sphere in Figure 3.

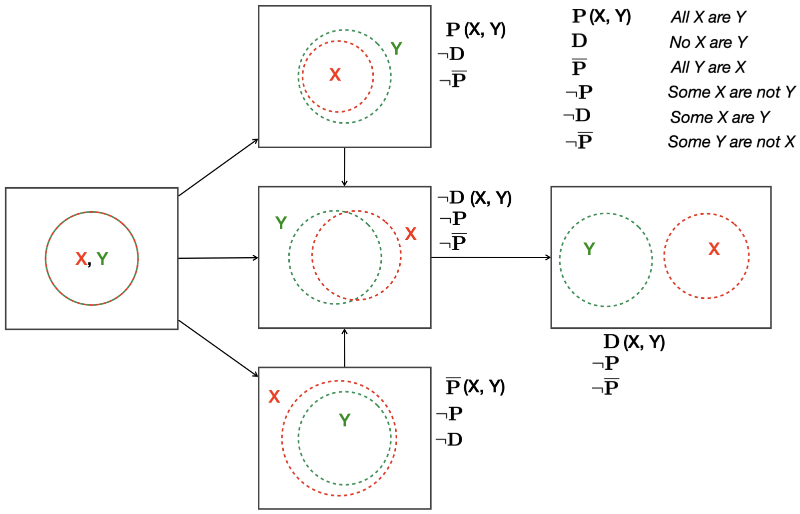


Figure 4: Sphere X and Sphere Y are coincided, and Sphere Y is fixed. When X moves away from Y while changing the radius, all six target relations of syllogistic reasoning will be generated.

2. Intuitions behind HSphNN with its technical implementation

We describe the intuitions behind the design of HSphNN.

2.1. Intuitions

Intuition 1. *In traditional neural networks, gradual descent functions optimise vectors. An n dimensional sphere is a structure of a centre and a radius and can be structured by an $n+1$ dimensional vector, where the first n components represent the centre and the last component represents the radius. Thus, gradual descent functions can also optimise spheres.*

Intuition 2. *For any two coincided spheres, we fix one and let the other move away, with the allowance to change the size during the movement. This movement can generate all target relations for syllogistic reasoning, as illustrated in Figure 4.*

Intuition 3. *In order to use gradual descent operations for optimising qualitative spatial relations between spheres, we need a loss function for each neighbourhood transition between qualitative spatial relations. We need a criterion for each loss function to*

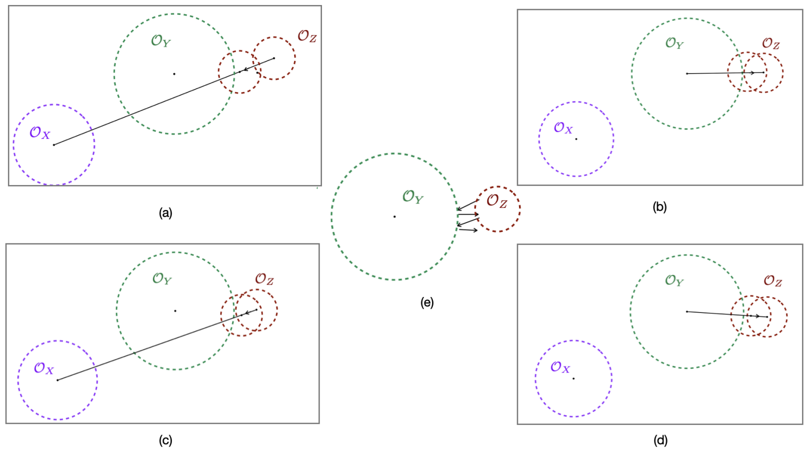


Figure 5: (a) $\mathcal{O}_Z$ is moving towards $\mathcal{O}_X$ following the line with the fastest falling gradient and hits $\mathcal{O}_Y$; (b) $\mathcal{O}_Z$ is moving away from $\mathcal{O}_Y$ following the line with the fastest falling gradient, before moving towards $\mathcal{O}_X$; (c) $\mathcal{O}_Z$ is moving towards $\mathcal{O}_X$ and hits $\mathcal{O}_Y$ again; (d) $\mathcal{O}_Z$ is moving away from $\mathcal{O}_Y$ again. (d) $\mathcal{O}_Z$ is moving away from $\mathcal{O}_Y$, before moving towards $\mathcal{O}_X$; (e) the repeated zigzag movement of $\mathcal{O}_Z$ successfully circumvents the obstacle sphere $\mathcal{O}_Y$.

examine whether the neighbourhood relation has arrived and to guarantee the arrival. We define each loss function is monotonous, and its value is always greater than zero, representing how close to the target relation. The value equals zero only when the neighbourhood relation is arrived. We enumerate these loss functions in Section 2.4 and integrate them into a transition system in Section 2.5.

Intuition 4. *With gradual descent operations, a sphere will optimise its relation with another sphere by moving along the shortest path. However, there may be other spheres on this path that the moving sphere should not touch. We need to invent a novel gradual descent operation that allows the moving sphere to circumvent obstacles. Our solution is that if the moving sphere hits a sphere that it should avoid, this moving sphere shall move away from this obstacle sphere first, then continue to move to the target. This zigzag movement will lead the sphere to circumvent the obstacle sphere, as illustrated in Figure 5. We prove that this process is a gradual descent operation from Section 3.5 to Section 3.7.*

Intuition 5. *Let* $f : \mathbb{R} \rightarrow \mathbb{R}^{+} \cup \{0\}$ *be a real function. If* $f(x)$ *is continuous and* *strictly monotonically decreasing, and there is x^* such that $f(x^*) = 0$, then the gradual descent operation on $f(x)$ will find x' such that $f(x') = 0$. In the setting of syllogistic reasoning, x is a configuration of spheres, and $f(x)$ is the metric to evaluate how close to the target configuration. We need to prove that all operations on $f(x)$ are continuous and strictly monotonically decreasing, and that target configuration exists. We diagram the procedure in Figure 14.*

2.2. Geometric operations

We introduce a set of geometric operations on a sphere $\mathcal{O}_X$ to update its relation to a fixed Sphere $\mathcal{O}_V$.

$$\delta(\mathcal{O}_X|\mathcal{O}_V) = \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}, \mathbb{D}(r_X)^{\downarrow}, d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\uparrow}, \mathbb{D}(r_X)^{\uparrow}\}$$

where $d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)$ is the hyperbolic distance between their centres, or shortened as $d_{\mathbb{D}}$, $\downarrow$ represents to decrease a value, $\uparrow$ represents to increase a value. The target relation $\mathbf{T}$ determines possible operations, either to preserve the already reached target relation or to transform it into a neighbourhood relation towards the target. For example, to preserve $\mathcal{O}_X$ being inside $\mathcal{O}_V$, $\mathbf{T}(\mathcal{O}_X,\mathcal{O}_V) = \mathbf{P}(\mathcal{O}_X,\mathcal{O}_V)$, the possible operations on $\mathcal{O}_X$ are either to decrease the distance between their centres, or to decrease the radius of $\mathcal{O}_X$, written as $\delta^{\mathbf{P}}(\mathcal{O}_X|\mathcal{O}_V) = \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}, \mathbb{D}(r_X)^{\downarrow}\}$; to transform $\mathcal{O}_X$ from being disjoint with $\mathcal{O}_V$ to partially overlapping with $\mathcal{O}_V$, the possible operations on $\mathcal{O}_X$ are either to decrease the distance, or to increase $\mathbb{D}(r_X)$, so, $\delta_{\mathbf{D}:\mathbf{PO}}(\mathcal{O}_X|\mathcal{O}_V) = \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}, \mathbb{D}(r_X)^{\uparrow}\}$. If this neighbourhood transition is targeted at making $\mathcal{O}_X$ be inside $\mathcal{O}_V$, the current operation of increasing $\mathbb{D}(r_X)$ will violate the possible operations of the target relation and may introduce unnecessary back-and-forth updates of $\mathcal{O}_X$, so, $\mathbb{D}(r_X)^{\uparrow}$ will not be selected. So, the set of possible operations are the intersections of $\delta^{\mathbf{P}}(\mathcal{O}_X|\mathcal{O}_V)$ and $\delta_{\mathbf{D}:\mathbf{PO}}(\mathcal{O}_X|\mathcal{O}_V)$.

$$\begin{aligned}
\delta^{\mathbf{P}}_{\mathbf{D}:\mathbf{PO}}(\mathcal{O}_X|\mathcal{O}_V) &\triangleq \delta^{\mathbf{P}}(\mathcal{O}_X|\mathcal{O}_V) \cap \delta_{\mathbf{D}:\mathbf{PO}}(\mathcal{O}_X|\mathcal{O}_V) \\
&= \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}, \mathbb{D}(r_X)^{\uparrow}\} \cap \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}, \mathbb{D}(r_X)^{\downarrow}\} \\
&= \{d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}\}
\end{aligned}$$

Possible operations are implemented by gradual descent functions as follows: $x^{\downarrow}$ is implemented by $+x$, written as $\zeta(x^{\downarrow}) = +x$; $x^{\uparrow}$ is implemented by $-x$, written

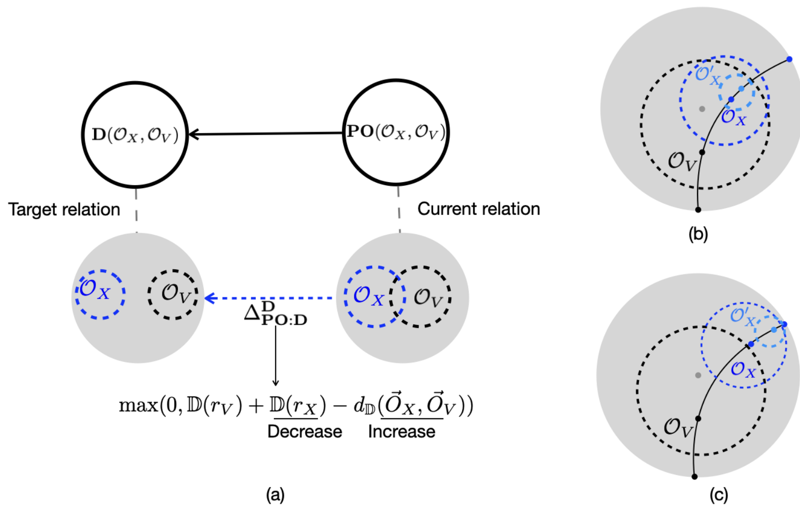


Figure 6: (a) $\Delta^{\mathbf{D}}_{\mathbf{PO:D}}(\mathcal{O}_X, \mathcal{O}_V)$ implements a neighbourhood transition from $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ to $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$. (b) If the centre of $\mathcal{O}_X$ is inside $\mathcal{O}_V$, reducing $\mathbb{D}(r_X)$ too fast will cause $\mathcal{O}_X$ being inside $\mathcal{O}_V$; (c) Solution: when the centre of $\mathcal{O}_X$ is inside $\mathcal{O}_V$, $\mathcal{O}_X$ will be moved away from $\mathcal{O}_V$ till its centre is at the boundary of $\mathcal{O}_V$.

as $\zeta(x^{\downarrow}) = -x$, where $x \in \{d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V), \mathbb{D}(r_X), \mathbb{D}(r_V)\}$. This transforms a set of operations into a gradual descent function. $\Delta^{\mathbf{T}}_{\mathbf{T}_1:\mathbf{T}_2}(\mathcal{O}_X, \mathcal{O}_V)$ is implemented by $\max\{0, C + \sum \zeta(op)\}$, where $op \in \delta^{\mathbf{T}}_{\mathbf{T}_1:\mathbf{T}_2}(\mathcal{O}_X|\mathcal{O}_V)$ and C is a constant such that $\mathbf{T}_2$ is reached, exactly when $C + \sum \zeta(op) = 0$.

2.3. Atomic neighbourhood transition

As a neighbourhood transition, $\Delta^{\mathbf{T}}_{\mathbf{T}_1:\mathbf{T}_2}(\mathcal{O}_X, \mathcal{O}_V)$ needs to guarantee that during the transition, there will not appear a third relation $\mathbf{T}_3(\mathcal{O}_X, \mathcal{O}_V)$, where $\mathbf{T}_3 \notin \{\mathbf{T}_1, \mathbf{T}_2\}$. That is, neighbourhood transitions should be *atomic*. However, as gradual descent functions update independently the centre and the radius of $\mathcal{O}_X$, a neighbourhood transition may not be *atomic*. For example, the transition from the partial overlapping relation $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ to the disconnectedness relation $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$ is realised by the gradual descent function, as follows, shown in Figure 6(a).

$$\Delta^{\mathbf{D}}_{\mathbf{PO:D}}(\mathcal{O}_X, \mathcal{O}_V) = \Delta^{\mathbf{D}}_{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V) = \max\{0, \mathbb{D}(r_X) + \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)\}$$

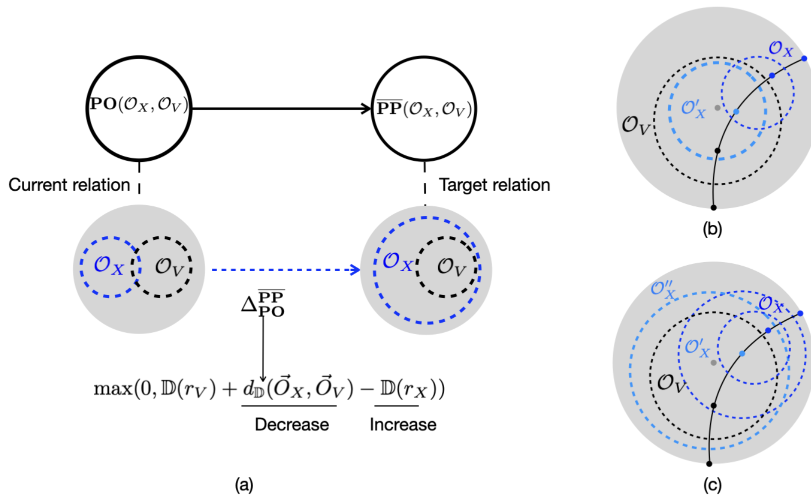


Figure 7: (a) $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}:\overline{\mathbf{PP}}}(\mathcal{O}_X, \mathcal{O}_V)$ implements a neighbourhood transition $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ to $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_V)$. (b) If $\mathbb{D}(r_X) < \mathbb{D}(r_V)$, enlarging $\mathbb{D}(r_X)$ too slow will cause $\mathcal{O}_X$ being inside $\mathcal{O}_V$; (c) Solution: when $\mathbb{D}(r_X) < \mathbb{D}(r_V)$, firstly enlarge $\mathbb{D}(r_X)$ to $\mathbb{D}(r_V)$ while fixing the centre of $\mathcal{O}_X$.

To reach the target, $\Delta^{\mathbf{D}}_{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$ will either reduce $\mathbb{D}(r_X)$ or increase $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$ or both ($\mathcal{O}_V$ is fixed). When the centre of $\mathcal{O}_X$ is inside $\mathcal{O}_V$, reducing $\mathbb{D}(r_X)$ too fast may cause $\mathcal{O}_X$ being inside $\mathcal{O}_V$, as shown in Figure 6(b). To avoid this, we partition $\mathbf{PO}$ into two sub-relations: $\mathbf{PO}_1$ and $\mathbf{PO}_2$: $\mathbf{PO}_1(\mathcal{O}_X, \mathcal{O}_V)$ is the sub-relation of $\mathbf{PO}$ when the centre of $\mathcal{O}_X$ is outside $\mathcal{O}_V$; $\mathbf{PO}_2(\mathcal{O}_X, \mathcal{O}_V)$ is the sub-relation of $\mathbf{PO}$ when the centre of $\mathcal{O}_X$ is inside or at the border of $\mathcal{O}_V$. We define $\Delta^{\mathbf{D}}_{\mathbf{PO}_2:\mathbf{PO}_1}(\mathcal{O}_X, \mathcal{O}_V)$ as moving $\mathcal{O}_X$ away from $\mathcal{O}_V$ till $\mathbf{PO}_1(\mathcal{O}_X, \mathcal{O}_V)$, while fixing $\mathbb{D}(r_X)$. $\Delta^{\mathbf{D}}_{\mathbf{PO}_1}(\mathcal{O}_X, \mathcal{O}_V)$ is *atomic* even if its centre and radius are optimised independently, as shown in Figure 6(c). We replace $\Delta^{\mathbf{D}}_{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$ with either $\Delta^{\mathbf{D}}_{\mathbf{PO}_1}(\mathcal{O}_X, \mathcal{O}_V)$ or $\Delta^{\mathbf{D}}_{\mathbf{PO}_2:\mathbf{PO}_1}(\mathcal{O}_X, \mathcal{O}_V)$ followed with $\Delta^{\mathbf{D}}_{\mathbf{PO}_1}(\mathcal{O}_X, \mathcal{O}_V)$. Each case is *atomic*.

Another case is the transition from the partial overlapping relation to the containing relation, $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}:\overline{\mathbf{PP}}}(\mathcal{O}_X, \mathcal{O}_V) = \Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$, whose geometric operations are enlarging $\mathbb{D}(r_X)$ and decreasing $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$, as shown in Figure 7(a). If $\mathbb{D}(r_X) < \mathbb{D}(r_V)$ and $\mathbb{D}(r_X)$ is enlarged too slow, $\mathcal{O}_X$ will be part of $\mathcal{O}_V$, instead of containing $\mathcal{O}_V$, as shown in Figure 7(b). To avoid this situation, we split the $\mathbf{PO}$ relation into $\mathbf{PO}_3$ and $\mathbf{PO}_4$: $\mathbf{PO}_3(\mathcal{O}_X, \mathcal{O}_V)$ is the sub-relation of $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ with the condition that

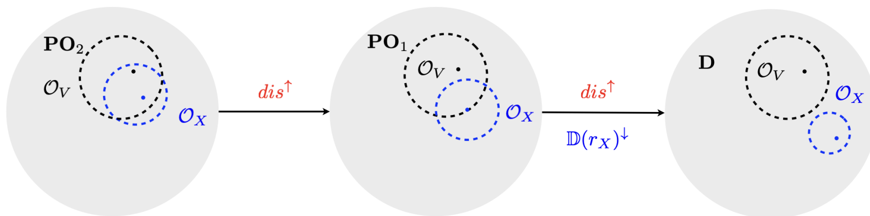


Figure 8: If the centre of $\mathcal{O}_X$ is inside $\mathcal{O}_V$, HSphNN will move $\mathcal{O}_X$ away from $\mathcal{O}_V$ while fixing $\mathbb{D}(r_X)$, till the centre of $\mathcal{O}_X$ is located at the boundary of $\mathcal{O}_V$. Then, HSphNN will continue to move $\mathcal{O}_X$ away from $\mathcal{O}_V$ while independently decreasing $\mathbb{D}(r_X)$, till reaching the target relation $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$. dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$.

$\mathbb{D}(r_X) < \mathbb{D}(r_V)$; $\mathbf{PO}_4(\mathcal{O}_X, \mathcal{O}_V)$ is the sub-relation of $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ with the condition that $\mathbb{D}(r_X) \geq \mathbb{D}(r_V)$. If $\mathbf{PO}_3(\mathcal{O}_X, \mathcal{O}_V)$ holds, $\mathbb{D}(r_X)$ will be enlarged to reach the same length as $\mathbb{D}(r_V)$, while fixing the centre of $\mathcal{O}_X$, resulting in $\mathbf{PO}_4(\mathcal{O}_X, \mathcal{O}_V)$. After that, $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}_4}$ will transform $\mathbf{PO}_4(\mathcal{O}_X, \mathcal{O}_V)$ into $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_V)$, as illustrated in Figure 7(c). Therefore, we replace $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}:\overline{\mathbf{PP}}}(\mathcal{O}_X, \mathcal{O}_V)$ with either $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V)$ or $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}_3:\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V)$ followed with $\Delta^{\overline{\mathbf{PP}}}_{\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V)$. Each case is *atomic*.

2.4. List of transition functions between two spheres

An inspection function $\mathcal{I}^{\mathbf{R}}(\mathcal{O}_X, \mathcal{O}_V)$ signals whether the relation $\mathbf{R}$ is held between $\mathcal{O}_X$ and $\mathcal{O}_V$. It returns zero, if and only if the relation $\mathbf{R}(\mathcal{O}_X, \mathcal{O}_V)$ is held; otherwise, it returns a positive real number. So, a target configuration is reached when the sum of all inspection functions equals zero. We list all transition functions for $\mathcal{O}_X$ and $\mathcal{O}_V$, where $\mathcal{O}_V$ is fixed.

Targeting at $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$. Let the target be $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$. Whether this target relation is satisfied can be measured geometrically by the truth value of

$$\mathbb{D}(r_X) + \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) \leq 0$$

To make the formula true, HSphNN can either gradually descent r_X or gradually ascent $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$, therefore, the possible operations are $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\uparrow}$ and $\mathbb{D}(r_X)^{\downarrow}$, $\delta^{\mathbf{D}}(\mathcal{O}_X | \mathcal{O}_V) = \{d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\uparrow}, \mathbb{D}(r_X)^{\downarrow}\}$. The inspection function is defined as

$$\mathcal{I}^{\mathbf{D}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_X) + \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)\}$$

If $\mathcal{O}_X$ and $\mathcal{O}_V$ are currently partially overlapped and the centre of $\mathcal{O}_X$ is inside $\mathcal{O}_V$, it may happen that uncoordinated optimising the centre and $\mathbb{D}(r_X)$ will not lead $\mathcal{O}_X$ to disconnect from $\mathcal{O}_V$. Following the early discussion, we partition the $\mathbf{PO}$ relation into $\mathbf{PO}_1$ and $\mathbf{PO}_2$ and list the related formulas as follows and illustrated in Figure 8.

$$
\begin{aligned}
&\mathbf{PO}_1(\mathcal{O}_X,\mathcal{O}_V) \triangleq \mathbf{PO}(\mathcal{O}_X,\mathcal{O}_V) \wedge d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) > \mathbb{D}(r_V)\\
&\mathbf{PO}_2(\mathcal{O}_X,\mathcal{O}_V) \triangleq \mathbf{PO}(\mathcal{O}_X,\mathcal{O}_V) \wedge d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) \le \mathbb{D}(r_V)\\
&\mathcal{I}^{\mathbf{PO}}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \max\{0, |\mathbb{D}(r_X) - \mathbb{D}(r_V)| - d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) + \epsilon\} +\\
&\qquad\qquad\qquad\qquad \max\{0, d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) - \mathbb{D}(r_V) - \mathbb{D}(r_X) + \epsilon\}\\
&\mathcal{I}^{\mathbf{PO}_1}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \mathcal{I}^{\mathbf{PO}}(\mathcal{O}_X,\mathcal{O}_V) + \max\{0, \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) + \epsilon\}\\
&\mathcal{I}^{\mathbf{PO}_2}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \mathcal{I}^{\mathbf{PO}}(\mathcal{O}_X,\mathcal{O}_V) + \max\{0, d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) - \mathbb{D}(r_V)\}\\
&\Delta^{\mathbf{D}}_{\mathbf{PO}_2:\mathbf{PO}_1}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)\}\\
&\Delta^{\mathbf{D}}_{\mathbf{PO}_1}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_X) + \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)\}
\end{aligned}
$$

Targeting at $\overline{\mathbf{P}}(\mathcal{O}_X,\mathcal{O}_V)$. Let the target be $\overline{\mathbf{P}}(\mathcal{O}_X,\mathcal{O}_V)$. Whether this target relation is satisfied can be measured geometrically by the truth value of

$$d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) + \mathbb{D}(r_V) \le \mathbb{D}(r_X)$$

The inspection function is as follows.

$$\mathcal{I}^{\overline{\mathbf{P}}}(\mathcal{O}_X,\mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V) + \mathbb{D}(r_V) - \mathbb{D}(r_X)\}$$

When the value is 0, $\mathcal{O}_X$ contains $\mathcal{O}_V$; otherwise, $\mathcal{O}_X$ does not contain $\mathcal{O}_V$. To reduce the value of $\mathcal{I}^{\overline{\mathbf{P}}}(\mathcal{O}_X,\mathcal{O}_V)$, HSphNN shall increase $\mathbb{D}(r_X)$ or reduce $d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)$. Therefore, the allowed operations are $\mathbb{D}(r_X)^{\uparrow}$ and $d_{\mathbb{D}}(\vec{O}_X,\vec{O}_V)^{\downarrow}$.

If $\mathcal{O}_X$ and $\mathcal{O}_V$ are partially overlapped, and $\mathbb{D}(r_X)$ is shorter than $\mathbb{D}(r_V)$, it may happen that uncoordinated optimising the centre and the radius of $\mathcal{O}_X$ will not lead $\mathcal{O}_X$ to contain $\mathcal{O}_V$. Following the early analysis, we partition the $\mathbf{PO}$ relation into $\mathbf{PO}_3$ and $\mathbf{PO}_4$ and list the related formulas as follows and illustrated in Figure 9.

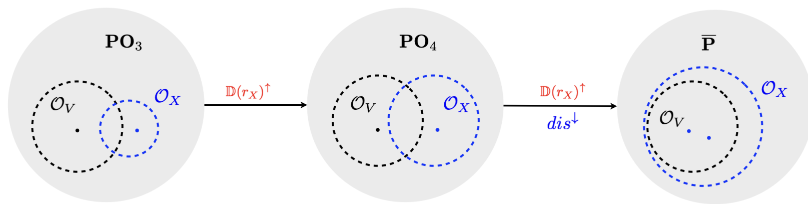


Figure 9: If $\mathbb{D}(r_X)$ is shorter than $\mathbb{D}(r_V)$, HSphNN will enlarge $\mathbb{D}(r_X)$ till $\mathbb{D}(r_X) = \mathbb{D}(r_V)$. Then, HSphNN will continue to move $\mathcal{O}_X$ towards $\mathcal{O}_V$ while independently increasing $\mathbb{D}(r_X)$, till reaching the target relation $\overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V)$. dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$.

$$
\begin{aligned}
&\mathbf{PO}_3(\mathcal{O}_X, \mathcal{O}_V) \triangleq \mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V) \wedge \mathbb{D}(r_V) < \mathbb{D}(r_X)\\
&\mathcal{I}^{\mathbf{PO}_3}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \mathcal{I}^{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V) + \max\{0, \mathbb{D}(r_V) - \mathbb{D}(r_X) + \epsilon\}\\
&\mathbf{PO}_4(\mathcal{O}_X, \mathcal{O}_V) \triangleq \mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V) \wedge \mathbb{D}(r_V) \geq \mathbb{D}(r_X)\\
&\mathcal{I}^{\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \mathcal{I}^{\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V) + \max\{0, \mathbb{D}(r_X) - \mathbb{D}(r_V)\}\\
&\Delta^{\overline{\mathbf{P}}}_{\mathbf{PO}_3:\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_V) - \mathbb{D}(r_X)\}\\
&\Delta^{\overline{\mathbf{P}}}_{\mathbf{PO}_4}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) + \mathbb{D}(r_V) - \mathbb{D}(r_X)\}
\end{aligned}
$$

Targeting at $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$. Two spheres being partially overlapped means that the distance between their centres is (1) shorter than the sum of their radii, and (2) longer than the difference between their radii. That is

$$
|\mathbb{D}(r_X) - \mathbb{D}(r_V)| < d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) < \mathbb{D}(r_X) + \mathbb{D}(r_V)
$$

The partial overlapping relation $\mathbf{PO}$ is an intermediate relation to reach a target relation. It can be reached from four other relations. If $\mathcal{O}_X$ currently disconnects from $\mathcal{O}_V$, HSphNN can perform both $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\downarrow}$ and $\mathbb{D}(r_X)^{\uparrow}$ operations; if $\mathcal{O}_X$ currently is a proper part of $\mathcal{O}_V$, HSphNN can perform $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\uparrow}$ and $\mathbb{D}(r_X)^{\uparrow}$ operations; if $\mathcal{O}_V$ is a proper part of $\mathcal{O}_X$, HSphNN can perform $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\uparrow}$ and $\mathbb{D}(r_X)^{\downarrow}$ operations; if $\mathcal{O}_X$ coincides with $\mathcal{O}_V$, HSphNN only need to randomly shift $\mathcal{O}_X$ away from

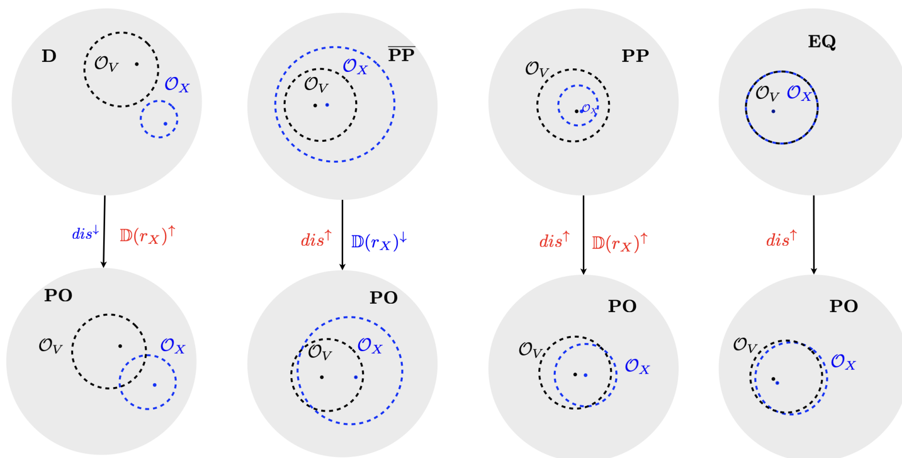


Figure 10: The target relation $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_V)$ can be reached from $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$, $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_V)$, and $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_V)$ by independently optimising $\mathbb{D}(r_X)$ and $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$. If the current relation is $\mathbf{EQ}(\mathcal{O}_X, \mathcal{O}_V)$, HSphNN will increase $d_{\mathbb{D}}$ by slightly changing the centre of $\mathcal{O}_X$ to reach the target relation. dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$.

its current location, as illustrated in Figure 10.

$$\Delta^{\mathbf{PO}}_{\mathbf{D}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) - \mathbb{D}(r_V) - \mathbb{D}(r_X)\}$$
$$\Delta^{\mathbf{PO}}_{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_V) - \mathbb{D}(r_X) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)\}$$
$$\Delta^{\mathbf{PO}}_{\overline{\mathbf{PP}}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_X) - \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)\}$$
$$\Delta^{\mathbf{PO}}_{\mathbf{EQ}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \vec{O}_X + \vec{\epsilon}$$

Target at $\mathbf{P}(\mathcal{O}_X, \mathcal{O}_V)$. Let the target be $\mathbf{P}(\mathcal{O}_X, \mathcal{O}_V)$. Whether this target relation is held can be measured geometrically by the truth value of

$$d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) + \mathbb{D}(r_X) \leq \mathbb{D}(r_V)$$

with the inspection function

$$\mathcal{I}^{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) + \mathbb{D}(r_X) - \mathbb{D}(r_V)\}$$

To reach the target, HSphNN can perform $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\downarrow}$ and $\mathbb{D}(r_X)^{\downarrow}$ operations. If $\mathbf{PO}_2(\mathcal{O}_X, \mathcal{O}_V)$ holds, performing either $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\downarrow}$ or $\mathbb{D}(r_X)^{\downarrow}$ will lead to the target status, as illustrated in Figure 11. We define

$$\Delta^{\mathbf{P}}_{\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) + \mathbb{D}(r_X) - \mathbb{D}(r_V)\}$$

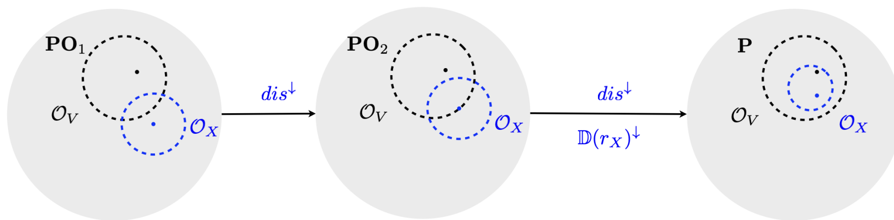


Figure 11: If the centre of $\mathcal{O}_X$ is outside $\mathcal{O}_V$, HSphNN will move $\mathcal{O}_X$ towards $\mathcal{O}_V$ while fixing $\mathbb{D}(r_X)$, till the centre of $\mathcal{O}_X$ is located at the boundary of $\mathcal{O}_V$. Then, HSphNN will continue to move $\mathcal{O}_X$ towards $\mathcal{O}_V$ while independently decreasing $\mathbb{D}(r_X)$, till reaching the target relation $\mathbf{P}(\mathcal{O}_X, \mathcal{O}_V)$. dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$.

In $\mathbf{PO}_1$ status, decreasing $\mathbb{D}(r_X)$ too fast may lead $\mathcal{O}_X$ to disconnect from $\mathcal{O}_V$. To prevent this situation, HSphNN fixes $\mathbb{D}(r_X)$ and only performs the $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)^{\downarrow}$ operation. We introduce $\color{red}{\Delta}^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)\}$ to transform the relation from $\mathbf{PO}_1$ to $\mathbf{PO}_2$. The condition for $\color{red}{\Delta}^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V)$ to reach $\mathbf{PO}_2(\mathcal{O}_X, \mathcal{O}_V)$ is that the radius of $\mathcal{O}_X$ should be less than the diameter of $\mathcal{O}_V$, that is, $\mathbb{D}(r_X) < 2\mathbb{D}(r_V)$. If the condition is not satisfied, repeated operations of $\color{red}{\Delta}^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V)$ will push $\mathcal{O}_X$ to contain $\mathcal{O}_V$, $\overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V)$, and trigger the function $\Delta^{\mathbf{P}}_{\overline{\mathbf{PP}}:\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$, whose possible operation is $\mathbb{D}(r_X)^{\downarrow}$. This operation is the intersection of the possible operations from $\overline{\mathbf{PP}}$ to $\mathbf{PO}_1$ and the possible operations of the target relation $\mathbf{P}$, as shown in Figure 12. This works like that $\color{red}{\Delta}^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V)$ borrows $\Delta^{\mathbf{P}}_{\overline{\mathbf{PP}}:\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$ to reduce the radius of $\mathcal{O}_X$. $\color{red}{\Delta}^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}(\mathcal{O}_X, \mathcal{O}_V)$ is the only Δ function with a condition. So, we use the red colour to demarcate this feature. To avoid this loop and make all Δ operations atomic, we can introduce an additional operation: set $\mathbb{D}(r_X)$ to $\mathbb{D}(r_V)$, if $\mathbb{D}(r_X) > \mathbb{D}(r_V)$ and targeting at $\mathbf{P}(\mathcal{O}_X, \mathcal{O}_V)$.

Targeting at negative relations. If the target is a negative relation $\mathbf{R} \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$, there will be only one non-target relation $\neg\mathbf{R} \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$. If the relation between $\mathcal{O}_X$ and $\mathcal{O}_V$ is $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_V)$, $\mathbf{P}(\mathcal{O}_X, \mathcal{O}_V)$, or $\overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V)$, following three transition

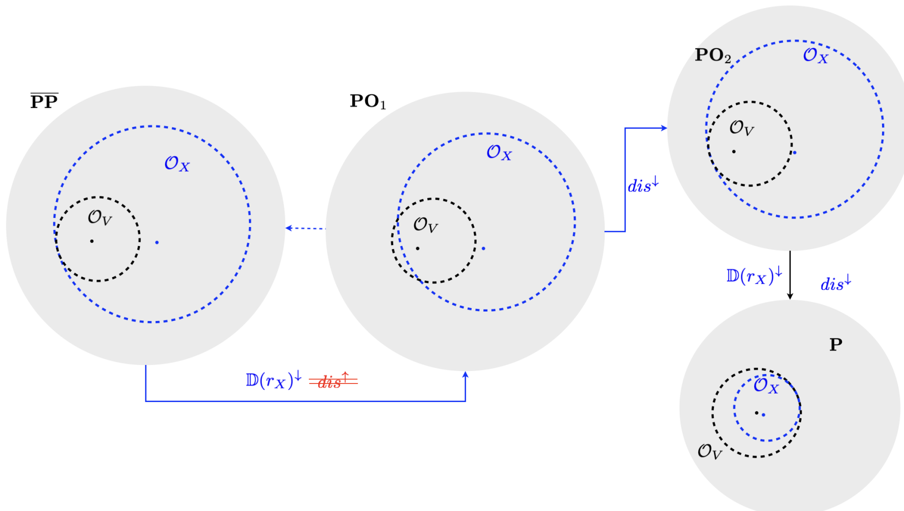


Figure 12: If $\mathbb{D}(r_X) > 2\mathbb{D}(r_V)$, it may happen that reducing the distance between their centres causes $\mathcal{O}_X$ containing $\mathcal{O}_V$, as shown by the blue dotted line. Then, $\Delta^{\mathbf{P}}_{\overline{\mathbf{PP}}:\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$ will reduce the value of $\mathbb{D}(r_X)$. This loop repeats till $\mathbf{PO}_2(\mathcal{O}_X, \mathcal{O}_V)$ is reached. dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$.

functions will optimise $\mathcal{O}_X$ to reach the target relation with $\mathcal{O}_V$.

$$\Delta^{\neg\mathbf{D}}_{\mathbf{D}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) - \mathbb{D}(r_V) - \mathbb{D}(r_X) + \epsilon\}$$
$$\Delta^{\neg\mathbf{P}}_{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_V) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) - \mathbb{D}(r_X) + \epsilon\}$$
$$\Delta^{\neg\overline{\mathbf{P}}}_{\overline{\mathbf{P}}}(\mathcal{O}_X, \mathcal{O}_V) \triangleq \max\{0, \mathbb{D}(r_X) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V) - \mathbb{D}(r_V) + \epsilon\}$$

2.5. *Model construction guided by a transition map*

For syllogistic reasoning, there are six target spatial relations $\mathcal{T} \triangleq \{\mathbf{D}, \neg\mathbf{D}, \mathbf{P}, \neg\mathbf{P}, \overline{\mathbf{P}}, \neg\overline{\mathbf{P}}\}$. A target relation determines its qualitative partition of the space. For a negative target, namely, $\neg\mathbf{D}, \neg\mathbf{P}$, and $\neg\overline{\mathbf{P}}$, HSphNN only needs to partition the space into two parts: $\mathbf{D}$ and $\neg\mathbf{D}$, or $\mathbf{P}$ and $\neg\mathbf{P}$, or $\overline{\mathbf{P}}$ and $\neg\overline{\mathbf{P}}$. Each case only needs one transition function $\Delta^{\neg\mathbf{D}}_{\mathbf{D}}(\mathcal{O}_X, \mathcal{O}_V)$, $\Delta^{\neg\mathbf{P}}_{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_V)$, and $\Delta^{\neg\overline{\mathbf{P}}}_{\overline{\mathbf{P}}}(\mathcal{O}_X, \mathcal{O}_V)$, respectively. When the target relation is $\mathbf{D}$, $\mathbf{P}$, or $\overline{\mathbf{P}}$, the space will be partitioned hierarchically into two layers, at the top layer are five *jointly-exhaustive-and-pairwise-disjoint* relations: $\{\mathbf{D}, \mathbf{EQ}, \mathbf{PO}, \overline{\mathbf{PP}}, \mathbf{PP}\} = \mathcal{T}_5$; at the second layer, $\mathbf{PO}$ will be partitioned either into

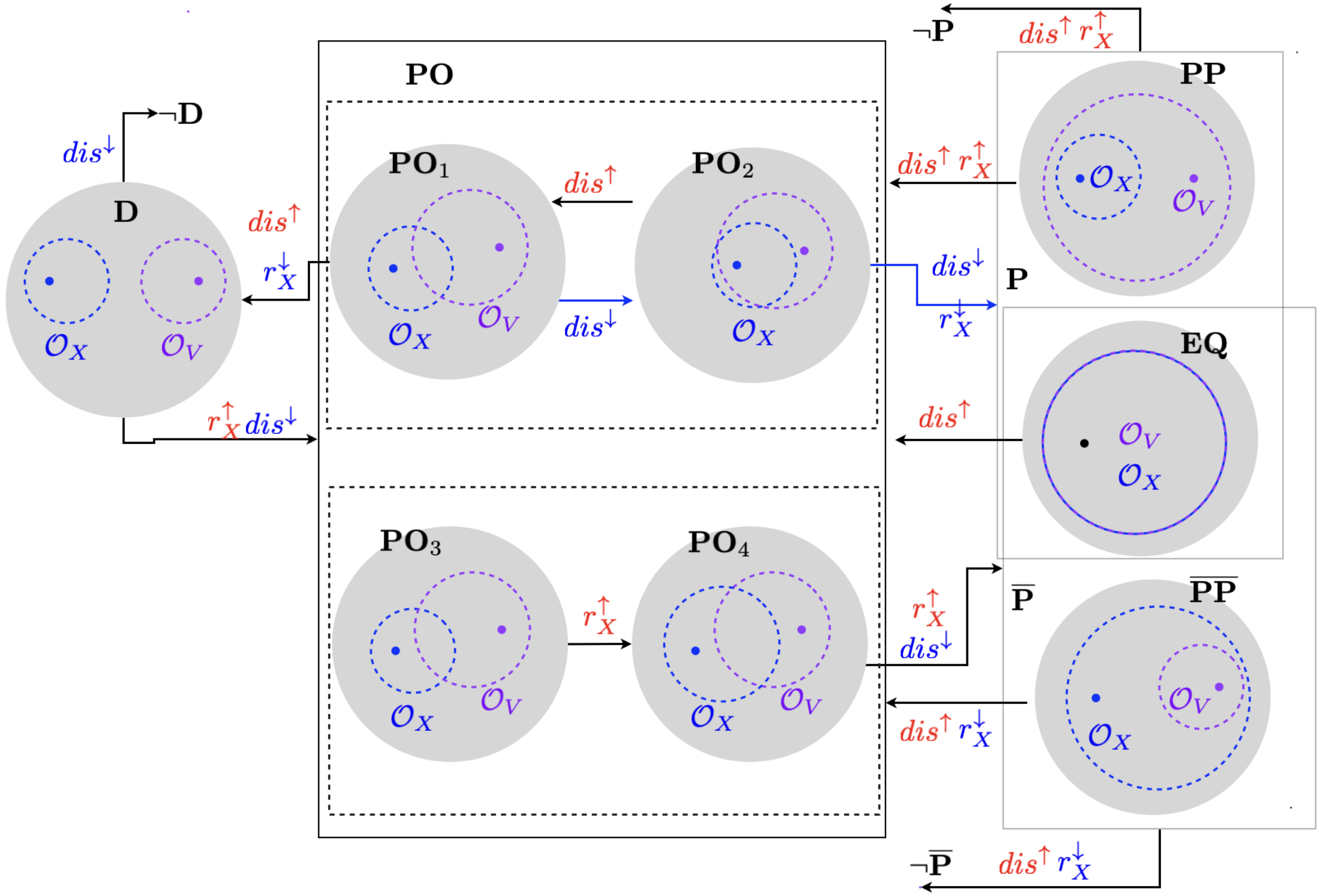


Figure 13: The full version of the neuro-symbolic transition map of neighbourhood spatial relations. $\mathcal{O}_V$ is fixed, $d_{\mathbb{D}}$ is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$. Here, $\mathbf{EQ}$ is set as the initial status, and only takes $\mathbf{PO}$ as its neighbour. Two ways to partition $\mathbf{PO}$: $\mathbf{PO}_1 \cup \mathbf{PO}_2$ and $\mathbf{PO}_3 \cup \mathbf{PO}_4$. Possible operations between neighbourhood relations are labelled. The target relation determines which of them will be chosen. Here, dis is shortened for $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$; r_X is shortened for $\mathbb{D}(r_X)$.

Table 1: Target oriented neighbourhood transition table. '$\varnothing$' means that the target relation is reached; '-' means that the current relation is not in the domain of spatial partition.

$(\mathcal{O}_X, \mathcal{O}_V)$	$\mathbf{D}$	$\mathbf{P}$	$\overline{\mathbf{P}}$	$\neg\mathbf{D}$	$\neg\mathbf{P}$	$\neg\overline{\mathbf{P}}$
$\mathbf{D}$	$\varnothing$	$\Delta^{\mathbf{P}}_{\mathbf{D:PO}}$	$\Delta^{\overline{\mathbf{P}}}_{\mathbf{D:PO}}$	$\Delta^{\neg\mathbf{D}}_{\mathbf{D}}$	$\varnothing$	$\varnothing$
$\mathbf{PO}_1$	$\Delta^{\mathbf{D}}_{\mathbf{PO}_1}$	$\Delta^{\mathbf{P}}_{\mathbf{PO}_1:\mathbf{PO}_2}$	-	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{PO}_2$	$\Delta^{\mathbf{D}}_{\mathbf{PO}_2:\mathbf{PO}_1}$	$\Delta^{\mathbf{P}}_{\mathbf{PO}_2}$	-	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{PO}_3$	-	-	$\Delta^{\overline{\mathbf{P}}}_{\mathbf{PO}_3:\mathbf{PO}_4}$	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{PO}_4$	-	-	$\Delta^{\overline{\mathbf{P}}}_{\mathbf{PO}_4}$	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{PP}$	$\Delta^{\mathbf{D}}_{\mathbf{PP:PO}}$	$\varnothing$	$\Delta^{\overline{\mathbf{P}}}_{\mathbf{PP:PO}}$	$\varnothing$	-	$\varnothing$
$\mathbf{EQ}$	$\Delta^{\mathbf{D}}_{\mathbf{EQ:PO}}$	$\varnothing$	$\varnothing$	$\varnothing$	-	-
$\overline{\mathbf{PP}}$	$\Delta^{\mathbf{D}}_{\overline{\mathbf{PP}}:\mathbf{PO}}$	$\Delta^{\mathbf{P}}_{\overline{\mathbf{PP}}:\mathbf{PO}}$	$\varnothing$	$\varnothing$	$\varnothing$	–
$\mathbf{P}$	-	$\varnothing$	-	$\varnothing$	$\Delta^{\neg\mathbf{P}}_{\mathbf{P}}$	-
$\overline{\mathbf{P}}$	-	-	$\varnothing$	$\varnothing$	-	$\Delta^{\neg\overline{\mathbf{P}}}_{\overline{\mathbf{P}}}$
$\neg\mathbf{D}$	-	-	-	$\varnothing$	-	-
$\neg\mathbf{P}$	-	-	-	-	$\varnothing$	-
$\neg\overline{\mathbf{P}}$	-	-	-	-	-	$\varnothing$

$\mathbf{PO}_1$ and $\mathbf{PO}_2$, or $\mathbf{PO}_3$ and $\mathbf{PO}_4$, as illustrated in Figure 13. The whole can be organised into a transition map among neighbourhood relations and formalised as a six-tuple $\mathcal{M} \triangleq (\mathcal{T}, f_{tsp}, \mathcal{I}, \mathcal{S}, f_{tn}, \Delta)$ as follows.

- $\mathcal{T}$: the set of six target relations;
- f_{tsp}: the function that maps a target relation to a set of *jointly-exhaustive-and-pairwise-disjoint* qualitative spatial partitions, where tsp stands for *t*arget-oriented *s*patial *p*artitions. For example, $f_{tsp}(\mathbf{D}) \triangleq \{\mathbf{D}, \mathbf{EQ}, \mathbf{PO}_1, \mathbf{PO}_2, \overline{\mathbf{PP}}, \mathbf{PP}\}$;
- $\mathcal{I}$: a family of inspection functions. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two spheres. We distinguish three kinds of inspection functions.
    1. inspecting relations with an explicit target relation. $\mathcal{I}(\mathcal{O}_1, \mathcal{O}_2|\mathbf{T})$ returns the relation $\mathbf{R} \in f_{tsp}(\mathbf{T})$ and $\mathbf{T} \in \mathcal{T}$;

2. inspecting relations with default target relations. $\mathcal{I}(\mathcal{O}_1, \mathcal{O}_2)$ returns the relation $\mathbf{R} \in \mathcal{T}_5 = \{\mathbf{D}, \mathbf{EQ}, \mathbf{PO}, \overline{\mathbf{PP}}, \mathbf{PP}\}$;

3. inspecting whether a given relation holds. $\mathcal{I}^{\mathbf{R}}(\mathcal{O}_1, \mathcal{O}_2)$ returns 0 if $\mathbf{R}(\mathcal{O}_1, \mathcal{O}_2)$ holds, otherwise, returns a positive real number.

- $\mathcal{S}$: the set of all relations between two spheres. $\mathcal{S} \triangleq \bigcup f_{tsp}(\mathbf{T})$, $\mathbf{T} \in \mathcal{T}$;

- f_{tn}: the function maps the current relation $\mathbf{R}$ to its neighbourhood relation $\mathbf{R}_2$, towards the target $\mathbf{T}$, namely, $\mathbf{R}_2 = f_{tn}(\mathbf{T}, \mathbf{R})$, tn stands for *t*arget-oriented *n*eighbourhood;

- Δ: the set of neighbourhood transition functions. Let $\mathbf{T} \in \mathcal{T}$ be the target relation, $\mathcal{O}_1$ and $\mathcal{O}_2$ be two spheres. $\mathbf{R} = \mathcal{I}(\mathcal{O}_1, \mathcal{O}_2 | \mathbf{T})$, where $\mathbf{R} \in f_{tsp}(\mathbf{T})$. The neighbourhood transition function will be $\Delta^{\mathbf{T}}_{\mathbf{R}:f_{tn}(\mathbf{T},\mathbf{R})}(\mathcal{O}_1, \mathcal{O}_2)$ or $\Delta^{\mathbf{T}}_{\mathbf{R}}(\mathcal{O}_1, \mathcal{O}_2)$ for short. All neighbourhood transition functions are listed in Table 1.

3. The proofs of the theorems

In this section, we present theorems and proofs. The dependency relations among theorems are diagrammed in Figure 14. Notions are in hyperbolic metrics, for example, $\vec{O}_X$, $\vec{O}_Y$, and $\vec{O}_V$ refer to Poincaré centres of $\mathcal{O}_X$, $\mathcal{O}_Y$, and $\mathcal{O}_V$, respectively.

3.1. Basic theorems

Corollary 1. *Each Δ function is* linear *concerning the radius and* monotonic *concerning the distance between the centres.*

Proof (corollary) 1. *Each Δ function is a function of the hyperbolic radius $\mathbb{D}(r_X)$ and the hyperbolic distance between the Poincaré centres $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$. So, it is linear concerning the radius r_X and monotonic concerning the distance between the centres $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$. When $\mathcal{O}_X$ coincides with $\mathcal{O}_V$ (*$\mathbf{EQ}$*), one step of update the length of $\|\vec{O}_X\|$ ($\|\vec{O}_X\| \neq 0$) will push $\mathcal{O}_X$ to partially overlap with $\mathcal{O}_V$, so $\Delta^{\mathbf{T}}_{\mathbf{EQ}:\mathbf{PO}}(\mathcal{O}_X, \mathcal{O}_V)$ can also be understood as monotonic.* □

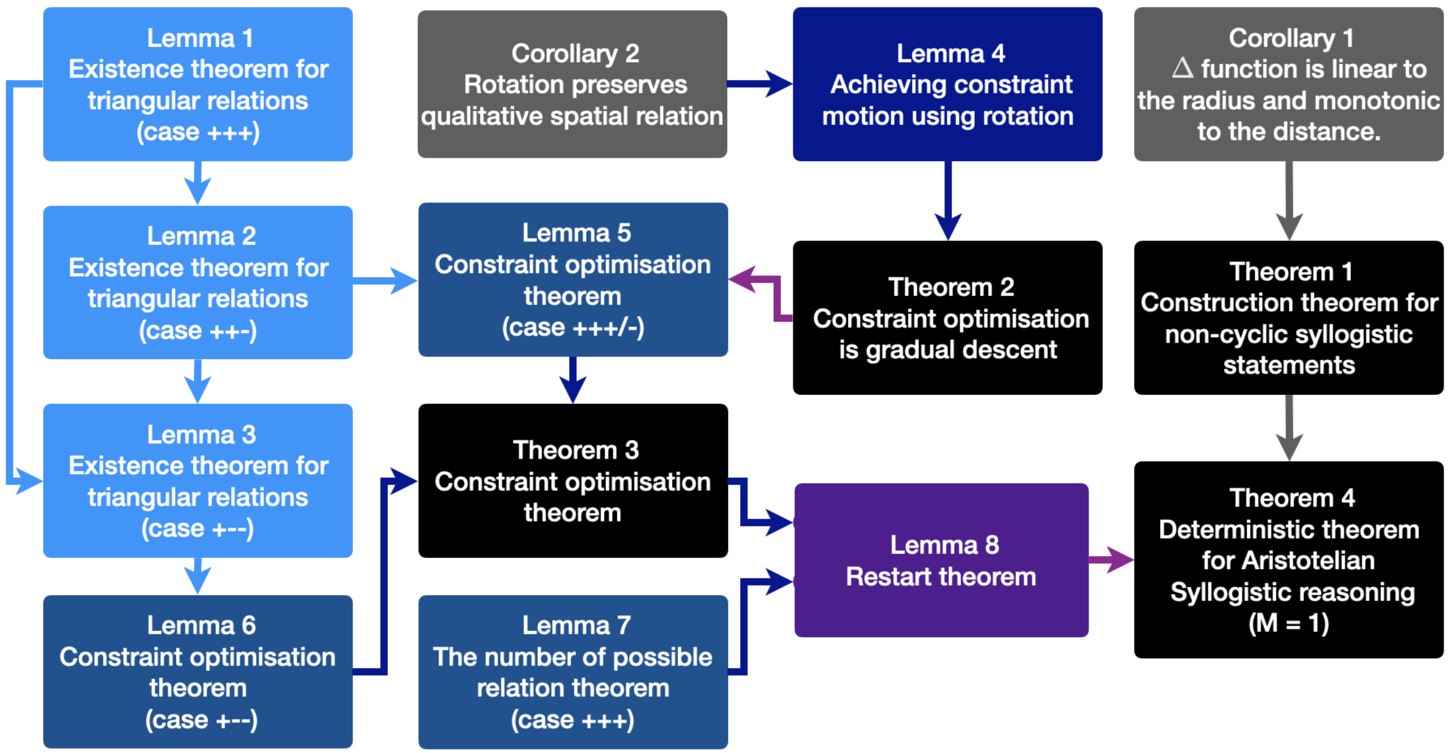


Figure 14: The dependency relation among 16 theorems. Theorem 4 is the main theorem. The construction process initialises all spheres to be coinciding and transforms to the target configuration by gradual descent operations (Theorem 1, Theorem 2, with a maximum of one restart (The Restart Lemma 8). The constraint optimisation theorem guarantees the transformation process will gradually reduce the global loss without violating the constraints (Corollary 2 and Lemma 4); the restart theorem guarantees that if there is a satisfiable configuration of three spheres for any Aristotelian syllogistic reasoning, HSphNN can successfully construct by first constructing and fixing two spheres, with a maximum of one restart with another sphere (Lemma 1- 3, 5- 7, where '+' and '−' represent a positive relation, e.g. $\mathbf{D}$, and a negative relation), e.g., $\neg\mathbf{P}$. Each $\triangle$ function (gradual descent function) of two spheres is linear to the radius and monotonic to the distance between the centres of spheres (Corollary 1), so gradual descent operation can construct sphere configuration for non-cyclic syllogistic statements in one epoch.

3.2. The satisfiability theorem for non-cyclic statements

Theorem 1. *Let $p_1, \ldots, p_{N-1}$ be $N-1$ premises of a long-chained syllogistic reasoning, where p_i can be either $r_i(X_i, X_{i+1})$ or $r_i(X_{i+1}, X_i)$, $(1 \le i \le N-1)$, $r_i \in \{all, some, no, some\_not\}$. HSphNN can construct a configuration of N spheres as an Euler diagram of the $N-1$ syllogistic statements, such that X_i maps to $\mathcal{O}_i$, and p_i maps to $\psi_i(\mathcal{O}_i, \mathcal{O}_{i+1})$, where $\psi_i = \psi(r_i)$ if $r_i(X_i, X_{i+1})$ or $\psi_i = \psi^{-1}(r_i)$ if $r_i(X_{i+1}, X_i)$, and $\psi_i \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$.*

Proof 1. *We show $\psi_1(\mathcal{O}_1, \mathcal{O}_2), \ldots, \psi_{N-1}(\mathcal{O}_{N-1}, \mathcal{O}_N)$ are* satisfiable. *We prove this by inducting on the length of the sequence.*

1. *$N = 1$. For any initial relation between $\mathcal{O}_1$ and $\mathcal{O}_2$, HSphNN can realise the target relation by using Δ functions in the neural transition map of qualitative spatial relations.*

2. *Suppose that it holds for $N \le K-1$.*

3. *$N = K$. Assume that HSphNN has constructed $K-1$ spheres $\mathcal{O}_1, \ldots, \mathcal{O}_{K-1}$ satisfying first $K-2$ constraints. To optimise $\mathcal{O}_K$, HSphNN repeats the method used for $N = 1$, as optimising $\psi_{K-1}(\mathcal{O}_{K-1}, \mathcal{O}_K)$ will not hurt other relations.* □

3.3. Existence theorem

Lemma 1. *Given $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in \{\mathbf{D}, \mathbf{EQ}, \mathbf{PO}, \mathbf{PP}, \overline{\mathbf{PP}}\}$. If the three relations are satisfiable, then for any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$ satisfying $\mathbf{R}_1(\mathcal{O}_X, \mathcal{O}_Y)$, there will be $\mathcal{O}_Z$ such that $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ and $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$.*

Proof (lemma) 1. *We enumerate the combination of values of $\mathbf{R}_2$ and $\mathbf{R}_3$.*

1. *$\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{EQ}(\mathcal{O}_Z, \mathcal{O}_X)$. A trivial case.*

2. *$\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{EQ}(\mathcal{O}_Y, \mathcal{O}_Z)$. A trivial case.*

3. *$\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$ and $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$. For any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$, if*

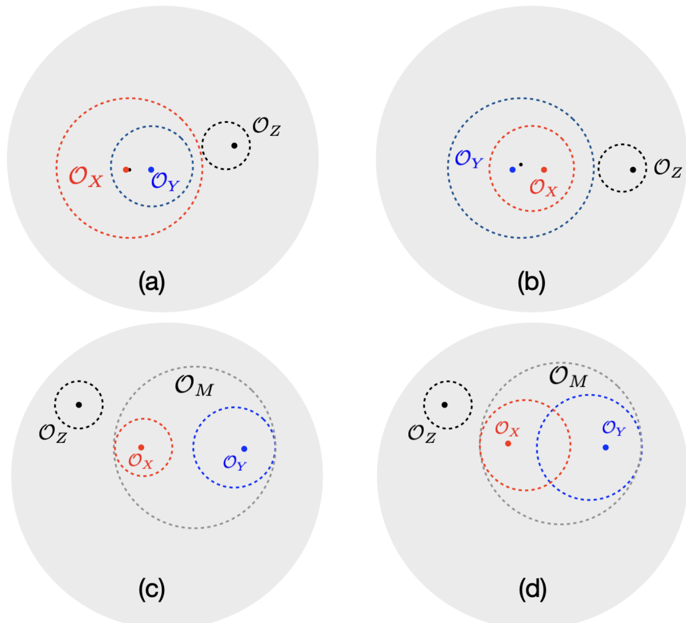


Figure 15: (a) $\mathcal{O}_Y$ is part of $\mathcal{O}_X$; (b) $\mathcal{O}_X$ is part of $\mathcal{O}_Y$; (c) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$; (d) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$. In any situation, there is $\mathcal{O}_Z$ disconnecting from $\mathcal{O}_X$ and $\mathcal{O}_Y$.

(a) $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. *Any* $\mathcal{O}_Z$ *disconnecting from* $\mathcal{O}_X$ *disconnects from* $\mathcal{O}_Y$*, shown in Figure 15(a).*

(b) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *Any* $\mathcal{O}_Z$ *disconnecting from* $\mathcal{O}_Y$ *disconnects from* $\mathcal{O}_X$*, shown in Figure 15(b).*

(c) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let both* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *be inside* $\mathcal{O}_M$*, any* $\mathcal{O}_Z$ *disconnecting from* $\mathcal{O}_M$ *disconnects from* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, shown in Figure 15(c).*

(d) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. *The same as (c), shown in Figure 15(d).*

4. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PO}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any fixed* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, if*

(a) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let* $\vec{O}_Z$ *(the centre of* $\mathcal{O}_Z$*) be located at the boundary of* $\mathcal{O}_Y$ *and be the apogee to* $\mathcal{O}_X$*. As* $\mathcal{O}_X$ *partially overlaps with* $\mathcal{O}_Y$*,* $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$*, they cannot be concentric, so there is* $\mathbb{D}(r_Z) = \epsilon$ *such that*

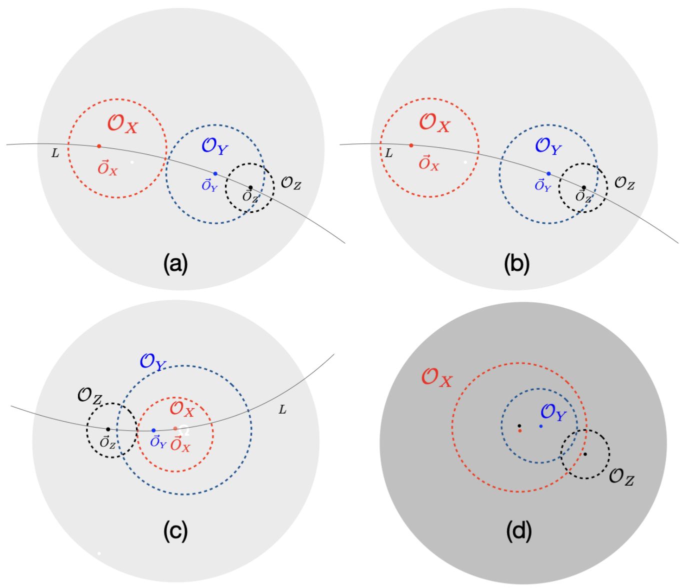


Figure 16: (a-c) $\mathcal{O}_Y$ partially overlaps with $\mathcal{O}_Z$ and $\mathcal{O}_Z$ disconnects from $\mathcal{O}_X$; (d) $\mathcal{O}_X$ contains $\mathcal{O}_Y$, therefore, if $\mathcal{O}_Z$ connects with $\mathcal{O}_Y$, it will connect with $\mathcal{O}_X$. The grey background means an unsatisfiable case.

$\mathcal{O}_Z$ disconnects from $\mathcal{O}_X$ and partially overlaps with $\mathcal{O}_Z$, shown in Figure 16(a).

(b) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. The same as (a), shown in Figure 16(b).

(c) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. If $\mathcal{O}_X$ and $\mathcal{O}_Y$ are not concentric, the case is the same as (a); otherwise, $\mathcal{O}_X$ is a proper part of $\mathcal{O}_Y$ ($\mathbb{D}(r_Y) > \mathbb{D}(r_X)$), let $\mathbb{D}(r_Z) < \mathbb{D}(r_Y) - \mathbb{D}(r_X)$. shown in Figure 16(c).

(d) $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. Any $\mathcal{O}_Z$ connecting with $\mathcal{O}_Y$ connects with $\mathcal{O}_X$. This contradicts with $\mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 16(d).

5. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{PP}}(\mathcal{O}_Y, \mathcal{O}_Z)$ and $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$. For any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$, if

(a) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. As $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$, let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with the boundary of $\mathcal{O}_X$ at $\vec{P}_0$ ($\vec{P}_0$ is inside $\mathcal{O}_Y$), intersect with the boundary of $\mathcal{O}_Y$ at $\vec{P}_1$ ($\vec{P}_1$ is outside $\mathcal{O}_X$).

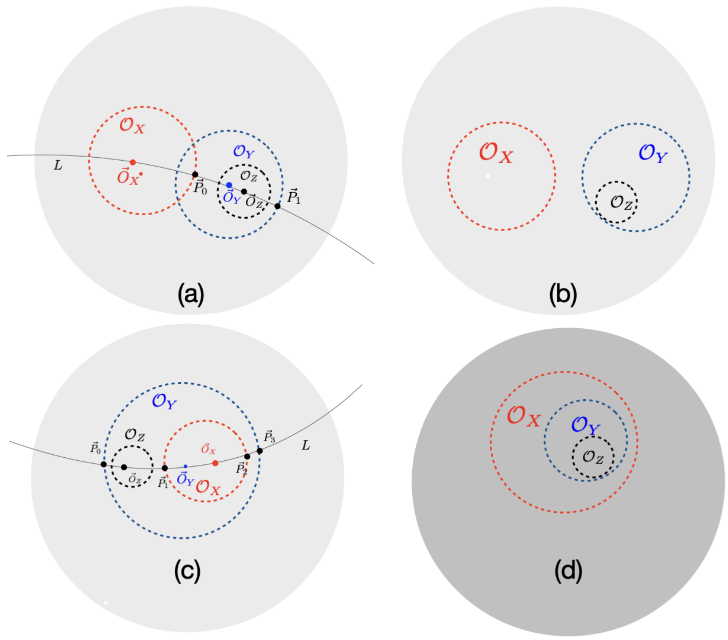


Figure 17: (a-c) As long as $\mathcal{O}_X$ does not totally cover $\mathcal{O}_Y$, there will be $\mathcal{O}_Z$ that is part of $\mathcal{O}_Y$, and disconnects from $\mathcal{O}_X$; (d) $\mathcal{O}_X$ totally covers $\mathcal{O}_Y$, if $\mathcal{O}_Z$ is inside $\mathcal{O}_Y$, it will be inside $\mathcal{O}_X$.

Let $\mathcal{O}_Z$ be a sphere whose diameter is a segment between $\vec{P}_0$ and $\vec{P}_1$, shown in Figure 17(a).

(b) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let $\mathcal{O}_Z$ be any sphere inside $\mathcal{O}_Y$, shown in Figure 17(b).*

(c) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *As $\mathcal{O}_X$ is proper part of $\mathcal{O}_Y$, let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with the boundary of $\mathcal{O}_X$ at $\vec{P}_1$ and $\vec{P}_2$, intersect with the boundary of $\mathcal{O}_Y$ at $\vec{P}_0$ and $\vec{P}_3$. Without the loss of generality, let $d_{\mathbb{D}}(\vec{P}_0, \vec{P}_1) \geq d_{\mathbb{D}}(\vec{P}_2, \vec{P}_3) \geq 0$. As $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$, it is not possible that $d_{\mathbb{D}}(\vec{P}_0, \vec{P}_1) = d_{\mathbb{D}}(\vec{P}_2, \vec{P}_3) = 0$. Let $\mathcal{O}_Z$ be a sphere whose diameter is a segment between $\vec{P}_0$ and $\vec{P}_1$, shown in Figure 17(c).*

(d) $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. *As $\mathcal{O}_Z$ is part of $\mathcal{O}_Y$, $\mathcal{O}_Z$ will be inside $\mathcal{O}_X$, which contradicts with $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Z)$, shown in Figure 17(d).*

6. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PP}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any fixed*

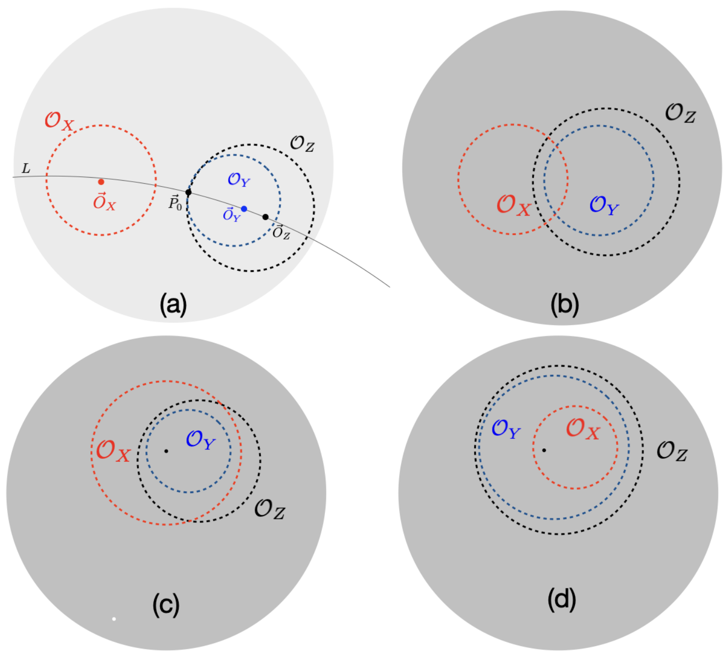


Figure 18: $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$ and $\mathcal{O}_Z$ disconnects from $\mathcal{O}_X$. This case is only possible when $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$ (a).

$\mathcal{O}_X$ and $\mathcal{O}_Y$, if

(a) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with the boundary of $\mathcal{O}_Y$ at $\vec{P}_0$, the perigee of $\mathcal{O}_X$. Let $\mathcal{O}_Z$ be the sphere that tangentially contains $\mathcal{O}_Y$ and $\vec{P}_0$ be the tangential point, shown in Figure 18(a).*

(b) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y) \vee \overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y) \vee \mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *As $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$, any sphere $\mathcal{O}_X$, if $\mathcal{O}_X$ connects with $\mathcal{O}_Y$, $\mathcal{O}_X$ connects with $\mathcal{O}_Z$, which contradicts with $\mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 18(b-d).*

7. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 4.*

8. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PO}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$, if*

(a) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let the boundaries of $\mathcal{O}_X$ and $\mathcal{O}_Y$ intersect at $\vec{P}_0$ and $\vec{P}_1$.*

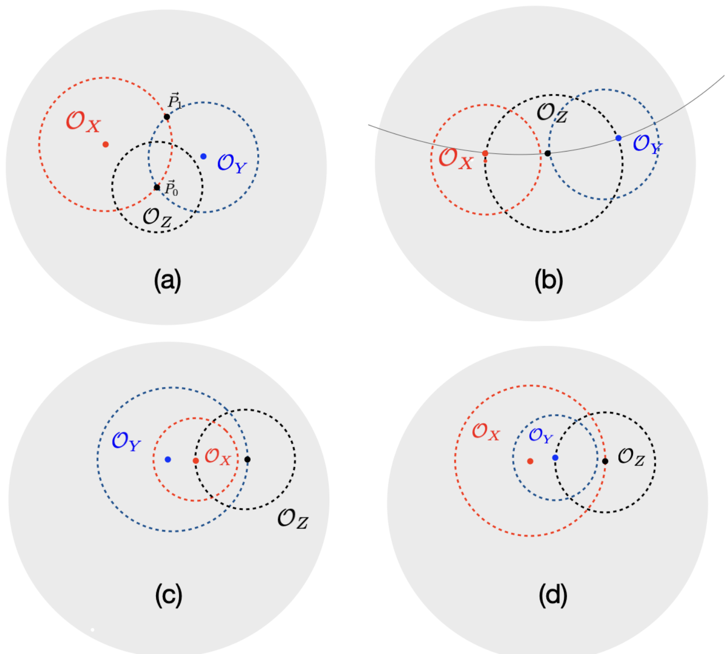


Figure 19: (a) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$, and their boundaries intersect at $\vec{P}_0$ and $\vec{P}_1$; (b) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$; (c) $\mathcal{O}_X$ is part of $\mathcal{O}_Y$; (d) $\mathcal{O}_Y$ is part of $\mathcal{O}_X$.

Any sphere $\mathcal{O}_Z$ with $\vec{P}_0$ as the centre and with $\mathbb{D}(r_Z)$ less than $\min\{\mathbb{D}(r_X), \mathbb{D}(r_Y)\}$ will partially overlap with $\mathcal{O}_X$ and $\mathcal{O}_Y$, shown in Figure 19(a).

(b) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let $\mathcal{O}_Z$ be the sphere with the segment $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_Y)$ as the diameter, shown in Figure 19(b).*

(c) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let $\mathcal{O}_Z$ be a sphere whose centre is at the boundary of $\mathcal{O}_Y$ and whose boundary passes the centre of $\mathcal{O}_X$, shown in Figure 19(c).*

(d) $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let $\mathcal{O}_Z$ be a sphere whose centre is at the boundary of $\mathcal{O}_X$ and whose boundary passes the centre of $\mathcal{O}_Y$, shown in Figure 19(d).*

9. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{PP}}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$, if*

(a) $\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with the boundary of $\mathcal{O}_X$ at points $\vec{P}_0$ and $\vec{P}_3$, and intersect with the bound-*

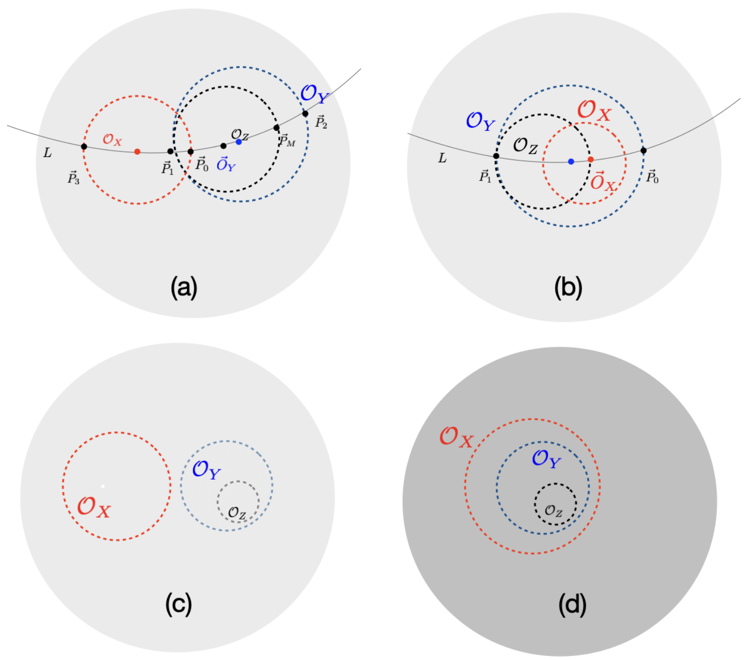


Figure 20: (a) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$; (b) $\mathcal{O}_X$ is proper part of $\mathcal{O}_Y$; (c) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$; (d) $\mathcal{O}_Y$ is part of $\mathcal{O}_X$.

ary of $\mathcal{O}_Y$ at points $\vec{P}_1$ and $\vec{P}_2$, $\vec{P}_M$ is located between $\vec{P}_0$ and $\vec{P}_2$. $\mathcal{O}_Z$ is the sphere with $d_{\mathbb{D}}(\vec{P}_1, \vec{P}_M)$ as the diameter. It is easy to prove that $\mathcal{O}_Z$ partially overlaps with $\mathcal{O}_X$ and is a proper part of $\mathcal{O}_Y$, as shown in Figure 20(a).

(b) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$ and intersect with the boundary of $\mathcal{O}_Y$ at points $\vec{P}_0$ and $\vec{P}_1$. Let $\vec{O}_X$ be closer to $\vec{P}_0$ than to $\vec{P}_1$. Let $\mathcal{O}_Z$ be the sphere whose diameter is $d_{\mathbb{D}}(\vec{P}_1, \vec{P}_X)$, shown in Figure 20(b).*

(c) $\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. *As $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$, $\mathcal{O}_X$ will disconnect from any sphere inside $\mathcal{O}_Y$. This contradicts with $\mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 20(c).*

(d) $\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. *As $\mathcal{O}_X$ contains $\mathcal{O}_Y$, $\mathcal{O}_X$ will contains any sphere inside $\mathcal{O}_Y$. This contradicts with $\mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 20(d).*

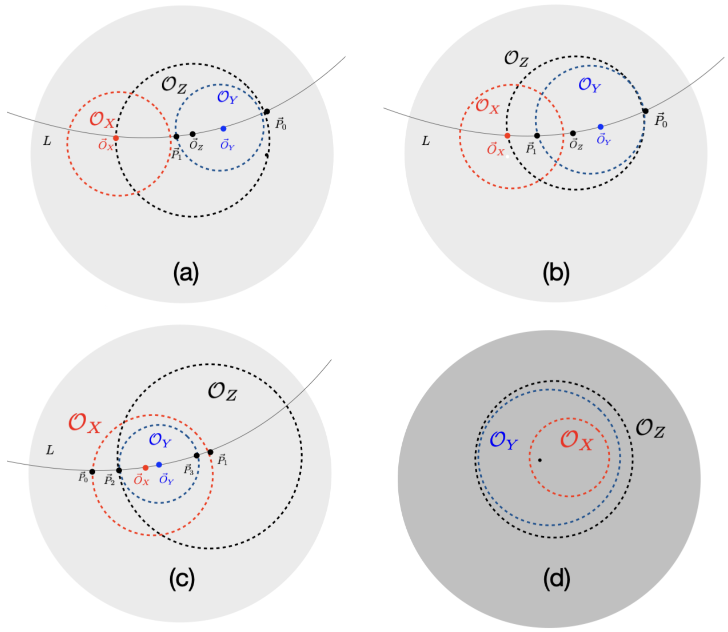


Figure 21: (a) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$; (b) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$; (c) $\mathcal{O}_Y$ is proper part of $\mathcal{O}_X$; (d) $\mathcal{O}_X$ is part of $\mathcal{O}_Y$.

10. *$\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PP}(\mathcal{O}_Y, \mathcal{O}_Z)$ and $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$. For any fixed $\mathcal{O}_X$ and $\mathcal{O}_Y$, if*

    (a) *$\mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$. Let $\vec{P}_0$ be located at the boundary of $\mathcal{O}_Y$ and the apogee to $\mathcal{O}_X$. Let $\mathcal{O}_Z$ be the sphere takes the segment $|\vec{O}_X \vec{P}_0|$ as the diameter. Then, $\mathcal{O}_Z$ contains $\mathcal{O}_Y$ and partially overlaps with $\mathcal{O}_X$, shown in Figure 21(a).*

    (b) *$\mathbf{PO}(\mathcal{O}_X, \mathcal{O}_Y)$. The same as (a), shown in Figure 21(b).*

    (c) *$\overline{\mathbf{PP}}(\mathcal{O}_X, \mathcal{O}_Y)$. Let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with the boundary of $\mathcal{O}_X$ at $\vec{P}_1$ and $\vec{P}_2$, and intersect with the boundary of $\mathcal{O}_Y$ at $\vec{P}_3$ and $\vec{P}_4$, as shown in Figure 21(c). Let $\mathcal{O}_Z$ take $\vec{P}_2$ as the centre, and $d_{\mathbb{D}}(\vec{P}_2, \vec{P}_3)$ as the radius.*

    (d) *$\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. For any $\mathcal{O}_Z$ containing $\mathcal{O}_Y$, $\mathcal{O}_Z$ will contain $\mathcal{O}_X$. This contradicts with $\mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 21(d).*

11. *$\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$ and $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \overline{\mathbf{PP}}(\mathcal{O}_Z, \mathcal{O}_X)$. Case 5.*

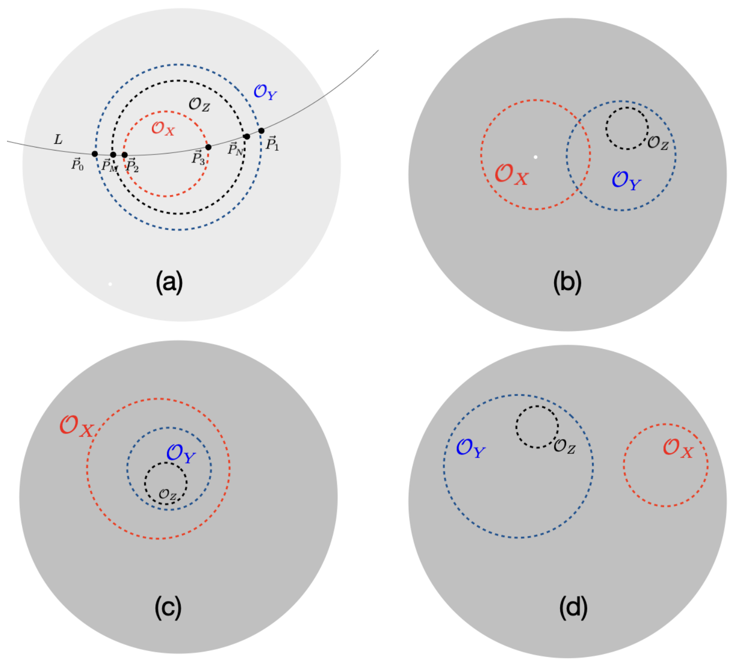


Figure 22: (a) $\mathcal{O}_X$ is proper part of $\mathcal{O}_Y$; (b) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Y$; (c) $\mathcal{O}_Y$ is part of $\mathcal{O}_X$; (d) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$.

12. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PO}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \overline{\mathbf{PP}}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 9.*

13. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{PP}}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \overline{\mathbf{PP}}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any fixed* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, if*

    (a) $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$. *Let the line* L *pass the centres of* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, intersect with the boundary of* $\mathcal{O}_X$ *at* $\vec{P}_2$ *and* $\vec{P}_3$*, and intersect with the boundary of* $\mathcal{O}_Y$ *at* $\vec{P}_0$ *and* $\vec{P}_1$*. Let* $\vec{P}_M$ *be a point between* $\vec{P}_0$ *and* $\vec{P}_2$*;* $\vec{P}_N$ *be a point between* $\vec{P}_1$ *and* $\vec{P}_3$*. Let* $\mathcal{O}_Z$ *be the sphere with* $d_{\mathbb{D}}(\vec{P}_M, \vec{P}_N)$ *as the diameter, as shown in Figure 22(a).*

    (b) *All other relations between* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *contradict with* $\mathbf{PP}(\mathcal{O}_X, \mathcal{O}_Y)$*, shown in Figure 22(b-d).*

14. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PP}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \overline{\mathbf{PP}}(\mathcal{O}_Z, \mathcal{O}_X)$. *For any* $\mathcal{O}_X$*,* $\mathcal{O}_Y$*, let* $\mathcal{O}_Z$ *be large enough to contain both* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, as shown in*

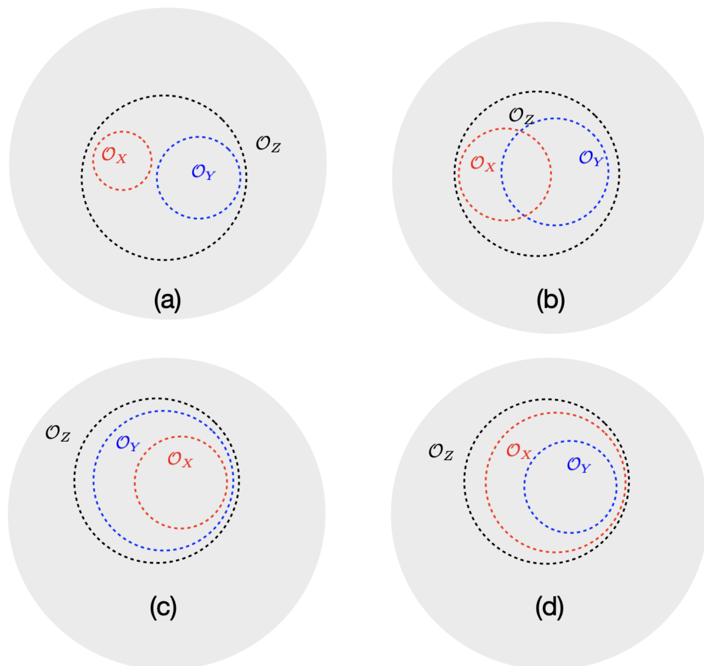


Figure 23: When $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$ and $\mathcal{O}_Z$ contains $\mathcal{O}_X$, $\mathcal{O}_Y$ and $\mathcal{O}_X$ can be of any relation.

Figure 23.

15. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PP}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 6.*

16. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PO}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PP}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 10.*

17. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{PP}}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PP}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 14.*

18. $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{PP}(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{PP}(\mathcal{O}_Z, \mathcal{O}_X)$. *Case 13.* ■

Lemma 2. *Given* $\mathbf{R}_1, \mathbf{R}_2 \in \{\mathbf{D}, \mathbf{EQ}, \mathbf{PO}, \mathbf{PP}, \overline{\mathbf{PP}}\}$ *and* $\mathbf{R}_3 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. *If the three relations are satisfiable, that is,* $\exists \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3[\mathbf{R}_1(\mathcal{O}_1, \mathcal{O}_2) \wedge \mathbf{R}_2(\mathcal{O}_2, \mathcal{O}_3) \wedge \mathbf{R}_3(\mathcal{O}_3, \mathcal{O}_1)]$, *for any fixed* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *satisfying* $\mathbf{R}_1(\mathcal{O}_X, \mathcal{O}_Y)$, *there will be* $\mathcal{O}_Z$ *such that* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$.

Proof (lemma) 2. *We enumerate the values of* $\mathbf{R}_1$ *and* $\mathbf{R}_2$. *The negative value of* $\mathbf{R}_3$

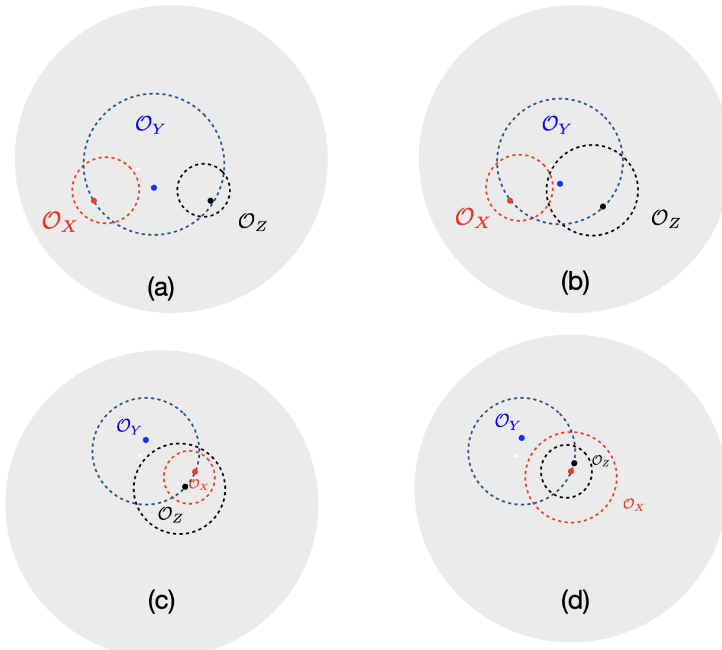


Figure 24: $\mathcal{O}_Y$ partially overlaps with $\mathcal{O}_X$ and $\mathcal{O}_Z$, when (a) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Z$, (b) $\mathcal{O}_X$ partially overlaps with $\mathcal{O}_Z$, (c) $\mathcal{O}_X$ is part of $\mathcal{O}_Z$, (d) $\mathcal{O}_Z$ is part of $\mathcal{O}_X$.

can be understood as the grouping of several positive relations, as follows.

$$\neg\mathbf{D} = \mathbf{EQ} \vee \mathbf{PO} \vee \mathbf{PP} \vee \overline{\mathbf{PP}}$$

$$\neg\mathbf{P} = \mathbf{D} \vee \mathbf{PO} \vee \overline{\mathbf{PP}}$$

$$\neg\overline{\mathbf{P}} = \mathbf{D} \vee \mathbf{PO} \vee \mathbf{PP}$$

The rest of the part is similar to the proof of Lemma 1. ■

Lemma 3. *Relations $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ are satisfiable, where $\mathbf{T}_1, \mathbf{T}_2 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$ and $\mathbf{T}_3 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \mathbf{PO}\}$. For any fixed $\mathcal{O}_Z$ and $\mathcal{O}_X$ satisfying $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$, there is $\mathcal{O}_Y$ satisfying $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y)$, $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$.*

Proof (lemma) 3. *1.* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X) = \mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$.

Let $\mathcal{O}_Y$ be such a sphere whose centre is outside $\mathcal{O}_X$ and $\mathcal{O}_Z$ and whose boundary passes the centres of $\mathcal{O}_X$ and $\mathcal{O}_Z$. In this way, $\mathcal{O}_Y$ partially overlaps with

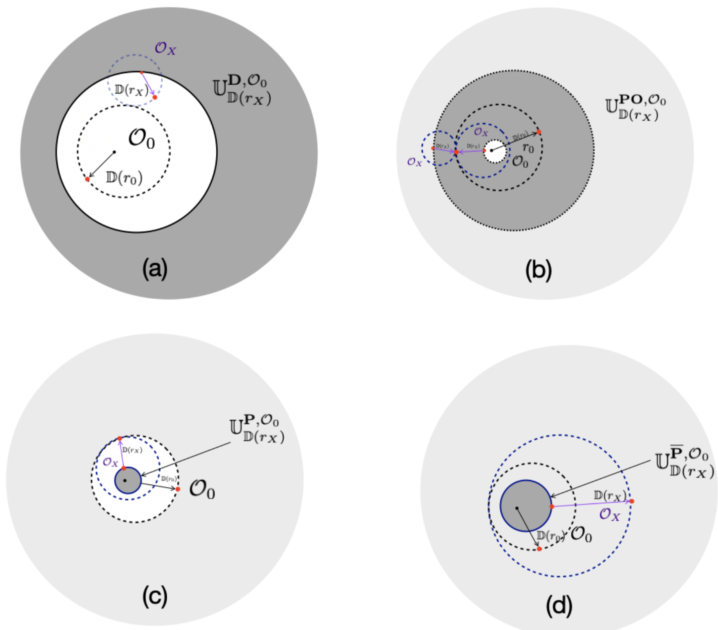


Figure 25: (a) $\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ occupies the whole space except an open sphere with the centre $\vec{O}_0$ and the radius of $\mathbb{D}(r_0)+\mathbb{D}(r_X)$. $\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is concave; (b) $\mathbb{U}^{\mathbf{PO},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is an open ring concentric with $\mathcal{O}_0$ whose radius within the range of $|\mathbb{D}(r_0)-\mathbb{D}(r_X)|$ and $\mathbb{D}(r_0)+\mathbb{D}(r_X)$. $\mathbb{U}^{\mathbf{PO},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is concave; (c) $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is a closed sphere with the centre $\vec{O}_0$ and the radius of $\mathbb{D}(r_0)-\mathbb{D}(r_X)$. It is convex; (d) $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is a closed sphere that is concentric with $\mathcal{O}_0$ and with the radius of $\mathbb{D}(r_X)-\mathbb{D}(r_0)$. It is convex.

$\mathcal{O}_X$ and $\mathcal{O}_Z$. Therefore, for all $\mathbf{T}_1,\mathbf{T}_2\in\{\neg\mathbf{D},\neg\mathbf{P},\neg\overline{\mathbf{P}}\}$, we have $\mathbf{T}_1(\mathcal{O}_X,\mathcal{O}_Y)$ and $\mathbf{T}_2(\mathcal{O}_Y,\mathcal{O}_Z)$, as illustrated in Figure 24(a).

2. *$\mathbf{T}_3(\mathcal{O}_Z,\mathcal{O}_X)=\mathbf{PO}(\mathcal{O}_Z,\mathcal{O}_X)\vee\mathbf{P}(\mathcal{O}_Z,\mathcal{O}_X)\vee\overline{\mathbf{P}}(\mathcal{O}_Z,\mathcal{O}_X)$. The same as case 1, illustrated in Figure 24(b-d).* ■

3.4. The relative qualitative space $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r_X)}$

Definition 1. *Let $\mathcal{O}_0$ be a fixed sphere with radius $\mathbb{D}(r_0)$ and the centre $\vec{O}_0$, and let $\mathcal{O}_X$ be a moving sphere with fixed radius $\mathbb{D}(r_X)$, satisfying $\mathbf{R}(\mathcal{O}_X,\mathcal{O}_0)$, where $\mathbf{R}\in\{\mathbf{D},\mathbf{PO},\mathbf{P},\overline{\mathbf{P}}\}$. All possible locations of the centre of $\mathcal{O}_X$ form a relative qualitative space $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ as follows.*

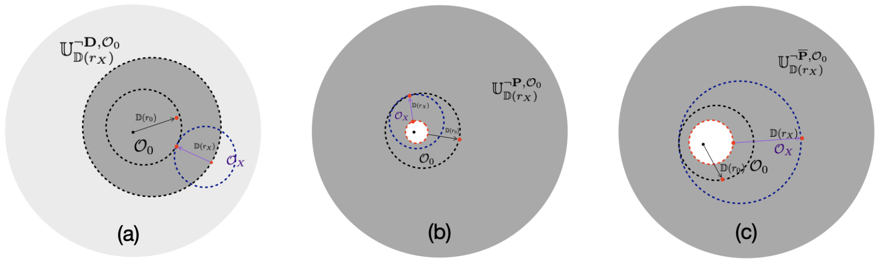


Figure 26: (a) $\mathbb{U}^{\neg\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ occupies the complement space of $\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$; (b) $\mathbb{U}^{\neg\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ occupies the complement space of $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$; (c) $\mathbb{U}^{\neg\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ occupies the complement space of $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$.

1. $\mathbf{R}(\mathcal{O}_X,\mathcal{O}_0) = \mathbf{D}(\mathcal{O}_X,\mathcal{O}_0)$. *$\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is greater than or equal to $\mathbb{D}(r_X)+\mathbb{D}(r_0)$. $\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X,\vec{O}_0) \geq \mathbb{D}(r_X)+\mathbb{D}(r_0)\}$, as illustrated in Figure 25(a).*

2. $\mathbf{R}(\mathcal{O}_X,\mathcal{O}_0) = \mathbf{PO}(\mathcal{O}_X,\mathcal{O}_0)$. *$\mathbb{U}^{\mathbf{PO},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is less than $\mathbb{D}(r_X)+\mathbb{D}(r_0)$ and greater than the absolute difference between $\mathbb{D}(r_X)$ and $\mathbb{D}(r_0)$. $\mathbb{U}^{\mathbf{PO},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : |\mathbb{D}(r_X)-\mathbb{D}(r_0)| < d_{\mathbb{D}}(\vec{O}_X,\vec{O}_0) < \mathbb{D}(r_X)+\mathbb{D}(r_0)\}$, as illustrated in Figure 25(b).*

3. $\mathbf{R}(\mathcal{O}_X,\mathcal{O}_0) = \mathbf{P}(\mathcal{O}_X,\mathcal{O}_0)$. *If $\mathbb{D}(r_0) \geq \mathbb{D}(r_X)$, $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is less than or equal to $\mathbb{D}(r_0)-\mathbb{D}(r_X)$. $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X,\vec{O}_0) \leq \mathbb{D}(r_0)-\mathbb{D}(r_X)\}$, as illustrated in Figure 25(c). If $\mathbb{D}(r_0) < \mathbb{D}(r_X)$, $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is empty $\varnothing$.*

4. $\mathbf{R}(\mathcal{O}_X,\mathcal{O}_0) = \overline{\mathbf{P}}(\mathcal{O}_X,\mathcal{O}_0)$. *If $\mathbb{D}(r_X) \geq \mathbb{D}(r_0)$, $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is less than or equal to $\mathbb{D}(r_X)-\mathbb{D}(r_0)$. $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X,\vec{O}_0) \leq \mathbb{D}(r_X)-\mathbb{D}(r_0)\}$, as illustrated in Figure 25(d). If $\mathbb{D}(r_X) < \mathbb{D}(r_0)$, $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is empty $\varnothing$.*

Definition 2. *$\mathbb{U}^{\neg\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$, $\mathbb{U}^{\neg\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$, and $\mathbb{U}^{\neg\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ are complement regions of $\mathbb{U}^{\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$, $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$, and $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$, respectively.*

1. *$\mathbb{U}^{\neg\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is less than $\mathbb{D}(r_X)+\mathbb{D}(r_0)$. $\mathbb{U}^{\neg\mathbf{D},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X,\vec{O}_0) < \mathbb{D}(r_X)+\mathbb{D}(r_0)\}$, as illustrated in Figure 26(a).*

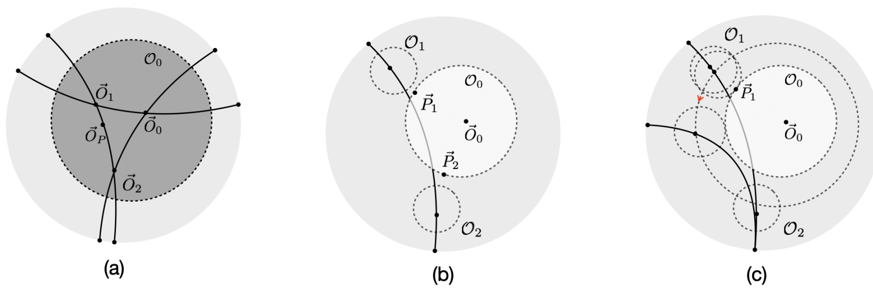


Figure 27: (a) $\mathbb{U}^{\mathbf{P},\mathcal{O}}_{\mathbb{D}(r_X)}$ and $\mathbb{U}^{\overline{\mathbf{P}},\mathcal{O}}_{\mathbb{D}(r_X)}$ are convex spheres; (b) $\mathbb{U}^{\mathbf{D},\mathcal{O}}_{\mathbb{D}(r_X)}$ and $\mathbb{U}^{\mathbf{PO},\mathcal{O}}_{\mathbb{D}(r_X)}$ are concave with a spherical hole; (c) this hole can be circumvented by rotating around it.

2. *If $\mathbb{D}(r_0) \geq \mathbb{D}(r_X)$, $\mathbb{U}^{\neg\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is greater than $\mathbb{D}(r_0) - \mathbb{D}(r_X)$. $\mathbb{U}^{\neg\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X, \vec{O}_0) > \mathbb{D}(r_0) - \mathbb{D}(r_X)\}$, as illustrated in Figure 26(b). If $\mathbb{D}(r_0) < \mathbb{D}(r_X)$, $\mathbb{U}^{\neg\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the whole space $\mathbb{U}$.*

3. *If $\mathbb{D}(r_X) \geq \mathbb{D}(r_0)$, $\mathbb{U}^{\neg\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the space of all points $\vec{O}_X$ whose distance to $\vec{O}_0$ is greater than $\mathbb{D}(r_X) - \mathbb{D}(r_0)$. $\mathbb{U}^{\neg\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)} = \{\vec{O}_X : d_{\mathbb{D}}(\vec{O}_X, \vec{O}_0) > \mathbb{D}(r_X) - \mathbb{D}(r_0)\}$, as illustrated in Figure 26(c). If $\mathbb{D}(r_X) < \mathbb{D}(r_0)$, $\mathbb{U}^{\neg\overline{\mathbf{P}},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is $\mathbb{U}^{\mathbf{P},\mathcal{O}_0}_{\mathbb{D}(r_X)}$ is the whole space $\mathbb{U}$.*

3.5. The rotation theorem in a relative qualitative space

Corollary 2. *For any spheres $\mathcal{O}_X$ with fixed $\mathbb{D}(r_X)$ and $\mathcal{O}_V$, rotating $\mathcal{O}_X$ around the Poincaré centre of $\mathcal{O}_V$ preserves the qualitative spatial relation between them.*

Proof (corollary) 2. *Each qualitative spatial relation is a function of the distance between their centres, $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$, and the radii. Rotating $\mathcal{O}_X$ around the centre of $\mathcal{O}_V$ preserves $d_{\mathbb{D}}(\vec{O}_X, \vec{O}_V)$, and their radii $\mathbb{D}(r_X)$ and $\mathbb{D}(r_V)$. Therefore, it preserves the qualitative spatial relation.* ■

Lemma 4. *For any two different spheres $\mathcal{O}_1$ and $\mathcal{O}_2$ with the same radius $\mathbb{D}(r)$ and satisfying $\mathbf{R}(\mathcal{O}_1, \mathcal{O}_0)$ and $\mathbf{R}(\mathcal{O}_2, \mathcal{O}_0)$, if $\mathbf{R} \in \{\mathbf{P}, \overline{\mathbf{P}}\}$, directly move $\mathcal{O}_1$ to $\mathcal{O}_2$, the relation $\mathbf{R}$ will always hold during the movement process; if $\mathbf{R} \in \{\mathbf{D}, \mathbf{PO}\}$, directly shifting $\mathcal{O}_1$ to $\mathcal{O}_2$ may violate the relation $\mathbf{R}$ during the movement process. To preserve*

$\mathbf{R}(\mathcal{O}_1, \mathcal{O}_0)$ *during the process of shifting,* $\mathcal{O}_1$ *may need to rotate around the centre of* $\mathcal{O}_0$.

Proof (lemma) 4.

1. $\mathbf{R} \in \{\mathbf{P}, \overline{\mathbf{P}}\}$. $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r)}$ *is a sphere, thus convex. Therefore, when* $\mathcal{O}_1$ *moving from* $\vec{O}_1$ *to* $\vec{O}_2$*, its centre point will be inside* $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r)}$.

2. $\mathbf{R} \in \{\mathbf{D}, \mathbf{PO}\}$. $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r)}$ *encompasses a concentric sphere that does not belong to* $\mathbf{R} \in \{\mathbf{D}, \mathbf{PO}\}$. $\mathbb{U}^{\mathbf{R},\mathcal{O}_0}_{\mathbb{D}(r)}$. *Suppose that direct shifting* $\mathcal{O}_1$ *to* $\mathcal{O}_2$ *would intersect with this concentric sphere at points* $\vec{P}_1$ *and* $\vec{P}_2$*, as illustrated in Figure 27(b). At point* $\vec{P}_1$ *the relation* $\mathbf{R}$ *exactly holds, with Corollary 2, rotating* $\mathcal{O}_1$ *at point* $\vec{P}_1$ *around the centre of* $\mathcal{O}_0$ *to point* $\vec{P}_2$ *will preserve the relation* $\mathbf{R}$*, as illustrated in Figure 27(c).* ■

3.6. The constraint optimisation is gradual descent

Theorem 2. *Let* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *be two fixed non-concentric spheres;* $\mathcal{O}_Z$ *be a movable sphere;* $\mathbf{T}_{ZY}$ *and* $\mathbf{T}_{ZX}$ *be the target relations of* $\mathcal{O}_Z$ *to* $\mathcal{O}_Y$ *and* $\mathcal{O}_X$*, respectively,* $\mathbf{T}_{ZY}, \mathbf{T}_{ZX} \in \mathcal{T} = \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. $COP^{\mathbf{T}_{ZX}}_{\mathbf{T}_{ZY}}(\mathcal{O}_Z|\mathcal{O}_X;\mathcal{O}_Y)$ *is gradual descent.*

Proof 2. $\mathbf{S}_{ZY}$ *and* $\mathbf{S}_{ZX}$ *be the actual relations of* $\mathcal{O}_Z$ *to* $\mathcal{O}_Y$ *and* $\mathcal{O}_X$*, respectively.* $\mathbf{S}_{ZX} \in f_{tsp}(\mathbf{T}_{ZX})$ *and* $\mathbf{S}_{ZY} \in f_{tsp}(\mathbf{T}_{ZY})$*; the relation* $\tilde{\mathbf{R}}_2(\mathcal{O}_Z, \mathcal{O}_Y)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$ *be the next target relations of* $\mathcal{O}_Z$ *to* $\mathcal{O}_Y$ *and to* $\mathcal{O}_X$*, respectively, where* $\tilde{\mathbf{R}}_2$, $\mathbf{R}_3 \in \mathcal{S}$. $COP(\mathcal{O}_Z|\mathcal{O}_X;\mathcal{O}_Y)$ *repeatedly performs two steps as follows: (1) it gradually decreases the value of the function* $\Delta^{\mathbf{R}_3}_{\mathbf{S}_{ZX}}(\mathcal{O}_Z, \mathcal{O}_X) + \Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z, \mathcal{O}_Y)$*; (2) while* $\Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z, \mathcal{O}_Y) > 0$*, gradual descent operations will be applied for* $\Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z, \mathcal{O}_Y)$.

Consider the case that the radius $\mathbb{D}(Z)$ *is fixed, and repeatedly perform gradual descent operation on* $\Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z, \mathcal{O}_Y)$ *until it equals zero, at this time* $\Delta^{\mathbf{R}_3}_{\mathbf{S}_{ZX}}(\mathcal{O}_Z, \mathcal{O}_X)$ *may increase a value* δ_0^**. We need to prove that decreased value from the relation to* $\mathcal{O}_Y$*, is no less than* δ_0^*.

When $\mathcal{O}_Y$ *is fixed, the gradual descent operation on* $\Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z, \mathcal{O}_Y)$ *will move* $\mathcal{O}_Z$ *along the geodesic* $\vec{O}_Y\vec{O}_Z$*, with the decrease of* $\delta_1' = d_{\mathbb{D}}(\vec{O}_Z, \vec{O}'_Z)$*. This may cause a maximum increase of* $\Delta^{\mathbf{R}_3}_{\mathbf{S}_{ZX}}(\mathcal{O}_Z, \mathcal{O}_X)$ *with the value of*

$$\delta_0^* = |d_{\mathbb{D}}(\vec{O}_X, \vec{O}'_Z) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_Z)|$$

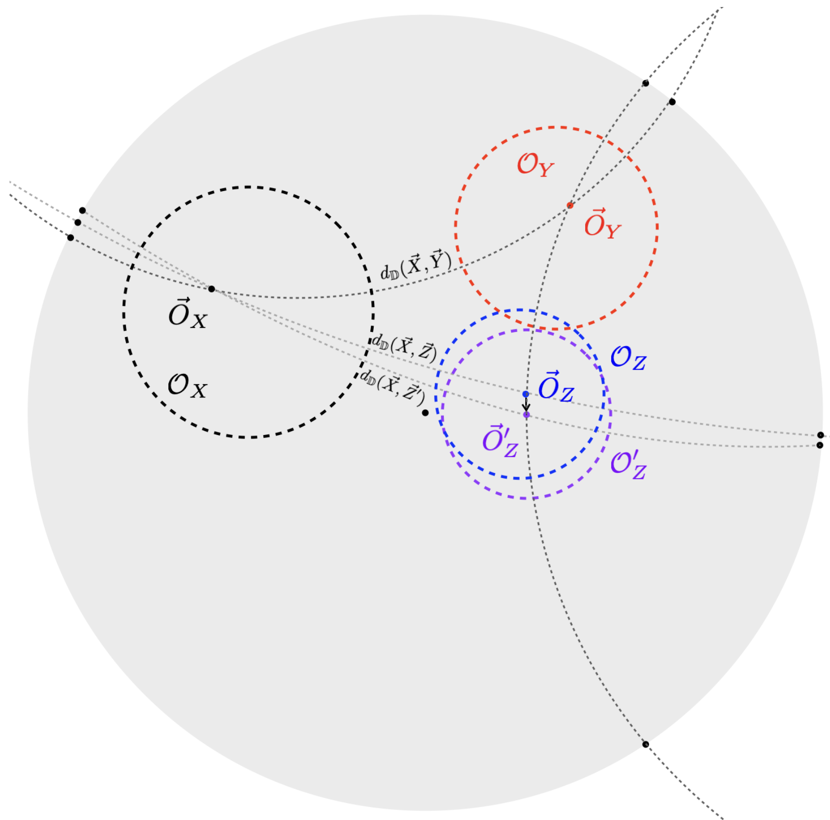


Figure 28: The gradual descent along the line segment $d_{\mathbb{D}}(\vec{Z},\vec{Z}')$ may cause the gradual ascent of $d_{\mathbb{D}}(\vec{X},\vec{Z}') - d_{\mathbb{D}}(\vec{X},\vec{Z})$, however, $d_{\mathbb{D}}(\vec{X},\vec{Z}') - d_{\mathbb{D}}(\vec{X},\vec{Z}) \le d_{\mathbb{D}}(\vec{Z},\vec{Z}')$.

shown in Figure 28. We have

$$\delta_1^{'} = d_{\mathbb{D}}(\vec{O}_Z, \vec{O}'_Z) \ge |d_{\mathbb{D}}(\vec{O}_X, \vec{O}'_Z) - d_{\mathbb{D}}(\vec{O}_X, \vec{O}_Z)| = \delta_0^*$$

The equal relation holds if $\vec{O}_X$, $\vec{O}_Z$, and $\vec{O}'_Z$ are collinear.

Consider the case that the gradual descent operation on $\Delta^{\tilde{\mathbf{R}}_2}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z,\mathcal{O}_Y)$ also update $\mathbb{D}(Z)$, with the change of $\delta_{\mathbb{D}(r_Z)}$. This value helps to reduce the value of $\Delta^{\mathbf{R}_3}_{\mathbf{S}_{ZY}}(\mathcal{O}_Z,\mathcal{O}_Y)$. In this case, the same amount of the value may increase the value of $\Delta^{\mathbf{R}_3}_{\mathbf{S}_{ZX}}(\mathcal{O}_Z,\mathcal{O}_X)$, which exactly counteracts the decreased value from the relation to $\mathcal{O}_Y$. Therefore, $COP^{\mathbf{T}_{ZX}}_{\mathbf{T}_{ZY}}(\mathcal{O}_Z|\mathcal{O}_X;\mathcal{O}_Y)$ is gradual descent. ■

3.7. Theorems about constraint optimisation

Lemma 5. *Let* $\mathbf{R}_1$*,* $\mathbf{R}_2$*, and* $\mathbf{R}_3$ *be* satisfiable*, where* $\mathbf{R}_1, \mathbf{R}_2 \in \{\mathbf{D}, \mathbf{P}, \mathbf{PO}, \overline{\mathbf{P}}\}$*,* $\mathbf{R}_3 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*. Let spheres* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *be fixed and satisfying the relation* $\mathbf{R}_1(\mathcal{O}_X, \mathcal{O}_Y)$*. HSphNN can construct* $\mathcal{O}_Z$ *such that* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$*.*

Proof (lemma) 5. *HSphNN can construct* $\mathcal{O}_Z$ *such that* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z)$*. With Theorem 2,* $COP(\mathcal{O}_Z|\mathcal{O}_X; \mathcal{O}_Y)$ *is gradual descent.*

1. $\mathbf{R}_3 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$.

   With Lemma 1, there is $\mathcal{O}^*_Z$ *satisfying* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}^*_Z)$ *and* $\mathbf{R}_3(\mathcal{O}^*_Z, \mathcal{O}_X)$*. That is,* $COP(\mathcal{O}^*_Z|\mathcal{O}_X; \mathcal{O}_Y) = 0$.

2. $\mathbf{R}_3 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$.

   With Lemma 2, there is $\mathcal{O}'_Z$ *satisfying* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}'_z)$ *and* $\mathbf{R}_3(\mathcal{O}'_Z, \mathcal{O}_X)$*. That is,* $COP(\mathcal{O}'_Z|\mathcal{O}_X; \mathcal{O}_Y) = 0$.

In both cases, $COP(\mathcal{O}_Z|\mathcal{O}_X; \mathcal{O}_Y)$ *will reach 0. Therefore, HSphNN can construct* $\mathcal{O}_Z$ *such that* $\mathbf{R}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$ *by gradual descending the function* $COP(\mathcal{O}_Z|\mathcal{O}_X; \mathcal{O}_Y)$. ■

Lemma 6. *Let* $\mathcal{O}_X$*,* $\mathcal{O}_Y$*, and* $\mathcal{O}_Z$ *be spheres that satisfy three relations* $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y)$*,* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ *and* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$*, where* $\mathbf{T}_1, \mathbf{T}_2 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*,* $\mathbf{T}_3 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \mathbf{PO}\}$*. HSphNN can construct an Euler Diagram by first realising* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$*, then fix* $\mathcal{O}_Z$ *and* $\mathcal{O}_X$*, and constructs* $\mathcal{O}_Y$ *to satisfy both* $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$*.*

Proof (lemma) 6. *Let* $\mathbf{S}_1$ *be the actual relation between* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*, and* $\mathbf{S}_2$ *be the actual relation between* $\mathcal{O}_Y$ *and* $\mathcal{O}_Z$*, where* $\mathbf{S}_1 \in f_{tsp}(\mathbf{T}_1) = \{\mathbf{T}_1, \neg\mathbf{T}_1\}$*,* $\mathbf{S}_2 \in f_{tsp}(\mathbf{T}_2) = \{\mathbf{T}_2, \neg\mathbf{T}_2\}$*,* $\mathbf{T}_1, \mathbf{T}_2 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*. With Lemma 3, for any fixed* $\mathcal{O}_Z$ *and* $\mathcal{O}_X$ *satisfying* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$*, there exists* $\mathcal{O}^*_Y$ *satisfying* $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}^*_Y)$ *and* $\mathbf{T}_2(\mathcal{O}^*_Y, \mathcal{O}_Z)$*, in which* $\mathbf{T}_1, \mathbf{T}_2 \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*. So,* $COP(\mathcal{O}^*_Y|\mathcal{O}_X; \mathcal{O}_Z) = 0$*. Therefore, HSphNN can realise* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$*, then fix* $\mathcal{O}_Z$ *and* $\mathcal{O}_X$*, and constructs* $\mathcal{O}_Y$ *to satisfy both* $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$ *by gradual descending the function* $COP(\mathcal{O}_Y|\mathcal{O}_X; \mathcal{O}_Z)$. ■

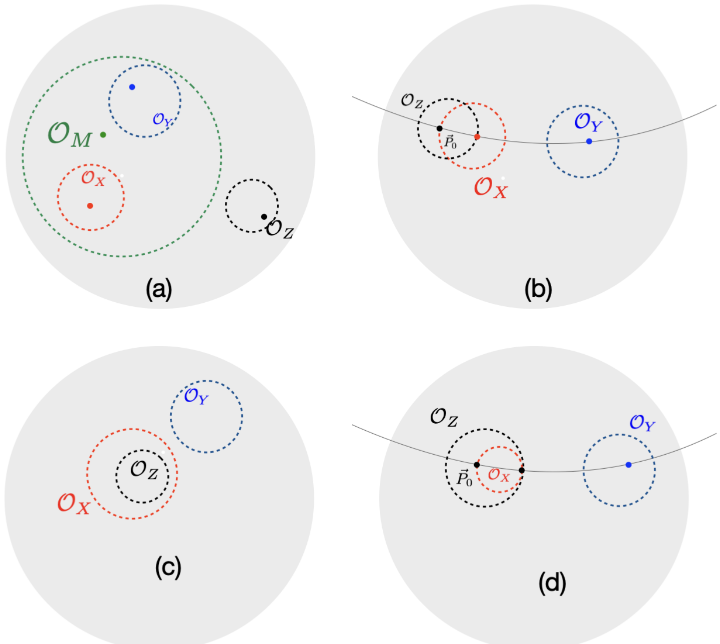


Figure 29: When $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$, and $\mathcal{O}_Y$ disconnects from $\mathcal{O}_Z$, there are 4 possible relations between $\mathcal{O}_X$ and $\mathcal{O}_Z$.

Theorem 3. *Let* $\mathbf{R}_1$*,* $\mathbf{T}_2$*, and* $\mathbf{T}_3$ *be* satisfiable*, where* $\mathbf{R}_1 \in \{\mathbf{D}, \mathbf{P}, \mathbf{PO}, \overline{\mathbf{P}}\}$*,* $\mathbf{T}_2, \mathbf{T}_3 \in \mathcal{T} = \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*. Let* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *be fixed and satisfying* $\mathbf{R}_1(\mathcal{O}_X, \mathcal{O}_Y)$*. HSphNN can construct* $\mathcal{O}_Z$ *such that* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$*, and* $\mathbf{T}_3(\mathcal{O}_Z, \mathcal{O}_X)$*.*

Proof 3. *Lemma 5 and Lemma 6.* ■

3.8. The restart theorem

Lemma 7. *Let* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *be fixed, satisfying* $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y)$*, and* $\mathcal{O}_Z$ *be movable, satisfying* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z)$*, where* $\mathbf{T}_1, \mathbf{T}_2 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$*. Let the relation between* $\mathcal{O}_Z$ *and* $\mathcal{O}_X$ *be* $\mathbf{R}_3(\mathcal{O}_Z, \mathcal{O}_X)$ *and the three relations* $\mathbf{T}_1, \mathbf{T}_2, \mathbf{R}_3$ *are satisfiable, where* $\mathbf{R}_3 \in \{\mathbf{D}, \mathbf{PO}, \mathbf{P}, \overline{\mathbf{P}}\}$*. The number of possible relations of* $\mathbf{R}_3$ *can not be 2.*

Proof (lemma) 7. *We enumerate values of* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*.*

1. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$.

(a) *Let $\mathcal{O}_M$ be a sphere that contains $\mathcal{O}_X$ and $\mathcal{O}_Y$. Any $\mathcal{O}_Z$ that disconnects from $\mathcal{O}_M$ disconnects from $\mathcal{O}_X$ and $\mathcal{O}_Y$, $\mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$ shown in Figure 29(a).*

(b) *$\vec{P}_0$ be the apogee to $\mathcal{O}_Y$ at the boundary of $\mathcal{O}_X$. Let $\mathcal{O}_Z$ take $\vec{P}_0$ as the centre and have the same radius as $\mathcal{O}_X$, then $\mathcal{O}_Z$ partially overlaps with $\mathcal{O}_X$ and disconnects from $\mathcal{O}_Y$, $\mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$ shown in Figure 29(b).*

(c) *Let $\mathcal{O}_Z$ be part of $\mathcal{O}_X$, $\mathcal{O}_Z$ will disconnect from $\mathcal{O}_Y$, $\mathbf{P}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 29(c).*

(d) *$\vec{P}_0$ be the apogee to $\mathcal{O}_Y$ at the boundary of $\mathcal{O}_X$. Let $\mathcal{O}_Z$ take $\vec{P}_0$ as the centre and the diameter of $\mathcal{O}_X$ as the radius, then $\mathcal{O}_X$ is part of $\mathcal{O}_Z$ and disconnects from $\mathcal{O}_Y$, $\overline{\mathbf{P}}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 29(d).*

So, the number of possible values of $\mathbf{R}_3$ is 4.

2. *$\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$ and $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{P}(\mathcal{O}_Y, \mathcal{O}_Z)$. Let the line L pass the centres of $\mathcal{O}_X$ and $\mathcal{O}_Y$, intersect with boundaries of $\mathcal{O}_X$ and $\mathcal{O}_Y$ at $\vec{P}_0$, $\vec{P}_1$, $\vec{P}_2$, and $\vec{P}_3$, respectively, shown in Figure 30(a).*

   (a) *Let $\mathcal{O}_Z$ be the sphere with $\vec{P}_0\vec{P}_3$ as diameter, $\overline{\mathbf{P}}(\mathcal{O}_Z, \mathcal{O}_X)$;*

   (b) *Let $\mathcal{O}_Z$ take $\vec{P}_3$ as the centre and $\mathcal{O}_Z$'s boundary pass the centre of $\mathcal{O}_X$, then $\mathcal{O}_Z$ contains $\mathcal{O}_Y$ and partially overlaps with $\mathcal{O}_X$, $\mathbf{PO}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 30(b);*

   (c) *Let $\mathcal{O}_Z$ take $\vec{P}_3$ as the centre and take the diameter of $\mathcal{O}_Y$ as the radius, then $\mathcal{O}_Z$ contains $\mathcal{O}_Y$ and disconnects from $\mathcal{O}_X$, $\mathbf{D}(\mathcal{O}_Z, \mathcal{O}_X)$, shown in Figure 30(c).*

   (d) *If $\mathcal{O}_Z$ is part of $\mathcal{O}_X$, $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$, $\mathcal{O}_Z$ will disconnect from $\mathcal{O}_Y$. This contradicts with the relation that $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$, $\mathbf{P}(\mathcal{O}_Y, \mathcal{O}_Z)$.*

   So, the number of possible values of $\mathbf{R}_3$ is 3.

3. *$\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{D}(\mathcal{O}_X, \mathcal{O}_Y)$ and $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{P}}(\mathcal{O}_Y, \mathcal{O}_Z)$. As $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$ and $\mathcal{O}_Z$ is inside $\mathcal{O}_Y$, so $\mathcal{O}_X$ disconnects from $\mathcal{O}_Z$, as shown in Figure 30(d). So, the number of possible values of $\mathbf{R}_3$ is 1.*

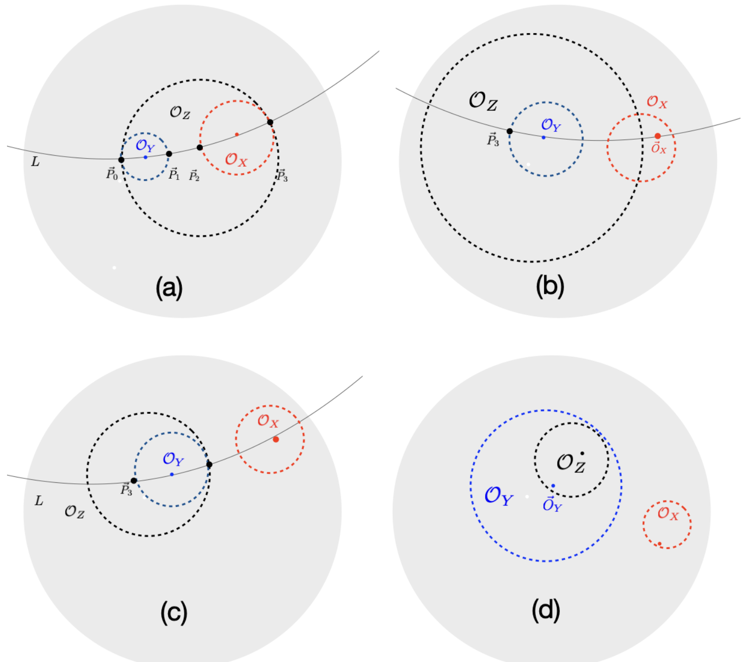


Figure 30: (a-c) $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$ and $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$, $\mathcal{O}_Z$ cannot be part of $\mathcal{O}_X$, other relations between $\mathcal{O}_X$ and $\mathcal{O}_Z$ are possible; (d) if $\mathcal{O}_X$ disconnects from $\mathcal{O}_Y$ and $\mathcal{O}_Z$ is part of $\mathcal{O}_Y$, $\mathcal{O}_X$ will disconnect from $\mathcal{O}_Z$.

4. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{P}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$. *Case 2.*

5. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{P}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{P}}(\mathcal{O}_Y, \mathcal{O}_Z)$. $\mathcal{O}_X$ *and* $\mathcal{O}_Z$ *are part of* $\mathcal{O}_Y$, $\mathcal{O}_X$ *and* $\mathcal{O}_Z$ *can be of any relations, as shown in Figure 31. So, the number of possible values of* $\mathbf{R}_3$ *is 4.*

6. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \mathbf{P}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{P}(\mathcal{O}_Y, \mathcal{O}_Z)$. *When* $\mathcal{O}_X$ *is part of* $\mathcal{O}_Y$ *and* $\mathcal{O}_Y$ *is part of* $\mathcal{O}_Z$, $\mathcal{O}_X$ *will be part of* $\mathcal{O}_Z$, *as shown in Figure 32(a). So, the number of possible values of* $\mathbf{R}_3$ *is 1.*

7. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{D}(\mathcal{O}_Y, \mathcal{O}_Z)$. *Case 3.*

8. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \mathbf{P}(\mathcal{O}_Y, \mathcal{O}_Z)$. *Case 5.*

9. $\mathbf{T}_1(\mathcal{O}_X, \mathcal{O}_Y) = \overline{\mathbf{P}}(\mathcal{O}_X, \mathcal{O}_Y)$ *and* $\mathbf{T}_2(\mathcal{O}_Y, \mathcal{O}_Z) = \overline{\mathbf{P}}(\mathcal{O}_Y, \mathcal{O}_Z)$. *This is equivalent to Case 6, as shown in Figure 32(b). So, the number of possible values of*

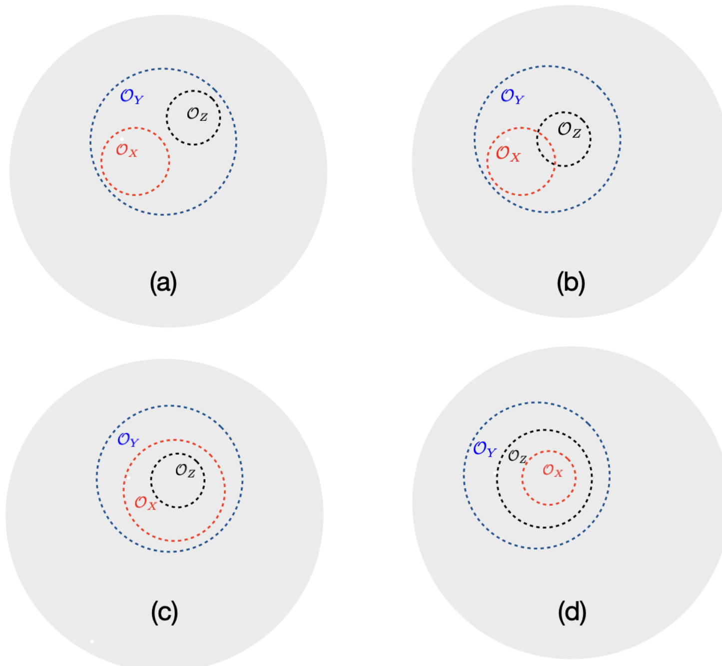


Figure 31: If $\mathcal{O}_Y$ contains both $\mathcal{O}_X$ and $\mathcal{O}_Z$, $\mathcal{O}_X$ and $\mathcal{O}_Z$ can be of any relations.

$\mathbf{R}_3$ *is 1.*

Therefore, the number of possible relations of $\mathbf{R}_3$ *can not be 2.* ■

Lemma 8. *Let three relations* $\mathbf{T}_0$*,* $\mathbf{T}_1$*, and* $\mathbf{T}_2$ *be* satisfiable*, which means that there are three spheres* $\mathcal{O}_0$*,* $\mathcal{O}_1$*, and* $\mathcal{O}_2$ *satisfying the relations* $\mathbf{T}_0(\mathcal{O}_0, \mathcal{O}_1)$*,* $\mathbf{T}_1(\mathcal{O}_1, \mathcal{O}_2)$*, and* $\mathbf{T}_2(\mathcal{O}_2, \mathcal{O}_0)$*, where* $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2 \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*. Let* $i \in \{0, 1, 2\}$*,* $j = (i+1) \mod 3$*,* $k = (j+1) \mod 3$*. If HSphNN fixes* $\mathcal{O}_i$ *and* $\mathcal{O}_j$ *(their relation is* $\mathbf{T}_i(\mathcal{O}_i, \mathcal{O}_j)$*) and can not find* $\mathcal{O}_k$ *that satisfies* $\mathbf{T}_j(\mathcal{O}_j, \mathcal{O}_k)$ *and* $\mathbf{T}_k(\mathcal{O}_k, \mathcal{O}_i)$*, HSphNN will update* $\mathcal{O}_j$ *and* $\mathcal{O}_k$ *to the relation* $\mathbf{T}_j(\mathcal{O}_j, \mathcal{O}_k)$ *and fix* $\mathcal{O}_j$ *and* $\mathcal{O}_k$ *to find* $\mathcal{O}_i$ *that satisfies* $\mathbf{T}_k(\mathcal{O}_k, \mathcal{O}_i)$ *and* $\mathbf{T}_i(\mathcal{O}_i, \mathcal{O}_j)$*.*

Proof (lemma) 8. *1.* $\mathbf{T}_i \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$*.*

(a) at most one of $\mathbf{T}_j$ *and* $\mathbf{T}_k$ *is a member of* $\{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$*.* $\mathbf{T}_0$*,* $\mathbf{T}_1$*, and* $\mathbf{T}_2$ *are* satisfiable*, with Lemma 5, HSphNN will construct an Euler diagram.*

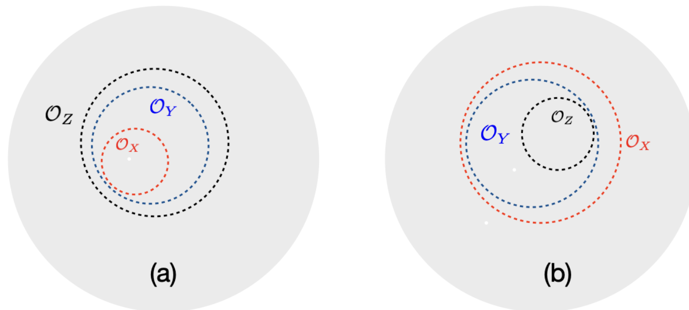


Figure 32: (a) if $\mathcal{O}_X$ is part of $\mathcal{O}_Y$ and $\mathcal{O}_Y$ is part of $\mathcal{O}_Z$, $\mathcal{O}_X$ will be part of $\mathcal{O}_Z$; switching $\mathcal{O}_X$ and $\mathcal{O}_Z$ will result in the case (b).

(b) $\mathbf{T}_j, \mathbf{T}_k \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. *With Lemma 6, HSphNN will construct an Euler diagram.*

2. $\mathbf{T}_i \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$.

$\mathbf{T}_i$ *is consistent with three relations in the set* $\{\mathbf{D}, \mathbf{PO}, \mathbf{P}, \overline{\mathbf{P}}\}$. *Let* consis$(\mathbf{T}_i)$ *denote the three consistent relations:* consis$(\neg\mathbf{D}) = \{\mathbf{PO}, \mathbf{P}, \overline{\mathbf{P}}\}$, consis$(\neg\mathbf{P}) = \{\mathbf{PO}, \mathbf{D}, \overline{\mathbf{P}}\}$, *and* consis$(\neg\overline{\mathbf{P}}) = \{\mathbf{PO}, \mathbf{P}, \mathbf{D}\}$.

Let the relation between $\mathcal{O}_i$ *and* $\mathcal{O}_j$ *be* $\mathbf{R}_i(\mathcal{O}_i, \mathcal{O}_j)$, $\mathbf{R}_i \in \{\mathbf{D}, \mathbf{PO}, \mathbf{P}, \overline{\mathbf{P}}\}$ *and* $\mathbf{R}_i$ *is consistent with* $\mathbf{T}_i$.

(a) *If* $\mathbf{R}_i$, $\mathbf{T}_j$, *and* $\mathbf{T}_k$ *are* satisfiable, *the same proof structure as case 1, as Lemma 5 and Lemma 6 hold for* $\{\mathbf{D}, \mathbf{P}, \mathbf{PO}, \overline{\mathbf{P}}\}$.

(b) *If* $\mathbf{R}_i$, $\mathbf{T}_j$, *and* $\mathbf{T}_k$ *are* unsatisfiable, *then "*$\mathbf{T}_j$, $\mathbf{T}_k \therefore \neg\mathbf{R}_i$*" will be valid.*

i. $\mathbf{T}_j, \mathbf{T}_k \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$.

$\mathbf{R}_i \in$ consis$(\mathbf{T}_i)$. *So, only relations in* consis$(\mathbf{T}_i)/\mathbf{R}_i$ *can be consistent with* $\mathbf{T}_j$ *and* $\mathbf{T}_k$. *The size of* consis$(\mathbf{T}_i)/\mathbf{R}_i$ *is less than or equals to 2. With Lemma 7, the value cannot be 2, so, exactly one relation* $\mathbf{R}_i^*$ *is consistent with* $\mathbf{T}_i$ *and* $\mathbf{T}_j$ *and* $\mathbf{T}_k$. *Let HSphNN fix* $\mathcal{O}_k$, *then optimise* $\mathcal{O}_j$ *to the relation* $\mathbf{T}_j(\mathcal{O}_j, \mathcal{O}_k)$, *and optimise* $\mathcal{O}_i$ *to the relation* $\mathbf{T}_k(\mathcal{O}_k, \mathcal{O}_i)$, *then the relation between* $\mathcal{O}_i$ *and* $\mathcal{O}_j$ *can only be* $\mathbf{R}_i^*$.

ii. $\mathbf{T}_j, \mathbf{T}_k \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$.

That is, "$\mathbf{T}_j$, $\mathbf{T}_k$ $\therefore\neg\mathbf{R}_i$" is valid, where all premises and the conclusion are in the negative form. However, there is no valid syllogism with three negative forms (check Table 2). So, $\mathbf{T}_j, \mathbf{T}_k \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$ is not possible.

iii. exactly one of $\mathbf{T}_j$ and $\mathbf{T}_k$ is the member of $\{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. Without the loss of generality, let $\mathbf{T}_j \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. As "$\mathbf{T}_j$, $\mathbf{T}_k$ $\therefore\neg\mathbf{R}_i$" is valid, therefore, $\mathbf{T}_k \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}\}$, and $\mathbf{T}_i, \mathbf{T}_j \in \{\neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. With Lemma 6, HSphNN will construct an Euler diagram. ■

Theorem 4. *Let p_0, p_1, p_2 be three syllogistic statements, where p_0 can be either $r_0(X_0, X_1)$ or $r_0(X_1, X_0)$, p_1 can be either $r_1(X_1, X_2)$ or $r_1(X_2, X_1)$, and p_2 can be either $r_2(X_0, X_2)$ or $r_2(X_2, X_0)$, $r_0, r_1, r_2 \in \{all, some, no, some\_not\}$. HSphNN can determine the satisfiability of p_0, p_1, p_2 in the first epoch, with maximum once restart.*

Proof 4. *We map X_i to $\mathcal{O}_i$ ($i = 0, 1, 2$) and map p_i to $\mathbf{T}_{ij}(\mathcal{O}_i, \mathcal{O}_j)$, where $i, j = 0, 1, 2$, $i \neq j$, $\mathbf{T}_{ij} = \psi(r_i)$ if $r_i(X_i, X_j)$ or $\mathbf{T}_{ij} = \overline{\psi}(r_i)$ if $r_i(X_j, X_i)$, and $\mathbf{T}_{ij} \in \{\mathbf{D}, \mathbf{P}, \overline{\mathbf{P}}, \neg\mathbf{D}, \neg\mathbf{P}, \neg\overline{\mathbf{P}}\}$. HSphNN first initialises three coincided spheres; if this configuration is a model that satisfies the three target relations, done. If not, HSphNN fixes $\mathcal{O}_0$, then updates $\mathcal{O}_1$ and $\mathcal{O}_2$, to satisfy $\mathbf{T}_{01}(\mathcal{O}_0, \mathcal{O}_2)$ and $\mathbf{T}_{20}(\mathcal{O}_2, \mathcal{O}_0)$, respectively, then HSphNN performs $COP^{\mathbf{T}_{21}}_{\mathbf{T}_{20}}(\mathcal{O}_2|\mathcal{O}_1, \mathcal{O}_0)$. If the global loss reaches zero, done; otherwise, HSphNN repeats the process by fixing $\mathcal{O}_2$ and updating $\mathcal{O}_0$ and $\mathcal{O}_1$ ..., finally, $COP^{\mathbf{T}_{02}}_{\mathbf{T}_{01}}(\mathcal{O}_0|\mathcal{O}_2, \mathcal{O}_1)$ will reach zero, if the input is satisfiable (Theorem 2 and Lemma 8).* ■

4. 24 valid types of Aristotelian syllogistic reasoning

Among 256 types of Aristotelian syllogistic reasoning, only 24 types are valid. Each valid syllogism is given a name whose vowels indicate the type (or mood) of a valid syllogistic reasoning, e.g., 'C$\underline{\text{E}}$L$\underline{\text{A}}$R$\underline{\text{E}}$NT' indicates types of moods are 'E', 'A', 'E', respectively. 'A' for *universal affirmative*, all X are Y, 'I' for *particular affirmative*, some X are Y; 'E' for *universal negative*, no X are Y, 'O' for *particular negative*, some X are not Y.

Table 2: List of all valid syllogisms, each is mapped to a spatial statement.

Num	Name	Premise	Conclusion	spatial relations statement
1	BARBARA	all s are m, all m are p	all s are p	$\mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
2	BARBARI	all s are m, all m are p	some s are p	$\mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
3	CELARENT	no m is p, all s are m	no s is p	$\mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \wedge \mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
4	CESARE	no p is m, all s are m	no s is p	$\mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
5	CALEMES	all p are m, no m is s	no s is p	$\mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
6	CAMESTRES	all p are m, no s is m	no s is p	$\mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
7	DARII	all m are p, some s are m	some s are p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \wedge \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
8	DATISI	all m are p, some m are s	some s are p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \wedge \neg\mathbf{D}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
9	DARAPTI	all m are s, all m are p	some s are p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
10	DISAMIS	some m are p, all m are s	some s are p	$\neg\mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
11	DIMATIS	some p are m, all m are s	some s are p	$\neg\mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
12	BAROCO	all p is m, some s are not m	some s are not p	$\mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \wedge \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
13	CESARO	no p is m, all s are m	some s are not p	$\mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
14	CAMESTROS	all s are m, no m is p	some s are not p	$\mathbf{P}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
15	CELARONT	no s is m, all p are m	some s are not p	$\mathbf{D}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
16	CALEMOS	all p are m, no m is s	some s are not p	$\mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
17	BOCARDO	some m are not p, all m are s	some s are not p	$\neg\mathbf{P}(\mathcal{O}_m, \mathcal{O}_p) \wedge \mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
18	BAMALIP	all m are s, all p are m	some s are p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{P}(\mathcal{O}_p, \mathcal{O}_m) \rightarrow \neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_p)$
19	FERIO	some s are m, no m is p	some s are not p	$\neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
20	FESTINO	some s are m, no p is m	some s are not p	$\neg\mathbf{D}(\mathcal{O}_s, \mathcal{O}_m) \wedge \mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
21	FERISON	some m are s, no m is p	some s are not p	$\neg\mathbf{D}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
22	FRESISON	some m are s, no p is m	some s are not p	$\neg\mathbf{D}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
23	FELAPTON	all m are s, no m is p	some s are not p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{D}(\mathcal{O}_m, \mathcal{O}_p) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$
24	FESAPO	all m are s, no p is m	some s are not p	$\mathbf{P}(\mathcal{O}_m, \mathcal{O}_s) \wedge \mathbf{D}(\mathcal{O}_p, \mathcal{O}_m) \rightarrow \neg\mathbf{P}(\mathcal{O}_s, \mathcal{O}_p)$

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