MATRIX ALGEBRA

CLPS 2908 | Lecture 3 | January 29, 2019

0 Background

- Matrix, *sing.*; matrices, *pl.*
- Origin: solving simultaneous equations in *arrays*
  - Chinese text from 200?BC (yield bundles → area)
  - Determinant ($16^{th}/17^{th}$ century, explicitly Gauss 1801)
  - Term *matrix*: 1850 by **James Joseph Sylvester**
- Rise to prominence in 20^{th} century
  - **Olga Taussky-Todd** (1906-1995)
    - Correcting Hilbert's math errors
    - Analyzing airplane vibrations using matrix theory
    - "How I became a torchbearer for matrix theory," *American Mathematical Monthly, 95,* 1988.

http://www-history.mcs.st-and.ac.uk/history/HistTopics/Matrices_and_determinants.html
http://www.sosmath.com/matrix/matrix.html

1 Basics

- What are matrices?
  - Calendars, tables, EXCEL files, data files

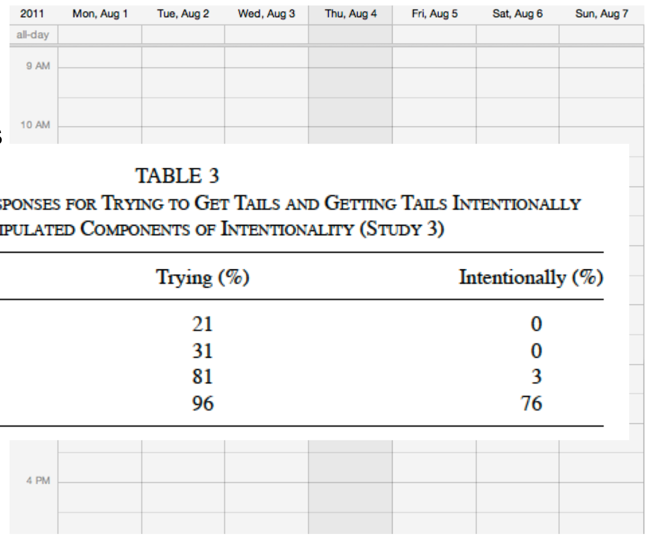
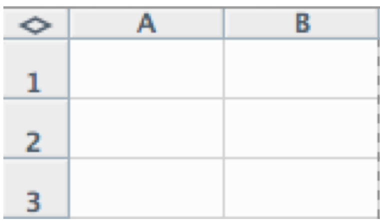


TABLE 3
PERCENTAGE OF "YES" RESPONSES FOR TRYING TO GET TAILS AND GETTING TAILS INTENTIONALLY WITH MANIPULATED COMPONENTS OF INTENTIONALITY (STUDY 3)

Components present	Trying (%)	Intentionally (%)
Desire	21	0
Belief	31	0
Desire + belief	81	3
Desire + belief + skill	96	76



1 Basics

- More formally, matrices are:
  - **Any array of numbers [...] in rows and columns**
- **Dimensionality** = numbers of rows and columns: r x c, n x p

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \boxed{5} & 6 \end{bmatrix}$$

- Elements of a matrix **A**: a_{ij} $(i = 1 \ldots r, j = 1 \ldots c)$
- **Vectors** are matrices with r = 1 or c = 1
  - Column vector, row vector
- **Scalar** = single number (1 x 1 matrix)

$$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \ldots \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}$$

2 Special Matrices

- **Square matrix**
  $r = c$

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

- **Diagonal matrix**
  off-diagonal elements are 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Main diagonal:
$a_{i=j}$

- **Identity matrix, I**

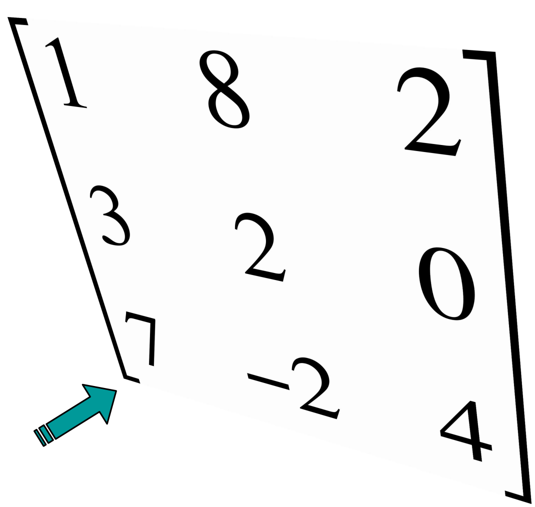
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

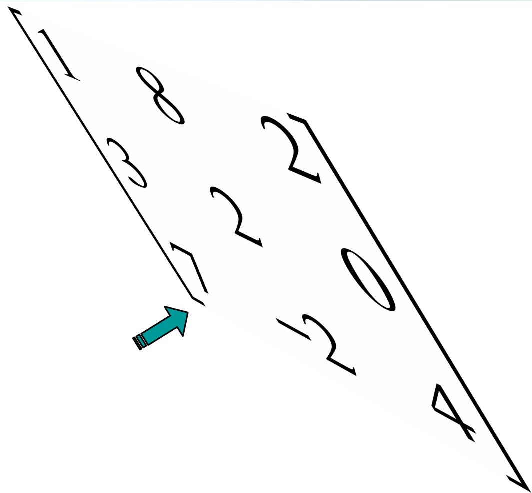
3 Transpose

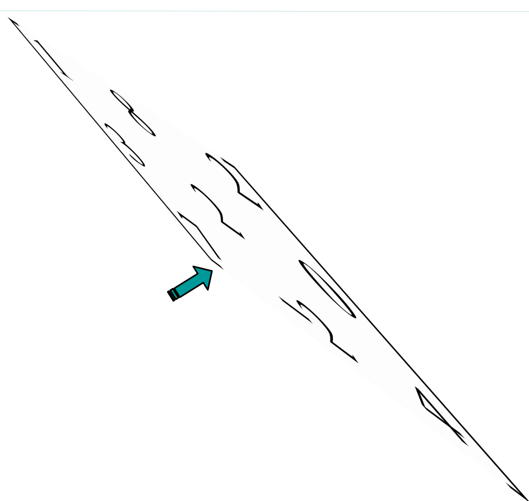
(= flip r and c)

Transposing, Animated

$$\begin{bmatrix} 1 & 8 & 2 \\ 3 & 2 & 0 \\ 7 & -2 & 4 \end{bmatrix}$$







$$\begin{bmatrix} 1 & 3 & 7 \\ 8 & 2 & -2 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 7 \\ 8 & 2 & -2 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 7 \\ 8 & 2 & -2 \\ 2 & 0 & 4 \end{bmatrix}$$

4 Matrix Operations: +, –

- **Addition, subtraction**
  - Requires matrices of equal dimensionality

A + **B** = **C**

$$\underset{3 \times 2}{\begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 6 & 3 \end{bmatrix}} + \underset{3 \times 2}{\begin{bmatrix} 7 & 5 \\ -1 & 2 \\ 4 & 3 \end{bmatrix}} = \underset{3 \times 2}{\begin{bmatrix} 9 & 10 \\ 0 & 6 \\ 10 & 6 \end{bmatrix}}$$

- Scalars
  - cannot be added or subtracted unless they are "expanded" to the appropriate dimensionality

$$\cancel{\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} + 1} \qquad \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$$

4 Matrix Operations: ×

- **Scalar multiplication**: 6 × 5 = 35
- **Scalar by matrix multiplication**: k × **A**

$$6 \times \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ 24 & 6 \end{bmatrix}$$

- **Matrix by matrix multiplication A × B**
  1. Match up $r_A \times \boxed{c_A \quad r_B} \times c_B$

     2 x 3 3 x 2
  2. Multiply rows of **A** with columns of **B** and add up

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \times \underset{3 \times 2}{\begin{bmatrix} 5 & 2 \\ -1 & 7 \\ 1 & 3 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 6 & 25 \\ 21 & 61 \end{bmatrix}}$$

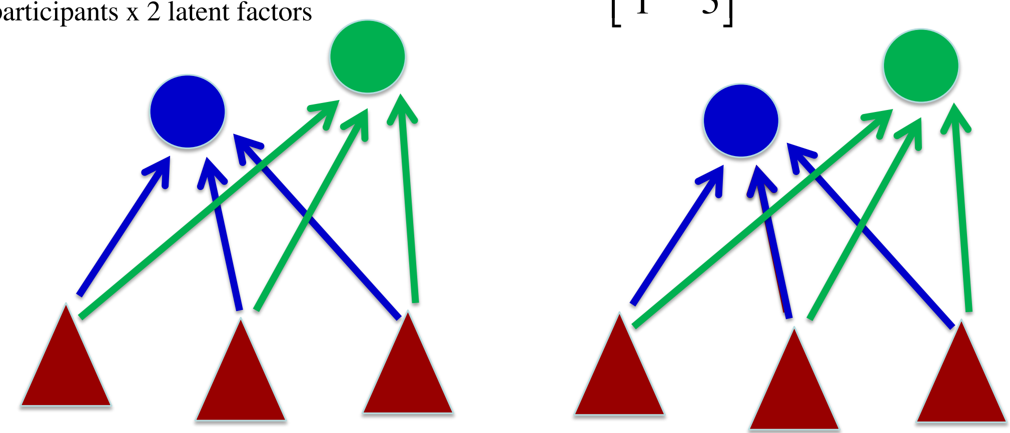
$1 \times 5 + 2 \times (-1) + 3 \times 1 = 6$

$4 \times 2 + 5 \times 7 + 6 \times 3 = 61$

Example

2 participants x 3 variables
×
3 variables x 2 latent factors
→
2 participants x 2 latent factors

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 2 \\ -1 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 25 \\ 21 & 61 \end{bmatrix}$$



4 Matrix Operations: ×

- The dimensions of the new matrix are a strict function of the dimensions of the multiplied matrices:

$r_A \times \quad \times c_B \Rightarrow r_A \times c_B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 2 \\ -1 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 25 \\ 21 & 61 \end{bmatrix}$$

The inside dimensions are "swallowed up"

- Because of the $r \times c$ matching, **A** x **B** ≠ **B** x **A** if r ≠ c
- But (**A** x **B**) x **C** = **A** x (**B** x **C**)
- Vectors can be dimensionally reduced or expanded:

$$\underset{1\text{x}3}{\boldsymbol{v}} \times \underset{3\text{x}1}{\boldsymbol{w}} = \underset{1\text{x}1}{\mathbf{k}} \qquad \underset{3\text{x}1}{\boldsymbol{w}} \times \underset{1\text{x}3}{\boldsymbol{v}} = \underset{3\text{x}3}{\mathbf{Z}}$$

Practice

- **A**(1x4) × **B**(4x2) ✔
- **C**(3x2) × **A** (1x4) ☒
- **B**(4x2) × **C'**, **B** × **B'**, or **B'** × **A'**

AA' and **A' A** always possible

$$\begin{bmatrix} 1 & 3 & -2 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ -1 & 5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -9 & 21 \\ 13 & -8 \end{bmatrix}$$

Summing matrix columns/rows

Why? *Means, variance…*

- How?
- *We know:* Matrix multiplication = row x column cross-multiplication and adding up.

$$\underset{(1\times3)}{\underset{\mathbf{1}'}{\begin{bmatrix}1 & 1 & 1\end{bmatrix}}} \times \underset{(3\times2)}{\underset{\mathbf{A}}{\begin{bmatrix}5 & 2\\ -1 & 7\\ 1 & 3\end{bmatrix}}} = \underset{(1\times2)}{\underset{\mathbf{1}'\mathbf{A}}{\begin{bmatrix}5 & 12\end{bmatrix}}}$$

$$\underset{(3\times2)}{\underset{\mathbf{A}}{\begin{bmatrix}5 & 2\\ -1 & 7\\ 1 & 3\end{bmatrix}}} \text{ x } \underset{(2\times1)}{\underset{\mathbf{1}}{\begin{bmatrix}1\\ 1\end{bmatrix}}} = \underset{(3\times1)}{\underset{\mathbf{A}\,\mathbf{1}}{\begin{bmatrix}7\\ 6\\ 4\end{bmatrix}}}$$

- Summing **R**ows = p**R**e-multiplying with $\mathbf{1}'$
- Summing c**O**lumns = p**O**st-multiplying with $\mathbf{1}$

Weighting matrix columns/rows

- If a multiplying vector has elements other than 1, the target elements are differentially ***weighted*** before being added up.

$$\underset{(1\times3)}{\underset{w'}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}} \times \underset{(3\times2)}{\underset{\mathbf{A}}{\begin{bmatrix}5 & 2\\ -1 & 7\\ 1 & 3\end{bmatrix}}} = \underset{(1\times2)}{\underset{w'\mathbf{A}}{\begin{bmatrix}6 & 25\end{bmatrix}}}$$

$$\underset{(3\times2)}{\underset{\mathbf{A}}{\begin{bmatrix}5 & 2\\ -1 & 7\\ 1 & 3\end{bmatrix}}} \text{ x } \underset{(2\times1)}{\underset{w}{\begin{bmatrix}1\\ 2\end{bmatrix}}} = \underset{(3\times1)}{\underset{\mathbf{A}w}{\begin{bmatrix}9\\ 13\\ 7\end{bmatrix}}}$$

- Apply weights to **R**ows = p**R**e-multiply with w'
- Apply weights to c**O**lumns = p**O**st-multiply with w

Rescaling

- How can we ***rescale*** a matrix's column (or row) elements while maintaining the matrix's overall dimensionality?
- Goal: Multiply elements by weight but *don't* sum.

$$\underset{(3x2)}{\underset{\mathbf{A}}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}}} \times \underset{(2x2)}{\underset{\mathbf{D}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}} = \underset{(3x2)}{\underset{\mathbf{AD}}{\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}}}$$

Diagonal Matrix **D**

- (Differentially) rescale c**O**lumns = p**O**st-multiply w/ **D**
- (Differentially) rescale **R**ows = p**R**e-multiply with **D**