Harmony, Stability, and Identity

On the possibility of an intensional account in proof-theoretic semantics

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Table of contents

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- In truth-theoretic semantics, truth is considered as a primitive and non-analyzed notion, and meaning is then explained in terms of it.

- In proof-theoretic semantics, meaning is explained in terms of (our) inferential abilities, and truth is then explained in terms of proofs (where a proof is a concatenation of inferences).

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- In truth-theoretic semantics the interpretation of a sentence A corresponds to a truth-value: either the true or the false.

$$[\![A]\!] = \mathcal{B}, \text{ where } \mathcal{B} \in \{\mathtt{T}, \mathtt{F}\}$$

- In proof-theoretic semantics the interpretation of a sentence A corresponds to a set of proofs sharing a bunch of properties $\mathcal{P}$ saying that these proofs possess some *common inferential structure* (e.g. canonicity or canonisability).

$$[\![A]\!] = \{\pi \,|\, \mathcal{P}(\pi)\}$$

- But what happens if the principles used for obtaining these properties $\mathcal{P}$ are somehow mutually incompatible and lead to the identification all proofs of A?

$$[\![A]\!] = \mathcal{B}, \text{ where } \mathcal{B} \in \{\mathfrak{P}_A, \emptyset\} \text{ and } \mathfrak{P}_A \text{ is the set of all proofs of } A$$

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Proof-theoretic semantics *vs* provability

- In this case, proof-theoretic semantics does not necessarily boil down to truth-theoretic semantics.
- However, the meaning of a certain sentence A would be explained without any reference to the inferential structure of its proofs: it is sufficient that *there exists* a proof of A, but it is not necessary to know how this proof is built.
- In other words, what counts for the meaning of a sentence A is not the way in which it is proved, but just the fact it is *provable*.
- Provability is not enough for having a genuine proof-theoretic semantics:
  - a theory based on provability is *extensional*: what counts is just the *relation* between premisses and conclusions;
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Two principles under analysis: harmony and stability

- The proof-theoretic debate is centered around the study of the principles used for obtaining the properties $\mathcal{P}$ (harmony, conservativeness, uniqueness, stability, deducibility of identicals, etc.).
- However, little is said about whether these principles entail the collapse of certain sets of proofs or not.
- Our aim is to focus the attention on two of these principles – *harmony* and *stability* – and to show that even if they are very natural to demand, they could lead to the collapse of the set of proofs of certain specific sentences.
- We will not reduce our analysis to sentences constructed with propositional operators; we will also look at first-order predicates, like the identity predicate.

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Harmony and the inversion principle

- Harmony is usually understood as the idea that there should be a balance between the conditions for the (correct) assertion of a certain (complex) sentence and what can be drawn from this assertion.

- Usually, this is understood in terms of Prawitz's *inversion principle*, according to which,

  > *[...] by an application of an elimination rule one essentially only restores what had been established if the major premiss of the application was inferred by an application of an introduction rule.* (Prawitz 1965: 33)

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Harmonious rules

- The inference rules of standard *intuitionistic* connectives are harmonious.
- Consider, for example, the rules of conjunction:

$$\frac{A_1 \qquad A_2}{A_1 \wedge A_2}\wedge_I \qquad \frac{A_1 \wedge A_2}{A_i}\wedge_{E_i}\ (i \in \{1,2\})$$

- These rules satisfy Prawitz's inversion principle, since a *hillock*, or (local) *complexity peak*, of the following kind

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Not harmonious rules

- A typical example of inference rules which fail to satisfy harmony are those of `tonk` (Prior 1960):

$$\dfrac{A_1}{A_1 \,\texttt{tonk}\, A_2}\,\texttt{tonk}_I \qquad\qquad \dfrac{A_1 \,\texttt{tonk}\, A_2}{A_2}\,\texttt{tonk}_E$$

- These rules do not satisfy Prawitz's inversion principle since they do not allow one to reduce the following hillock, or (local) complexity peak:

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- Harmony guarantees that what can be drawn from a complex sentence A by means of the elimination rules for its main connective † is *no more* than what can be drawn from the premisses of the introduction rules for † (used in order to obtain A).

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- In order to fully capture the idea that the rules of inference *define* the meaning of linguistic expressions, it is natural to ask also to fulfill the *no less* condition.

- A principle that has been proposed in order to capture this condition is the principle of *deducibility of identicals* (see Hacking 1979, Pfenning & Davies 2001, Francez & Dyckhoff 2012, Naibo & Petrolo 2015).

Deducibility of identicals: An example

- Deducibility of identical says that given a complex sentence A, it is possible to derive A from itself using only the inference rules of the principal connective of A.

- The idea is that what can be drawn from the elimination rules of a certain connective † is *enough* in order to re-establish the conclusion of the introduction rules of †.

- The inference rules of standard intuitionistic connectives satisfy the deducibility of identicals. Consider, for example, the rules of conjunction. We have that:

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$$\begin{array}{c}\mathcal{D}\\ A_1 \wedge A_2\end{array} \quad \text{expands to} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A_1 \wedge A_2\end{array}}{A_1}\,\wedge_{E_1} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A_1 \wedge A_2\end{array}}{A_2}\,\wedge_{E_2}}{A_1 \wedge A_2}\,\wedge_I$$

generating a *valley*, or (local) *complexity riff*.

Deducibility of identicals: An example

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Deducibility of identicals: A counter-example

- Consider a binary connective $\star$ having the following inference rules:

$$\frac{A \qquad B}{A \star B}\,\star_I \qquad\qquad \frac{A \star B \qquad A}{B}\,\star_E$$

- These rules satisfy harmony, since the hillock, or (local) complexity peak,

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \star B}\,\star_I \qquad \begin{matrix}\mathcal{D}_3 \\ A\end{matrix}}{B}\,\star_E \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}$$

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But an extra-assumption A has been used.

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This means that what can be drawn from the elimination rule of $\star$ is less than what has to be established in order to infer $A \star B$ using the introduction rules of $\star$.

Univocity and stability

- However, deducibility of identicals is not yet enough in order to *univocally* determine the set of elimination rules associated to the introduction rules of a certain given connective.

- In order to solve this problem, Dummett asks for an additional principle, that of *stability*.

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- Consider the rules for standard disjunction:

$$\frac{A_i}{A_1 \vee A_2}\vee_{I_i}\ (i \in \{1,2\}) \qquad \frac{A_1 \vee A_2 \quad \begin{array}{c}[A_1]\\ C\end{array} \quad \begin{array}{c}[A_2]\\ C\end{array}}{C}\vee_E$$

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- Consider then the rules for *quantum disjunction*:

$$\frac{A_i}{A_1 \bar{\triangledown} A_2}\ \bar{\triangledown}_{I_i}\ (i \in \{1,2\}) \qquad \frac{A_1 \bar{\triangledown} A_2 \quad \begin{matrix}[A_1]\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ C\end{matrix}}{C}\ \bar{\triangledown}_E$$

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Differently from the standard disjunction, the elimination rule of the quantum disjunction can be applied only in certain conditions, i.e. when there is *no side assumptions* in the sub-derivations of the two minor premisses.

Standard disjunction and quantum disjunction: Analogies

- Standard disjunction satisfies both harmony and deducibility of identicals:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A_i\end{matrix}}{A_1 \vee A_2}\,\vee_{I_i} \qquad \begin{matrix}\overset{m}{[A_1]}\\ \mathcal{D}_1\\ C\end{matrix} \qquad \begin{matrix}\overset{n}{[A_2]}\\ \mathcal{D}_2\\ C\end{matrix}}{C}\,\vee_E(m,n) \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}\\ [A_i]\\ \mathcal{D}_i\\ C\end{matrix}$$

and

$$\begin{matrix}\mathcal{D}\\ A_1 \vee A_2\end{matrix} \qquad \text{expands to} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A_1 \vee A_2\end{matrix} \qquad \dfrac{\overset{m}{A_1}}{A_1 \vee A_2}\,\vee_{I_1} \qquad \dfrac{\overset{n}{A_2}}{A_1 \vee A_2}\,\vee_{I_2}}{A_1 \vee A_2}\,\vee_E(m,n)$$

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- Since standard disjunction and quantum disjunction have the same set of introduction rules, what we showed is that this set is compatible with two different elimination rules.

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$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A_i\end{matrix}}{A_1 \bar{\vee} A_2}\,\bar{\vee}_{I_i} \qquad \begin{matrix}\overset{m}{[A_1]}\\ \mathcal{D}_1\\ C\end{matrix} \qquad \begin{matrix}\overset{n}{[A_2]}\\ \mathcal{D}_2\\ C\end{matrix}}{C}\,\bar{\vee}_E(m,n) \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}\\ [A_i]\\ \mathcal{D}_i\\ C\end{matrix}$$

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- Since standard disjunction and quantum disjunction have the same set of introduction rules, what we showed is that this set is compatible with two different elimination rules. This could give rise to ambiguities concerning the use of these rules.

Generalized expansion

- However, consider now the case in which the expansion of a disjunctive sentence is operated in the middle of a derivation, and not just at the conclusion of it:

$$
\begin{array}{c}\mathcal{D}\\ [A_1 \vee A_2]\\ \mathcal{D}'\\ C\end{array}
\quad \text{expands to} \quad
\begin{array}{c}
\dfrac{\begin{array}{c}\mathcal{D}\\ A_1 \vee A_2\end{array} \qquad \dfrac{\overset{m}{A_1}}{A_1 \vee A_2}\,\vee_{I_1} \qquad \dfrac{\overset{n}{A_2}}{A_1 \vee A_2}\,\vee_{I_2}}{[A_1 \vee A_2]}\,\vee_E(m,n)\\
\mathcal{D}'\\
C
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\qquad \text{expands to} \qquad
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\color{red}{\mathcal{D}'} \\
\color{red}{C}
\end{array}
$$

the red highlighted part can be *permuted* upward in the sub-derivations of the minor premisses.

Generalized expansion

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$$
\begin{array}{c}
\mathcal{D} \\
[A_1 \vee A_2] \\
\mathcal{D}' \\
C
\end{array}
\quad \text{expands to} \quad
\dfrac{\begin{array}{c} \mathcal{D} \\ A_1 \vee A_2 \end{array} \qquad \begin{array}{c} \dfrac{\overset{m}{A_1}}{\color{red}{[A_1 \vee A_2]}}\,\vee_{I_1} \\ \color{red}{\mathcal{D}'} \\ \color{red}{C} \end{array} \qquad \begin{array}{c} \dfrac{\overset{n}{A_2}}{\color{red}{[A_1 \vee A_2]}}\,\vee_{I_2} \\ \color{red}{\mathcal{D}'} \\ \color{red}{C} \end{array}}{C}\,\vee_E(m, n)
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\end{array}
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$$

This can be considered as a form of *generalized expansion*.

A failure of generalized expansion

- This generalized expansion fails in the case of quantum disjunction:

$$
\begin{array}{c} \mathcal{D} \\ [A_1 \bar{\triangledown} A_2] \\ \mathcal{D}' \\ C \end{array}
\quad \text{does not expands to} \quad
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$$

since the sub-derivation $\mathcal{D}'$ could contain side assumptions, and thus the application of the rule $\bar{\triangledown}_E$ would not be correct.

- The generalized expansion can be seen as a way of capturing Dummett's stability (Jacinto & Read 2016, Tranchini 2016).

- Moreover, the no less condition can be generalized: what can be drawn from a complex sentence A by means of the elimination rules for its main connective † is no less than what can be drawn from the premisses of the introduction rules for † (used in order to obtain A).

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Generalized expansion and general elimination rules

- The generalized expansion works only in the case of general elimination rules, but not when they are *flattened* elimination rules *à la* Negri & von Plato (2001).
- Consider the flattened rules for implication:

$$\dfrac{\begin{array}{c}[A]\\ B\end{array}}{A \to B}\to_I \qquad\qquad \dfrac{A \to B \qquad A \qquad \begin{array}{c}[B]\\ C\end{array}}{C}\to_E$$

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- If we try to expand

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But this is not a generalized expansion, since we added an extra-hypothesis A.

Generalized expansion and general elimination rules

- On the contrary, general expansions can be obtained by working with general elimination rules having the form of *higher level* rules *à la* Schroeder-Heister (1984).
- Consider the higher level rules for implication:

$$\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \to B} \to_I \qquad\qquad \dfrac{A \to B \quad \begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{C} \to_E$$

- With these rules we have that:

$$\begin{matrix}\mathcal{D}\\ [A \to B]\\ \mathcal{D}'\\ C\end{matrix} \quad \text{expands to} \quad \dfrac{\begin{matrix}\mathcal{D}\\ A \to B\end{matrix} \quad \begin{matrix}\dfrac{\dfrac{\overset{m}{A}}{B}\, \overset{n}{A \Rightarrow B}}{[A \to B]} \to_I (m)\\ \mathcal{D}'\\ C\end{matrix}}{C} \to_E (n)$$

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- With these rules we have that:

$$\begin{matrix}\mathcal{D}\\ [A \to B]\\ \mathcal{D}'\\ C\end{matrix} \quad \text{expands to} \quad \dfrac{\begin{matrix}\mathcal{D}\\ A \to B\end{matrix} \qquad \begin{matrix}\dfrac{\dfrac{\overset{m}{A}}{B}\, \overset{n}{A \Rightarrow B}}{[A \to B]}\to_I (m)\\ \mathcal{D}'\\ C\end{matrix}}{C}\to_E (n)$$

Generalized expansion and general elimination rules

- On the contrary, general expansions can be obtained by working with general elimination rules having the form of *higher level* rules *à la* Schroeder-Heister (1984).
- Consider the higher level rules for implication:

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A limit case

- A limit case of the satisfaction of harmony and stability is represented by the 0-ary connective $\perp$:

no introduction rules $\qquad \dfrac{\perp}{C}\perp_E$

- The elimination rule allows one to obtain:

(i) no more than what has to be established in order to infer $\perp$ from its introduction rules, since nothing is enough to infer $\perp$ by an introduction rule.

(ii) no less than what has to be established in order to infer any C obtainable from $\perp$, since

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This is a *sui generis* expansion, since the derivation on the right corresponds to a sort of simplified version of the derivation on the left.

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This is due to the fact that $\bot_E$ has no minor premisses where to permute upward $\begin{array}{c} \mathcal{D}' \\ C \end{array}$.

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$$\begin{array}{c}\mathcal{D}\\ \bot\end{array} \qquad \text{expands to} \qquad \begin{array}{c}\mathcal{D}\\ \dfrac{\bot}{\bot}\,\bot_E\end{array}$$

The sense of the word 'expansion' becomes clear when we consider the simple expansion (i.e. deducibility of identicals).

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In fact, the same expanded proof is obtained when C is $\bot$.

Inductive definitions: natural numbers

- Harmony and stability are satisfied by those expressions defined by (generalized) inductive definitions (see Martin-Löf 1971).

- Consider the rules for the predicate N for natural numbers (Martin-Löf 1984: 71):

$$\frac{}{N0}\,N_{I_1} \qquad \frac{Nt}{Nt'}\,N_{I_2} \qquad \frac{Nt \quad C(0/y) \quad \begin{array}{c}[Nx/y],[C(x/y)]\\ C(x'/y)\end{array}}{C(t/y)}\,N_E$$

where x is an eigenvariable.

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- These rules satisfy harmony:

$$\frac{\dfrac{}{N0}\,N_{I_1} \quad \begin{array}{c}\mathcal{D}_1(0/y)\\ C(0/y)\end{array} \quad \begin{array}{c}[N\overset{m}{x}/y],[C(\overset{n}{x}/y)]\\ \mathcal{D}_2(x/y)\\ C(x'/y)\end{array}}{C(0/y)}\,N_E(m,n) \quad \text{reduces to} \quad \begin{array}{c}\mathcal{D}_1(0/y)\\ C(0/y)\end{array}$$

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Here, it is crucial the fact that we presented a N_E as corresponding to the operation of *recursion*.

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  Take C as N and $\mathcal{D}'$ as a rule of N introduction.

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Take C as N and $\mathcal{D}'$ as a rule of N introduction.
For satisfying only deducibility of identical it is sufficient to consider a N_E which corresponds to the operation of *iteration*.

Inductive definitions: identity

- Consider the following rules of identity (Martin-Löf 1971):

$$\frac{}{t = t} =_I \qquad\qquad \frac{t = s \qquad C(x/y, x/z)}{C(t/y, s/z)} =_E$$

- The usual rule of indiscernibility of identicals

$$\frac{t = s \qquad A(t/z)}{A(s/z)}$$

can be obtained by taking $C(x/y, x/z)$ as $A(x/y) \to A(x/z)$ and considering then the following derivation:

$$\dfrac{\dfrac{t = s \qquad \dfrac{\overset{1}{A(x/z)}}{A(x/y) \to A(x/z)} \to_I (1)}{A(t/y) \to A(s/z)} =_E \qquad A(t/z)}{A(s/z)} \to_E$$

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Inductive definitions: identity

- The rules of identity satisfy harmony:

$$\dfrac{\dfrac{}{t=t}{=_I} \qquad \begin{array}{c}\mathcal{D}(x/y,x/z)\\ C(x/y,x/z)\end{array}}{C(t/y,t/z)}{=_E} \qquad \text{reduces to} \qquad \begin{array}{c}\mathcal{D}(t/y,t/z)\\ C(t/y,t/z)\end{array}$$

- And they also satisfy stability:

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- And they also satisfy stability:

$$\begin{array}{c}\mathcal{D}\\ [t = s]\\ \mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array} \qquad \text{expands to} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \begin{array}{c}\dfrac{}{[x = x]}{=_I}\\ \mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}}{C(t/y, s/z)}{=_E}$$

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$$\dfrac{\dfrac{}{t = t}=_I \qquad \begin{matrix}\mathcal{D}(x/y, x/z) \\ C(x/y, x/z)\end{matrix}}{C(t/y, t/z)} =_E \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}(t/y, t/z) \\ C(t/y, t/z)\end{matrix}$$

- And they also satisfy stability:

$$\begin{matrix}\mathcal{D} \\ t = s\end{matrix} \qquad \text{expands to} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ t = s\end{matrix} \qquad \dfrac{}{x = x}=_I}{t = s} =_E$$

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Where

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{ is } t/y=s/z \text{ and } \left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } x/y=x/z$$

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Hence, deducibility of identicals becomes a particular case of generalized expansion.

Table of contents

Equivalence relations and the identification of proofs

- The reflexive, symmetric, and transitive closure of
  - the *reduction* relation corresponding to harmony,
  - the *standard expansion* relation corresponding to deducibility of identicals

  yields two equivalence relations on derivations: $\equiv_\beta$ and $\equiv_\eta$.

- Now, if we want to take proofs as the fundamental entities on which founding a semantics (as it is the idea of proof-theoretic semantics), then we have to say what proofs are, that is, we have to assign them a genuine ontological status.

- A first crucial step in this direction is to possess a criterion of identification for proofs, in agreement with the Quinean epistemological principle that there is no entity without identity.

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Equivalence relations and the identification of proofs

- The reflexive, symmetric, and transitive closure of
  - the *reduction* relation corresponding to harmony,
  - the *standard expansion* relation corresponding to deducibility of identicals

  yields two equivalence relations on derivations: $\equiv_\beta$ and $\equiv_\eta$.

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- In particular, this extension does not yield the identification of all the derivations of a certain sentence. For example, the two following derivations cannot be identified:

$$\dfrac{\dfrac{\overset{1}{A \to A}}{A \wedge B \to (A \to A)}\ \to_I}{(A \to A) \to (A \wedge B \to (A \to A))}\ \to_I (1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{A \wedge B}}{A}\ \wedge_{E_1}}{A \to A}\ \to_I}{A \wedge B \to (A \to A)}\ \to_I (1)}{(A \to A) \to (A \wedge B \to (A \to A))}\ \to_I$$

- However, we have seen that stability seems to be preferable to simple deducibility of identicals, within a proof-theoretic semantics setting. What happens if we consider a generalized form of η, "induced" by stability? And what happens if we go outside the fragment $\{\to, \wedge\}$?

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Collapse on negative sentences*

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$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix}\quad\dfrac{\overset{2}{A}}{\bot}\; {\scriptstyle A\overset{1}{\Rightarrow}\bot}}{\bot}\;\to_E (1)}{\neg A}\;\to_I (2) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix}\quad\dfrac{\overset{2}{A}}{\bot}\; {\scriptstyle A\overset{1}{\Rightarrow}\bot}}{\bot}\;\to_E (1)}{\neg A}\;\to_I (2)$$

$$\underset{(\to)_{Sym}}{\rightsquigarrow_\eta}$$

$$\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix}$$

*Thanks to a hint of Luiz Carlos Pereira.

Collapse on negative sentences*

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \quad \dfrac{\overset{4}{A}}{\bot}\;A \overset{1}{\Rightarrow} \bot}{\bot}\;\to_E\ (1)}{\neg B}\;\to_I \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \quad \dfrac{\overset{4}{A}}{\bot}\;A \overset{2}{\Rightarrow} \bot}{\bot}\;\to_E\ (2)}{B}\;\bot_E}{\bot}\;B \overset{3}{\Rightarrow} \bot}{\bot}\;\to_E\ (3)}{\neg A}\;\to_I\ (4) \qquad \underset{(\bot)}{\rightsquigarrow_\eta} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \quad \dfrac{\overset{2}{A}}{\bot}\;A \overset{1}{\Rightarrow} \bot}{\bot}\;\to_E\ (1)}{\bot}\;\bot_E}{\neg A}\;\to_I\ (2)$$

$$\underset{(\to)}{\rightsquigarrow_\beta} \qquad\qquad\qquad \underset{(\bot)_{Sym}}{\rightsquigarrow_\eta}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \quad \dfrac{\overset{2}{A}}{\bot}\;A \overset{1}{\Rightarrow} \bot}{\bot}\;\to_E\ (1)}{\neg A}\;\to_I\ (2) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \quad \dfrac{\overset{2}{A}}{\bot}\;A \overset{1}{\Rightarrow} \bot}{\bot}\;\to_E\ (1)}{\neg A}\;\to_I\ (2)$$

$$\underset{(\to)_{Sym}}{\rightsquigarrow_\eta} \qquad\qquad\qquad \underset{(\to)_{Sym}}{\rightsquigarrow_\eta}$$

$$\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \qquad\qquad\qquad \begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix}$$

*Thanks to a hint of Luiz Carlos Pereira.

Collapse on negative sentences*

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{4}{A}$}
\RightLabel{$A \overset{1}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}
\BinaryInfC{$\bot$}
\RightLabel{$\to_I$}
\UnaryInfC{$\neg B$}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{4}{A}$}
\RightLabel{$A \overset{2}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (2)}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{B}
\RightLabel{$B \overset{3}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (3)}
\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (4)}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\eta$$

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}
\RightLabel{$A \overset{1}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\beta$$

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}
\RightLabel{$A \overset{1}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}
\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\eta$$

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}
\RightLabel{$A \overset{1}{\Rightarrow} \bot$}
\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}
\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}_1 \\ \neg A\end{array}$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}_2 \\ \neg A\end{array}$$

*Thanks to a hint of Luiz Carlos Pereira.

Collapse on negative sentences*

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{4}{A}$}\RightLabel{$A \overset{1}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}\BinaryInfC{$\bot$}
\RightLabel{$\to_I$}\UnaryInfC{$\neg B$}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{4}{A}$}\RightLabel{$A \overset{2}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (2)}\BinaryInfC{$\bot$}
\RightLabel{$\bot_E$}\UnaryInfC{B}
\RightLabel{$B \overset{3}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (3)}\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (4)}\UnaryInfC{$\neg A$}
\end{prooftree}
\quad \equiv_\eta \quad
\begin{prooftree}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}\RightLabel{$A \overset{1}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}\BinaryInfC{$\bot$}
\RightLabel{$\bot_E$}\UnaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\beta \qquad\qquad \equiv_\eta$$

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}\RightLabel{$A \overset{1}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}\UnaryInfC{$\neg A$}
\end{prooftree}
\qquad\qquad
\begin{prooftree}
\AxiomC{$\mathcal{D}_2$}\noLine\UnaryInfC{$\neg A$}
\AxiomC{$\overset{2}{A}$}\RightLabel{$A \overset{1}{\Rightarrow} \bot$}\UnaryInfC{$\bot$}
\RightLabel{$\to_E$ (1)}\BinaryInfC{$\bot$}
\RightLabel{$\to_I$ (2)}\UnaryInfC{$\neg A$}
\end{prooftree}
$$

$$\equiv_\eta \qquad\qquad \equiv_\eta$$

$$
\begin{array}{c}\mathcal{D}_1\\ \neg A\end{array}
\quad \equiv_{\beta\eta} \quad
\begin{array}{c}\mathcal{D}_2\\ \neg A\end{array}
$$

*Thanks to a hint of Luiz Carlos Pereira.

Collapse on negative sentences*

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \qquad \dfrac{\overset{4}{A}}{\bot}}{\bot}\;\begin{matrix}A \overset{1}{\Rightarrow} \bot\\ \to_E\ (1)\end{matrix}}{\neg B}\;\to_I \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \qquad \dfrac{\overset{4}{A}}{\bot}}{\bot}\;\begin{matrix}A \overset{2}{\Rightarrow} \bot\\ \to_E\ (2)\end{matrix}}{B}\;\bot_E}{\bot}\;\begin{matrix}B \overset{3}{\Rightarrow} \bot\\ \to_E\ (3)\end{matrix}}{\bot}}{\neg A}\;\to_I\ (4) \qquad \equiv_\eta \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \qquad \dfrac{\overset{2}{A}}{\bot}}{\bot}\;\begin{matrix}A \overset{1}{\Rightarrow} \bot\\ \to_E\ (1)\end{matrix}}{\bot}\;\bot_E}{\neg A}\;\to_I\ (2)$$

$$\equiv_\beta \qquad\qquad\qquad\qquad \equiv_\eta$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \qquad \dfrac{\overset{2}{A}}{\bot}}{\bot}\;\begin{matrix}A \overset{1}{\Rightarrow} \bot\\ \to_E\ (1)\end{matrix}}{\neg A}\;\to_I\ (2) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix} \qquad \dfrac{\overset{2}{A}}{\bot}}{\bot}\;\begin{matrix}A \overset{1}{\Rightarrow} \bot\\ \to_E\ (1)\end{matrix}}{\neg A}\;\to_I\ (2)$$

$$\equiv_\eta \qquad\qquad\qquad\qquad \equiv_\eta$$

$$\begin{matrix}\mathcal{D}_1\\ \neg A\end{matrix} \qquad \equiv_{\beta\eta} \qquad \begin{matrix}\mathcal{D}_2\\ \neg A\end{matrix}$$

Any two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\neg A$ are $\beta\eta$-equivalent, and can thus be identified.

*Thanks to a hint of Luiz Carlos Pereira.

Collapse on identity sentences (1)

$$\dfrac{\begin{matrix}\mathcal{D} \\ t = t\end{matrix} \qquad \dfrac{}{x/w = x/w}\,{=_I}}{t/w = t/w}\,{=_E}$$

Collapse on identity sentences (1)

$$
\dfrac{\begin{array}{c}\mathcal{D}\\ t = t\end{array} \qquad \dfrac{}{x/w = x/w}\,{=_I}}{t/w = t/w}\,{=_E}
$$

$$\rightsquigarrow_{\eta}$$

$$(=)_{Sym}$$

$$
\begin{array}{c}\mathcal{D}\\ t = t\end{array}
$$

Collapse on identity sentences (1)

$$\dfrac{\begin{matrix}\mathcal{D} \\ t = t\end{matrix} \qquad \dfrac{}{x/w = x/w}=_I}{t/w = t/w}=_E$$

$$\rightsquigarrow_\eta$$
$$(=)_{Sym}$$

$$\begin{matrix}\mathcal{D} \\ t = t\end{matrix}$$

$$\left.\begin{matrix}\mathcal{D}'(t/w, t/w) \\ C(t/w, t/w)\end{matrix}\right\} \text{ is } t/w = t/w$$

$$\left.\begin{matrix}\mathcal{D}'(x/w, x/w) \\ C(x/w, x/w)\end{matrix}\right\} \text{ is } x/w = x/w$$

Collapse on identity sentences (1)

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = t\end{array} \qquad \dfrac{}{x/w = x/w}=_I}{t/w = t/w}=_E \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/w = t/w}=_I$$

$$\underset{(=)_{Sym}}{\rightsquigarrow_\eta}$$

$$\begin{array}{c}\mathcal{D}\\ t = t\end{array}$$

$$\left.\begin{array}{r}\mathcal{D}'(t/w, t/w)\\ C(t/w, t/w)\end{array}\right\} \text{ is } t/w = t/w$$

$$\left.\begin{array}{r}\mathcal{D}'(x/w, x/w)\\ C(x/w, x/w)\end{array}\right\} \text{ is } x/w = x/w$$

Collapse on identity sentences (1)

$$\dfrac{\begin{matrix}\mathcal{D}\\ t = t\end{matrix} \qquad \dfrac{}{x/w = x/w}{=_I}}{t/w = t/w}{=_E} \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/w = t/w}{=_I}$$

$$\underset{(=)_{Sym}}{\rightsquigarrow_\eta}$$

$$\begin{matrix}\mathcal{D}\\ t = t\end{matrix}$$

$$\left.\begin{matrix}\mathcal{D}'(t/w, t/w)\\ C(t/w, t/w)\end{matrix}\right\} \text{is } \dfrac{}{t/w = t/w}{=_I}$$

$$\left.\begin{matrix}\mathcal{D}'(x/w, x/w)\\ C(x/w, x/w)\end{matrix}\right\} \text{is } \dfrac{}{x/w = x/w}{=_I}$$

$$\left.\begin{matrix}\mathcal{D}'(t/w, t/w)\\ C(t/w, t/w)\end{matrix}\right\} \text{is } t/w = t/w$$

$$\left.\begin{matrix}\mathcal{D}'(x/w, x/w)\\ C(x/w, x/w)\end{matrix}\right\} \text{is } x/w = x/w$$

Collapse on identity sentences (1)

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = t\end{array} \qquad \dfrac{}{x/w = x/w}{=_I}}{t/w = t/w}{=_E} \quad \equiv_\eta \quad \dfrac{}{t/w = t/w}{=_I}$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}\\ t = t\end{array}$$

Collapse on identity sentences (1)

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = t\end{array} \qquad \dfrac{}{x/w = x/w}\,{=_I}}{t/w = t/w}\,{=_E} \qquad \equiv_\eta \qquad \dfrac{}{t/w = t/w}\,{=_I}$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}\\ t = t\end{array}$$

- This means that any derivation $\mathcal{D}$ of $t = t$ can be identified with the derivation obtained just by an application of $=_I$, i.e. by a reflexivity proof.

Collapse on identity sentences (1)

$$
\dfrac{\dfrac{\mathcal{D}}{t = t} \qquad \dfrac{}{x/w = x/w}\,{=_I}}{t/w = t/w}\,{=_E} \quad \equiv_\eta \quad \dfrac{}{t/w = t/w}\,{=_I}
$$

$$
\equiv_\eta
$$

$$
\begin{array}{c} \mathcal{D} \\ t = t \end{array}
$$

- This means that any derivation $\mathcal{D}$ of $t = t$ can be identified with the derivation obtained just by an application of $=_I$, i.e. by a reflexivity proof.

- At first glance, this could seem to be an odd situation.

Collapse on identity sentences (1)

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = t\end{array} \qquad \dfrac{}{x/w = x/w}\,{=_I}}{t/w = t/w}\,{=_E} \quad \equiv_\eta \quad \dfrac{}{t/w = t/w}\,{=_I}$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}\\ t = t\end{array}$$

- This means that any derivation $\mathcal{D}$ of $t = t$ can be identified with the derivation obtained just by an application of $=_I$, i.e. by a reflexivity proof.
- At first glance, this could seem to be an odd situation. However, it can become acceptable if we think of it in the following way:

Collapse on identity sentences (1)

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t=t\end{array} \qquad \dfrac{}{x/w=x/w}=_I}{t/w=t/w}=_E \quad \equiv_\eta \quad \dfrac{}{t/w=t/w}=_I$$

$$\equiv_\eta$$

$$\begin{array}{c}\mathcal{D}\\ t=t\end{array}$$

- This means that any derivation $\mathcal{D}$ of $t=t$ can be identified with the derivation obtained just by an application of $=_I$, i.e. by a reflexivity proof.
- At first glance, this could seem to be an odd situation. However, it can become acceptable if we think of it in the following way: there is only one canonical proof of $t=t$ – its reflexivity proof – and all the other proofs of $t=t$ correspond to this canonical proof.

Collapse on identity sentences (1)

$$\dfrac{\begin{matrix}\mathcal{D}\\ t = t\end{matrix} \qquad \dfrac{}{x/w = x/w}{=_I}}{t/w = t/w}{=_E} \quad \equiv_\eta \quad \dfrac{}{t/w = t/w}{=_I}$$

$$\equiv_\eta$$

$$\begin{matrix}\mathcal{D}\\ t = t\end{matrix}$$

- This means that any derivation $\mathcal{D}$ of $t = t$ can be identified with the derivation obtained just by an application of $=_I$, i.e. by a reflexivity proof.
- At first glance, this could seem to be an odd situation. However, it can become acceptable if we think of it in the following way: there is only one canonical proof of $t = t$ – its reflexivity proof – and all the other proofs of $t = t$ correspond to this canonical proof.
- This is an acceptable formulation since it says that all the proofs of a *trivial* identity sentence like $t = t$ are in fact themselves trivial since they are simple reflexivity proofs.

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\dfrac{\mathcal{D}}{t = s} \qquad \dfrac{}{x/y = x/z}{=_I}}{t/y = s/z}{=_E}$$

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}=_I}{t/y = s/z}=_E$$

$$\rightsquigarrow_\eta$$
$$(=)_{Sym}$$

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}=_I}{t/y = s/z}=_E$$

$$\rightsquigarrow_\eta$$

$$(=)_{Sym}$$

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\}\text{ is } t/y = s/z$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\}\text{ is } x/y = x/z$$

Collapse on identity sentences (2)

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$$
\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}=_I}{t/y = s/z}=_E \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/y = s/z}=_I
$$

$$
\underset{(=)_{Sym}}{\rightsquigarrow_\eta}
$$

$$
\begin{array}{c}\mathcal{D}\\ t = s\end{array}
$$

$$
\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{ is } t/y = s/z
$$

$$
\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } x/y = x/z
$$

Collapse on identity sentences (2)

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$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}=_I}{t/y = s/z}=_E \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/y = s/z}=_I$$

$$\underset{(=)_{Sym}}{\rightsquigarrow_\eta}$$

t rewrites on s or s rewrites on t

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{is } \dfrac{}{t/y = s/z}=_I$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{is } t/y = s/z$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{is } \dfrac{}{x/y = x/z}=_I$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{is } x/y = x/z$$

Collapse on identity sentences (2)

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$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}\,=_I}{t/y = s/z}\,=_E \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/y = t/z}\,=_I$$

$$\underset{(=)_{Sym}}{\rightsquigarrow_\eta}$$

t rewrites on s or s rewrites on t

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, t/z)\\ C(t/y, t/z)\end{array}\right\} \text{ is } \dfrac{}{t/y = t/z}\,=_I$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{ is } t/y = s/z$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } \dfrac{}{x/y = x/z}\,=_I$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } x/y = x/z$$

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}{=_I}}{t/y = s/z}{=_E} \quad \underset{(=)_{Sym}}{\rightsquigarrow_\eta} \quad \dfrac{}{t/y = t/z}{=_I}$$

$$\underset{(=)_{Sym}}{\rightsquigarrow_\eta}$$

t rewrites on s or s rewrites on t

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

$$\left.\begin{array}{c}\mathcal{D}'(s/y, s/z)\\ C(s/y, s/z)\end{array}\right\} \text{ is } \dfrac{}{s/y = s/z}{=_I}$$

$$\left.\begin{array}{c}\mathcal{D}'(t/y, s/z)\\ C(t/y, s/z)\end{array}\right\} \text{ is } t/y = s/z$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } \dfrac{}{x/y = x/z}{=_I}$$

$$\left.\begin{array}{c}\mathcal{D}'(x/y, x/z)\\ C(x/y, x/z)\end{array}\right\} \text{ is } x/y = x/z$$

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}{=_I}}{t/y = s/z}{=_E} \quad \equiv_\eta \quad \dfrac{}{t/y = s/z}{=_I}$$

$\equiv_\eta$ $\qquad$ t rewrites on s or s rewrites on t

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\genfrac{}{}{0pt}{}{\mathcal{D}}{t = s} \qquad \dfrac{}{x/y = x/z} =_I}{t/y = s/z} =_E \quad \equiv_\eta \quad \dfrac{}{t/y = s/z} =_I$$

$$\equiv_\eta \qquad\qquad t \text{ rewrites on } s \text{ or } s \text{ rewrites on } t$$

$$\genfrac{}{}{0pt}{}{\mathcal{D}}{t = s}$$

- This situation becomes particularly problematic if, for every t and s, and every derivation $\mathcal{D}$ of $t = s$, it is always possible to make that t rewrite on s, and *vice versa*.

Collapse on identity sentences (2)

- Suppose now to consider a richer syntactical setting allowing one to transform a term t intro another term s – and *vice versa* – by mean of some set of rewriting rules.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ t = s\end{array} \qquad \dfrac{}{x/y = x/z}{=_I}}{t/y = s/z}{=_E} \quad \equiv_\eta \quad \dfrac{}{t/y = s/z}{=_I}$$

$$\equiv_\eta \qquad\qquad\qquad t \text{ rewrites on } s \text{ or } s \text{ rewrites on } t$$

$$\begin{array}{c}\mathcal{D}\\ t = s\end{array}$$

- This situation becomes particularly problematic if, for every t and s, and every derivation $\mathcal{D}$ of $t = s$, it is always possible to make that t rewrite on s, and *vice versa*.

- In this way, every proof of a every identity sentence would be identified with a reflexivity proof.

Identity sentences in Martin-Löf's ITT

- This situation can be better explained by considering Martin-Löf's intuitionistic type theory (1984).
- In Martin-Löf's type theory proofs are denoted – or better, *coded* – by λ-terms, i.e. computable functions.
  In this way one can produces *judgements* like:
  - $t : A$, which says that t codes a proof of the proposition A, i.e. t is a proof-term of type A;
  - $t \equiv s : A$, which says that (the codes of A) t and s are equivalent with respect to certain transformations (viz. rewrite operations); since t and s are λ-terms, the most fundamental of these transformations is the β-reduction (t and s are thus computationally equivalent codes).
- The rules for identity are the following:*

*Note that in Martin-Löf-s type theory identity is not absolute, but relative to a type A.

Identity sentences in Martin-Löf's ITT

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- These rules allow one to pass from an equivalence by rewriting ($\equiv$) to a propositional identity ($=_A$), and *vice versa*.
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- This corresponds to an *extensional* version of Martin-Löf's intuitionistic type theory.

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- The (generalized) η-equivalence

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[t = s] \\
\mathcal{D}'(t/y, s/z) \\
C(t/y, s/z)
\end{array}
\quad \equiv_\eta \quad
\dfrac{\begin{array}{c} \mathcal{D} \\ t = s \end{array} \qquad \begin{array}{c} \dfrac{}{[x = x]}{=_I} \\ \mathcal{D}'(x/y, x/z) \\ C(x/y, x/z) \end{array}}{C(t/y, s/z)}{=_E}
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can be internalized at the level of proof-terms by the following η_{Conv} rule:

$$\dfrac{d : t =_A s \quad \begin{array}{c}[x : A], [y : A, z : A, u : y =_A z] \\ d'(x/y, x/z, \mathbf{r}(x)/u) : C(x/y, x/z)\end{array}}{J(t, s, d, \hat{x} : A.d'(x/y, x/z, \mathbf{r}(x)/u)) \equiv d'(t/y, s/z, d/u) : C(t/y, s/z)}\eta_{Conv}$$

where d corresponds to $\mathcal{D}$, and d' corresponds to $\mathcal{D}'$.

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\dfrac{\dfrac{\dfrac{d : t =_A s \qquad [x \overset{1}{:} A]}{J(t, s, d, \hat{x} : A.x) \equiv t : A}\,\eta_{Conv}\,(1)}{t \equiv J(t, s, d, \hat{x} : A.x) : A}\,Sym \qquad \dfrac{d : t =_A s \qquad [x \overset{2}{:} A]}{J(t, s, d, \hat{x} : A.x) \equiv s : A}\,\eta_{Conv}\,(2)}{t \equiv s : A}\,Trans
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- As shown by Streicher (1993) these two rules are in fact inter-derivable.

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$$\dfrac{\dfrac{\dfrac{d : t =_A s \qquad [x \overset{1}{:} A]}{J(t, s, d, \hat{x} : A.x) \equiv t : A}\ \eta_{Conv}\ (1)}{t \equiv J(t, s, d, \hat{x} : A.x) : A}\ Sym \qquad \dfrac{d : t =_A s \qquad [x \overset{2}{:} A]}{J(t, s, d, \hat{x} : A.x) \equiv s : A}\ \eta_{Conv}\ (2)}{t \equiv s : A}\ Trans$$

- As shown by Streicher (1993) these two rules are in fact inter-derivable.

- The generalized expansion seems then to be the main responsable of the collapsing phenomena.

Possible solutions (1)

- Which solution should we adopt in order to avoid collapses?

- Drop stability, and work with harmony and deducibility of identical, i.e. work modulo $\equiv_{\beta\eta}$, where $\equiv_{\eta}$ is induced by simple expansion and not the generalized one.

  Problem 1: certain expressions would be governed by inference rules which could be considered as ambiguous definitions of these expressions, since different sets of elimination rules could be associated to the same set of introduction rules (cf. the problem of quantum disjunction).

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Possible solutions (2): Propositional case

- In order to work within a genuine intensional account, it seems then natural to ban any kind of expansion, and consider only $\equiv_\beta$.
- This solution is not so restrictive as it could seem. Consider the following situation where $\mathcal{D}$ is a closed derivation:

$$\underbrace{\dfrac{\begin{matrix}\mathcal{D}\\ A\to B\end{matrix} \qquad \dfrac{\dfrac{\overset{m}{A}}{B}\,A \overset{n}{\Rightarrow} B}{A\to B}\to_I (m)}{A\to B}\to_E (n)}_{\mathcal{D}_1} \rightsquigarrow_\beta \underbrace{\dfrac{\dfrac{\begin{matrix}\overset{l}{A}\\ \mathcal{D}^*\\ B\end{matrix}}{A\to B}\to_I (l) \qquad \dfrac{\dfrac{\overset{m}{A}}{B}\,A \overset{n}{\Rightarrow} B}{A\to B}\to_I (m)}{A\to B}\to_E (n)}_{\mathcal{D}_2} \rightsquigarrow_\beta \underbrace{\dfrac{\begin{matrix}\overset{l}{A}\\ \mathcal{D}^*\\ B\end{matrix}}{A\to B}\to_I (l)}_{\mathcal{D}_3}$$

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- Thus, with β-reduction it is possible to recover some instances of η-expansion which do not lead to any collapse.
- This imposes a hierarchical order on the principles used in proof-theoretic semantics: deducibility of identicals (and possibly stability) has not to be taken as an independent principle, but has to be induced by harmony.

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- In other words, the no less conditions have to be induced after having established the no more conditions.

Possible solutions (2): Identity case

- Not imposing η-equivalence from the beginning corresponds – in the framework of Martin-Löf's ITT – to drop the η_{Conv} rule allowing one to conclude:

$$J(t, s, d, \hat{x} : A.d'(x, x, \mathbf{r}(x))) \equiv d'(t, s, d) : C(t, s)$$

from a proof d of $t =_A s$ and a proof $d'(x, x, \mathbf{r}(x))$ of $C(x, x)$.

- The collapse of all proofs of $t =_A s$, for all t and s, is thus avoided.

- However, even without the η_{Conv}, it is still possible to have a *propositional version* of the η relation, that is, to find a p such that

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- Not imposing η-equivalence from the beginning corresponds – in the framework of Martin-Löf's ITT – to drop the η_{Conv} rule allowing one to conclude:

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Intensionality and extensionality again

- Similarly, we have that the judgment

$$\lambda u.u \equiv \lambda u.\mathtt{It}(u, \mathsf{O}, \hat{v}.\mathsf{S}(v)) : \forall x(Nx \to Nx)$$

  is not assertable, since without simple expansion there is no way to rewrite $\lambda u.u$ on $\lambda u.\mathtt{It}(u, \mathsf{O}, \hat{v}.\mathsf{S}(v))$, and *vice versa* (as we said, the two terms are not β-equivalent).

- On the contrary, the judgment

$$p : \lambda u.u =_{\forall x(Nx \to Nx)} \lambda u.\mathtt{It}(u, \mathsf{O}, \hat{v}.\mathsf{S}(v))$$

  is assertable, for a certain p.

- This means that even if we do not have a procedure for transforming $\lambda u.u$ into $\lambda u.\mathtt{It}(u, \mathsf{O}, \hat{v}.\mathsf{S}(v))$, and *vice versa*, we can however *say* that the are identical.
- The reason is that $\lambda u.u$ and $\lambda u.\mathtt{It}(u, \mathsf{O}, \hat{v}.\mathsf{S}(v))$ correspond to different algorithms, but they are denotationally equivalent.
- Thus, while the relation $\equiv$ is intensional, the identity predicate $=_A$ (for a certain A) is extensional.

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The beginning of homotopy type theory

- More precisely, Hoffman & Streicher (1998) set up a groupoid interpretation of type theory in which they showed that there exists an equivalent relation $\simeq$ on the set of terms of type $t =_A s$ such that

$$f \simeq g \text{ iff there exists a term } p \text{ such that } p : f =_{(t =_A s)} g.$$

- In terms of spaces, f and g correspond to paths from the point a to the point b in the space A and p corresponds to a homotopy relative to their endpoints from f to g.

- These ideas have been one of the starting points of the project of homotopy type theory.

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