

Preliminaries

Evolution of Gene
Function

Mechanistic Machine
Learning

Proof of Concept

Predicting of Gene Functions by Leveraging Biological Insights with Mechanistic Machine Learning



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You can download the slides from <https://ggyv.cl/UofUDS2023>

But first...



George G. Vega Yon's GitHub Stats

⭐ Total Stars Earned:	247
⌚ Total Commits (2022):	2.4k
👉 Total PRs:	111
❗ Total Issues:	228
💻 Contributed to:	33



- ▶ Research Assistant Professor in Epidemiology since Nov. 2021.
- ▶ A methodologist researching Statistical Computing applied to Complex Systems Modeling.
- ▶ Got a Ph.D. in Biostats from USC and an M.Sc. in Economics from Caltech.
- ▶ Working with networks + programming since ~2010.
- ▶ Scientific software developer (R/C++) with (~ 500K downloads + 9 CRAN packages).

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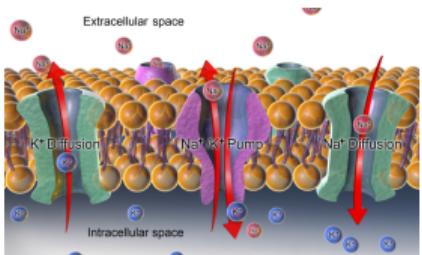
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Gene Function

Encode the synthesis of genetic products that ultimately are related to a particular aspect of life, for example

Molecular function

Active transport
GO:0005215



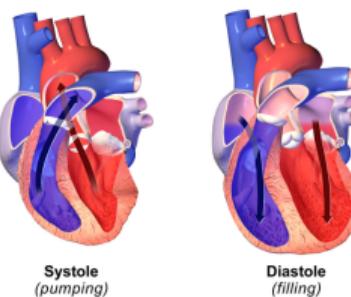
Cellular component

Mitochondria GO:0004016



Biological process

Heart contraction
GO:0060047



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Gene Function: the Gene Ontology Project



GENEONTOLOGY
Unifying Biology

- ▶ The GO project has \sim 43,000 validated terms, \sim 7.4M annotations on \sim 5,200 species.
- ▶ About \sim 700,000 annotations are on human genes.
- ▶ Only half of the human gene annotations are based on experimental evidence.
- ▶ About \sim 173,000 publications have used the GO.

source: Statistics from <http://pantherdb.org/panther/summaryStats.jsp> and <http://geneontology.org/stats.html>

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Predicting Gene Function: State-of-the-art

Sequences, phylogenomics, and ML.

- ▶ **BLAST**:² Prediction by sequence homology ($\sim 105,000$ citations).
- ▶ **SIFTER**:^{6,5} An evolutionary model of gene function/loss using phylogenetics.
- ▶ **aphylo**¹⁵ (by yours truly): Another phylo-based method. Leverages negative annotations and pooled trees.
- ▶ **Phylo-PFP**:¹⁰ A BLAST-based adding phylogenetic based distances.
- ▶ **DeepGOPlus**:¹² One of the top-performing models in the literature, uses a 2D convolutional neural network on gene sequences.
- ▶ **GOLabeler**:¹⁷ Top performing tool according to the *Critical Assessment of Function Annotation* [CAFA] challenge,¹⁸ is an ensemble of various simple ML methods, including K-means and logistic regression.
- ▶ **DeepFRI**:⁸ Uses Graph Convolutional Neural Networks (GCNs) to predict function based on protein structure and genetic sequence.

In the latest CAFA, **none** of the top-performing methods scored an AUC above 0.60, and most were outperformed by BLAST,¹⁹ which annotates using homology based on sequence similarity.

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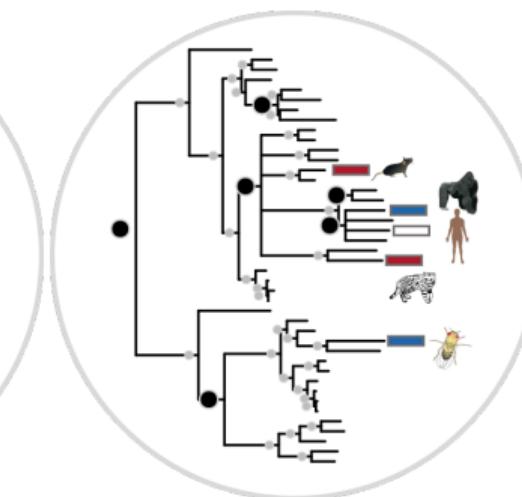
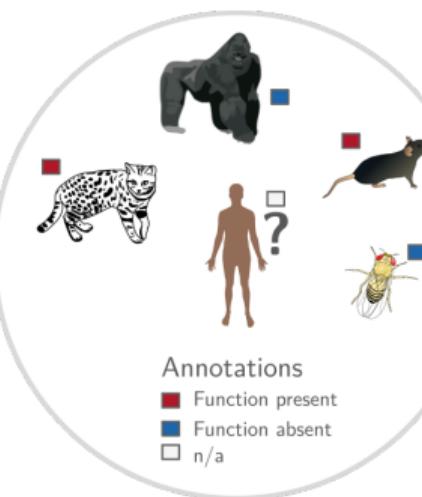
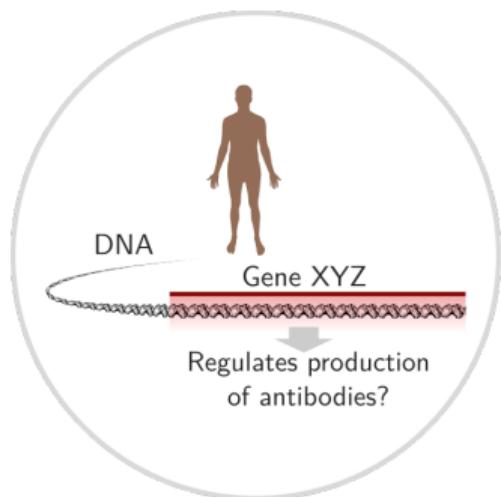
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Is gene XYZ involved in process ABC?



Complex to directly assess

But we may know from other species

And we further know how these are *evolutionary* connected

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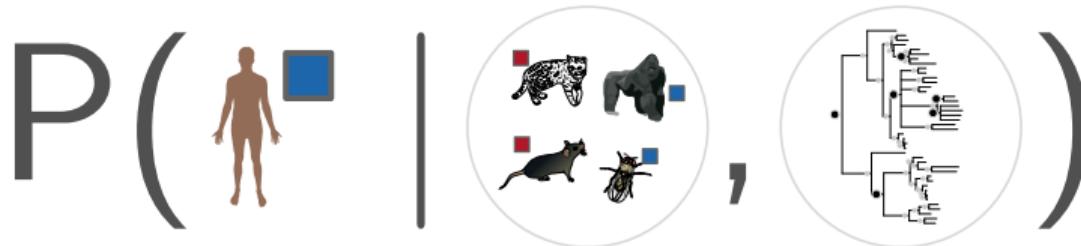
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Is the human gene **XYZ** involved in process **ABC**, given what we know about that for
other related species?



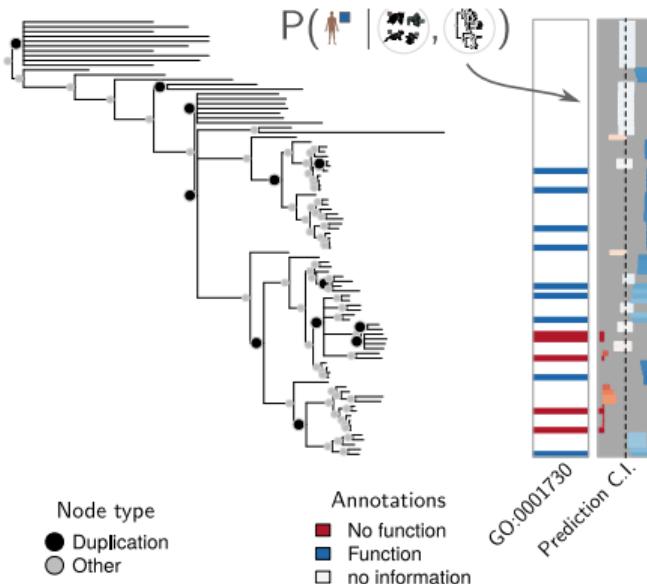
Annotations

- Function present
- Function absent
- n/a

Evolution of Gene function (of one function)

Built a big model (lots of trees and annotations) called aphylo:

- ▶ Only two sources of data:
Phylogenetic tree
(pantherdb.org) and functional annotations
(geneontology.org).
- ▶ Leverage negative annotation of GO terms (NOT).
- ▶ Use Felsenstein's tree pruning algorithm to compute tree likelihood.
- ▶ Fit pooled models featuring thousands of annotations in hundreds of trees (with split-second prediction capability).



... But what if we wanted to deal with multiple functions?

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Evolution of Gene function (multiple functions)

Tapping into Evol. Theory

- ▶ A fundamental part of Fun. Evol. is Duplication Events.
- ▶ Furthermore, knowing what happened to gene A (e.g., neofunctionalization) is highly informative to learn about the functional state of B.
- ▶ One way to model this is using a Markov Transition Model (as in SIFTER).

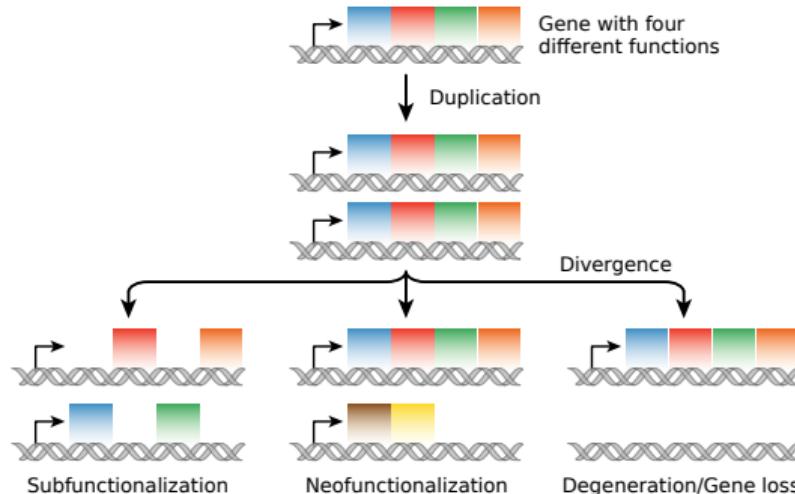


Figure: A key part of molecular innovation, gene duplication provides an opportunity for new functions to emerge (wikimedia)

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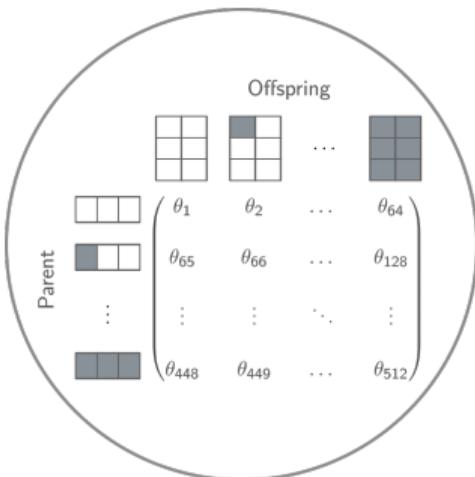
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Evolution of Gene function (multiple functions)

If we wanted to build a model with 3 functions, we would need to estimate...

Full Markov Transition Matrix



- ▶ 512 parameters
- ▶ Finding this many parameters is not easy.
- ▶ Even if you can, interpretation is awkward.

Social Network Analysis may help us...

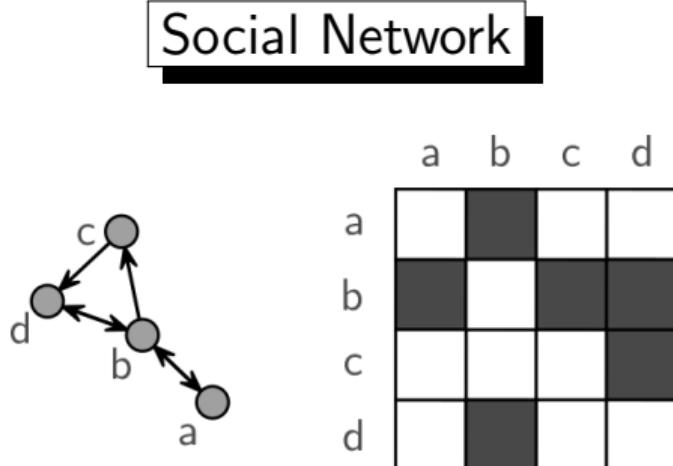
Exponential Random Graph Models (ERGMs)

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- ▶ Not about individual ties.
- ▶ Statistical inference on *motifs* (triangles, dyads, homophily, etc.)
- ▶ Literature about ERGMs is vast, a.k.a. a low-hanging fruit.

Ultimately...

ERGM ≡ **Modeling binary arrays**

Exponential Random Graph Models (ERGMs)

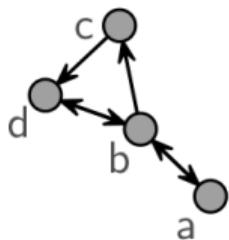
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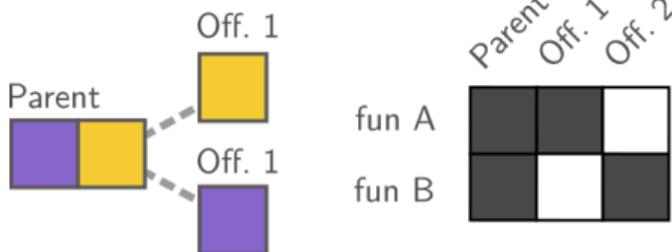
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Social Network



	a	b	c	d
a				
b				
c				
d				

Evolutionary Event

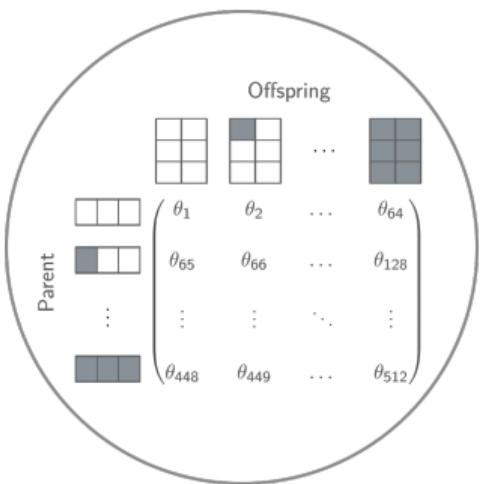


Social Networks are usually represented as **adjacency matrices**, and so can evolutionary events!

Evolution of Gene function (multiple functions) (cont.)

If we wanted to build a model with 3 functions, we would need to estimate...

Full Markov Transition Matrix

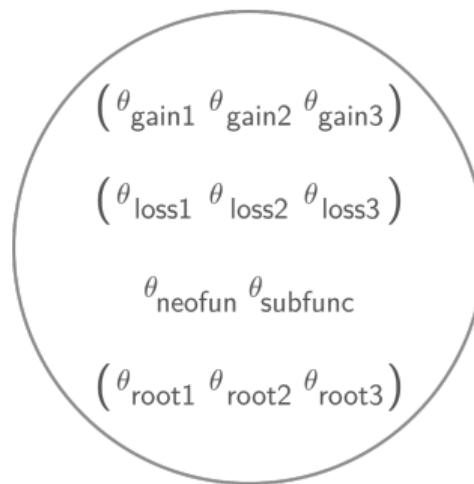


512 parameters

Sufficient statistics

$$\begin{pmatrix} \theta_{\text{gain}1} & \theta_{\text{gain}2} & \theta_{\text{gain}3} \\ \theta_{\text{loss}1} & \theta_{\text{loss}2} & \theta_{\text{loss}3} \\ \theta_{\text{neofun}} & \theta_{\text{subfunc}} \\ \theta_{\text{root}1} & \theta_{\text{root}2} & \theta_{\text{root}3} \end{pmatrix}$$

Easier to fit
Easier to interpret



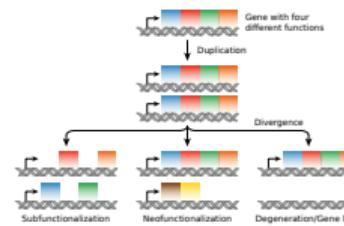
11 parameters (for example)

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Rep.	Description	Definition
	Gain of function	$(1 - x_p) \sum_{n:n \in Off} x_n$
	Loss of function	$x_p \sum_{n:n \in Off} (1 - x_n)$
	Subfunctionalization	$x_p^k x_p^j \sum_{n \neq m} x_n^k (1 - x_n^j) (1 - x_m^k) x_m^j$
	Neofunctionalization	$x_p^k (1 - x_p^j) \sum_{n \neq m} x_n^k (1 - x_n^j) (1 - x_m^k) x_m^j$
	Longest branch gains	$(1 - x_p^k) \mathbf{1} (x_m^k : m = \text{argmax}_n \text{blength}_n)$

Table: Example of sufficient statistics for evolutionary transitions.

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GEESE: GEne functional Evolution using SuficiEncy

I implemented what I just described in a C++ library with a companion R package called `geese`. The question is: How much do we earn by using these motifs?

- ▶ Using 37 phylogenetic trees featuring 401 go annotations.
- ▶ `aphylo`: Fitted a *simple gain/loss* of function model.
- ▶ **GEESE**: Fitted an evolutionary model controlling for *functional preservation* (i.e., like neofun or subfun.)
- ▶ Fitted both of them using MCMC.
- ▶ Used LOO cross-validation to compute aggregated AUCs and MAE.

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GEESE for predicting gene function (cont.)

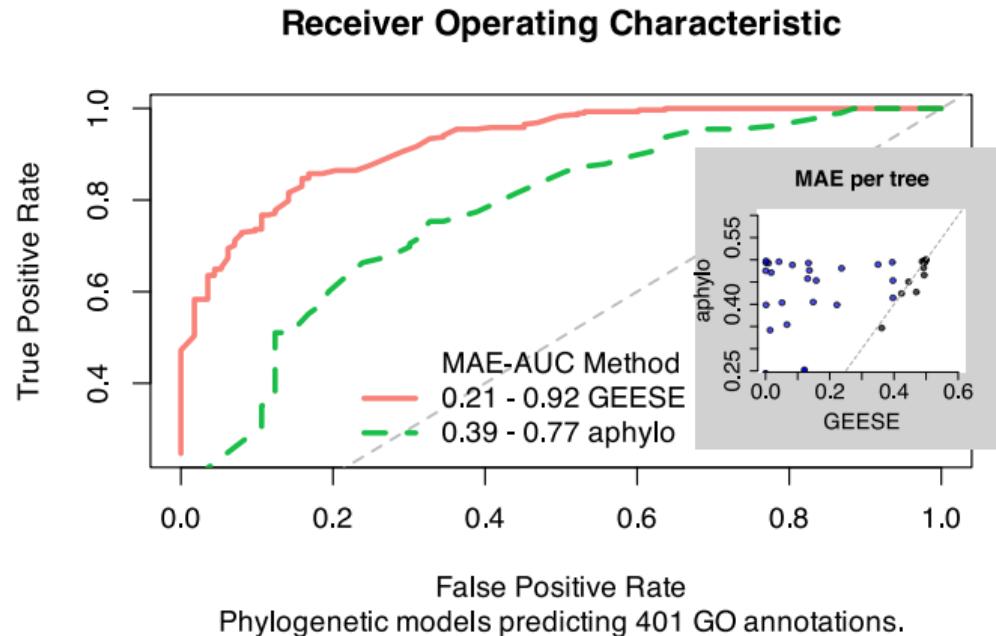
How much can we gain from a joint dist. model?

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Just controlling for preservation (having only one duplicate changing) significantly improves our predictions.

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Mechanistic Machine Learning: State-of-the-art

- ▶ After all the data pouring, attention to causal inference and mechanistic models is coming back^{3,14}
- ▶ Applications in Physics, Chemistry, Biomedical Imaging, and Biology^{16,11,7,1} show the benefits of combining the two approaches.

Mechanistic Models

- ▶ Inference driven (causality)
- ▶ Great for small datasets
- ▶ Not the most accurate

Machine Learning Models

- ▶ Data-driven (prediction)
- ▶ Lots of points to “learn”
- ▶ Great for big data

Important: Mechanistic Machine Learning **is not** domain-knowledge aided feature engineering. You need a whole other model to complement the ML algorithm.

Three strategies

- a. **ML Correction:** Use machine learning to learn the errors of a mechanistic model.
- b. **Mechanistic Feature:** Add mechanistic predictions as a feature of a machine learning model.
- c. **Mechanistic Penalty:** Add constraints to the ML algorithm based on a mechanistic model.

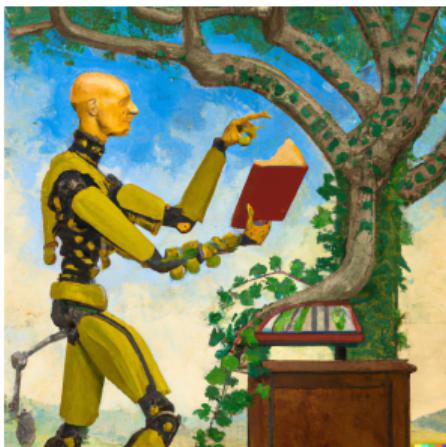


Figure: “A van Gogh-style painting of an android holding a large biology book in one hand and a computer in another, examining an evolutionary tree that, instead of leaves, have genes.”—DALL-E’s interpretation of my description ([link](#))

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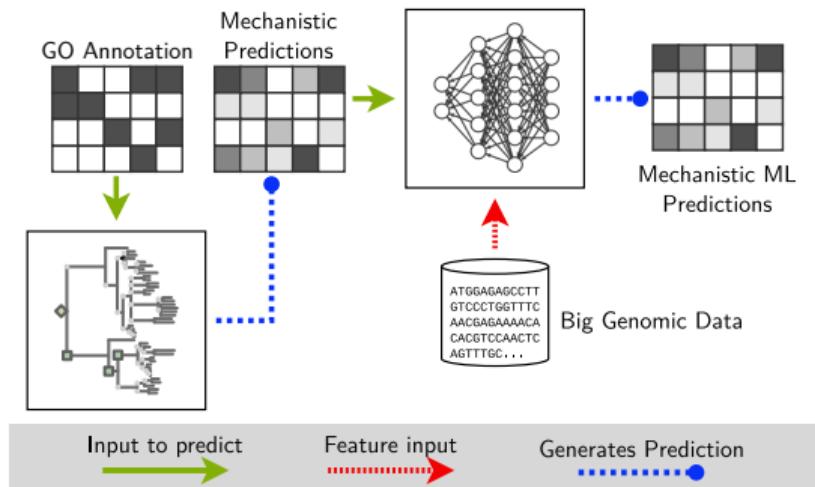
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Three strategies

a. ML Correction

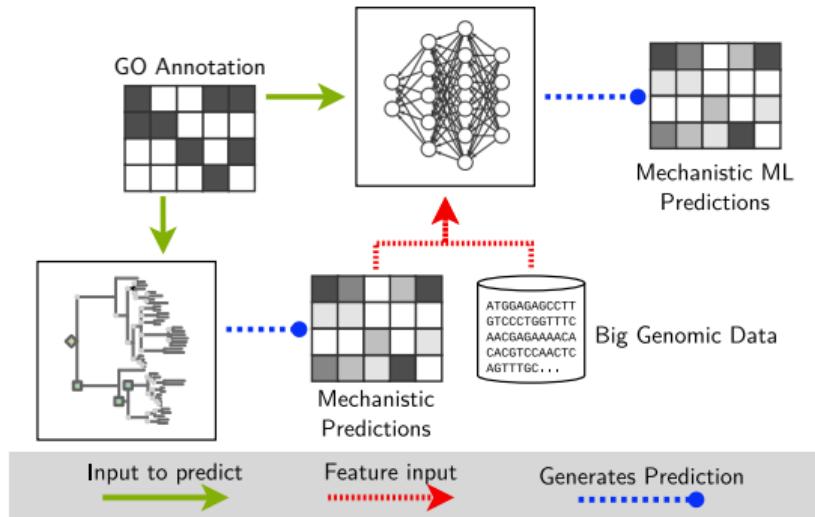
1. Fit the mechanistic model using GEESE
2. Generate the mechanistic-based predictions, $\hat{\mathbf{y}}^{GEESE}$,
3. fit an ML model $f(X, \Omega)$ to predict $\varepsilon \equiv (\mathbf{y} - \hat{\mathbf{y}}^{GEESE})$,
4. generate the predictions of $\hat{\varepsilon}$, and
5. Compute the Mechanistic-ML predictions as $\hat{\mathbf{y}}^{MML1} \equiv \hat{\mathbf{y}}^{GEESE} + \hat{\varepsilon}$



Three strategies

b. Mechanistic Feature

1. Fit the mechanistic model using GEESE,
2. generate the mechanistic-based predictions, \hat{y}^{GEESE} ,
3. fit an ML model that uses the mechanistic predictions as features, $f(X, \Omega, \hat{y}^{GEESE})$, and
4. Compute the Mechanistic-ML predictions as $\hat{y}^{MML2} \equiv f(X, \Omega, \hat{y}^{GEESE})$



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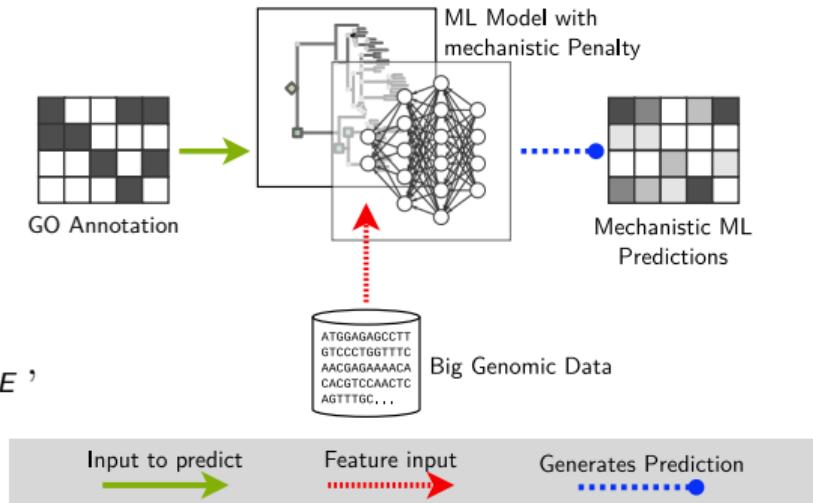
Three strategies

c. Mechanistic Penalty

1. Fit the mechanistic model using GEESE and store the parameter estimates $\hat{\theta}$,
2. minimize the following loss function:

$$L(y_{,}^{obs} X, \Omega) - \mathcal{L}(f(y_{,}^{obs} X, \Omega))_{GEESE},$$

where $\mathcal{L}(\cdot)$ is the likelihood function under GEESE.



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Beyond GO and Trees... Bgee

The **Bgee** project “is a **database** for retrieval and **comparison of gene expression** patterns **across multiple animal species**. It provides an intuitive answer to the question ‘where is a gene expressed?’ and supports research in cancer and agriculture as well as evolutionary biology.” – Bastian et al. [4]

- ▶ Raw expression annotations.
- ▶ Standardized expression scores (so can compare across species/tissues).
- ▶ And also yes/no expression annotations based on the standardized scores.

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Beyond GO and Trees... Bgee (cont.)

Divergence across species in gene expression levels has been linked to evolutionary events,^{13,9} i.e., expression levels clustered phylogenies.

Thinking of different ways to use it as:

- ▶ As an additional feature for our model: “Given the phylo, observed annotations, **and expression levels in n tissues**, . . .”
- ▶ As 0/1 variable (expression is present/absent) to predict in our model: “Model the evolution of gene function **and expression**.”
- ▶ As part of a prediction model in, say, a Machine Learning Model.

Data Feats

- ▶ Bgee 15 dataset: approx 7 billion annotations for 1.5 million genes.
- ▶ Our dataset: 1,484 predictions for 1,318 genes.
- ▶ Search by Gene name: 9,923,427 Bgee annotations.

Final model

- ▶ 10 GO terms (in a full-Markov model, this is 1 MM params).
- ▶ 278 annotations for 256 genes.
- ▶ 10 GEESE predictions for each gene.
- ▶ 46 Bgee score for gene expression computed as **mean expression score by gene-genus**

GO terms: GO:0004672, GO:0004713, GO:0004867, GO:0005730, GO:0005829, GO:0005886, GO:0006468, GO:0009408, GO:0015020, GO:0060070

Genus: Anguilla, Anolis, Astatotilapia, Astyanax, Bos, Branchiostoma, Caenorhabditis, Callithrix, Canis, Capra, Cavia, Cercocetus, Chlorocebus, Danio, Drosophila, Equus, Esox, Felis, Gadus, Gallus, Gasterosteus, Gorilla, Heterocephalus, Homo, Latimeria, Lepisosteus, Macaca, Manis, Meleagris, Microcebus, Monodelphis, Mus, Neolamprologus, Nothobranchius, Ornithorhynchus, Oryctolagus, Oryzias, Ovis, Pan, Papio, Poecilia, Rattus, Salmo, Scophthalmus, Sus, Xenopus

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Mechanistic ML

We are comparing three models:

GEESE

Bgee

GEESE + Bgee

Phylogenetic based
predictions (evolution of
gene function)

Linear Prob. model using
expression as predictors.

Linear Prob. model using
expression as predictors
and predictions made by
GEESE.

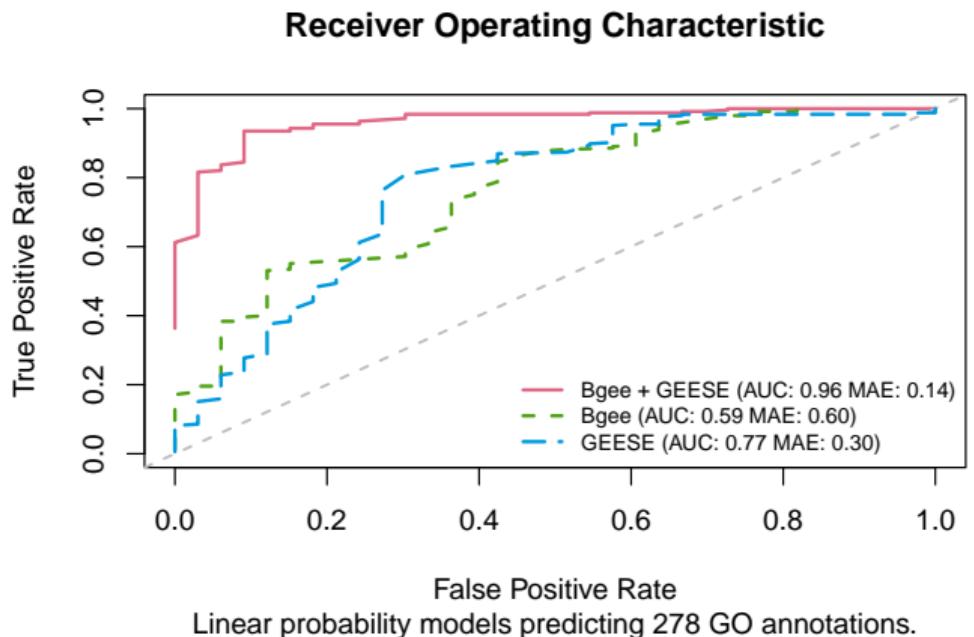
Mechanistic ML (prelim res.)

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Both AUC and MAE were computed only using predictions for which we knew the true value.

Discussion

Gene function

- ▶ We are racing to discover what genes do.
- ▶ Experimental assessment is expensive (money and time,) → automatic annotations.
- ▶ Many ways to do it (seq. homology, evolutionary theory, ML, etc.)
- ▶ The best methods use ML (pattern discovery)... but none (AFAIK) are based on bio. theory.

Evol. Model

- ▶ We proposed an Evolutionary model of Gene Function.
- ▶ This new model, GEESE, uses sufficiency to reduce “Markov complexity.”
- ▶ We showed it really helps.

Mechanistic ML

- ▶ Mechanistic Machine Learning (mixing theory-based models with ML) promises improved predictions.
- ▶ I showed an application using gene expression (Bgee).
- ▶ Adding our mechanistic predictions (based on GEESE) boosted quality

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Thank you!

Predicting of Gene Functions by Leveraging Biological Insights with Mechanistic Machine Learning



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Jan 11th, 2023 @ Data Science Seminar

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Tree likelihoods: Felsenstein's Pruning algorithm

All possible transitions from x_n → Transition Probability (ERGM)

$$\mathbb{P}(\tilde{D}_n \mid x_n, \Theta) = \sum_x \mathbb{P}(x \mid x_n) \prod_{m \in O(n)} \mathbb{P}(\tilde{D}_m \mid x_m)$$

Model Parameters → Vector of Sufficient Statistics

$$\mathbb{P}(x \mid x_n) = \frac{\exp\{\Theta^t s(x, x_n)\}}{\sum_{x'} \exp\{\Theta^t s(x', x_n)\}}$$

Normalizing Constant → the *lingua franca* of SNA

... I implemented this (and more) on **barry**

Some computational features of barry

