PROBLEM 1

A dice is thrown twice (random experiment). Let A,B,C be random events in sample space generated by this experiment. Event A occurs then both results are the same, event B occurs then the sum of results is divisible by 3, event C occurs then result of first roll is bigger than result of second roll.

a) calculate the probability for all events,
b) calculate the probability of events: $A \cap B$, $A \cap C$, $B \cap C$, $A \cap B \cap C$, $A \cup B$, A', B'.
c) are events A and B independent?

Elementary outcome (example) (2,3) – result of first thrown is number "2" and result of second thrown is number "3". Each thrown brings 6 possible options. Thus sample space

$\Omega=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),\ldots,(6,6)\}$

Number of elementary outcomes in sample space Ω:

$|\Omega|=6\cdot 6=6^2=36$

Events (subsets) in sample space Ω

$A=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

Second roll / First roll	1	2	3	4	5	6
1	X					
2		X				
3			X			
4				X		
5					X	
6						X

$B=\{(1,2),(2,1),(1,5),(2,4),(3,3),(4,2),(1,5),(3,6),(4,5),(5,4),(6,3),(6,6)\}$

Second roll / First roll	1	2	3	4	5	6
1		3			6	
2	3			6		
3			6			9
4		6			9	
5	6			9		
6			9			12

C={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5)}

Second roll / First roll	1	2	3	4	5	6
1						
2	X					
3	X	X				
4	X	X	X			
5	X	X	X	X		
6	X	X	X	X	X	

a) Number of elementary outcomes and probabilities for events A,B,C

|A|=6

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

|B|=12

$$P(B) = \frac{|B|}{|\Omega|} = \frac{12}{36} = \frac{1}{3}$$

|C|=15

$$P(C) = \frac{|C|}{|\Omega|} = \frac{15}{36} = \frac{5}{12}$$

b) A∩B={(3,3),(6,6)}

|A∩B|=2

$$P(\mathrm{A} \cap \mathrm{B}) = \frac{|\mathrm{A} \cap \mathrm{B}|}{|\Omega|} = \frac{2}{36} = \frac{1}{18}$$

A∩C= Ø

|A∩C|=0

$$P(\mathrm{A} \cap \mathrm{C}) = \frac{|\mathrm{A} \cap \mathrm{C}|}{|\Omega|} = \frac{0}{36} = 0$$

B∩C={(4,2),(5,1),(5,4),(6,3)}

|B∩C|=4

$$P(\mathrm{B} \cap \mathrm{C}) = \frac{|\mathrm{B} \cap \mathrm{C}|}{|\Omega|} = \frac{4}{36} = \frac{1}{9}$$

A∩B∩C= Ø

| A∩B∩C |=0

$$P(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) = \frac{|\mathrm{A} \cap \mathrm{C}|}{|\Omega|} = \frac{0}{36} = 0$$

AUB={ **(1,1)**,(1,2),(2,1),(1,5),(2,4),**(3,3)**,(4,2),(1,5),(3,6),(4,5),(5,4),(6,3),**(6,6)**, **(2,2)**,**(4,4)**,**(5,5)**}

|AUB|=15

$P(\mathrm{AUB}) = \frac{|\mathrm{AUB}|}{|\Omega|} = \frac{15}{36} = \frac{5}{12}$

A’=Ω-A

P(A’)=1-P(A)=$1 - \frac{1}{6} = \frac{5}{6}$

B’= Ω-B.

P(B’)=1-P(B)=$1 - \frac{1}{3} = \frac{2}{3}$

c) independence of events A and B

Definition of independence $P(A \cap \mathrm{B}) = \mathrm{P(A)} \cdot \mathrm{P(B)}$

L=$\frac{1}{18}$

R=$\frac{1}{3} \cdot \frac{1}{6}$=$\frac{1}{18}$

L=R thus events A and B are independent.

PROBLEM 2

A survey of students was carried out before an exam. 10% of students stated that they were very confident, 60% stated that they were confident and 30% were unconfident. 70% of those who said they were very confident, 40% of those who said they were confident and 10% of those who said they were unconfident got at least a 3.0 grade. Calculate

a) the probability that a randomly picked student got at least a grade 3.0.
b) given that the student got at least a 3.0 grade, what is the probability that he/she was very confident or confident.
c) given that the student got less than a 3.0 grade, what is the probability that he/she was unconfident.

Sample space Ω (without details of structure) consists of events:
A – randomly pick up student is (was) very confident,
B – randomly pick up student is (was) confident,
C – randomly pick up student is (was) unconfident,
D – randomly pick up student got at least 3.0 grade.

The following decomposition of Ω is obvious.

$\Omega = A \cup B \cup C$, moreover $A \cap B = \emptyset$, $A \cap C = \emptyset$, $B \cap C = \emptyset$ (A,B,C – disjoined events).

Probabilities described in the content of the problem:

P(A)=0,10; P(B)=0,6; P(C)=0,3

P(D/A)=0,7; P(D/B)=0,4; P(D/C)=0,1

a) probability of passing exam

$P(D)=P(A)\cdot P(D/A)+ P(B)\cdot P(D/B)+ P(C)\cdot P(C/A)=0,1\cdot 0,7+0,6\cdot 0,4+0,3\cdot 0,1=0,34$

b) Events A and B are disjoined events ($A \cap B = \emptyset$), then

$P(A \cup B/D)=P(A/D)+P(B/D)$

Conditional probability (definition)

$P(A/D)=\frac{P(A \cap D)}{P(D)}$

and

$P(D/A)=\frac{P(D \cap A)}{P(A)}$

Obviously

$A \cap D = D \cap A$

thus

P(A/D)=$\frac{P(A \cap D)}{P(D)} = \frac{P(D/A) \cdot P(A)}{P(D)} = \frac{0{,}7 \cdot 0{,}1}{0{,}34}$.

Similar calculation is possible in the case of P(B/D)

P(B/D)=$\frac{P(B \cap D)}{P(D)} = \frac{P(D/B) \cdot P(B)}{P(D)} = \frac{0{,}4 \cdot 0{,}6}{0{,}34}$.

Finally

P(AUB/D)=P(A/D)+P(B/D)= $\frac{0{,}7 \cdot 0{,}1 + 0{,}4 \cdot 0{,}6}{0{,}34}$

c) D' – opposite event for event D (student got less than 3.0 grade)

P(C/D')= $\frac{P(C \cap D')}{P(D')} = \frac{P(D'/C) \cdot P(C)}{1 - P(D)} = \frac{(1 - P(D/C)) \cdot P(C)}{1 - P(D)} = \frac{(1 - 0{,}1) \cdot 0{,}3}{1 - 0{,}34}$

PROBLEM 3

A dice is rolled until the first number divisible by 3 appears. Calculate the probability that

a) the dice is rolled exactly once,

b) the dice is rolled exactly three times,

c) the dice is rolled at least three times,

d) the number of rolls is even.

Sample space

Ω={1,2,3,4,5,6,…} (set of natural numbers)

a) the dice is rolled exactly once,

A={1}

The "easiest" solution P(A)=$\frac{1}{|\Omega|}$ doesn't fit to the assumption about classical concept of the probability (the problem with number of elementary outcomes in the sample space |Ω|=?)

P(A)=$\frac{2}{6}$, because condition "the dice is rolled exactly once" means that number 3 or number 6 appears in first roll.

Condition "the dice is rolled exactly once" means that there is possibility to calculate conditional probability. In that case sample space Ω={1,2,3,4,5,6} corresponds to the consideration of results for first roll only. In above mentioned sample space there are two numbers divisible by 3 and 6 numbers in total.

In that case A={3,6}.

b) Condition "the dice is rolled exactly three times" let to consider sample space

Ω={(1,1,1), (1,1,2), (1,1,3),…., (6,6,6)} as a set of outcomes of three consecutive rolls.

$|\Omega|=6^3$.

B={(1,1,3),(1,2,3),(1,4,3),(1,5,3),….,(5,5,6)} (description of event B – first roll, result is not 3 and result is not 6, second roll, result is not 3 and result is not 6, third roll, result is 3 or 6).

|B|=4·4·2

P(B)= $\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{2}{6}$

c) C – event the dice is rolled at least three times.

P(C)=1-P(C')

C' – opposite event for event C, the dice is rolled less than 3 times (it means the dice is rolled 1 or 2 times).

P(C')=$\frac{2}{6}+\frac{4}{6}\cdot\frac{2}{6}$

P(C)=$1-\frac{2}{6}-\frac{4}{6}\cdot\frac{2}{6}$

d) D- event, the number of rolls is even.

Extended description of event D: number of rolls equals 2 or equals 4 or equals 6 or equals 8 or .....

P(D)=$\frac{4}{6}\cdot\frac{2}{6}+\left(\frac{4}{6}\right)^3\cdot\frac{2}{6}+\left(\frac{4}{6}\right)^5\cdot\frac{2}{6}+\left(\frac{4}{6}\right)^7\cdot\frac{2}{6}+\cdots=$

$=\frac{2}{6}\cdot\left(\frac{4}{6}+\left(\frac{4}{6}\right)^3+\left(\frac{4}{6}\right)^5+\left(\frac{4}{6}\right)^7+\cdots\right)=\frac{1}{3}\left(\frac{\frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}\right)$

The concept of classical geometric sequence was used for calculation of sum:

$$\frac{4}{6}+\left(\frac{4}{6}\right)^3+\left(\frac{4}{6}\right)^5+\left(\frac{4}{6}\right)^7+\cdots$$

Additional question what is probability of having odd number of rolls in the same random experiment?

PROBLEM 4

Consider choosing a card from a well-shuffled standard deck of 52 playing cards.

a) Suppose that, after the first extraction, the card is not reinserted in the deck. What is the probability that the second card is an ace?

b) Suppose that, after the first extraction, the card is reinserted in the deck. What is the probability that the second card is an ace?

Two events:

A – the result of first extraction is an ace,

B – the result of the second extraction is an ace.

a) card is not reinserted, after first extraction the total number of cards in the deck is 51.

First extraction

P(A)=$\mathbf{\frac{4}{52}}$ (number of aces is 4, total number of cards in the deck is 52)

Conditional probability (card in not reinserted)

P(B/A)=$\frac{3}{51}$ (number of aces is 3, number of cards in the deck is 51, one card – exactly one ace was extracted)

P(B/A')=$\frac{4}{51}$ (number of aces is 4, number of cards in the deck is 51, one card – not an ace was extracted)

P(B)=P(A)·P(B/A)+P(A')·P(B/A')=$\mathbf{\frac{4}{52}\cdot\frac{3}{51}+\frac{48}{52}\cdot\frac{4}{51}=\frac{(48+3)\cdot 4}{52\cdot 51}=\frac{51\cdot 4}{52\cdot 51}=\frac{4}{52}}$

b) card is reinserted, no matter first or second extraction – number of aces and total number of cards are the same.

P(B)=$\frac{4}{52}$

Conclusion – result doesn't dependent on the procedure, reinserted card or not reinserted card after first extraction doesn't have impact on the probability of having an ace in the second extraction.

PROBLEM 5.

In the lottery 6 numbers are chosen without replacement from 49. Calculate the probability of

a) winning the jackpot (choosing all 6 numbers correctly),
b) winning the smallest prize (choosing 3 of the 6 numbers correctly),
c) choosing at least one of the numbers correctly.

Elementary outcome – single result in the lottery, for example {1,2,3,4,5,6} or {2,19,21,35,37,42} etc.

Sample space Ω consists of 6 items sets, in each set there are natural numbers from range 1,2,….,49.

Ω={(1,2,3,4,5,6);{1,2,3,4,5,7};{1,2,3,4,5,8};…;{44,45,46,47,48,49}}

$|\Omega|=\binom{49}{6}=\frac{49!}{6!\cdot(49-6)!}=13\,983\,816$

a) A={one winning combination of numbers from range 1,2,…,49}

|A|=1 (one winning set)

$P(A)=\frac{1}{13983816}$

b) B – event, winning of exactly the smallest prize (choosing exactly 3 numbers correctly)

$P(B)=\frac{\binom{6}{3}\cdot\binom{49-6}{3}}{\binom{49}{6}}$

c) C – event, at least one number is chosen correctly

P(C)=1-P(C’)

C’ – none of numbers is selected correctly, no winning numbers.

$P(C')=\frac{\binom{49-6}{6}}{\binom{49}{6}}$

$P(C)=1-\frac{\binom{49-6}{6}}{\binom{49}{6}}$

PROBLEM 6

A lift cabin in given building, which has 9 floors is starting from ground level and inside there are 5 passengers. Calculate the probability, that each passenger will leave the lift on different floor. In additional calculate the probability that all passengers will leave the lift on one given floor.

Elementary outcome – sequence of 5 numbers (selection of floors) from range 1,2,….,9. For example sequence (1,2,1,5,5) means: first passenger selected 1st floor, second passenger selected 2nd floor, third passenger selected 1st floor, fourth passenger selected 5th floor, fifth passenger selected 5th floor.
Sample space
$\Omega=\{(1,1,1,1,1),(1,1,1,1,2);\dots;(9,9,9,9,9)\}$
$|\Omega|=9^5$

A – event, everybody leaves on different floor.
$A=\{(1,2,3,4,5),(1,2,3,4,6);\dots;(5,6,7,8,9)\}$
$|A|=9\cdot 8\cdot 7\cdot 6\cdot 5$

$$P(A)=\frac{|A|}{|\Omega|}=\frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{9^5}$$

B – event, everybody leaves the lift on the same floor
$B=\{(1,1,1,1,1);(2,2,2,2,2);\dots;(9,9,9,9,9)\}$
$|B|=9$

$$P(B)=\frac{9}{9^5}=\frac{1}{9^4}$$

PROBLEM 7

A machine is composed of 3 components, which function independently of each other with probabilities p_1, p_2 and p_3, respectively. Calculate the probability that the machine works when

a) the machine only works when all the components are working,

b) the machine works when at least one of the component works.

a) A – event, machine works, all components work

$$P(A)=p_1 \cdot p_2 \cdot p_3$$

(common part of independent events: component 1 works and component 2 works and component 3 works)

b) B – event, machine works, at least one of the component works.

$P(B)=1-P(B')$

B' – event, none of the component works

B' is common part of independent events: component 1 doesn't work and component 2 doesn't work and component 3 doesn't work.

$$P(B)=1-P(B')=1-(1-p_1)\cdot(1-p_2)\cdot(1-p_3)$$

PROBLEM 8

I throw a coin 10 times. Calculate the probability that

a) I throw exactly 10 heads (is that result possible in the case when coin is 100 thrown),
b) I throw at least 2 heads,
c) calculate the probability that I need to wait 4 trials to receive first time the head,
d) calculate the probability that it necessary to wait for first head more than 2 trials.

Sample space

Ω={(H,H,H,T,T,T,T,T,T,T);(T,T,H,H,T,T,H,H,T,T);…;(H,H,H,H,H,H,H,H,H,H)}

$|\Omega|=2^{10}$

a**) 10 heads in 10 trials**

A={(H,H,H,H,H,H,H,H,H,H)}

|A|=1

P(A)=$\frac{1}{2^{10}}$ (in the case of 100 coin tosses P(A)=$\frac{1}{2^{100}} > 0$)

b) **at least 2 heads in 10 trials**

B={(H,H,T,T,H,T,T,T,T,T); (H,T,T,T,T,H,T,T,T);…; (T,T,H,H,T,T,T,T,H,H)}

B' occurs in case there is one head in 10 trials or no heads at all.

B'={(T,T,T,T,T,T,T,T,T,T);(H,T,T,T,T,T,T,T,T,T); (T,H,T,T,T,T,T,T,T,T);…; (T,T,T,T,T,T,T,T,T,H)}

|B|=1+10=11

1 corresponds to the case – only tails in 10 trials, 10 corresponds to the case – one head and nine tails in 10 trials.

P(B)=$\mathbf{1-\frac{11}{2^{10}}}$

c) **head first time in the 4th trial.**

C={(T,T,T,**H**,T,T,T,T,T,T);(T,T,T,**H**,T,H,T,T,T,T); (T,T,T,**H**,T,H,H,T,T,T);…; (T,T,T,**H**,H,H,H,H,H,H)}

$|C|=2^6$

P(C)=$\mathbf{\frac{2^6}{2^{10}}=\frac{1}{2^4}}$

d) head first time in the third trial or after third trial.

D' – opposite event for event D, head occurs in first or in second trial.

D'={(H,T,T,H,T,T,T,T,T,T);(T,H,T,H,T,H,T,T,T,T); (T,H,T,H,T,H,H,T,T,T);…; (T,H,T,H,H,H,H,H,H,H)}

$|D'|=2^9+2^8$

$$P(D)=1-\frac{2^9+2^8}{2^{10}}$$

PROBLEM 9

The probability that a player could win in some game is equal to *p* for each round of the game. She/he will stop the game if she/he will win or after 3 trials (not enough time to play). Calculate probabilities for each possible number of trials.

X – random variable, number of games

$S_X=\{1,2,3\}$

$P(X=1)=p$

$P(X=2)=(1-p)\cdot p$ (to take part in the second game player has to lose in the first game, results of games are independent – the same probability of being winner in all games)

$P(X=3)=(1-p)^2\cdot p+(1-p)^2\cdot(1-p)= (1-p)^2\cdot(p+1-p)= (1-p)^2$ (the third game is the last one, there are two options – player wins or player loses. The third game happens if player loses first and second game)

Validation of the solution:

Condition: **$P(X=1)+P(X=2)+P(X=3)=1$** has to be hold.

$\mathbf{L}=p+(1-p)\cdot p+(1-p)^2=p+p-p^2+1-2\cdot p+p^2=1=\mathbf{R}$

The case - unlimited number of games

$S_x=\{1,2,3,4,5,\ldots.\}$

$P(X=1)=p$

$P(X=2)=(1-p)\cdot p$

$P(X=3)=(1-p)^2\cdot p$

$P(X=4)=(1-p)^3\cdot p$

$P(X=5)=(1-p)^4\cdot p$

.....

Probability mass function for random variable X:

$P(X=k)=(1-p)^k\cdot p$, for k=1,2,3,4,…

Validation:

$P(X=1)+P(X=2)+P(X=3)+P(X=4)+\ldots=p+(1-p)\cdot p+(1-p)^2\cdot p+(1-p)^3\cdot p+(1-p)^4\cdot p+\ldots=$

$=p\cdot(1+(1-p)+(1-p)^2+(1-p)^3+(1-p)^4+\ldots.)=p\cdot\frac{1}{1-(1-p)} = 1$