1. Suppose you flip a fair dice 300 times and you let X equal the number of result "six". What's the probability that X is greater than 60?

X – random variable, Bernoulli (binomial) probability distribution.

P(X>60)=1-P(X<=60)=1-P(X<61)=1-$P\left(\frac{X-300\cdot\frac{1}{6}}{\sqrt{300\cdot\frac{1}{6}(1-\frac{1}{6})}}<\frac{61-300\cdot\frac{1}{6}}{\sqrt{300\cdot\frac{1}{6}(1-\frac{1}{6})}}\right)\overset{CLT}{\approx}1-\Phi\left(\frac{61-300\cdot\frac{1}{6}}{\sqrt{300\cdot\frac{1}{6}(1-\frac{1}{6})}}\right)=$

=1- Φ(1,7)=1-0,96=**0,04**.

2. Sixty two percent of six years old children attend preliminary school in a particular urban school district. If a sample of 500 six years old children are selected, find the probability that at least 300 are actually enrolled in preliminary school. Additionally find the shortest interval, which consists 99% of results from the sample (in term of number children actually enrolled in preliminary school).

X – number of 6 years old children enrolled in preliminary school out of 500 children (the sample).

Support of random variable X, S_x={0,1,2,3,…,500}.

Random variable X has Bernoulli distribution with parameters p=0,62 (probability of success), n=500 (number of trials). To compute probability of the event {X>=300} normal approximation of binomial probability distribution is going to be applied.

P(X>=300)=1-P(X<300)=1-$P\left(\frac{X-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}<\frac{300-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\right)\overset{CLT}{\approx}$ 1-
$\Phi\left(\frac{300-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\right)$=

=1-Φ(-0,92)=1-(1- Φ(0,92))= Φ(0,92)=**0,82**

Searching of interval which consists 99% or results (number of children enrolled preliminary school in age of 6).

The interval has end *a* and *b*. Unknown parameters *a* and *b* hold condition

P(a<=X<b)=0,99

First inequality is weak type inequality because of a need to adjust the type of inequality to the definition of cumulative distribution function (Let F denote CDF, then P(*a*<=X<*b*)=F(*b*)-F(*a*)).

The task is to find values of both parameters *a* and *b*.

The computation starts from the left side of above equation.

P(a<=X<b)= $P\left(\frac{a-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\leq\frac{X-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}<\frac{b-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\right)\overset{CLT}{\approx}$

=$\Phi\left(\frac{b-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\right)$- $\Phi\left(\frac{a-500\cdot0,62}{\sqrt{500\cdot0,62\cdot(1-0,62)}}\right)$.

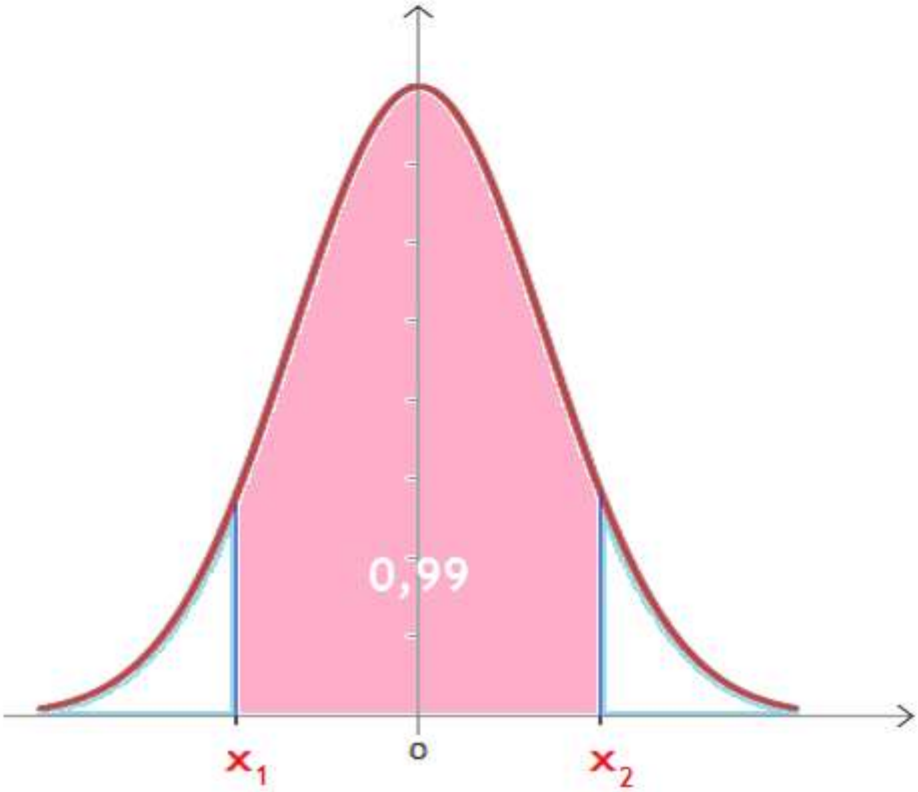
Let x_1, x_2 denote ends of searched interval after standardization.

$x_2 = \frac{b-500\cdot 0{,}62}{\sqrt{500\cdot 0{,}62\cdot(1-0{,}62)}}$, $x_1 = \frac{b-500\cdot 0{,}62}{\sqrt{500\cdot 0{,}62\cdot(1-0{,}62)}}$,

In that notation the equation has a form

$\Phi(x_2)$- $\Phi(x_1)$ ≈0,99

The shortest interval (the smallest difference $x_2 - x_1$) occurs then $|x_2|$=$|x_1|$ because of



symmetry of density curve. The condition $|x_2|$=$|x_1|$ means that $x_1 = -x_2$. Let denote $x = x_2$.

Then starting equation has a form

$\Phi(x) - \Phi(-x)$ ≈0,99.

It's possible to substitute $\Phi(-x) = 1 - \Phi(x)$ on the left side of above equation.

$\Phi(x) -$ (1− $\Phi(x)$)=0,99

Then

$\Phi(x) -$ 1+ $\Phi(x)$=0,99

And finally

$2\cdot \Phi(x) = 1{,}99$

$\Phi(x) = 0{,}995$

$x = \Phi^{-1}(0{,}995)$

$x = 2{,}58$ (because of $\Phi(2{,}58)\approx 0{,}995$)

Going back to the starting equation we have

$\Phi(2{,}58)-\Phi(-2{,}58)\approx 0{,}99$

It means that lower limit of normalized interval fulfils the condition:

$$\frac{a - 500 \cdot 0{,}62}{\sqrt{500 \cdot 0{,}62 \cdot (1 - 0{,}62)}} = -2{,}58$$

Thus

$$a = 500 \cdot 0{,}62 - 2{,}58\sqrt{500 \cdot 0{,}62 \cdot (1 - 0{,}62)}$$

Similar attempt to upper limit gives equation

$$\frac{b - 500 \cdot 0{,}62}{\sqrt{500 \cdot 0{,}62 \cdot (1 - 0{,}62)}} = 2{,}58$$

quite easy to solve

$$b = 500 \cdot 0{,}62 + 2{,}58\sqrt{500 \cdot 0{,}62 \cdot (1 - 0{,}62)}.$$

After obvious computation *a*=282,04 and *b*=337,96. Hence the interval may be written in two forms (open or closed interval):

$P(282<X<338)=P(283\leq X\leq 337)\approx 0{,}99$.

3. Suppose that probability of win "jackpot" in some game is equal to 1/13 983 816. The player takes part in the game n times. Find the lowest value of n which let to win "jackpot" at least once with probability greater than 99%.

Let S_n denotes number of jackpots in n games (n is unknown) (S_n has Bernoulli probability distribution with parameters n – number of games and $p=\frac{1}{13983816}$ – probability of success)

The condition described in the content of the 3rd task has a form

$P(S_n\geq 1)=0{,}99$.

Left side of above equation may be transformed by applying of Central Limit Theorem.

$$P(S_n\geq 1)=1-P(S_n<1)=1-P\left(\frac{S_n-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}}<\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}}\right)\overset{CLT}{\approx}1-\Phi\left(\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}}\right).$$

Then our equation

$$1-\Phi\left(\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}}\right)=0{,}99$$

let us to extract n.

$\Phi\left(\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}}\right)=0{,}01.$

$\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}} = \Phi^{-1}(0{,}01)$

$\frac{1-n\cdot p}{\sqrt{n\cdot p\cdot(1-p)}} \approx -2{,}33$ (because $\Phi(-2{,}33) \approx 0{,}01, \Phi(2{,}33) \approx 0{,}99$)

$1 - n \cdot p = -2{,}33 \cdot \sqrt{n \cdot p \cdot (1-p)}$

$-n \cdot p + 2{,}33 \cdot \sqrt{n \cdot p \cdot (1-p)} + 1 = 0$

Then after substitution $t = \sqrt{n}$ we have regular quadratic equation

$-pt^2 + 2{,}33 \cdot \sqrt{p \cdot (1-p)} \cdot t + 1 = 0$ (p has fixed value, is not parameter)

Above equation has two solutions:

$t_1 = 10085{,}84$ and $t_2 = -1386{,}48$. Solution $t_2 = -1386{,}48$ has to be rejected because of condition $t = \sqrt{n}$.

Then finally

$$n = 10085{,}84^2 = 101\ 724\ 236{,}9$$

The final answer is that there is a need to be in game at least 101 724 237 times in order to have at least one jackpot with probability at least 99%.

4. *In a survey of a IT companies in Wrocław, mean salary of employees is 4275,00 zł per month with standard deviation (S) 2120 złoty. Consider the sample of 100 employees and find the probability their mean salary will be less than 4000,00 złoty. Find the interval for mean of salary which consists of the unknown value of salary with probability 95%.*

Let X_i denote the salary of i^{th} employee. We have the sample of 100 employees.

$X_1, X_2, X_3, \dots, X_{100}$ – salaries of employees from the sample.

Let $\bar{X}_n$ denote mean salary.

$$\bar{X}_n = \frac{X_1 + X_2 + X_3 + \cdots + X_{100}}{100}$$

Mean salary considered as random variable has expected value

$$E\bar{X}_n = 4275$$

because of

$$E\bar{X}_n = \frac{1}{100}E(X_1 + X_2 + X_3 + \cdots + X_{100}) = \frac{1}{100}(EX_1 + EX_2 + EX_3 + \cdots + EX_{100}) =$$

$\frac{1}{100}100 \cdot EX_1 = EX_1$

(all expected values for random variables $X_1, X_2, X_3, \ldots, X_{100}$ are the same, instead of sum of 100 components it is possible to take 100·*one component*)

For variance there is similar condition

$$Var\bar{X}_n = \frac{1}{100^2}Var(X_1 + X_2 + X_3 + \cdots + X_{100}) = \frac{1}{100^2}(VarX_1 + VarX_2 + VarX_3 + \cdots + VarX_{100})$$

$= \frac{1}{100^2}100 \cdot VarX_1 = \frac{1}{100}VarEX_1$

(For variance we have $Var(a \cdot X) = a^2 \cdot Var(X)$)

Hence standard deviation for mean salary is σ=$\frac{2120}{10}$.

$P(\bar{X}_n < 4000) = P\left(\frac{\bar{X}_n - 4275}{212} < \frac{4000-4275}{212}\right) \overset{CLT}{\approx} \Phi\left(\frac{4000-4275}{212}\right)$= $\Phi(-1{,}3)$=0,0973

Regarding the interval the procedure is similar to the procedure presented in solution of the 2nd task.

Mean salary has continuous probability distribution, it means that type of inequality doesn't have any impact on the results, so we can start from open interval

$$P(a < \bar{X}_n < b) = 0{,}95.$$

To find limits of our interval it will be necessary to compute $\Phi^{-1}(0{,}975)$=1,96.

Using the same procedure (task 2) we have limits for mean salary in the sample of 100 employees given by

$a = 4275 - 1{,}96 \cdot 212$ and $b = 4275 + 1{,}96 \cdot 212$.