Problem 5.

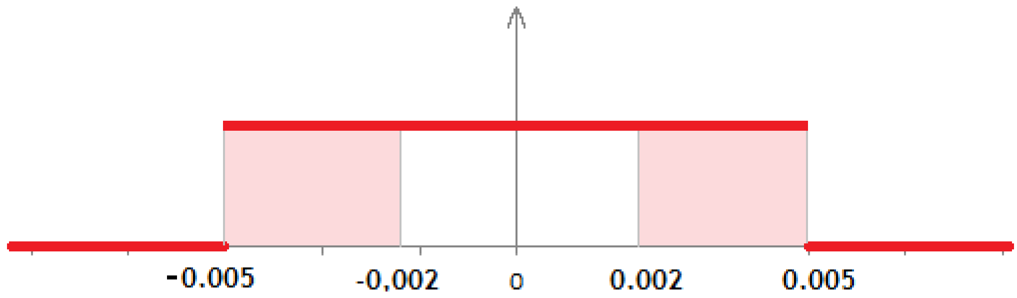
Suppose a calculator calculates to k decimal places. The rounding error involved in a calculation may be assumed to be uniform on the proper interval (depended on number of decimal places). Calculate probability that error is bigger than 0,002 for two-decimal places calculation, bigger than 0,0003 for three-decimal places calculation.

General form of uniform probability density function:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & x \in (a,b) \\ 0 & in\ other\ cases \end{cases}$$

Case 1. Two decimal places.

Let X denote rounding error (considered as difference between exact result and output of calculator). Then support of random variable X is given by S_X=(-0,005;0,005) (a=-0,005 b=0,005)



Probability density of random variable X:

$$f(x) = \begin{cases} \dfrac{1}{0{,}005-(-0{,}005)} & x \in (-0{,}005; 0{,}005) \\ 0 & in\ other\ cases \end{cases}$$

Then

$$f(x) = \begin{cases} 100 & x \in (-0{,}005; 0{,}005) \\ 0 & in\ other\ cases \end{cases}$$

To calculate probability that rounding error is bigger than 0,002 is enough to apply above form of density function.

P(|X|>0,002)=1-P(|X|<=0,002)=1-P(-0,002<=X<=0,002)=1-$\int_{-0{,}002}^{0{,}002} f(x)dx$=

=$1 - \int_{-0{,}002}^{0{,}002} 100dx = 1 - 100\int_{-0{,}002}^{0{,}002} 1dx = 1 - 100x|_{-0{,}002}^{0{,}002}$=

=1-100·(0,002-(-0,002))=1-100·0,004=1-0,4=0,6

Of course definite integral may be substituted by formula for a rectangle area. Then

$P(|X|>0{,}002)=2\cdot 100\cdot 0{,}003=0{,}6$

Problem 6.

The average number of calls coming into a call centre is 5 per minute. Calculate the probability that the time between two calls is greater than k minutes. Additionally calculate value of parameter t, where t is the time such that the length of time between two calls is less than t with probability 90%.

Let X denote the random variable - number of incoming calls per 1 minute (Poisson distribution) with parameter λ=5.

$$P(X=m)=\frac{5^m}{m!}\cdot e^{-5},\ m=1,2,3,4,5,6....$$

$$P(X=0)=\frac{5^0}{0!}\cdot e^{-5}=e^{-5}$$

For the period of k minutes the average number of calls per k minutes is 5·k.

Let Y denote random variable - number of incoming calls per k minutes. Y has Poisson probability distribution with parameter λ=5·k.

$$P(Y=0)=\frac{(5k)^0}{0!}\cdot e^{-5k}=e^{-5k}$$

Let T denote time between two consecutive calls (in minutes). Definitely

$$P(T>k)\geq P(Y=0)=e^{-5k}$$

The **exponential distribution** may be used to model the time between the arrival of signals, λ is the rate at which signals arrive as well.

Random variable T has exponential probability distribution with parameter λ=5. General form of exponential probability density function.

$$f(x)=\begin{cases}\lambda\cdot e^{-\lambda x} & x\in(0,+\infty)\\ 0 & in\ other\ cases\end{cases}$$

P(T>k)=$\int_k^{+\infty} f(x)dx = \int_k^{+\infty} 5e^{-5x}dx = -e^{-5x}\Big|_k^{+\infty}$= $\lim_{x\to+\infty} e^{-5x} - \left(-e^{5k}\right) = 0 + e^{-5k}$

Let t denote the time such that the length of time between two calls is less than t with probability 90%. It means that t fulfils following condition.

P(T<t)=0,90

On the left side of above equation there is CDF for random variable T (by definition).

P(T<t)=F(t) (where F(t) – CDF for random variable T)

Cumulative distribution function for exponential probability distribution with parameter λ=5 has a form:

$$F(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ -e^{-5x} + 1 & x \in (0, +\infty) \end{cases}$$

The value of CDF for point t (t>0)

F(t)= P(T<t)=$-e^{-5t} + 1$

Then the equation P(T<t)=0,90 has the form:

$$-e^{-5t} + 1 = 0{,}9$$

$$-e^{-5t} = -0{,}1$$

$$e^{-5t} = 0{,}1$$

$$-5t = \ln(0{,}1)$$

$$t = -\frac{1}{5}\ln(0{,}1)$$

$$\boldsymbol{t = 0{,}46\ \mathbf{minutes}}$$

The result in seconds

t=27,6 seconds.

PROBLEM 7.

The lifetime T (years) of an electronic component is a continuous random variable with a probability density function given by exponential distribution with λ=1, Find the lifetime L which a typical component is 60% certain to exceed. If five components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years.

Random variable T has exponential probability distribution with parameter λ. General form of exponential probability density function.

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \in (0, +\infty) \\ 0 & in\ other\ cases \end{cases}$$

Cumulative distribution function for exponential distribution

$$F(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ -e^{-\lambda x} + 1 & x \in (0, +\infty) \end{cases}$$

1-F(L)=0,6

F(L)=0,4

$$-e^{-L} + 1 = 0{,}4$$

$$-e^{-L} = -0{,}6$$

$$e^{-L} = 0{,}6$$

$$\ln(e^{-L}) = \ln(0{,}6)$$

$$-L = \ln(0{,}6)$$

$$L = -\ln(0{,}6)$$

$L = 0{,}51$ years

60% of components will work longer than 0,51 of the year

P(T<=L)=0,4

P(T>L)=0,6

T_1,T_2,T_3,T_4,T_5 – lifetimes of 5 components

P(at least one the component will work shorter than L)=

$=P(\text{minimum}\{T_1,T_2,T_3,T_4,T_5\}<L)=1-P(T_1>=L,T_2>=L,T_3>=L,T_4>=L,T_5>=L)=$

$=1-P(T_1>=L)\cdot P(T_2>=L)\cdot P(T_3>=L)\cdot P(T_4>=L)\cdot P(T_5>=L)=1-(0{,}6)^5$

PROBLEM 8.

The height of male students is normal with a mean of 180cm and variance of 169cm^2.

a) *what is the probability that a randomly picked male student is taller than 195cm,*
b) *what is the probability that height a randomly picked male student is between 175cm and 190cm,*
c) *calculate height of 10% and 25% of male students who are shorter than that height.*

a) X – denotes the height of randomly picked up student

P(X>195)=$P\left(\frac{X-m}{\sigma} > \frac{195-180}{13}\right) = 1 - P\left(\frac{X-m}{\sigma} \leq \frac{195-180}{13}\right) =$

$$= 1 - \Phi\left(\frac{195-180}{13}\right) = 1 - \Phi(1{,}15) = 1 - 0{,}874 = 0{,}126$$

b) P(175<X<190)=$P\left(\frac{190-180}{13} < \frac{X-180}{13} < \frac{175-180}{13}\right) = \Phi\left(\frac{190-180}{13}\right) - \Phi\left(\frac{175-180}{13}\right) =$

$$\Phi(0{,}769) - \Phi(-0{,}385) = \Phi(0{,}769) - \left(1 - \Phi(0{,}385)\right) = 0{,}779 - (1 - 0{,}649)$$

c) part of solution for 10% of the shortest male students

P(X<a)=0,1

Left side:

$$P(X < a) = P\left(\frac{X-180}{13} < \frac{a-180}{13}\right) = \Phi\left(\frac{a-180}{13}\right)$$

Going back to equation we have

$$\Phi\left(\frac{a-180}{13}\right) = 0{,}1$$

$$\frac{a-180}{13} = \Phi^{-1}(0{,}1)$$

$$\frac{a-180}{13} = -1{,}28$$

And finally:

$$a = 180 - 13 \cdot 1{,}28$$

10% of male students are shorter than 163,34 cm.

Part of solution for 25% of shortest male students:

P(X<a)=0,25

Left side:

$$P(X < a) = P\left(\frac{X-180}{13} < \frac{a-180}{13}\right) = \Phi\left(\frac{a-180}{13}\right)$$

Going back to equation we have

$$\Phi\left(\frac{a-180}{13}\right) = 0{,}25$$

$$\frac{a-180}{13} = \Phi^{-1}(0{,}25)$$

$$\frac{a-180}{13} = -0{,}674$$

And finally:

$$a = 180 - 13 \cdot 0{,}674$$

25% of male students are shorter than 171,23 cm.

Technical information about computation of values $\Phi(x)$ and $\Phi^{-1}(p)$ in Excel.

Syntax:

=NORM.DIST(x,mean,standard_dev,cumulative)

The NORM.DIST function syntax has the following arguments:

- **x** Required. The value for which you want the distribution.
- **Mean** Required. The arithmetic mean of the distribution.
- **Standard_dev** Required. The standard deviation of the distribution.
- **Cumulative** Required. A logical value that determines the form of the function. If cumulative is TRUE, NORM.DIST returns the cumulative distribution function; if FALSE, it returns the probability mass function.

$\Phi(x) =$ **NORM.DIST(x,0,1,1)**

Syntax

=NORM.INV (probability, mean, standard_dev)

Arguments

- **probability** - The probability of an event occurring below a threshold.
- **mean** - The mean of the distribution.
- **standard_dev** - The standard deviation of the distribution.

The NORM.INV function returns the inverse of the normal cumulative distribution. Given the probability of an event occurring below a threshold value, the function returns the threshold value associated with the probability. For example, NORM.INV(0.5, 3, 2) returns 3 since the probability of an event occurring below the mean of the distribution is 0.5. Note, the area under a normal distribution within an interval corresponds to the probability of an event occurring within that interval.

$\boldsymbol{\Phi^{-1}(p)}$ **=NORM.INV (p,0,1)**