Improper Integrals

Exercise 1. Using the definition evaluate the improper integral of Type 1.

a) $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{x^2-x}$; b) $\displaystyle\int_4^\infty \frac{\sqrt{x}\,\mathrm{d}x}{x+1}$; c) $\displaystyle\int_{2\pi}^\infty x\cos\ \mathrm{d}x$; d) $\displaystyle\int_0^\infty \frac{e^{-x}\,\mathrm{d}x}{\sqrt{e^{-x}+1}}$

Exercise 2. Use the Comparison Theorem to determine whether the improper integral of Type 1 is convergent or divergent.

a) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x(\sqrt{x}+1)}$; b) $\displaystyle\int_1^\infty \frac{x(x+1)\,\mathrm{d}x}{x^4+x+1}$; c) $\displaystyle\int_0^\infty \frac{(2^x+1)\,\mathrm{d}x}{3^x+1}$; d) $\displaystyle\int_\pi^\infty \frac{(x+\sin x)\,\mathrm{d}x}{x^3}$; e) $\displaystyle\int_4^\infty \frac{(3+\cos x)\,\mathrm{d}x}{\sqrt{x}+2}$.

Exercise 3. Use the Limit Comparison Test to determine whether the improper integral of Type 1 is convergent or divergent.

a) $\displaystyle\int_1^\infty \frac{(\sqrt{x}+1)\,\mathrm{d}x}{x(x+1)}$; b) $\displaystyle\int_5^\infty \frac{x^2\,\mathrm{d}x}{\sqrt{x^5-3}}$; c) $\displaystyle\int_2^\infty (e^{1/x}-1)\,\mathrm{d}x$; d) $\displaystyle\int_1^\infty \sin^2\frac{1}{x}\,\mathrm{d}x$; e) $\displaystyle\int_1^\infty \frac{x^2\,\mathrm{d}x}{x^3-\sin x}$.

Exercise 4. Using the definition evaluate the improper integral of Type 2.

a) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}(x+1)}$; b) $\displaystyle\int_0^e \frac{\ln x\,\mathrm{d}x}{x}$; c) $\displaystyle\int_{\frac{\pi}{2}}^\pi \frac{\mathrm{d}x}{\sin x}$; d) $\displaystyle\int_3^5 \frac{2^x\,\mathrm{d}x}{\sqrt{2^x-8}}$; e) $\displaystyle\int_{-1}^0 \frac{dx}{x(x+1)}$.

Exercise 5. Use the Comparison Theorem to determine whether the improper integral of Type 2 is convergent or divergent.

a) $\displaystyle\int_0^4 \frac{\arctan x\,\mathrm{d}x}{x\sqrt{x}}$; b) $\displaystyle\int_0^2 \frac{e^x\,\mathrm{d}x}{x^3}$; c) $\displaystyle\int_0^4 \frac{\mathrm{d}x}{x^2+\sqrt{x}}$.

Exercise 6. Use the Limit Comparison Test to determine whether the improper integral of Type 2 is convergent or divergent.

a) $\displaystyle\int_0^1 \frac{(x^3+1)\,\mathrm{d}x}{\sqrt{x}(x^2+1)}$; b) $\displaystyle\int_0^\pi \frac{\sin^3 x\,\mathrm{d}x}{x^4}$; c) $\displaystyle\int_0^1 \frac{(e^x-1)\,\mathrm{d}x}{\sqrt{x^3}}$.

Exercise 7. Find the Cauchy principal value of the integral.

a) $\displaystyle\int_{-\infty}^\infty \frac{x^3\cos x\,\mathrm{d}x}{x^2+4}$; b) $\displaystyle\int_{-\infty}^\infty \frac{e^x\,\mathrm{d}x}{e^x+1}$; c) $\displaystyle\int_{-\infty}^\infty e^{-|x+5|}\,\mathrm{d}x$.

Infinite series

Exercise 8. Find the sum of the series.

a) $\displaystyle\sum_{n=0}^\infty \left(\frac{5}{6}\right)^n$; b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+3n+2}$; c) $\displaystyle\sum_{n=2}^\infty \frac{n-1}{n!}$; d) $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Exercise 9. Use the Integral Test to determine whether the series is convergent or divergent.

a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+9}$; b) $\displaystyle\sum_{n=2}^\infty \frac{n-1}{n^2+n}$; c) $\displaystyle\sum_{n=2}^\infty \frac{\ln n}{n^2}$; d) $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n+1}}$; e) $\displaystyle\sum_{n=0}^\infty \frac{e^n}{e^{2n}+1}$.

[i] Exercises taken from the books *Analiza matematyczna 2 (Definicje, twierdzenia, wzory; Przykłady i zadania; Kolokwia i egzaminy)* and *Równania różniczkowe zwyczajne. Teoria, przykłady, zadania* by M. Gewert, Z. Skoczylas

Exercise 10. Use the Comparison Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2+2}$; c) $\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n}$; d) $\sum_{n=0}^{\infty} \frac{2^n+e^n}{e^n+4^n}$.

Exercise 11. Use the Limit Comparison Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{n^2+2}{\sqrt{2n^6-1}}$; b) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+1}$; c) $\sum_{n=1}^{\infty} \frac{e^n-1}{3^n-1}$; d) $\sum_{n=0}^{\infty} 4^n \ln(1+3^{-n})$; e) $\sum_{n=1}^{\infty} \frac{\sin(\pi/n)}{\sin(\pi/n^2)}$.

Exercise 12. Use the Ratio Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{2016^n}{(2n)!}$; b) $\sum_{n=1}^{\infty} \frac{5^n+1}{n^4+1}$; c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; d) $\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

Exercise 13. Use the Root Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$; b) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+5^n}$; c) $\sum_{n=1}^{\infty} \frac{3^n n^{n^2}}{(n+1)^{n^2}}$; d) $\sum_{n=1}^{\infty} \arctan\frac{n}{n+1})^n$.

Exercise 14. Using the appropriate series convergence test find the limits.

a) $\lim_{n\to\infty} \frac{n^{2020}}{3^n}$; b) $\lim_{n\to\infty} \frac{n^n}{(n!)^2}$; c) $\lim_{n\to\infty} \frac{n^n}{n!}$.

Exercise 15. Use the Alternating Series Test to determine whether the series is convergent or divergent.

a) $\sum_{n=0}^{\infty} (-1)^n(\sqrt{n^2+1}-n)$; b) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n+4^n}$; c) $\sum_{n=4}^{\infty} \sin\frac{(-1)^n}{n}$; d) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3^n}{n!}$.

Exercise 16. Find the radius of convergence and interval of convergence of the series.

a) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{ne^n}$; b) $\sum_{n=0}^{\infty} (4x-12)^n$; c) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$; d) $\sum_{n=0}^{\infty} \frac{(2x+6)^n}{3^n-2^n}$.

Exercise 17. Find the Maclaurin series of the function and and its radius of convergence.

a) $f(x) = \frac{4}{1-3x}$; b) $f(x) = \sin\frac{x}{4}$; c) $f(x) = x^2 e^{-x}$; d) $f(x) = \frac{x^3}{16+x^2}$; e) $f(x) = \sin^2 x$.

Exercise 18. Using the Differentiation Theorem and the Integration Theorem show that for $x \in (-1,1)$ we have

a) $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$; b) $\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$; c) $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$.

Exercise 19. Find the sum of the series.

a) $\sum_{n=0}^{\infty} \frac{1}{(n+1)3^n}$; b) $\sum_{n=2}^{\infty} \frac{2n-1}{2^n}$; c) $\sum_{n=1}^{\infty} \frac{n(n+1)}{5^n}$; d) $\sum_{n=1}^{\infty} \frac{n}{(n+1)4^n}$.

Functions of Two Variables

Exercise 20. Find and sketch the domain of the function.

$$\text{a) } f(x,y)=\ln(y-\sin x);\ \text{b) } f(x,y)=\sqrt{\frac{y-2}{x+1}};\ \text{c) } f(x,y)=\frac{x^2y}{\sqrt{x^2-y}};\ \text{d) } f(x,y)=\ln\frac{x^2+y^2-9}{16-x^2-y^2}.$$

Partial Derivatives

Exercise 21. Find the first partial derivatives of the function.

a) $f(x,y)=\dfrac{x^2+y^3}{xy^2}$; b) $f(x,y)=\arctan\dfrac{1-xy}{x+y}$; c) $f(x,y)=e^{\cos\frac{x}{y}}$;

d) $f(x,y)=y\sqrt{x^2+y^2}$; e) $f(x,y)=\ln(\sqrt{x^2+y^2}-x)$.

Exercise 22. Verify that the function is a solution of the given equation

a) $z=f(x^2+y^2)$, $yz_x-xz_y=0$;

b) $z=xf(\sin(x-y))$, $z_x+z_y=\dfrac{z}{x}$;

c) $z=x^n f\left(\dfrac{y}{x}\right)$, $xz_x+yz_y=nz$, $(n\in\mathbb{N})$.

Exercise 23. Find an equation of the tangent plane to the given surface at the specified point

a) $z=x^2\sqrt{y+1}$, $(1,3,z_0)$;

b) $z=e^{x+2y}$, $(2,-1,z_0)$;

c) $z=\dfrac{\arcsin x}{\arccos y}$, $\left(-\dfrac{1}{2},\dfrac{\sqrt{3}}{2},z_0\right)$;

d) $z=(2+x-3y)^4$, point of intersection with z-axis;

e) $z=e^{x+y}-e^{4-y}$, point of intersection with x-axis.

Exercise 24. At what points on the surface $z=\arctan\frac{x}{y}$ are the tangent planes parallel to $x+y-z=5$?

Exercise 25. At what points on the surface $z=x^2+y^2$ are the tangent planes perpendicular to the line $x=t, y=t, z=2t, t\in\mathbb{R}$?

Exercise 26. Using the definition find the directional derivative of f at the given point in the direction of a vector v

a) $f(x,\ y)=\sqrt{x^2+y^2}, (x_0,\ y_0)=(0,0)$, $v=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$;

b) $f(x,\ y)=\sqrt[3]{xy}, (x_0,\ y_0)=(1,0)$, $v=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

Exercise 27. Find the gradient vector of the function at the given point

a) $f(x,\ y)=x^3+xy^2+2, (1,\ -2)$; b) $f(x,\ y)=\ln(x+\ln y), (e,\ 1)$;

c) $f(x,\ y)=(1+xy)^y, (0,0)$; d) $g(x,\ y)=x\sqrt{y}-e^x\ln y, (0,1)$.

Exercise 28. Find the directional derivative of the function $f(x,y)=y-x^2+2\ln(xy)$ at the point $\left(-\frac{1}{2},-1\right)$ in the direction of the vector v forming the angle α with the positive x-axis. At what angle the directional derivative is equal to 0?

Exercise 29. Find unit vectors v such that the directional derivative of the function $f(x,y)=\sqrt{e^x}(x+y^2)$ at the point $(0,2)$ is equal to 0.

Maximum and Minimum Values

Exercise 30. Find the local maximum and minimum values

a) $f(x,y)=x^3+3xy^2-51x-24y$; b) $f(x,y)=xe^{-y}+\dfrac{1}{x}+e^y$; c) $f(x,y)=xy^2(12-x-y),(x,\ y>0)$

d) $f(x,y)=y\sqrt{x}-y^2-x+6y$; e) $f(x,y)=x^3+y^3-3xy$; f) $f(x,y)=\dfrac{8}{x}+\dfrac{x}{y}+y,(x,\ y>0)$;

g) $f(x,y)=xy+\ln y+x^2$; h) $f(x,y)=e^{x-2y}+e^{y-x}+e^{6+y}$; i) $f(x,y)=e^{x^2-y}(5-2x+y)$.

Exercise 31. Find the maximum and minimum values of the function subject to the given constraint(s)

a) $f(x,y)=x^2+y^2, 3x+2y=6$; b) $f(x,y)=x^2+y^2-8x+10, x-y^2+1=0$;
c) $f(x,y)=x^2y+\ln x, 8x+3y=0$; d) $f(x,y)=2x+3y, x^2+y^2=1$.

Exercise 32. Find the absolute maximum and minimum values of the function on the given set D

a) $f(x,y)=2x^3+4x^2+y^2-2xy, D=\{(x,y)\in\mathbb{R}^2 : x^2\leqslant y\leqslant 4\}$;
b) $f(x,\ y)=\sqrt{y-x^2}+\sqrt{x-y^2}$;
c) $f(x,\ y)=\sqrt{1-x^2}+\sqrt{4-(x^2+y^2)}$;
d) $f(x,\ y)=x^2-y^2, D$ – a triangle with vertices $(0,1),(0,2),(1,2)$;
e) $f(x,\ y)=x^4+y^4, D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 9\}$.

Exercise 33. In the triangle with vertices $A=(-1,5), B=(1,4), C=(2,-3)$ find the point $M=(x_0,y_0)$, for which the sum of the squares of its distances from the vertices is the smallest.
Exercise 34. A metal open tube is to have a volume of 32000 cm^3. Find the dimensions that minimize the amount of metal used.
Exercise 35. Find the distance between lines

$$k:\begin{cases} x+y-1=0,\\ z+1=0,\end{cases}\qquad l:\begin{cases} x-y+3=0,\\ z-2\ =0.\end{cases}$$

Exercise 36. Find the point on the parabola $y=\frac{x^2}{2}$ that is closest to the point $P=(4,1)$.
Exercise 37. The rectangular warehouse is to have a volume of $V=216\ \text{m}^3$. For the walls of the warehouse, slabs are used at a price of 30 PLN/m^2, for the construction of a floor - at 40 PLN/m^2, and the ceiling - at 20 PLN/m^2. Find the dimensions of the warehouse that minimize the cost of the materials.

Double Integrals

Exercise 38. Evaluate the double integral.

a) $\displaystyle\iint_R (x+xy-x^2-2y)\ \mathrm{d}x\ \mathrm{d}y, R=[0,1]\times[0,1]$, b) $\displaystyle\iint_R \frac{x\ \mathrm{d}x\ \mathrm{d}y}{y^2}, R=[1,2]\times[2,4])$,
c) $\displaystyle\iint_R (x\sin(xy))\ \mathrm{d}x\ \mathrm{d}y, R=[0,1]\times[\pi,\ 2\pi]$, d) $\displaystyle\iint_R e^{2x-y}\ \mathrm{d}x\ \mathrm{d}y$.

Exercise 39. Transform the integral $\iint\limits_D f(x,y)\ \mathrm{d}x\ \mathrm{d}y$ to the iterated integrals if D is bounded by

a) $y=x^2, y=x+2$ b) $x^2+y^2=4, y=2x-x^2, x=0\ (x,\ y\geqslant 0)$,
c) $x^2-4x+y^2+6y-51=0$, d) $x^2-y^2=1, x^2+y^2=3\ (x<0)$.

Exercise 40. Sketch the region of integration and change the order of integration.

a) $\displaystyle\int_{-1}^{1}\int_{-1}^{|x|} f(x,\ y)\ \mathrm{d}y\ \mathrm{d}x$, b) $\displaystyle\int_{-1}^{1} dx\int_{-\sqrt{2-x^2}}^{0} f(x,\ y)\ \mathrm{d}y\ \mathrm{d}x$,

c) $\displaystyle\int_{0}^{4}\int_{\sqrt{4x-x^2}}^{2\sqrt{x}} f(x,\ y)\ \mathrm{d}y\ \mathrm{d}x$, d) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2-1}^{\frac{y^2}{2}} f(x,\ y)\ \mathrm{d}x\ \mathrm{d}y.$

Exercise 41. Evaluate the given integral by changing to polar coordinates

a) $\displaystyle\iint_D xy\ \mathrm{d}x\ \mathrm{d}y, D: x^2+y^2\leqslant 1, \frac{x}{\sqrt{3}}\leqslant y\leqslant\sqrt{3}x$, b) $\displaystyle\iint_D xy^2\ \mathrm{d}x\ \mathrm{d}y, D: x\geqslant 0, 1\leqslant x^2+y^2\leqslant 2$,

c) $\displaystyle\iint_D y^2e^{x^2+y^2}\ \mathrm{d}x\ \mathrm{d}y, D: x\geqslant 0, y\geqslant 0, x^2+y^2\leqslant 1$, d) $\displaystyle\iint_D x^2\ \mathrm{d}x\ \mathrm{d}y, D: x^2+y^2\leqslant 2y$,

e) $\displaystyle\iint_D (x^2+y^2)\ \mathrm{d}x\ \mathrm{d}y, D: y\geqslant 0, y\leqslant x^2+y^2\leqslant x$, f) $\displaystyle\iint_D y\ \mathrm{d}x\ \mathrm{d}y, D: x^2+y^2\leqslant 2x (y\leqslant 0)$,

g) $\displaystyle\iint_D \sin(x^2+y^2)\ \mathrm{d}x\ \mathrm{d}y, D: x^2+y^2\leqslant \pi^2$, h) $\displaystyle\iint_D \ln(1+x^2+y^2)\ \mathrm{d}x\ \mathrm{d}y, D: 1\leqslant x^2+y^2\leqslant 9.$

Exercise 42. Find the volume of the given solid.

a) $z=\sqrt{25-(x^2+y^2)}, z=x^2+y^2-13$, b) $x^2+y^2+z^2=4, z=1\ (z\geqslant 1)$,

c) $x^2+y^2-2y=0, z=x^2+y^2, z=0$, d) $z=5-x^2-y^2, z=1, z=4.$

Laplace transform

Exercise 43. Determine the Laplace transform of each of the following functions.

a) $f(t)=2t-1$; b) $f(t)=\sin 2t$; c) $f(t)=t^2$; d) $f(t)=te^{-t}$; e) $f(t)=e^{2t}\cos 2t$.

Exercise 44. Determine the inverse Laplace transform of each of the following functions.

a) $F(s)=\dfrac{1}{s+2}$; b) $F(s)=\dfrac{s}{s^2+4s+5}$; c) $F(s)=\dfrac{1}{s^2-4s+3}$; d) $F(s)=\dfrac{s+2}{(s+1)(s-2)(s^2+4)}$.

Exercise 45. Solve each of the following initial-value problems.

a) $y'-y=1, y(0)=1$; b) $y'-2y=\sin t, y(0)=0$;

c) $y''+y'=0, y(0)=1, y'(0)=1$; d) $y''+3y'=e^{-3t}, y(0)=0, y'(0)=-1$;

e) $y''-2y'+2y=\sin t, y(0)=0, y'(0)=1$; f) $y''+4y+4y=t^2, y(0)=0, y'(0)=0$;

g) $y''-2y'+y=1+t, y(0)=0, y'(0)=0$; h) $y''+4y'+13y=te^{-t}, y(0)=0, y'(0)=2.$

Fourier transform

Exercise 46. Determine the Fourier transform of each of the following functions.

a) $f(t)=\begin{cases}\sin t, & |t|\leqslant\pi,\\ 0, & |t|>\pi;\end{cases}$ b) $f(t)=\begin{cases}\cos t, & |t|\leqslant\frac{\pi}{2},\\ 0, & |t|>\frac{\pi}{2};\end{cases}$ c) $f(t)=\begin{cases}t, & |t|\leqslant 1,\\ 0, & |t|>1;\end{cases}$

d) $f(t)=\begin{cases}t^2, & |t|\leqslant 1,\\ 0, & |t|>1;\end{cases}$ e) $f(t)=e^{-|t|}$.

Exercise 47. Determine the inverse Fourier transform of each of the following functions.

a) $\hat{f}(\xi)=\dfrac{1}{1+2i\xi}$; b) $\hat{f}(\xi)=\dfrac{1}{4+\xi^2}$; c) $\hat{f}(\xi)=\dfrac{e^{2i\xi}}{1+i\xi}$;

d) $\hat{f}(\xi)=\dfrac{\sin\xi\cos\xi}{2\xi}$; e) $\hat{f}(\xi)=\dfrac{1}{(1+\xi^2)(4+\xi^2)}$.