PROBLEM 3

Suppose the random variable X has distribution given by probability mass function

x	0	2	5	7	9
P(X=x)	$\frac{2}{15}$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	c

a) calculate the value of the constant c,
b) calculate the probability that $X^2-4>0$.
c) calculate expected value and standard deviation of random variable X,
d) derive the cumulative distribution function of random variable X,
e) calculate the median and mode for random variable X.

a) Support of random variable X - $S_X=\{0,2,5,7,9\}$

Condition for probability mass function (total probability)

P(X=0)+P(X=2)+P(X=5)+P(X=7)+P(X=9)=1

$$\frac{2}{15}+\frac{1}{3}+\frac{4}{15}+\frac{1}{5}+c=1$$

$$c=\frac{1}{15}$$

Examination of eligibility for parameter c, c>0 => both conditions for having probability mass function are hold.

b) Event $X^2-4>0$

$P(X^2-4>0)=P((X-2)(X+2)>0)=P(X<-2 \text{ or } X>2)=P(X<-2)+P(X>2)=0+P(X=5)+P(X=7)+P(X=9)=$

$=\frac{4}{15}+\frac{1}{5}+\frac{1}{15}=\frac{8}{15}$

d) Cumulative distribution function (CDF) of random variable X

CDF by definition: F(x)=P(X<x)

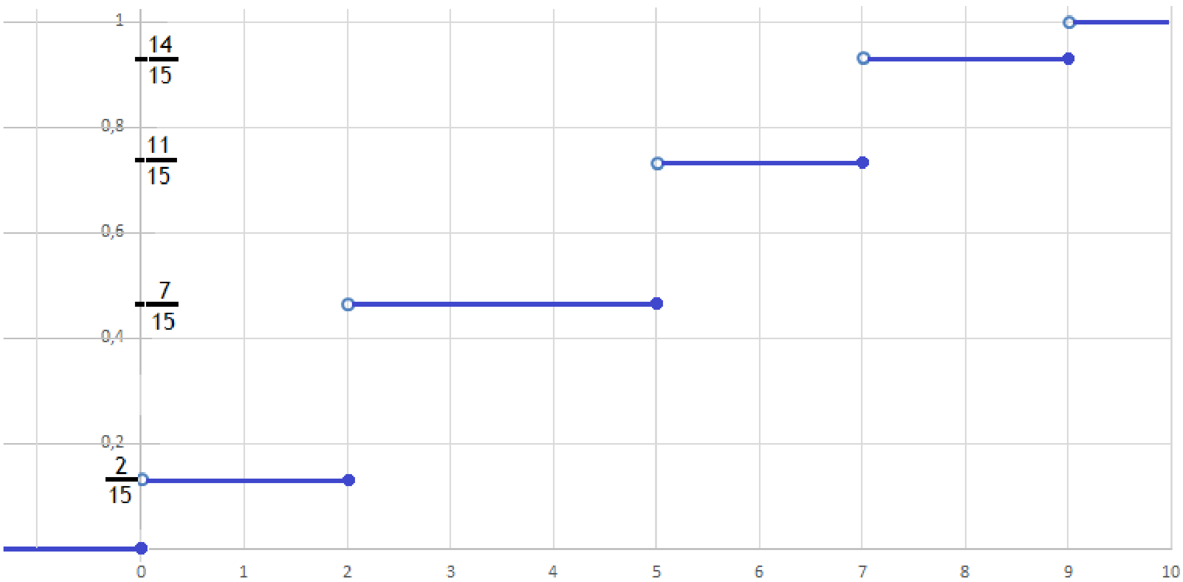
Supportive calculations:

x		0	2	5	7	9	
P(X=x)		$\frac{2}{15}$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{1}{15}$	
Intervals	-∞	0	2	5	7	9	-∞
Cumulative probabilities	0	$\frac{2}{15}$	$\frac{7}{15}=\frac{2}{15}+\frac{1}{3}$	$\frac{11}{15}=\frac{7}{15}+\frac{4}{15}$	$\frac{14}{15}=\frac{11}{15}+\frac{1}{5}$	$1=\frac{14}{15}+\frac{1}{15}$	1

Formal notation (CDF as function of variable x (x denotes real number):

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{2}{15} & 0 < x \le 2 \\ \frac{7}{15} & 2 < x \le 5 \\ \frac{11}{15} & 5 < x \le 7 \\ \frac{14}{15} & 7 < x \le 9 \\ 1 & x > 9 \end{cases}$$

PLOT of CDF:



c) expected value of random variable X

EX=0·P(X=0)+ 2·P(X=2)+ 5·P(X=5)+ 7·P(X=7)+ 9·P(X=9)= $0 \cdot \frac{2}{15} + 2 \cdot \frac{1}{3} + 5 \cdot \frac{4}{15} + 7 \cdot \frac{1}{5} + 9 \cdot \frac{1}{15} = 4$

second moment

$EX^2=0^2 \cdot P(X=0)+ 2^2 \cdot P(X=2)+ 5^2 \cdot P(X=5)+ 7^2 \cdot P(X=7)+ 9^2 \cdot P(X=9)=$

$$= 0^2 \cdot \frac{2}{15} + 2^2 \cdot \frac{1}{3} + 5^2 \cdot \frac{4}{15} + 7^2 \cdot \frac{1}{5} + 9^2 \cdot \frac{1}{15} = \frac{20 + 100 + 147 + 81}{15} = \frac{344}{15}$$

Variance of random variable X

$Var(X)=EX^2-(EX)^2=\frac{344}{15} - 4^2 \approx 6{,}93$

Standard deviation of random variable X

$\sigma=\sqrt{Var(X)} = \sqrt{6,93} \approx 2,68$

e) Median of random variable X

$$x_{\left(\frac{1}{2}\right)} = 5$$

(justification P(X≤5)=$\frac{11}{15} \geq \frac{1}{2}$ and P(X≥5)=$\frac{8}{15} \geq \frac{1}{2}$)

Mode x_{mod}=2 (P(X=2) is the biggest probability in the probability mass function).

PROBLEM 4

Suppose the random variable X has density function

$$f(x)=\begin{cases} c\cdot x & x\in\langle 2,5\rangle \\ 0 & x\notin\langle 2,5\rangle \end{cases}$$

a) calculate the value of the constant c,
b) calculate the probability that X^2-4>0.
c) calculate expected value and standard deviation of random variable X,
d) derive the cumulative distribution function of random variable X,
e) calculate the median and mode for random variable X.

a) Conditions for probability density function:

- f(x)>=0
- $\int_{-\infty}^{+\infty} f(x)dx = 1$

Calculation of definite integral:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{2} 0dx + \int_{2}^{5} cxdx + \int_{5}^{+\infty} 0dx = \int_{2}^{5} cxdx = c\frac{x^2}{2}\bigg|_{2}^{5} = c\left(\frac{5^2}{2}-\frac{2^2}{2}\right) = c\frac{21}{2}$$

leads to the simple equation:

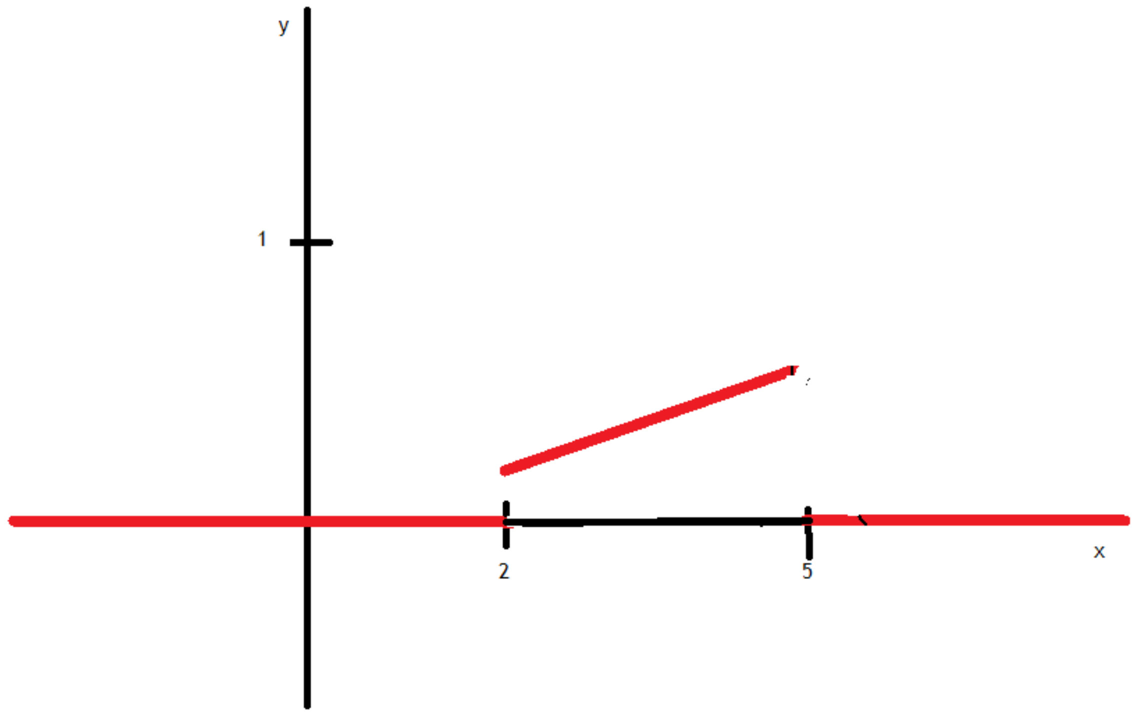
$$c\frac{21}{2} = 1$$

$$c = \frac{2}{21}$$

Then function

$$f(x) = \begin{cases} \frac{2}{21}x & x \in <2,5> \\ 0 & in\ other\ cases \end{cases}$$

fulfils both conditions for probability density function (c>0).



b) Event X^2-4>0

P(X^2-4>0)=P((X-2)(X+2)>0)=P(X<-2 or X>2)=P(X<-2)+P(X>2)= $\int_{-\infty}^{-2} 0dx + \int_{2}^{5} \frac{2}{21} x dx + \int_{5}^{+\infty} 0dx$=1

CUMULATIVE DISTRIBUTION FUNCTION FOR RANDOM VARIABLE X

$$f(x) = \begin{cases} \frac{2}{21}x & x \in< 2,5 > \\ 0 & in\ other\ cases \end{cases}$$

Definition of CDF for continuous type of probability distribution

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt$$

Calculation of CDF:

1) x<=2

$$F(x) = P(X < x) = \int_{-\infty}^{x} 0dt = 0$$

2) 2<x<5

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{2} 0dt + \int_{2}^{x} \frac{2}{21} tdt = 0 + \frac{2}{21}\int_{2}^{x} tdt =$$

$$= \frac{2}{21}\frac{t^2}{2}\Big|_{2}^{x} = \frac{1}{21}(x^2 - 2^2)$$

3) x>=5

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{2} 0dt + \int_{2}^{5} \frac{2}{21} tdt + \int_{5}^{x} 0dt = 1$$

Cumulative distribution function for random variable X

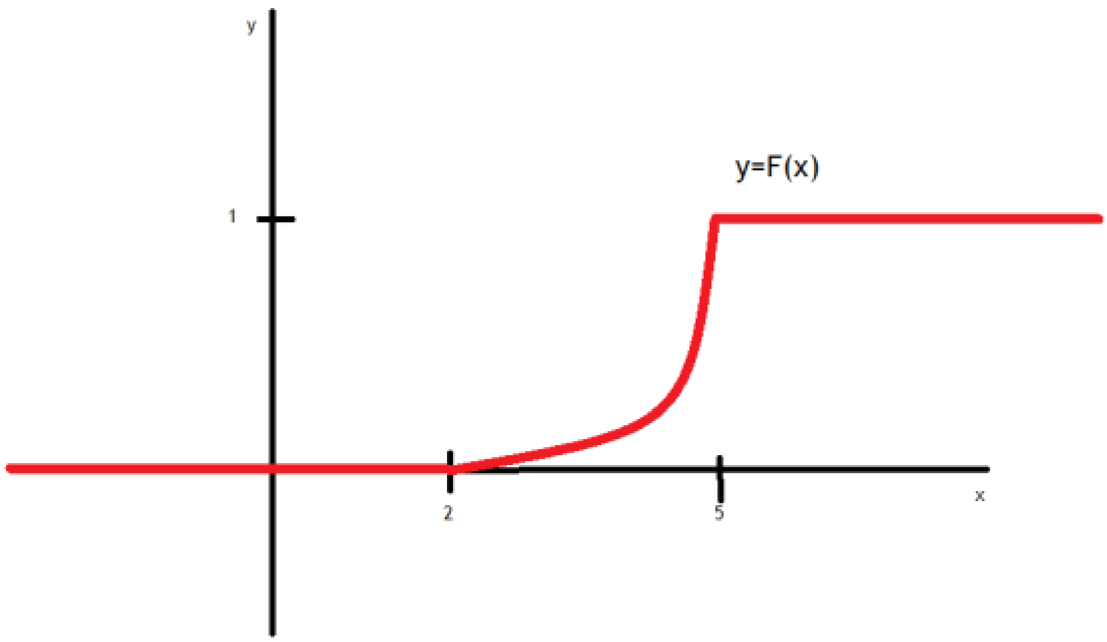
$$F(x) = \begin{cases} 0 & x \leq 2 \\ \frac{1}{21}(x^2 - 2^2) & 2 < x < 5 \\ 1 & x \geq 5 \end{cases}$$

Validation of CDF

a) limits

$\lim_{x \to -\infty} F(x)$ =0

$\lim_{x \to +\infty} F(x)$ =1



b) continuity – F(x) is continuous function
F(2)=0
$\lim_{x \to 2^+} F(x) = \frac{1}{21}(2^2 - 2^2)$=0

F(5)=1
$\lim_{x \to 5^-} F(x) = \frac{1}{21}(5^2 - 2^2)$=$\frac{21}{21} = 1$

c) F(x) non-decreasing function (look at the plot of CDF above)

Additional examples of CDF applications

$$F(x) = \begin{cases} 0 & x \le 2 \\ \frac{1}{21}(x^2 - 2^2) & 2 < x < 5 \\ 1 & x \ge 5 \end{cases}$$

PDF from CDF

$$f(x) = \begin{cases} 0 & x \le 2 \\ \frac{1}{21} 2x & 2 < x < 5 \\ 0 & x \ge 5 \end{cases}$$

P(X>4)= $\int_4^{+\infty} f(x)dx =$

P(X>4)=1-P(X<=4)=1-F(4)=1- $\frac{1}{21}(4^2 - 2^2)$=1-$\frac{12}{21}$=$\frac{9}{21}$

P(X<4)=F(4)

P(1<X<6)=F(6)-F(1)=1-0=1

Parameters of random variable X

$$f(x) = \begin{cases} \frac{2}{21}x & x \in < 2,5 > \\ 0 & in\ other\ cases \end{cases}$$

Expected value

EX=$\int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{-\infty}^{2} x \cdot 0dx + \int_2^5 x \cdot \frac{2}{21}xdx + \int_5^{+\infty} x \cdot 0dx$=$\int_{-\infty}^{2} 0dx + \int_2^5 x \cdot \frac{2}{21}xdx +$ $\int_5^{+\infty} 0dx$=$\int_2^5 x \cdot \frac{2}{21}xdx = \frac{2}{21}\int_2^5 x^2dx = \frac{2}{21}\frac{x^3}{3}\Big|_2^5 = \frac{2}{21}\left(\frac{5^3}{3} - \frac{2^3}{3}\right) \approx 3,71$

Second moment of random variable X

EX^2=$\int_{-\infty}^{+\infty} x^2 \cdot f(x)dx = \int_{-\infty}^{2} x^2 \cdot 0dx + \int_2^5 x^2 \cdot \frac{2}{21}xdx + \int_5^{+\infty} x^2 \cdot 0dx$=$\int_{-\infty}^{2} 0dx + \int_2^5 x^2 \cdot \frac{2}{21}xdx +$ $\int_5^{+\infty} 0dx$=$\int_2^5 x^2 \cdot \frac{2}{21}xdx = \frac{2}{21}\int_2^5 x^3dx = \frac{2}{21}\frac{x^4}{4}\Big|_2^5 = \frac{2}{21}\left(\frac{5^4}{4} - \frac{2^4}{4}\right) \approx 14,5$

Variance of random variable X

$Var(X)=EX^2-(EX)^2=14,5-(3,71)^2=0,7$

Standard deviation

$\sigma=\sqrt{0,7} = 0,84$

Median (continuous type of probability distribution)

$F\left(x_{\left(\frac{1}{2}\right)}\right)=0,5$

$\frac{1}{21}(x^2 - 2^2) = 0,5$

$(x^2 - 2^2) = 10,5$

$x^2 - 4 = 10,5$

$x^2 = 14,5$

x=3,8 (median)

Median is dividing probability into two equal parts in the sense of

$P(X<3,8)=P(X>3,8)=0,5$

MODE x=5 (the biggest value of PDF function is in point 5).