Liste der noch zu erledigenden Punkte

Kapitel 1

1.Overview

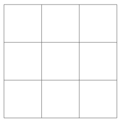
- „image society“ (webpages: 1995 text-based, 2005 image based, 2015 video based ...)

  - data transfer rates ↑, compression rates ↑

  critical shift: reading → watching
- „Photoshop“-ing (remove wrinkles, bumps, ...)
- Images in medicine („medical image proscessing“), x-ray, CT, MRI, ultrasound, ... („modalities“).
  different questions:

  1.) Layout!

  align bottom measurments $\overset{?}{\Rightarrow}$ image
  expl: tomography
  $\Rightarrow$ difficult mathematical problems

  2.) Image enhancements
  - denoising
    simple pixels/lines: „sandpaper“ interpolation so richtig?
    global noise: smoothing
  - grayscale
    histogramm balancing (spreading)
  - distortion
    makes straight lines (in real world) straight (in the images)
  - edge detection
    contour enhancement
  - segmentation
    detect and separate parts of the image
  - registration
    sequence of images of the same object ⇒ Wort? , compare Skizze
    ↗ object following in a movie

Our Focus:

- mathematical models/methods/ideas
- (algorthms)
- ((implementation))

skipped: Very fast intro: Matlab and images

Kapitel 2

2.What is an image?

2.1 Discrete and continuous images

There are (at least) two different points of view:

Discrete / Digital

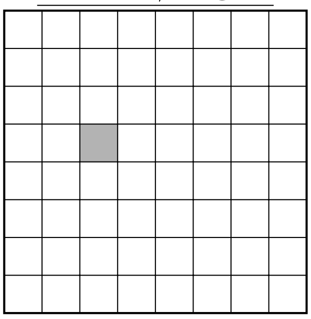


Abbildung 2.1: Discrete Image

Continous / Analogue

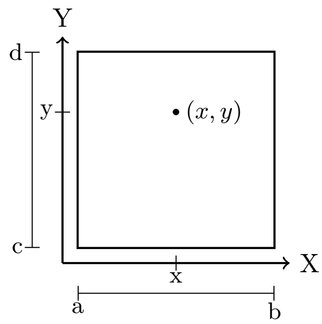


Abbildung 2.2: Continous Image

object:	matrix	function
tools:	linear algebra (SVD, ...)	analysis (differentrage, integrate, ...)
pros:	(finite storage) storage, complexity	freedom, tools, motions?P.4 (e.g. edge discontinuity)
cons:	limitations: zooming, rotations, ...	storage (infinite amout of data)

arguably, one has:

- real life $\Rightarrow$ continuous „images“ (objects)
- digital camers $\Rightarrow$ discrete images

In general we will say:

Definition 2.1 ((mathematical) image)**.** A (mathematical) *image* is a function

$$u : \Omega \to F,$$

where: $\Omega \subset \mathbb{Z}^d$ (discrete) or $\Omega \subset \mathbb{R}^d$ (continuous) ... *domain*
$d = 2$ (typical case 2D), $d = 3$ („3D image“ = body or $\underbrace{\text{2D + time}}_{\text{movie}}$)

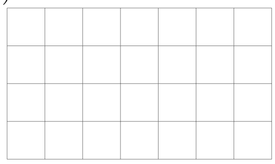
$d = 4$ (3D + time)

F ... *range of colours*

$F = \mathbb{R}$ or $[0, \infty]$ or $[0, 1]$ or $\{0, \dots 255\}$, ... grayscale (light intensity)

$F \subset \mathbb{R}^3$ ... RGB image (colored)

$F = \{0, 1\}$ ... black/white



3 Layers
$\Rightarrow$ colored images:w

Matlab stuff

Large parts of the course: analytical approach (i.e. continuous domain Ω)
Since we want to differentirate, ... the image u.

Still: need to assume that also F ist continuous (not as $\{0, 1\}$, $\{0, 1, \dots, 255\}$ or $\mathbb{N}$) since otherwise the only differentiable (actually, the only continuous) functions $u : \Omega \to F$ are *constant* functions $\Leftrightarrow$ single-colour images

Also: We usually take F one-dimensional ($F \subset \mathbb{R}$). Think of it as either

- gray scaled image, or
- treating R,G & B layer separately

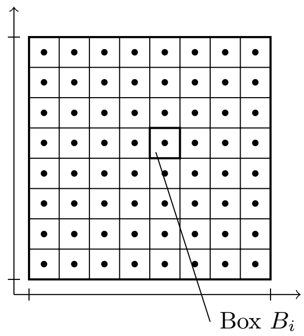
2.2 Switching between discrete and continuous images

continuous $\to$ discrete:

- divide the continuous image in small squared pieces (boxes) (superimpose grid)
- now: represent each box by *one* value
  - strategy 1: take function value $u(x_i)$ for x_i = midpoint of box B_i
  - strategy 2: use mean value

$$\frac{1}{|B_i|} \int_{B_i} u(x)dx$$

$\Omega =$



$\Rightarrow$ discrete image

strategy 1: simple (and quick) but problematic ($u(x_i)$ might represent $u|_{B_i}$ badly; for $u \in L^p$, single point evaluation not even defined)

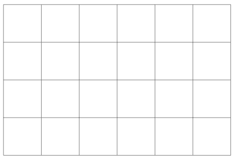
strategy 2: more complex but also more „democratic" (actually closer to the way how CCD Sensors in digital cameras work)

often the image value of the box B_i gets also digitized, i.e. fitted (by scaling & rounding) into range $\{0, 1, dots, 255\}$

discrete $\to$ continous

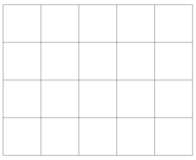
This is of course more tricky ...

- Again: each pixel of the discrete image corresponds to a „box" of the continuous image (that is still to be constructed)
- Usually: pixel value $\mapsto$ function value at the *midpoint* of the box
- Question: How to get the other function values (in the box)?



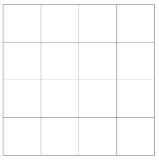
<u>idea 1:</u> just take the function value of the nearest midpoint („nearest neighbour interpolation“)

$$\text{For each } x \in B_i : u(x) := u(x_j) \quad \text{where } |x - x_j| = \min_k |x - x_k|$$



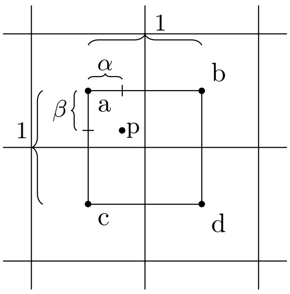
$\Rightarrow$ $u(x) = u(x_i)$ for all $x \in B_i$
$\Rightarrow$ each box is uni-color
$\Rightarrow$ the continuous image is essentially still discrete

<u>idea 2</u>: (bi-) linear interpolation



Let $a, b, c, d \ldots$ function values at 4 surrounding adjacent midpoints ($\nearrow$ figure)
$\alpha, \beta, 1-\alpha, 1-\beta \ldots$ distance to dotted lines ($\nearrow$ figure, w.l.o.g, bob is 1×1)

interpolation (linear) on the dotted line between a and b:



$e := a + \alpha(b - a) = (1 - \alpha)a + \alpha b$
(1D - interpolation, convex combination)

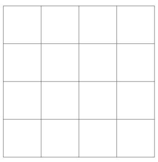
Similarly: $\quad f = (1 - \alpha)c + \alpha d$

Then: The same 1D-interpolation between e and f

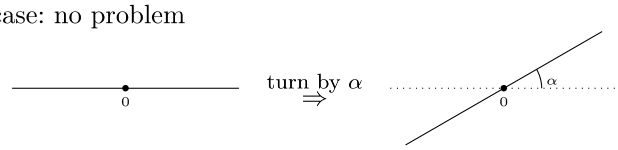
$$\begin{aligned} \Rightarrow u(x) :&= (1-\beta) \cdot e + \beta \cdot f \\ &= (1-\beta)[(1-\alpha)a + \alpha b] + \beta[(1-\alpha)c + \alpha d] \\ &= \underbrace{(1-\alpha)(1-\beta)}\, a + \underbrace{\alpha(1-\beta)}\, b + \underbrace{(1-\alpha)\beta}\, c + \underbrace{\alpha\beta}\, d \end{aligned}$$

$$\in [0,1] \wedge \sum = 1$$

$\Rightarrow$ <u>convex combination</u> of the function values a, b, c, d at the the surrounding 4 midpoints (on which points is the nearest, instead of taking just a, b, c or d - depending)
$\Rightarrow$ 2D linear interpolation, *bi-linear interpolation* (can be interpreted as spline interpolation with bilinear basis splines).

Beispiel 2.2. Rotate image  by angle $\phi \neq k \cdot \frac{\pi}{2}$

- continuous image case: no problem



$$x = D_\varphi\, y \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ D_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

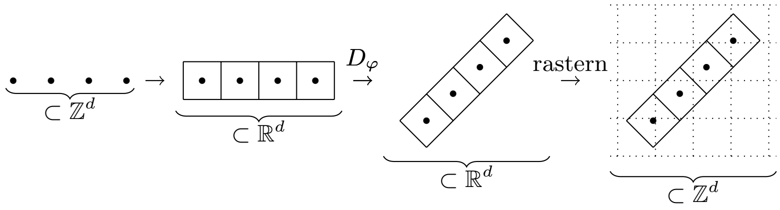
2D rotation matrix

$$y = D_\varphi^{-1}\, x = D_{-\varphi}\, x$$

$\Rightarrow v(x) := u(y) = u(D_{-\varphi}\, x) \quad \forall x \in$ domain of the rotated image

- discrete image case: problem !

  For $x \in$ domain of notated image, in general $D_{-\varphi}\, x \notin$ domain of original image[1]
  Way out: $v(x) :=$ *interpolation* between the $u(\cdot)$ of the 4 surrounding pixels of $D_{-\varphi}$



Something to think about:
What happens in the limit (?) if we, starting with an image (discrete or continuous), repeatedly switch between discrete and continuous, non-stop . . . ?
Does the answer depend on the way of switching ? (continuous $\rightarrow$ discrete: midpoint or average, discrete $\rightarrow$ continuous: nearest neighbour or bilinear?)

[1] it's not an integer

Kapitel 3

3.Histogramm and first applicatsion

3.1 The histogramm

Definition 3.1 (histogram)**.** Let $\Omega \subset \mathbb{Z}^d$, $F \subset \mathbb{R}$ discrete and $u : \Omega \to F$ a discrete discrete image. The function

$$H_u : F \to \mathbb{N}_0 \ (:= \mathbb{N} \cup \{0\})$$

with

$$H_u(k) := \# \{x \in \Omega : u(x) = k\}\,, \quad k \in F$$

is called *histogramm* of the image u.

$H_u(k)$ counts how often colour k appears in u.

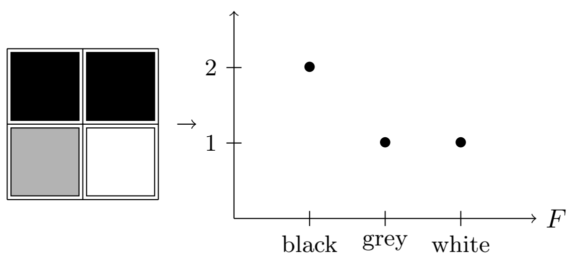
$$\sum_{k \in F} H_u(k) = |\Omega| = \text{number of pixels in the whole image}$$

or

$$\frac{H_u(k)}{|\Omega|} = \begin{array}{l}\text{relative frequence of colour } k \text{ in image } u \\ \text{(relative Häufigkeit)}\end{array}$$

Beispiel 3.2.

$u =$



If u ist a continous image, H_u can be understood as a measure (generalized function)[1]. Another way to write this:

$$H_u(k) = \sum_{x \in \Omega} \delta_{u(x)}(k),\ k \in F \qquad\qquad H_u(k) = \int_{\Omega} \delta_{u(x)}(k)dx,\ k \in F$$

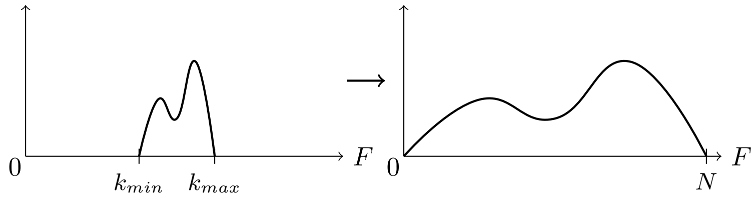
hier fehlt noch das Kronecker underarrow

Matlab-Code

[1] density of a probability distribution

3.2 Application: contrast enhancement

If the image only uses a small part of the available colour/grayscale „palette“ F, then its contrast can be improved by „spreading“ the histogramm over all of F.
Simple idea:



$$\begin{tikzcd} \Omega \arrow[r, bend left, "\text{originales Bild}"] \arrow[rr, bend right, "\text{Bild mit besserem Kontrast}"'] & F \arrow[r, bend left, "\text{Zauberei } T"] & F \end{tikzcd}$$

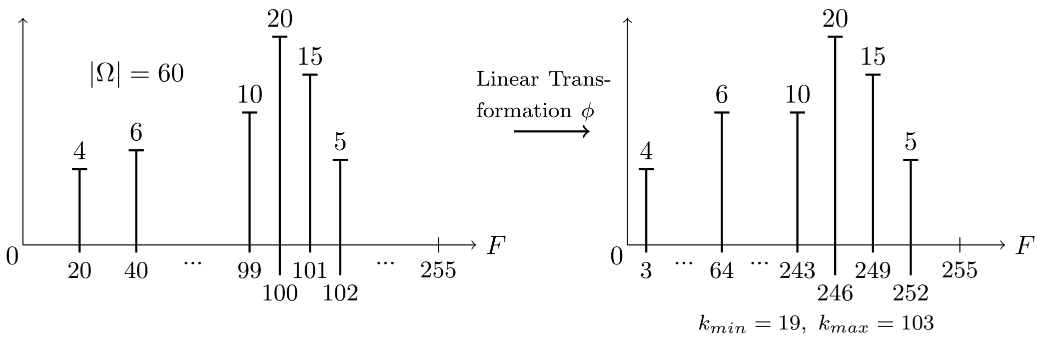
$$v = T \circ u$$
$$v(k) = T(u(k))$$

5

The above simple idea („contrast stretching“) corresponds to

$$\begin{aligned} \varphi : k_{\min} &\mapsto 0 \\ k_{\max} &\mapsto N \\ &\text{and linear in between} \\ \text{i.e} \qquad \varphi(k) &= \left[\frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot N \right] \end{aligned}$$

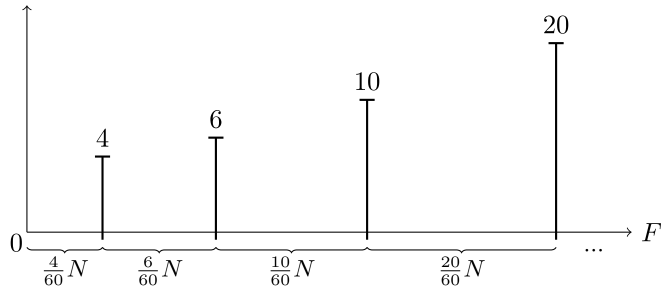
Where $[\quad \cdot \quad]$ means ... rounding to the nearest integer (assumuning that $F = \{0, 1, \ldots, N\}$).
Example histogram:



A bit more sophisticated:

$$\begin{aligned} \varphi :\ &(k_{\min} \mapsto 0) \\ &k_{\max} \mapsto N \\ &\text{and } \textbf{non} \text{ linear in between} \end{aligned}$$

such that colour ranges that occur more frequently in u can occupy a larger range of colours in u. ($\Rightarrow$ visibility $\uparrow$)
Example histogramm spread out according to frequency of occurence:



$\Rightarrow$ „density“ is equalized over $F = \{0, \dots, N\}$

Ideal would be:

Layout S.12 u

Note: The new colours (i.e the location of the bars in the histogramm of u) only depend on the frequencies / *height* of the bars in H_u but not on the colours/location of the bars in H_u

Finally: The formula

$$\varphi(k) = \left[\frac{N}{|\Omega|} \sum_{l=0}^{k} H_u(l) \right]$$

This process is called „histogramm equalization“

Exercise ?!

3.3 Another application: conversion to b/w

Task: convert grayscale image to black white
- interesting for object detection/*segmentation* ...!

Idea: Find a threshold $t \in T$ s.t. the histogramm splits into two „characteristic“ parts

For $t \in F$ put

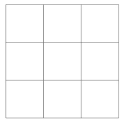
$$\begin{aligned} \text{black} &:= \{k \in F : k \leq t\} \\ \text{white} &:= \{k \in F : k > t\} \end{aligned}$$

and

$$\tilde{u} := \begin{cases} 0, & u(x) \in \text{black} \\ 1, & u(x) \in \text{white} \end{cases} \qquad \tilde{F} = \{0, 1\}$$

How to find the threshold t:

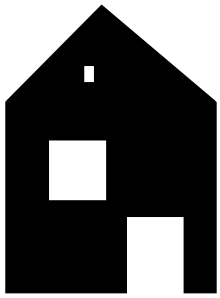
1.) Shape based methods
If the histogramm is „biomodal“
Put $t := \dfrac{k_{\max_1} + k_{\max_2}}{2}$
or $t := k_{\min}$



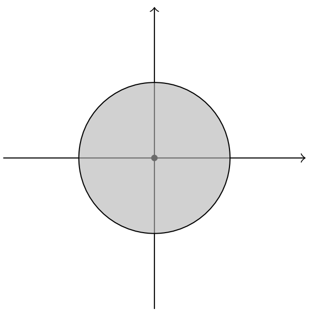
Kapitel 4

4.Basic Morphological Operations

B/W Bild:

$A =$ 

Structural element :

$B =$ 

4.1 Operations on A and B

$$A + B := \{a + b : a \in A, b \in B\}$$

This is called dilation.
You might imagine that at every dark point in the image A the Structurelement is applied.

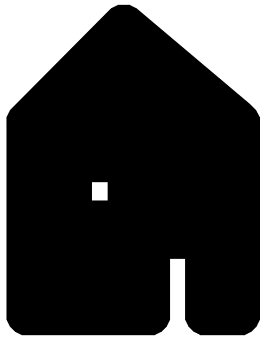
$A + B =$ 

Image created in Matlab through:

```
I=imread('Bild1.png');
se=strel('disk',40,8);
I2=imcomplement(imdilate(imcomplement(I),se));%I am using the complement of the image
    here so that the structural element is applied to the dark parts of the image
imshow(I2);
```

$$A - B := \{a : a + B \subset A\}$$

This is called erosion .
You can imagine that you search for the points in which the structural element fits.

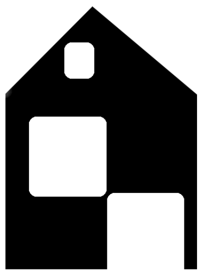
$A - B =$ 

Image created in Matlab thorugh:

```
I=imread('Bild1.png');
se=strel('disk',20,8);
I2=imcomplement(imerode(imcomplement(I),se));
imshow(I2);
```

One may quickly realize that $A \neq (A + B) - B$, so a new Operation is introduced:

$$A \bullet B := (A + B) - B$$

This is called <u>closing</u> and is used to e.g. remove noise. In the example image you might notice that the upper window is missing.

$A \bullet B =$

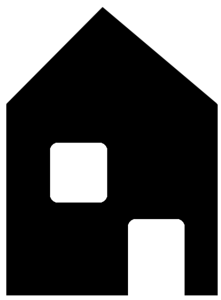


Image created in Matlab thorugh:

```
I=imread('Bild1.png');
se=strel('disk',20,8);
I2=imcomplement(imdilate(imcomplement(I),se));
I3=imcomplement(imerode(imcomplement(I2),se));
imshow(I3);
```

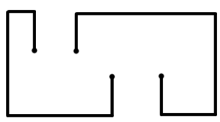
The inverse also exists:

$$A \circ B := (A - B) + B$$

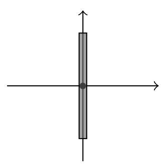
This is called <u>opening</u> .

This time with a new example:

$A =$



$B =$



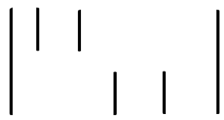
$A \circ B =$ 

Image created in Matlab thorugh:

```
I=imread('Bild2.png');
se=strel('line',10,90);
I2=imcomplement(imerode(imcomplement(I),se));
I3=imcomplement(imerode(imcomplement(I2),se));
imshow(I3);
```

Kapitel 5

5.Entrauschen: Filter und Co

5.1 Noise

<u>Noise</u> = Unwanted disturbances in an image. Mostly becaue of

- point wise
- random
- independent

We consider *noise* to be an additive disturbances (for multiplicative noise use *log*).
Notation:

$$\begin{array}{ccccc} \boxed{f_0} & \xrightarrow{+\text{ noise}} & \boxed{f} & \xrightarrow{\text{denoise}} & \boxed{u} \\ \text{clean} & & \text{disturbed image} & & \text{result} \end{array}$$

The quality of the denoised image u compared to the original image f_0 is described by norms:

$$||f - f_0|| \ldots \text{noise}$$
$$||u - f_0|| \ldots \; \underline{\text{absolute error}}$$
$$\frac{||u - f_o||}{||f - f_0||} \ldots \; \underline{\text{relative error}} \; \text{ compared to the noise}$$
$$\frac{||u - f_o||}{||f_0||} \ldots \text{relative error compared to the signal}$$

Typically the chosen norm is:

$$||f|| = ||f||_2 = \sqrt{\int_\Omega |f(x)|^2 \, dx}$$

or in the discrete:

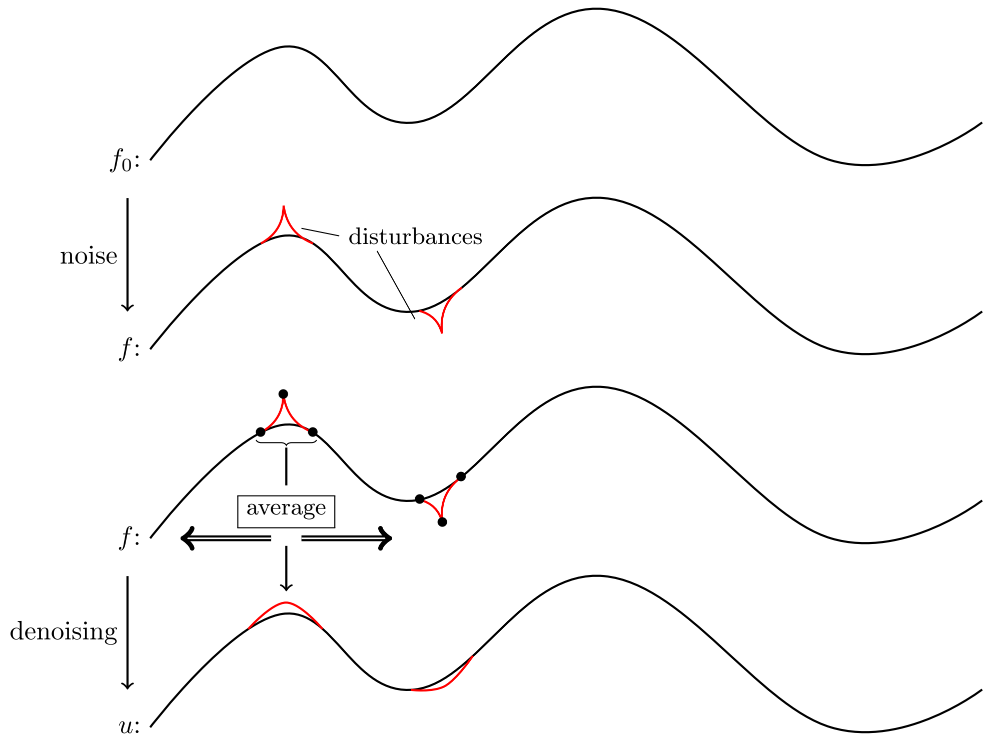
$$||f||_2 = \sqrt{\sum_{x \in \Omega} |f(x)|^2}$$

Closely connected is the <u>Signal to noise ratio</u> (SNR):

$$log(\underbrace{\frac{||f_0||_2}{||u - f_0||_2}}_{\in\ [1,\infty)}) \in [0, +\infty), \text{ where } 0 \text{ is bad and } +\infty \text{ is good.}$$

5.2 smoothing filter

Idea: (to simplify in 1D)

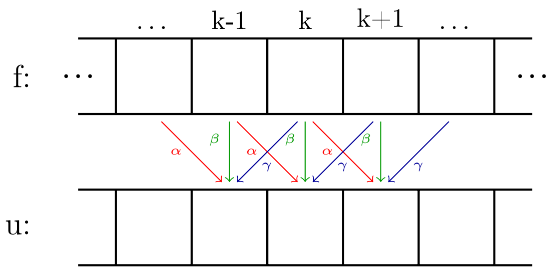


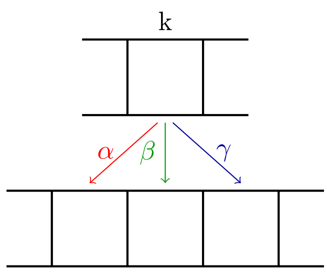
$$u(k) := \alpha \cdot f(k-1) + \beta \cdot f(k) + \gamma \cdot f(k+1) \tag{5.1}$$

where:

$$\alpha + \beta + \gamma = 1 \tag{5.2}$$

More precisely (5.1) means:





With (5.1) there is a mapping $f \mapsto u$, we write

$$u = m \circledast f, \text{ this is called } \underline{\text{Correlation}}.$$

where:

$$\boxed{(m \boxast f)(k) = \sum_{i \in supp(m)} m(i) f(k+i)} \tag{5.3}$$

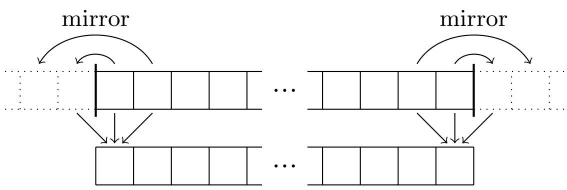
and:

$$m = \begin{array}{c|c|c|c|c} \dots & \text{-1} & 0 & 1 & \dots \\ \hline \cdots & \alpha & \beta & \gamma & \cdots \\ \hline \end{array}$$ called mask .

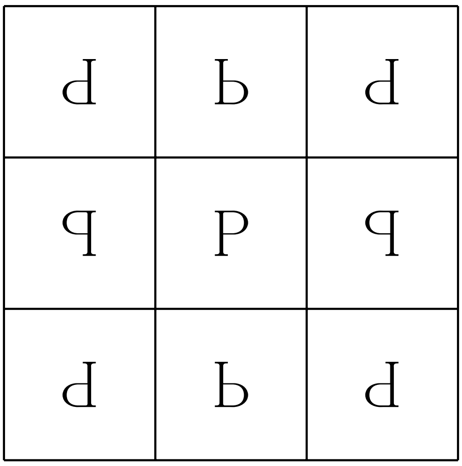
If you set $j := k + i$ in (5.1), then $i = j - k$, which means:

$$\boxed{(m \boxast f)(k) = \sum_{i \in supp(m)} m(j-k) f(j)} \tag{5.4}$$

To apply the mapping onto the boundary the image is reflected, in 1D:



in 2D:



Formula (5.4) might remind one of the convolution :

Layout!

$$\boxed{(g * f)(k) = \sum_{j \in \mathbb{Z}} g(\underbrace{k - j}_{\text{Difference to (5.4)}}) \cdot f(j)} \tag{5.5}$$

If you set $g(i) := m(-i) =: \tilde{m}(i)$, which corresponds to a reflection of the Mask, then

$$m \boxast f = g * f = \tilde{m} * f$$

Im Skript hier noch Beispiele und soetwas p. 32f

Properties of the convolution:

1. $(f * g) * h = f * (g * h)$, Associativity
2. $f * g = g * f$, Commutativity
3. $\tilde{f} * \tilde{g} = \widetilde{f * g}$, Compatibility with reflection

Properties of the correlation:

1. $f \boxast (g \boxast h) = \tilde{f} * (\tilde{g} * h) \overset{\boxed{1}}{=} (\tilde{f} * \tilde{g}) * h \overset{\boxed{3}}{=} (\widetilde{f * g}) * h = (f * g) \boxast h \neq (f \boxast g) \boxast h$, not associative!
2. $f \boxast g = \tilde{f} * g \overset{\boxed{2}}{=} g * \tilde{f} = \tilde{\tilde{g}} * \tilde{f} \overset{\boxed{3}}{=} (\widetilde{\tilde{g} * f}) = \widetilde{g \boxast f} \neq g \boxast f$, not commutative!
3. $\tilde{f} \boxast \tilde{g} = \tilde{\tilde{f}} * \tilde{g} \overset{\boxed{3}}{=} (\widetilde{\tilde{f} * g}) = \widetilde{f \boxast g}$, Compatibility with reflection

$\boxast$ und $*$ definiert man auf: $\ell^1(\mathbb{Z}^d) := \left\{ f = (f_i)_{i \in \mathbb{Z}^d} : \underbrace{\sum_{i \in \mathbb{Z}^d} |f_i|}_{:=||f||_1} < \infty \right\}$

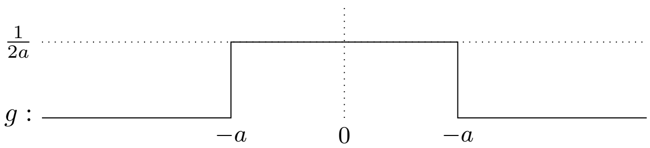
Man kann zeigen (Übung): $f, g \in \ell^1 \Rightarrow f * g \in \ell^1$ und $||f * g||_1 \leq ||f||_1 \cdot ||g||_1$. Wobei oft die Gleichheit gilt.

Alles gilt auch in der Kontinuierlichen Version:

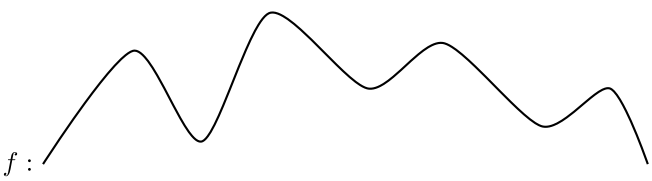
$$L^1(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{R} : \underbrace{\int_{\mathbb{R}^d} |f| \, dx}_{:=||f||_1} < \infty \right\}$$

$$f, g \in L^1(\mathbb{R}^d) : (g * f)(x) = \int_{\mathbb{R}^d} g(x - y) f(y) dy, \ y, x \in \mathbb{R}^d$$

Beispiel für den kontinueirlichen Fall:

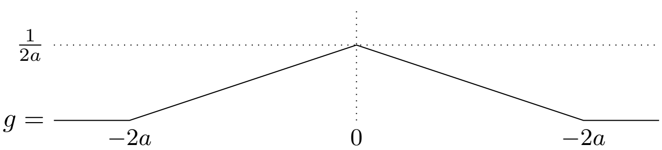


Hierbei gilt $\int_{\mathbb{R}} g(x) dx = 1$



$g \boxast f =$ gleitendes Mittel .

$g \boxast g = \tilde{g} * g = g * g =$



Layout!

Weitere Eigenschaften der Faltung:
Für alle $f, g \in L^1$ or ℓ^1

$$\left.\begin{aligned}(g_1 + g_2) * f &= (g_1 * f) + (g_2 * g)\\ (\alpha g) * f &= \alpha(g * f)\end{aligned}\right\} = \text{Linearität}$$

Somit ist:

$$g \mapsto f * g$$

ein linearer Operator.
Formt ℓ^1 bzw. L^1 eine Algebra mit neutralem Element δ?
ℓ^1?:

$$\delta: \quad \cdots \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 0 & 0 \\ \hline \end{array} \cdots$$

$$\begin{array}{c}\uparrow\\ \text{Pos } 0\end{array}$$

Ja!
L^1?: Für ein solches Element muss gelten:
$\forall f \in L^1 : d * f = f$
$\forall x \in \mathbb{R} : \displaystyle\int_{\mathbb{R}^d} \underbrace{\delta(x-y)}_{=0 \forall x \neq y} f(y)dy = f(x)$

Diese Funktion wird Dirac-Impuls gennant ist aber kein Element von L^1.
Nun zu Masken in 2D:

$$u = m \circledast f \text{ mit } m = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \alpha & \multicolumn{1}{c}{} \\ \hline \beta & \gamma & \delta \\ \hline \multicolumn{1}{c|}{} & \epsilon & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

wobei $\alpha + \beta + \gamma + \delta + \epsilon = 1$
Kurzschreibweise: $u_{ij} := u(x)$ wobei $x = \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2$, analog für f_{ij}.

$$\Rightarrow u_{ij} = \alpha f_{i-1,j} + \beta f_{i,j-i} + \gamma f_{ij} + \delta f_{i,j+1} + \epsilon f_{i+1,j}$$

$$u = m \circledast f = \tilde{m} * f \text{ mit } \tilde{m} = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \epsilon & \multicolumn{1}{c}{} \\ \hline \delta & \gamma & \beta \\ \hline \multicolumn{1}{c|}{} & \alpha & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

Symmetrischer Fall:

$$\tilde{m} = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \alpha & \multicolumn{1}{c}{} \\ \hline \alpha & \gamma & \alpha \\ \hline \multicolumn{1}{c|}{} & \alpha & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \text{ mit } \gamma = 1 - 4\alpha$$

$$u_{ij} = (1 - 4\alpha)f_{ij} + \alpha(f_{i-1,j} + f_{i,j-1} + f_{i,j+1} + f_{i+1,j}) \tag{5.6}$$

$$\text{Erinnerung:} \quad \underset{\text{Sauberes Bild}}{\boxed{f_0}} \xrightarrow{+\text{ Rauschen}} \underset{\text{Gestörtes Bild}}{\boxed{f}} \xrightarrow{\text{Entrauschen}} \underset{\text{Resultat}}{\boxed{u}}$$

Annahme: $f_{ij} = f_{ij} + r_{ij}$ mit $r_{ij} \sim N(0, \sigma^2)$ iid.
z.z.: $Var(u_{ij}) \leq Var(f_{ij})$

$$Var(f_{ij}) = E(\underbrace{f_{ij} - \overbrace{Ef_{ij}}^{f_{ij}^0}}_{r_{ij}})^2 = \sigma^2$$

Layout!

$$\begin{aligned}
Var(u_{ij}) = E(u_{ij} - Eu_{ij})^2 &= E((1-4\alpha)(\underbrace{f_{ij} - f_{ij}^0}_{r_{ij}}) + \alpha((\underbrace{f_{i-1,j} - f_{i-1,j}^0}_{r_{i-1,j}}) + \ldots + (\underbrace{f_{i+1,j} - f_{i+1,j}^0}_{r_{i+1,j}})))^2 \\
&= E((1-4\alpha)^2 r_{ij}^2 + \alpha^2(r_{i-1,j}^2 + r_{i,j-1}^2 + r_{i,j+1}^2 + r_{i+1,j}^2) + 2(1-4\alpha)\alpha r_{ij} r_{i-1,j} \ldots) \\
&= (1-4\alpha)^2 \underbrace{Er_{i,j}^2}_{\sigma^2} + \alpha^2(Er_{i-1,j}^2 + \ldots + Er_{i+1,j}^2) + 2(1-4\alpha)\alpha \underbrace{\underbrace{E(r_{ij} r_{i-1,j})}_{Er_{ij}Er_{i-1,j}}}_{0} + \underbrace{\ldots}_{0}) \\
&= (1-4\alpha)^2\sigma^2 + \alpha^2 4\sigma^2 = (1 - 8\alpha + 16\alpha^2 + 4\alpha^2)\sigma^2
\end{aligned}$$

Da $0 \leq \alpha$ und $0 \leq 1 - 4\alpha \Rightarrow 0 \leq \alpha \leq \frac{1}{4}$:

$$(1 - 8\alpha + 16\alpha^2 + 4\alpha^2)\sigma^2 = 1 + \underbrace{\underbrace{20\alpha}_{\geq 0}\underbrace{(\alpha - \frac{2}{5})}_{<0}}_{\leq 1}$$

$\Rightarrow Var(u_{ij}) \leq Var(f_{ij})$ für $\alpha \in [0, \frac{1}{4}]$

Dabei gilt: $Var(u_{ij}) \stackrel{\alpha}{\to} d\min \iff 1 - 8\alpha + 20\alpha^2 \stackrel{\alpha}{\to} \min \iff -8 + 40\alpha = 0 \iff \alpha = \frac{1}{5}$

$$\Rightarrow \text{bester Filter}: \begin{array}{|c|c|c|} \hline & \frac{1}{5} & \\ \hline \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \hline & \frac{1}{5} & \\ \hline \end{array}$$

Kapitel sollte noch fehlergelesen werden. Es könnte noch einiges aus dem Skript übernommen werden. Es braucht etwas Layout

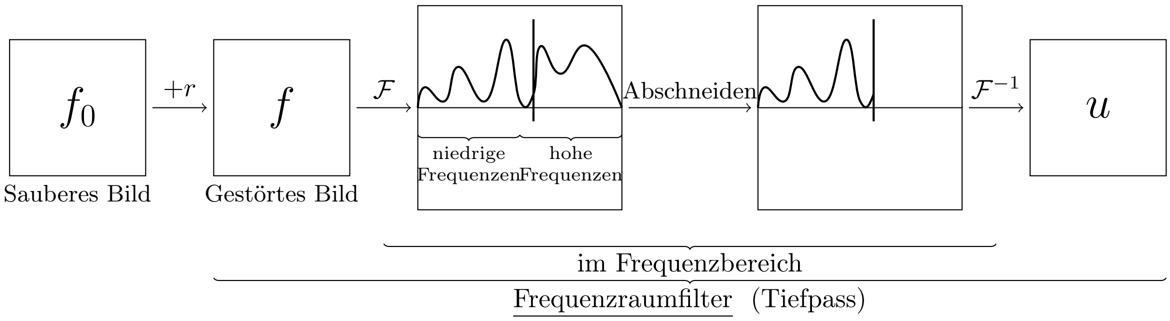
5.3 Frequenzfilter

Ansatz: Rauschen $\approx$ hochfrequente Anteile des Bildes/Signals
$\Rightarrow$ gezieltes entfernen
Wichtiges Instrument: Fouriertransformation (FT)

$$\mathcal{F} : f \mapsto \hat{f} \text{ mit } \hat{f}(z) = \frac{1}{(2\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} dx$$

hier fehlt der rest aus einer Vorlesung

siehe auch p. 41



Wobei $z \in \mathbb{R}^d, f \in L^1(\mathbb{R}^d)$.

Falls auch $\hat{f} \in L^1(\mathbb{R}^d)$ ist ,dann lässt sich f wie folgt mittels der inversen Fouriertransformation aus $\hat{f}$ rekonstruieren:

$$\mathcal{F}^{-1} : \hat{f} \mapsto f$$

$$\boxed{\hat{f}(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{i\langle z,x\rangle} dx} \tag{5.7}$$

Wobei $x \in \mathbb{R}^d$.

Man hat also $\mathcal{F}^{-1}\mathcal{F}f$, d.h.

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(y) e^{-i\langle z,y\rangle} dy \right) e^{i\langle z,x\rangle} dz$$

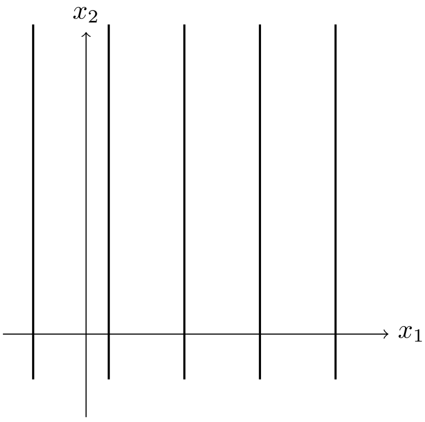
Sei nun $e_z(x) := e^{i\langle z,x\rangle},\ x \in \mathbb{R}^d$ mit Parameter $z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}$.

Also $e_z(x) = e^{i\langle \binom{z_1}{z_2}, \binom{x_1}{x_2} \rangle} = e^{i(z_1x_1 + z_2x_2)}$

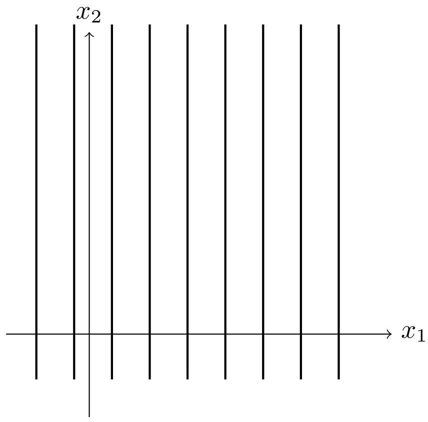
Beispiele in $2D$:

(Hier stellen die Linien, Punkte mit konstantem wert dar)

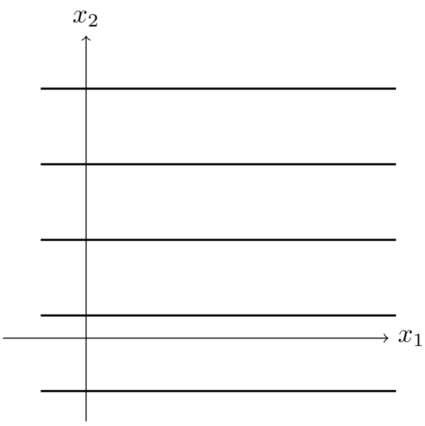
$z = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\ e_z(x) = e^{ix_1}$:



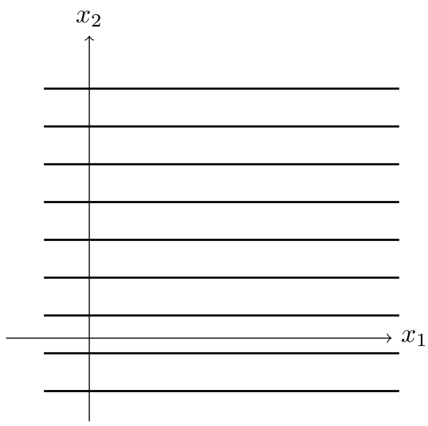
$z = \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right),\ e_z(x) = e^{i2x_1}$:



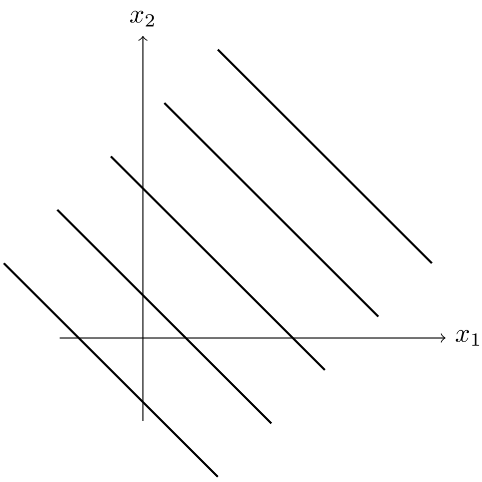
$z = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\ e_z(x) = e^{ix_2}$:



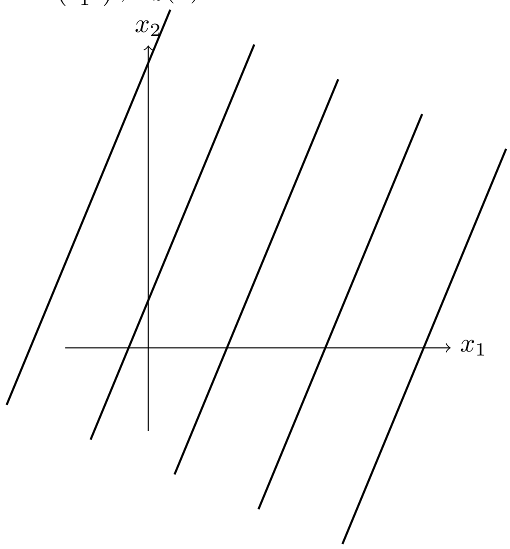
$z = \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right),\ e_z(x) = e^{i2x_2}$:



$z = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right),\ e_z(x) = e^{i(x_1+x_2}$:



$z = \left(\begin{smallmatrix}-2\\1\end{smallmatrix}\right),\ e_z(x) = e^{i(-2x_1+x_2)}$:



$f \in L^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} | \int_{\mathbb{R}^d} |f|^2 \, dx < \infty\}$ ist

- ein normierter Raum mit $+$, $\alpha\cdot$ und $||\cdot||_2 := \sqrt{\int_{\mathbb{R}^d} |f(x)|^2 \, dx}$

- ein Skalarproduktraum mit $\langle f, g\rangle := \int_{\mathbb{R}^d} f\bar{g} dx$, wobei $||f||_2^2 = \langle f, f\rangle$

- ein vollständiger Raum, also <u>Banachraum</u>

Ein vollständiger normierter Banachraum mit Skalarproduk heißt <u>Hilbertraum</u> .
$\mathcal{F}$ kann auch als Abbildung auf $L^2(\mathbb{R}^d)$ betrachtet werden. Dann gilt:

$$\hat{f} = \mathcal{F}f \in L^2(\mathbb{R}^d)$$

und

$$\left|\left|\hat{f}\right|\right|_2 = ||f||_2 \tag{5.8}$$

und sogar

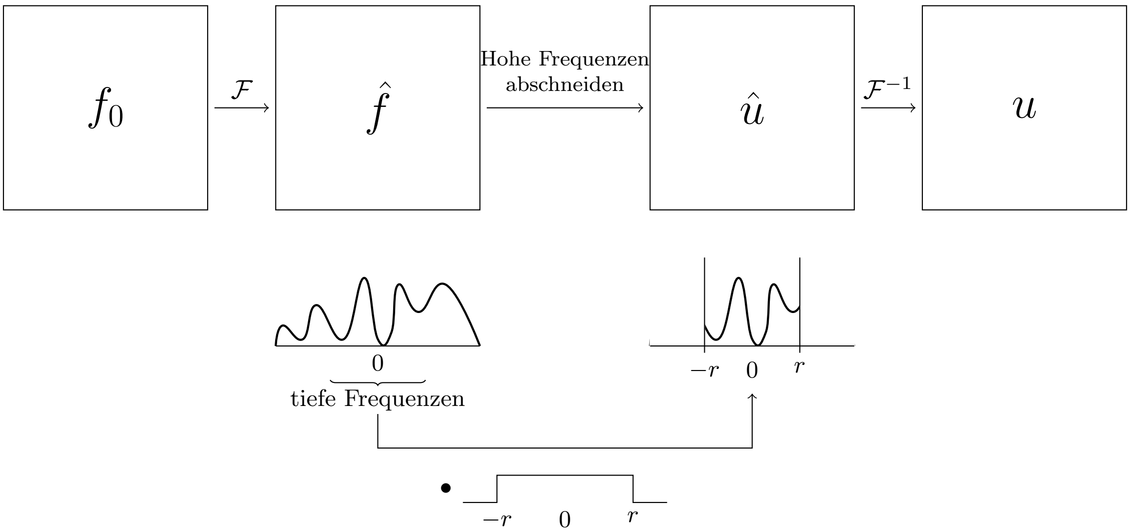
$$\left\langle \hat{f}, \hat{g} \right\rangle_2 = \langle f, g \rangle_2 \tag{5.9}$$

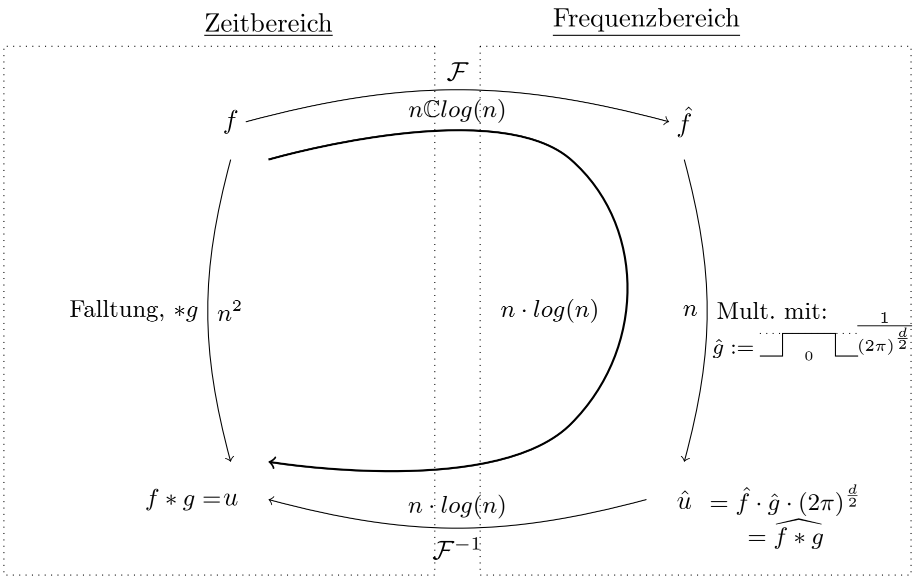
für alle $f, g \in L^2(\mathbb{R}^d)$.
Weitere Eigenschaften der Fouriertransformation:

- $f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f}$ stetig und $\lim\limits_{|z|\to\infty} \hat{f}(z) = 0$
- $\mathcal{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ ist eine lineare Abbildung
- $\mathcal{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ ist eine beschränkte/stetige Abbildung
- Verschiebung $\xrightarrow{\mathcal{F}}$ Modulation, d.h.
$$g(x) = f(x + a) \Rightarrow \hat{g}(z) = e^{i\langle a,z\rangle}\hat{f}(z)$$
- Modulation $\xrightarrow{\mathcal{F}}$ Verschiebung, d.h.
$$g(x) = e^{i\langle x,a\rangle}f(x) \Rightarrow \hat{g}(z) = \hat{f}(z - a)$$
- Skalierung $\xrightarrow{\mathcal{F}}$ inverse Skalierung, d.h.
$$g(x) = f(cx) \Rightarrow \hat{g}(z) = \frac{1}{|c|}\hat{f}(\frac{z}{|c|})$$
- Konjugation: $g(x) = \overline{f(x)} \Rightarrow \hat{g}(z) = \overline{\hat{f}(-z)}$
  Folglich: f reelwertig $\Rightarrow \hat{f}(z) = \overline{\hat{f}(-z)}$
-
$$\text{Grundmode: } \hat{f}(0) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)dx$$
$$\text{Analog: } f(0) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{f}(x)dx$$
- Differentation $\xrightarrow{\mathcal{F}}$ Multiplikation mit Potenzen von z, d.h.
$$g(x) = \frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}} f(x) \Rightarrow \hat{g}(z) = i^{\alpha_1+\cdots+\alpha_d} z_1^{\alpha_1}\cdots z_d^{\alpha_d}\hat{f}(z)$$
- Unkehrung des letzten Punktes:
$$g(x) = x_1^{\alpha_1}\cdots x_d^{\alpha_d} f(x) \Rightarrow \hat{g}(z) = i^{\alpha_1+\cdots+\alpha_d}\frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}}\hat{f}(z)$$
-
$$\text{Faltungssatz: } \mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\cdot\mathcal{F}(g),\ \widehat{f * g} = (2\pi)^{\frac{d}{2}}\hat{f}\cdot\hat{g}$$
$$\text{Analog: } \mathcal{F}(f \cdot g) = \frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}(f) * \mathcal{F}(g),\ \widehat{f \cdot g} = \frac{1}{(2\pi)^{\frac{d}{2}}}\hat{f} * \hat{g}$$

  d.h.: Faltung $\xrightarrow{\mathcal{F}}$ Multiplikation und umgekehrt

Zur Erinnerung:





Genauer:

$$\mathcal{F}u = \hat{v} =$$

$$\begin{aligned}
g(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}} (\mathcal{F}^{-1} \chi_{[-r,r]^d})(x) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \chi_{[-r,r]^d}(z) e^{i\langle z,x\rangle dz} \\
&\overset{1d}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_{[-r,r]} e^{izx} dz \\
&= \frac{1}{2\pi} \int_{-r}^{r} e^{izx} dz = \frac{1}{2\pi} \left. \frac{e^{izx}}{ix} \right|_{z=-r}^{r} \\
&= \frac{1}{2\pi i x} (e^{irx} - e^{-irx}) = \frac{1}{\pi x} \sin(rx)
\end{aligned}$$

$$\hat{g}(0) = (\mathcal{F}g)(0) = \frac{1}{2}$$

Es ist zu bemerken, dass g eine Art Tensor Struktur besitzt, was in etwa bedeutet das sich die Funktion in belibigen Dimensionen als Produkt der Funktion in einer Dimensionen darstellen lässt.
<u>Gauß-Kern</u> :

$$G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2}} \Rightarrow G\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-x_1^2 - x_2^2 + \cdots + x_d^2}{2}}$$
$$= \left(\frac{1}{(2\pi)^{\frac{1}{2}}} e^{\frac{-x_1^2}{2}}\right) \cdot \; \cdots \; \cdot \left(\frac{1}{(2\pi)^{\frac{1}{2}}} e^{\frac{-x_d^2}{2}}\right) = G(x_1) \cdot \; \cdots \; \cdot G(x_d)$$

allerhand noch im Skript und ein Tafelfoto

5.4 Filterbreite und Glättung

klar ist: $\frac{1}{25}$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

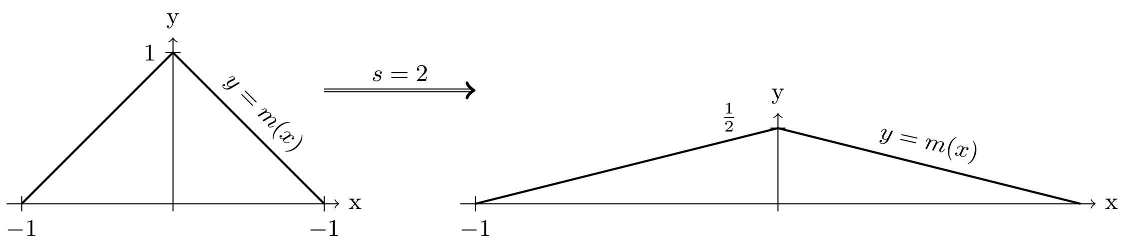
'glättet mehr als': $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

Im Kontinuierlichen: Sei $m \in L^1(\mathbb{R}^d)$ und $s > 0$. Setze

$$m_s(x) := \frac{1}{s^d} m(\frac{x}{s}), \quad x \in \mathbb{R}^d$$

Bsp (in $d = 1$):



Bsp: Gauß-Kern $G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2}}$
Skalierung mit Fehler $s > 0$

$$\Rightarrow G_s(x) = \frac{1}{s^d} G\left(\frac{x}{s}\right) = \frac{1}{s^d} \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|}{2}} = \frac{1}{(2\pi s^2)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2s^2}}$$

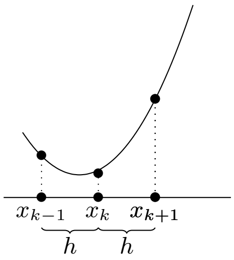
Skalierung $s \widehat{=}$ Standardabweichung σ

hier noch mehr im Skript p. 45

5.4.1 Differenzenfilter

Bisher: Glättung $\widehat{=}$ Mittelwert bilden $\widehat{=}$ Summe/Integrale
Jetzt: Schärfen $\widehat{=}$ Differenzen/Kontraste hervorheben $\widehat{=}$ Differenzen/Ableitungen

Diskretisierung von Ableitungen durch Differenzenquotienten



(hier bedeutet $f(k) = f(x_k)$)

Vorwärts: $u(h) = \dfrac{f(k+1) - f(k)}{h}$ $\qquad u = \dfrac{1}{h}\begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline\end{array} \circledast f$

Rückwärts: $u(h) = \dfrac{f(k) - f(k-1)}{h}$ $\qquad u = \dfrac{1}{h}\begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline\end{array} \circledast f$

Zentral: $u(h) = \dfrac{f(k+1) - f(k-1)}{2h}$ $\qquad u = \dfrac{1}{2h}\begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline\end{array} \circledast f$

2. Abbleitung:

$$\begin{aligned} u(h) &\approx \frac{f'(k+1) - f'(k)}{h} \text{(vorwärts)} \\ &\approx \frac{\frac{f(k+1)-f(k)}{h} - \frac{f(k)-f(k-1)}{h}}{h} \text{(rückwärts)} \\ &= \frac{f(k+1) - 2f(k) + f(k+1)}{h^2} \end{aligned}$$

Also folgt $u := \begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array} \circledast f$ und $\dfrac{1}{h^2}\begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array} = \dfrac{1}{h}\begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline\end{array} * \dfrac{1}{h}\begin{array}{|c|c|c|}\hline -1 & 1 & 0 \\ \hline\end{array}$

Denn:

$$\begin{aligned} &\frac{1}{h}\begin{array}{|c|c|c|}\hline -1 & 1 & 0 \\ \hline\end{array} \\ =&\frac{1}{h}\begin{array}{|c|c|c|}\hline 0 & 1 & -1 \\ \hline\end{array} * \left(\frac{1}{h}\begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline\end{array} * f\right) \\ =&\left(\frac{1}{h}\begin{array}{|c|c|c|}\hline 0 & 1 & -1 \\ \hline\end{array} * \frac{1}{h}\begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline\end{array}\right) * f \\ =&\left(\frac{1}{h}\begin{array}{|c|c|c|}\hline -1 & 1 & 0 \\ \hline\end{array} \circledast \frac{1}{h}\begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline\end{array}\right) * f \\ =&\frac{1}{h^2}\begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array} * f \\ =&\frac{1}{h^2}\begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array} \circledast f \end{aligned}$$

In 2D: $\dfrac{\partial}{\partial x} \mathrel{\widehat{=}} \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline\end{array}$, $\dfrac{\partial}{\partial y} \mathrel{\widehat{=}} \begin{array}{|c|}\hline 0 \\ \hline -1 \\ \hline 1 \\ \hline\end{array}$, $\dfrac{\partial^2}{\partial x^2} \mathrel{\widehat{=}} \begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array}$, $\dfrac{\partial^2}{\partial y^2} \mathrel{\widehat{=}} \begin{array}{|c|}\hline 1 \\ \hline -2 \\ \hline 1 \\ \hline\end{array}$.

<u>Diskreter Laplace Operator</u> :

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \mathrel{\widehat{=}} \begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array} + \begin{array}{|c|}\hline 1 \\ \hline -2 \\ \hline 1 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline\end{array}$$

5.4.2 Glättungsfilter und partielle Differentialgleichungen

Wir haben gesehen: $m = \dfrac{1}{5}\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline\end{array}$ ist unter allen 5-Punkt Filtern der am besten glättende.

Idee: Rauschen weiter verringern indem man $m\circledast$ wiederholt anwendet $\Rightarrow$ Folge von Bildern:

$$\boxed{\underset{:=u^{(0)}}{f}} \xrightarrow{m\boxast} \boxed{u^{(1)}} \xrightarrow{m\boxast} \boxed{u^{(2)}} \cdots$$

$$\begin{aligned}
\Rightarrow u^{(n+1)} - u^{(n)} &= (\text{Unterschied zwischen 'Zeit' Punkt } n \text{ und } n+1) \\
&= \underbrace{m \boxast u^{(n)}}_{u^{n+1}} - \underbrace{\delta \boxast u^{(n)}}_{u^{(n)}} \text{ mit } \delta = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \\
&= (m - \delta) \boxast u^{(n)} \\
&= \left(\frac{1}{5} \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} - \frac{1}{5} \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 5 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \right) \boxast u^{(n)} \\
&= \frac{1}{5} \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} u^{(n)}
\end{aligned}$$

Somit gilt insgesamt:

$$\underbrace{u^{(n+1)} - u^{(n)}}_{\widehat{=} \frac{\partial u}{\partial t}} = \underbrace{\frac{1}{5} \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}}_{\widehat{=} \Delta u} \tag{5.10}$$

Kontinuierlich: Funktion u

$$u(x,t) \quad x \in \mathbb{R}^2, \ t \text{ Zeit}$$

(5.10) ist eine Diskretisierung (1 Zeitschritt im Eulerverfahren) der partiellen Differentialgleichungen

$$\frac{\partial u}{\partial t} = \Delta u \tag{5.11}$$

Bekannt als Wärmegleichung oder Diffusionsgleichung .
Zum Zeitpunkt $t = 0$ möge die Anfangsbedingung

$$u(x,0) = u^{(0)} = f(x) \tag{5.12}$$

gelten. Vorranschreiten der Zeit t repräsentiert Diffusion.
Für einen stationären Zustand, also keine Änderung $\frac{\partial u}{\partial t}$ dann muss auch $\Delta u = 0$ gelten.
Diese wird unteranderem von konstanten Funktionen oder linearen Funktionen $u(x_1, x_2) = ax_1 + bx_2$ erfüllt.

Es existiert auch einen explizite Formel für die Lösung der Diffusionsgleichung (5.11) mit Anfangsbedingung (5.12):

$$u(x,t) = \left(G_{\sqrt{2t}} * u^{(0)} \right)(x)$$

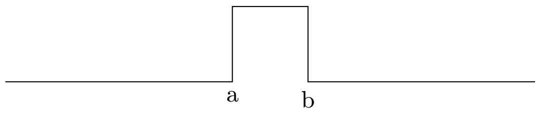
Wobei $\sqrt{2t}$ für eine Skalierung um diesen Wert steht.
Zu zeigen ist: $\dfrac{\partial u}{\partial t} = \Delta u$

$$\frac{\partial}{\partial t} \left(G_{\sqrt{2t}} * u^{(0)} \right) = \Delta \left(G_{\sqrt{2t}} * u^{(0)} \right)$$

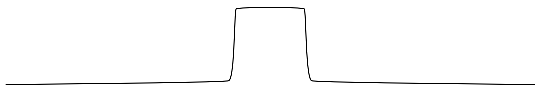
$$\overset{\text{mit Satz}}{\Longrightarrow} \left(\frac{\partial}{\partial t} G_{\sqrt{2t}} \right) * u^{(0)} = \left(\Delta G_{\sqrt{2t}} \right) * u^{(0)}$$

Es bleibt somit z.z.: $\frac{\partial}{\partial t} G_{\sqrt{2t}} = \Delta G_{\sqrt{2t}}$.

$t = 0$:



$t > 0$:



Bemerkenswert ist das, für $t = 0$ die Funktion nicht stetig ist, aber für alle $t > 0$ die Funktion beliebig oft differenzierbar ist.

Insgesamt lässt sich die Idee darstellen als: