Blatt 3. Aufgabe 2

1. $f \in L^1(\mathbb{R}^d)$, dann ist $\hat{f}$ beschränkt und stetig.
2. $c \in \mathbb{R} \setminus \{0\}$, $f \in L^1(\mathbb{R}^d)$. Es sei $g(x) := f(cx)$. Es gilt: $\hat{g}(z) = \frac{1}{|c|^d}\hat{f}(\frac{z}{c})$.
3. $f \in L^1(\mathbb{R}^d)$. Es sei $g(x) := \overline{f(x)}$. Es gilt: $\hat{g}(z) = \overline{\hat{f}(-z)}$.
4. Es sei $f \in L^1(\mathbb{R}^d)$ und $g(x) := f(x+a)$ für ein $a \in \mathbb{R}^d$. Dann gilt $\hat{g}(z) = e^{i\langle a,z\rangle}\hat{f}(z)$.

Blatt 3. Aufgabe 3. Gaußkern

Nur eindimensionalen Fall betrachet ... $G_{\sqrt{2t}} = \frac{1}{(4\pi t)^{d/2}} \cdot e^{\frac{-|x|^2}{4t}}$

$$\frac{\partial}{\partial t} G_{\sqrt{2t}}(x) = \Delta G_{\sqrt{2t}}(x)$$

$$\begin{aligned}
\frac{\partial}{\partial t} G_{\sqrt{2t}}(x) &= \frac{\partial}{\partial t} \frac{1}{(2\pi \cdot 2t)^{\frac{d}{2}}} \cdot exp(\frac{-|x|^2}{4t}) \\
&= \left(\frac{\partial}{\partial t} \frac{1}{(4\pi t)^{\frac{d}{2}}}\right) e^{-\frac{|x|^2}{4t}} + \frac{1}{(4\pi t)^{\frac{d}{2}}} \cdot \left(\frac{\partial}{\partial t} e^{-\frac{|x|^2}{4t}}\right) \\
&= \frac{-\frac{d}{2}4\pi}{(4\pi t)^{\frac{d}{2}-1}} \cdot e^{\frac{-|x|^2}{4t}} + \frac{1}{(4\pi t)^{\frac{d}{2}}} \cdot \left(\frac{\partial}{\partial t} \frac{-|x|^2}{4t}\right) e^{\frac{-|x|^2}{4t}} \\
&= \frac{-2d\pi}{(4\pi t)^{\frac{d}{2}-1}} \cdot e^{\frac{-|x|^2}{4t}} + \frac{1}{(4\pi t)^{d/2}} \cdot \left(\frac{|x|^2}{4t^2}\right) e^{\frac{-|x|^2}{4t}} \\
&= \left(\frac{-2d\pi}{(4\pi t)^{\frac{d-2}{2}}} + \frac{1}{(4\pi t)^{d/2}} \cdot \frac{|x|^2}{4t^2}\right) \cdot e^{\frac{-|x|^2}{4t}} \\
&= \left(\frac{-2d\pi}{\sqrt{(4\pi t)^2}} \cdot \frac{1}{(4\pi t)^{d/2}} + \frac{1}{(4\pi t)^{d/2}} \cdot \frac{|x|^2}{4t^2}\right) \cdot e^{\frac{-|x|^2}{4t}} \\
&= \left(\frac{-d}{2t} + \frac{|x|^2}{4t^2}\right) \frac{1}{(4\pi t)^{d/2}} \cdot e^{\frac{-|x|^2}{4t}}
\end{aligned}$$

$$\begin{aligned}
\Delta G_{\sqrt{2t}}(x) &= \Delta \frac{1}{(4\pi t)^{d/2}} \cdot e^{\frac{-|x|^2}{4t}} \\
&= \frac{1}{(4\pi t)^{d/2}} \Delta e^{\frac{-|x|^2}{4t}} \\
&= \frac{1}{(4\pi t)^{d/2}} \sum_{i\in\{1,..,d\}} \partial_i \left((\partial_i \frac{-|x|^2}{4t}) e^{\frac{-|x|^2}{4t}}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} \sum_{i\in\{1,..,d\}} \partial_i \left((\frac{-2x_i}{4t}) e^{\frac{-|x|^2}{4t}}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} \sum_{i\in\{1,..,d\}} \left((\partial_i \frac{-2x_i}{4t}) e^{\frac{-|x|^2}{4t}} + (\frac{-x_i}{4t}) \partial_i e^{\frac{-|x|^2}{4t}}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} \sum_{i\in\{1,..,d\}} \left((\frac{-1}{2t}) e^{\frac{-|x|^2}{4t}} + (\frac{-x_i}{4t})(\frac{-x_i}{4t}) e^{\frac{-|x|^2}{4t}}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}} \sum_{i\in\{1,..,d\}} \left(\frac{-1}{2t} + \frac{x_i^2}{(4t)^2}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}} \left(\left(\sum_{i\in\{1,..,d\}} \frac{-1}{2t}\right) + \frac{|x|^2}{(4t)^2}\right) \\
&= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}} \left(\frac{-d}{2t} + \frac{|x|^2}{(4t)^2}\right)
\end{aligned}$$