Liste der noch zu erledigenden Punkte

Kapitel 1

1.Overview

- „image society" (webpages: 1995 text-based, 2005 image based, 2015 video based ...)

  - data transfer rates ↑, compression rates ↑

  critical shift: reading → watching

- „Photoshop"-ing   (remove wrinkles, bumps, ...)

- Images in medicine („medical image proscessing"), x-ray, CT, MRI, ultrasound, ... („modalities").

  different questions:

  1.) Layout!

  align bottom $\quad$ measurments $\overset{?}{\Rightarrow}$ image

  expl: tomography

  $\Rightarrow$ difficult mathematical problems
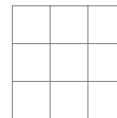
  2.) Image enhancements
  - denoising
    simple pixels/lines: „sandpaper" interpolation $\qquad$ so richtig?
    global noise: smoothing
  - grayscale
    histogramm balancing (spreading)
  - distortion
    makes straight lines (in real world) straight (in the images)
  - edge detection
    contour enhancement
  - segmentation
    detect and separate parts of the image
  - registration
    sequence of images of the same object $\Rightarrow$ Wort? , compare Skizze
    ↗ object following in a movie

Our Focus:

- mathematical models/methods/ideas

- (algorthms)

- ((implementation))

skipped: Very fast intro: Matlab and images

Kapitel 2

2.What is an image?

2.1 Discrete and continuous images

There are (at least) two different points of view:

Discrete / Digital

Abbildung 2.1: Discrete Image

Continous / Analogue
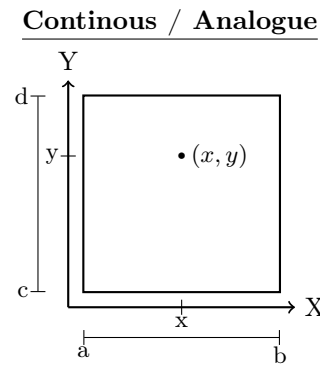
Abbildung 2.2: Continous Image

object:	matrix	function
tools:	linear algebra (SVD, . . .)	analysis (differentrage, integrate, . . .)
pros:	(finite storage) storage, complexity	freedom, tools, motions?P.4
		(e.g. edge discontinuity)
cons:	limitations: zooming, rotations, . . .	storage (infinite amout of data)

arguably, one has:

- real life $\Rightarrow$ continuous „images" (objects)

- digital camers $\Rightarrow$ discrete images

In general we will say:

Definition 2.1 ((mathematical) image)**.** A (mathematical) *image* is a function

$$u : \Omega \to F,$$

where: $\Omega \subset \mathbb{Z}^d$ (discrete) or $\Omega \subset \mathbb{R}^d$ (continuous) . . . *domain*

$\quad$ $d = 2$ (typical case 2D), $d = 3$ („3D image" = body or $\underbrace{\text{2D + time}}_{movie}$)

$\quad$ $d = 4$ (3D + time)

F ... *range of colours*
    $F = \mathbb{R}$ or $[0, \infty]$ or $[0, 1]$ or $\{0, \ldots 255\}$, ... grayscale (light intensity)
    $F \subset \mathbb{R}^3$ ... RGB image (colored)

$F = \{0, 1\}$ ... black/white
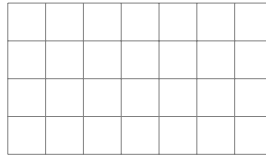
3 Layers
$\Rightarrow$ colored images:w

> Matlab stuff

Large parts of the course: analytical approach (i.e. continuous domain Ω)
Since we want to differentirate, ... the image u.

Still: need to assume that also F ist continuous (not as $\{0, 1\}$, $\{0, 1, \ldots, 255\}$ or $\mathbb{N}$)
 since otherwise the only differentiable (actually, the only continuous) functions $u : \Omega \to F$ are
 constant functions $\Leftrightarrow$ single-colour images

Also: We usually take F one-dimensional ($F \subset \mathbb{R}$). Think of it as either

 - gray scaled image, or

 - treating R,G & B layer separately

2.2   Switching between discrete and continuous images

continuous $\to$ discrete:

- divide the continuous image in small squared pieces (boxes) (superimpose grid)

- now: represent each box by *one* value

  - strategy 1: take function value $u(x_i)$
          for x_i = midpoint of box B_i

  - strategy 2: use mean value

$$\frac{1}{|B_i|} \int_{B_i} u(x) dx$$

$\Omega =$

Box B_i
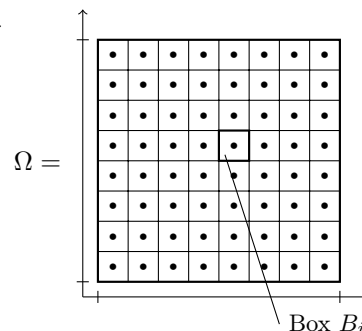
$\Rightarrow$ discrete image

strategy 1: simple (and quick) but problematic ($u(x_i)$ might represent $u|_{B_i}$ badly; for $u \in L^p$,
          single point evaluation not even defined)

strategy 2: more complex but also more „democratic" (actually closer to the way how CCD Sensors
          in digital cameras work)

often the image value of the box B_i gets also digitized, i.e. fitted (by scaling & rounding) into range
$\{0, 1, dots, 255\}$

discrete $\to$ continous

This is of course more tricky ...

- Again: each pixel of the discrete image corresponds to a „box" of the continuous image
          (that is still to be constructed)
- Usually: pixel value $\mapsto$ function value at the *midpoint* of the box
- Question: How to get the other function values (in the box)?
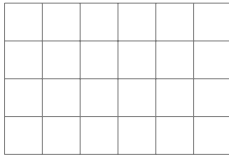
<u>idea 1:</u>   just take the function value of the nearest midpoint („nearest neighbour interpolation")

For each $x \in B_i : u(x) := u(x_j)$ where $|x - x_j| = \min_k |x - x_k|$
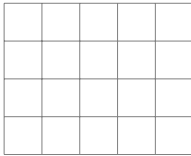
$\Rightarrow$ $u(x) = u(x_i)$ for all $x \in B_i$
$\Rightarrow$ each box is uni-color
$\Rightarrow$ the continuous image is essentially still discrete

<u>idea 2</u>: (bi-) linear interpolation
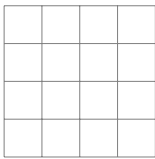
Let $a, b, c, d \ldots$ function values at 4 surrounding adjacent midpoints ($\nearrow$ figure)
$\alpha, \beta, 1 - \alpha, 1 - \beta \ldots$ distance to dotted lines ($\nearrow$ figure, w.l.o.g, bob is 1×1)

interpolation (linear) on the dotted line between a and b:

$$e := a + \alpha(b - a) = (1 - \alpha)a + \alpha b$$
(1D - interpolation, convex combination)

Similarly: $f = (1 - \alpha)c + \alpha d$

Then: The same 1D-interpolation between e and f
$$\Rightarrow u(x) := (1 - \beta) \cdot e + \beta \cdot f$$
$$= (1 - \beta)[(1 - \alpha)a + \alpha b] + \beta[(1 - \alpha)c + \alpha d]$$
$$= \underbrace{(1 - \alpha)(1 - \beta)}\, a + \underbrace{\alpha(1 - \beta)}\, b + \underbrace{(1 - \alpha)\beta}\, c + \underbrace{\alpha\beta}\, d$$
$$\in [0, 1] \wedge \sum = 1$$

$\Rightarrow$ <u>convex combination</u> of the function values a, b, c, d at the the surrounding 4 midpoints (on which points is the nearest, instead of taking just a, b, c or d - depending)
$\Rightarrow$ 2D linear interpolation, *bi-linear interpolation* (can be interpreted as spline interpolation with bilinear [basis] splines).
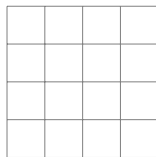
Beispiel 2.2. Rotate image   by angle $\phi \neq k \cdot \frac{\pi}{2}$

- continuous image case: no problem

turn by α
$\Rightarrow$
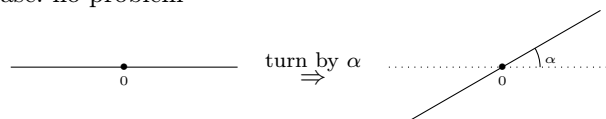
$$x = D_\varphi \, y \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; D_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

2D rotation matrix

$$y = D_\varphi^{-1} \, x = D_{-\varphi} \, x$$

$$\Rightarrow v(x) := u(y) = u(D_{-\varphi} \, x) \quad \forall x \in \text{domain of the rotated image}$$

- discrete image case: problem !

   For $x \in$ domain of notated image, in general $D_{-\varphi}\, x \notin$ domain of original image[1]
   Way out: $v(x) := interpolation$ between the $u(\cdot)$ of the 4 surrounding pixels of $D_{-\varphi}$
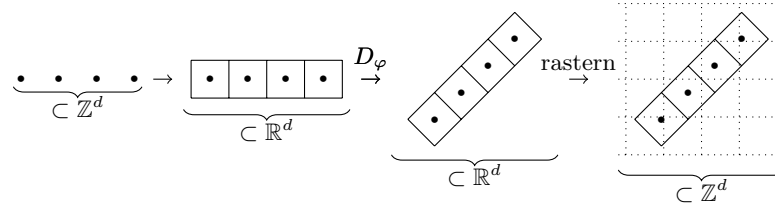
Something to think about:
 What happens in the limit (?) if we, starting with an image (discrete or continuous), repeatedly
 switch between discrete and continuous, non-stop . . . ?
 Does the answer depend on the way of switching ? (continuous $\to$ discrete: midpoint or average,
 discrete $\to$ continuous: nearest neighbour or bilinear?)

_____

[1] it's not an integer

Kapitel 3

3.Histogramm and first applicatsion

3.1 The histogramm

Definition 3.1 (histogram)**.** Let $\Omega \subset \mathbb{Z}^d$, $F \subset \mathbb{R}$ discrete and $u : \Omega \to F$ a discrete discrete image. The function

$$H_u : F \to \mathbb{N}_0 \ (:= \mathbb{N} \cup \{0\})$$

with

$$H_u(k) := \# \{x \in \Omega : u(x) = k\}, \quad k \in F$$

is called *histogramm* of the image u.

$H_u(k)$ counts how often colour k appears in u.

$$\sum_{k \in F} H_u(k) = |\Omega| = \text{number of pixels in the whole image}$$

or

$$\frac{H_u(k)}{|\Omega|} = \text{relative frequence of colour } k \text{ in image } u$$
$$\text{(relative Häufigkeit)}$$

Beispiel 3.2.

$$u = \begin{array}{|c|c|} \hline \blacksquare & \blacksquare \\ \hline \blacksquare & \square \\ \hline \end{array} \quad \rightarrow$$
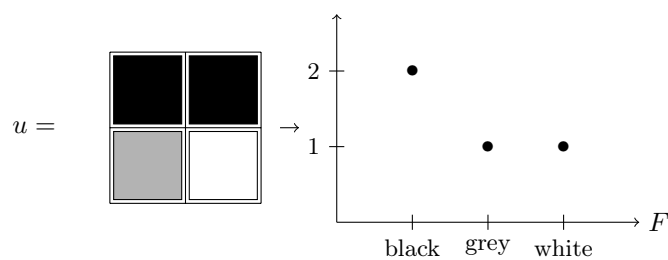
If u ist a continous image, H_u can be understood as a measure (generalized function)[1].
Another way to write this:

$$H_u(k) = \sum_{x \in \Omega} \delta_{u(x)}(k), \ k \in F \qquad H_u(k) = \int_{\Omega} \delta_{u(x)}(k) dx, \ k \in F$$

> hier fehlt noch das Kronecker underarrow

> Matlab-Code

---

[1]density of a probability distribution

3.2   Application: contrast enhancement

If the image only uses a small part of the available colour/grayscale „palette" F, then its contrast can be improved by „spreading" the histogramm over all of F.

Simple idea:

originales Bild   Zauberei T

$$\Omega \qquad F \qquad F$$

$$v = T \circ u$$
$$v(k) = T(u(k))$$

Bild mit besserem Kontrast
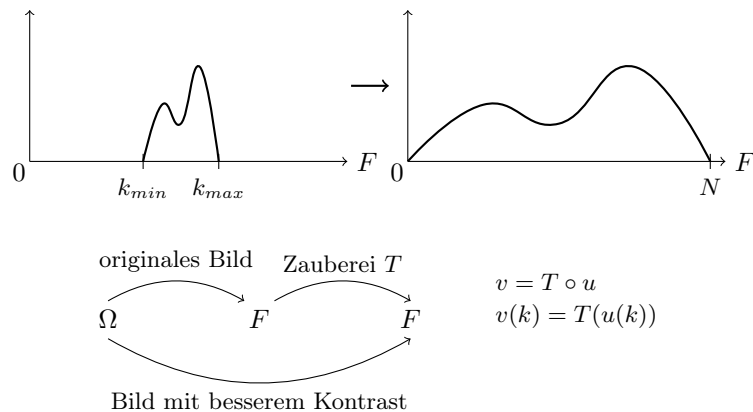
5

The above simple idea („contrast stretching") corresponds to

$$\varphi : k_{\min} \mapsto 0$$
$$k_{\max} \mapsto N$$
$$\text{and linear in between}$$

$$\text{i.e} \qquad \varphi(k) \quad = \left[\frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot N \right]$$

Where $[\quad \cdot \quad]$ means ... rounding to the nearest integer (assumuning that $F = \{0, 1, \ldots, N\}$).

Example histogram:

$$k_{min} = 19, \ k_{max} = 103$$
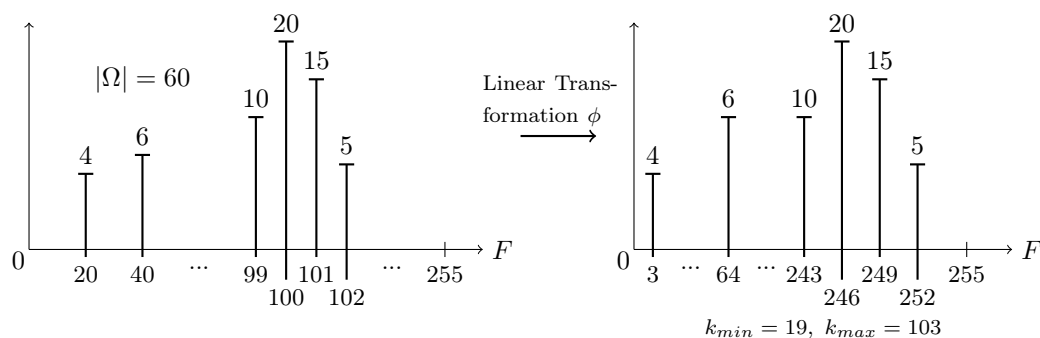
A bit more sophisticated:

$$\varphi : (k_{\min} \mapsto 0)$$
$$k_{\max} \mapsto N$$
$$\text{and } \textbf{non} \text{ linear in between}$$

such that colour ranges that occur more frequently in u can occupy a larger range of colours in u. ($\Rightarrow$ visibility $\uparrow$)

Example histogramm spread out according to frequency of occurence:
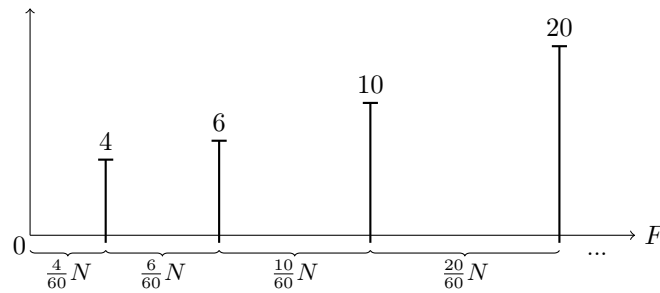
$\Rightarrow$ „density" is equalized over $F = \{0, \dots, N\}$

Ideal would be:
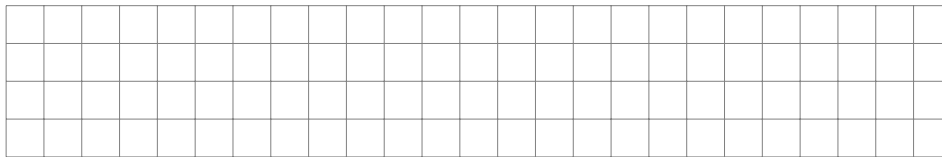
Layout S.12 u

Note: The new colours (i.e the location of the bars in the histogramm of u) only depend on the frequencies / *height* of the bars in H_u but not on the colours/location of the bars in H_u

Finally: The formula

$$\varphi(k) = \left\lceil \frac{N}{|\Omega|} \sum_{l=0}^{k} H_u(l) \right\rceil$$

This process is called „histogramm equalization"

Exercise ?!

3.3 Another application: conversion to b/w

Task: convert grayscale image to black white
 - interesting for object detection/*segmentation* ...!

 Idea: Find a threshold $t \in T$ s.t. the histogramm splits into two „characteristic" parts
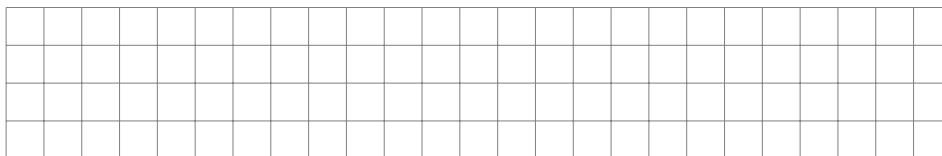
For $t \in F$ put

$$\text{black} := \{k \in F : k \leq t\}$$
$$\text{white} := \{k \in F : k > t\}$$

and

$$\tilde{u} := \begin{cases} 0, & u(x) \in \text{black} \\ 1, & u(x) \in \text{white} \end{cases} \qquad \tilde{F} = \{0, 1\}$$

How to find the threshold t:

1.) Shape based methods
   If the histogramm is „biomodal"
   Put $t := \dfrac{k_{\max_1} + k_{\max_2}}{2}$
   or $t := k_{\min}$
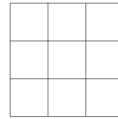
Kapitel 4

4.Basic Morphological Operations

B/W Bild:

$$A =$$

Structural element :

$$B =$$
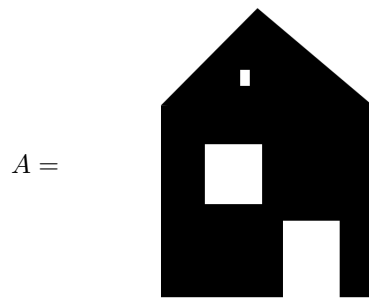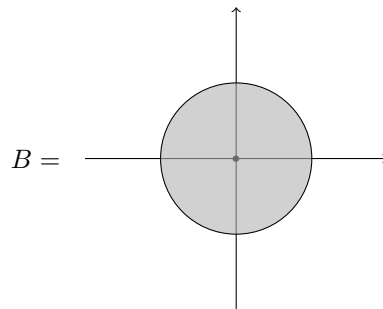
4.1 Operations on A and B

$$A + B := \{a + b : a \in A, b \in B\}$$

This is called <u>dilation</u>.
You might imagine that at every dark point in the image A the Structurelement is applied.

$$A + B =$$
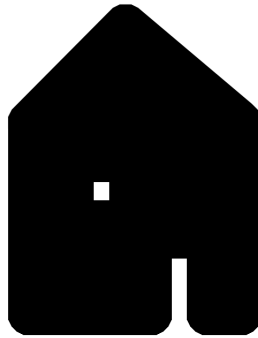
Image created in Matlab through:

```
1  I=imread('Bild1.png');
2  se=strel('disk',40,8);
3  I2=imcomplement(imdilate(imcomplement(I),se));%I am using the complement of the image
       here so that the structural element is applied to the dark parts of the image
4  imshow(I2);
```

$$A - B := \{a : a + B \subset A\}$$

This is called erosion .
You can imagine that you search for the points in which the structural element fits.

$$A - B =$$
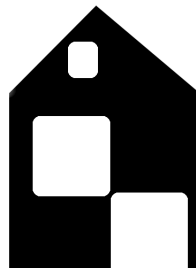
Image created in Matlab thorugh:

```
1  I=imread('Bild1.png');
2  se=strel('disk',20,8);
3  I2=imcomplement(imerode(imcomplement(I),se));
4  imshow(I2);
```

One may quickly realize that $A \neq (A + B) - B$, so a new Operation is introduced:

$$A \bullet B := (A + B) - B$$

This is called closing and is used to e.g. remove noise. In the example image you might notice that the upper window is missing.

$$A \bullet B =$$

Image created in Matlab thorugh:

```matlab
I=imread('Bild1.png');
se=strel('disk',20,8);
I2=imcomplement(imdilate(imcomplement(I),se));
I3=imcomplement(imerode(imcomplement(I2),se));
imshow(I3);
```

The inverse also exists:

$$A \circ B := (A - B) + B$$

This is called opening .

This time with a new example:
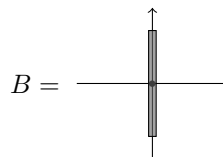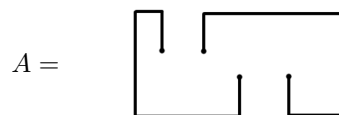
$$A =$$

$$B =$$

$$A \circ B = \quad \left| \begin{matrix} | & | & & & | \\ | & & | & | & | \\ | & & & & | \end{matrix} \right|$$

Image created in Matlab thorugh:

```matlab
I=imread('Bild2.png');
se=strel('line',10,90);
I2=imcomplement(imerode(imcomplement(I),se));
I3=imcomplement(imerode(imcomplement(I2),se));
imshow(I3);
```

Kapitel 5

5.Entrauschen: Filter und Co

5.1 Noise

<u>Noise</u> = Unwanted disturbances in an image. Mostly becaue of

- point wise

- random

- independent

We consider *noise* to be an additive disturbances (for multiplicative noise use *log*).
Notation:

$$\boxed{f_0} \xrightarrow{\;+\;\text{noise}\;} \boxed{f} \xrightarrow{\;\text{denoise}\;} \boxed{u}$$

$$\quad\text{clean}\qquad\qquad\text{disturbed image}\qquad\text{result}$$
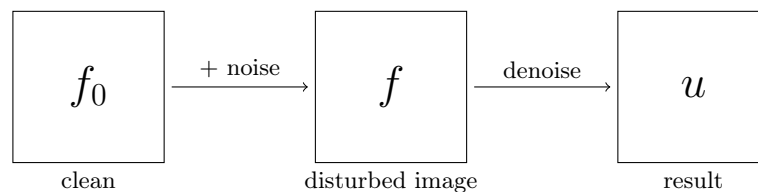
The quality of the denoised image u compared to the original image f_0 is described by norms:

$$||f - f_0|| \dots \text{noise}$$
$$||u - f_0|| \dots \underline{\text{absolute error}}$$
$$\frac{||u - f_o||}{||f - f_0||} \dots \underline{\text{relative error}} \text{ compared to the noise}$$
$$\frac{||u - f_o||}{||f_0||} \dots \text{relative error compared to the signal}$$

Typically the chosen norm is:

$$||f|| = ||f||_2 = \sqrt{\int_\Omega |f(x)|^2 \, dx}$$

or in the discrete:

$$||f||_2 = \sqrt{\sum_{x\in\Omega} |f(x)|^2}$$

Closely connected is the <u>Signal to noise ratio</u> (SNR):

$$log(\underbrace{\frac{||f_0||_2}{||u - f_0||_2}}_{\in\,[1,\infty)}) \in [0, +\infty), \text{ where 0 is bad and } +\infty \text{ is good.}$$

5.2 smoothing filter

Idea: (to simplify in 1D)
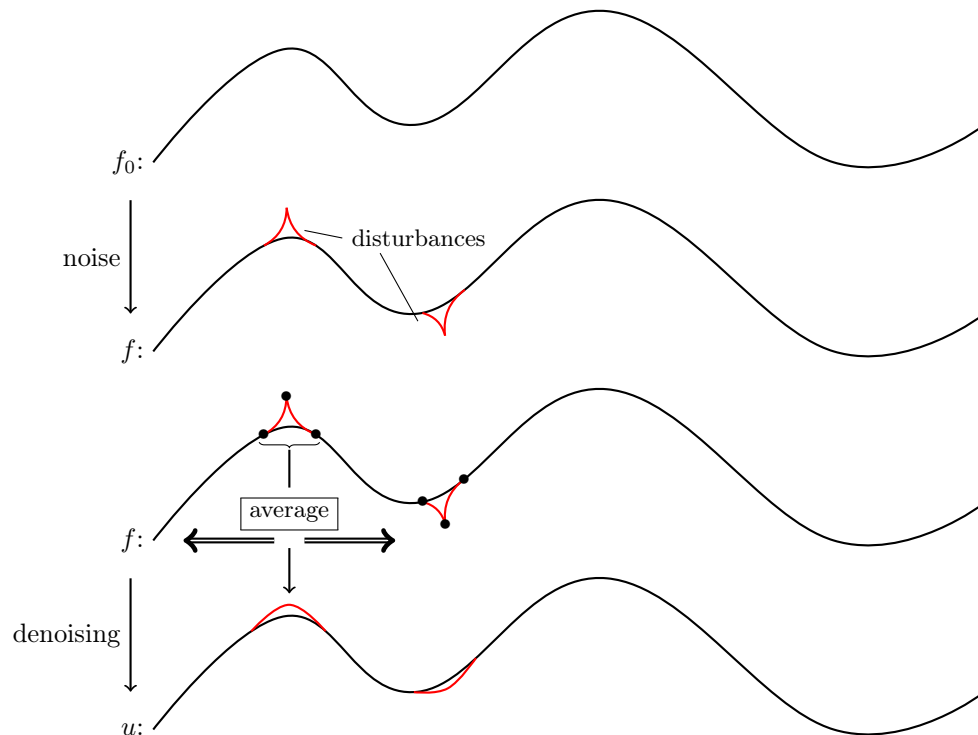
$$u(k) := \alpha \cdot f(k-1) + \beta \cdot f(k) + \gamma \cdot f(k+1) \tag{5.1}$$

where:

$$\alpha + \beta + \gamma = 1 \tag{5.2}$$

More precisely (5.1) means:
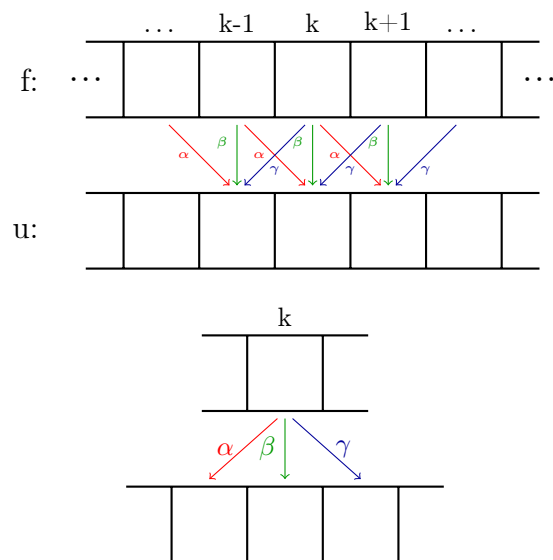
With (5.1) there is a mapping $f \mapsto u$, we write

$$u = m \boxast f, \text{ this is called } \underline{\text{Correlation}}.$$

where:

$$(m \boxplus f)(k) = \sum_{i \in supp(m)} m(i)f(k+i) \tag{5.3}$$

and:

$$m= \quad \cdots \quad \begin{array}{|c|c|c|} \hline \alpha & \beta & \gamma \\ \hline \end{array} \quad \cdots \quad \text{called } \underline{\text{mask}} .$$

with column labels $\cdots \quad -1 \quad 0 \quad 1 \quad \cdots$

If you set $j := k + i$ in (5.1), then $i = j - k$, which means:

$$(m \boxplus f)(k) = \sum_{i \in supp(m)} m(j - k)f(j) \tag{5.4}$$

To apply the mapping onto the boundary the image is reflected, in 1D:
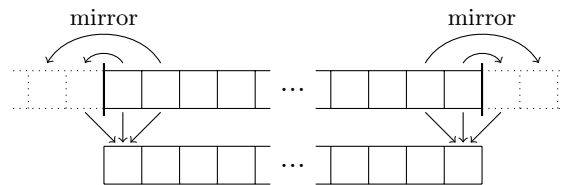
in 2D:

Formula (5.4) might remind one of the $\underline{\text{convolution}}$:

<div style="background-color:yellow">Layout!</div>

$$(g * f)(k) = \sum_{j \in \mathbb{Z}} g(\underbrace{k - j}_{\text{Difference to (5.4)}}) \cdot f(j) \tag{5.5}$$

If you set $g(i) := m(-i) =: \tilde{m}(i)$, which corresponds to a reflection of the Mask, then

$$m \boxplus f = g * f = \tilde{m} * f$$

<div style="background-color:orange">Im Skript hier noch Beispiele und soetwas p. 32f</div>

Properties of the convolution:

1. $(f * g) * h = f * (g * h)$, Associativity

2. $f * g = g * f$, Commutativity

3. $\tilde{f} * \tilde{g} = \widetilde{f * g}$, Compatibility with reflection

Properties of the correlation:

1. $f \boxplus (g \boxplus h) = \tilde{f} * (\tilde{g} * h) \stackrel{\boxed{1}}{=} (\tilde{f} * \tilde{g}) * h \stackrel{\boxed{3}}{=} (\widetilde{f * g}) * h = (f * g) \boxplus h \neq (f \boxplus g) \boxplus h$, not associative!

2. $f \boxplus g = \tilde{f} * g \stackrel{\boxed{2}}{=} g * \tilde{f} = \tilde{\tilde{g}} * \tilde{f} \stackrel{\boxed{3}}{=} (\widetilde{\tilde{g} * f}) = \widetilde{g \boxplus f} \neq g \boxplus f$, not commutative!

3. $\tilde{f} \boxplus \tilde{g} = \tilde{\tilde{f}} * \tilde{g} \stackrel{\boxed{3}}{=} (\widetilde{\tilde{f} * g}) = \widetilde{f \boxplus g}$, Compatibility with reflection

$\boxplus$ und $*$ definiert man auf: $\ell^1(\mathbb{Z}^d) := \left\{ f = (f_i)_{i \in \mathbb{Z}^d} : \underbrace{\sum_{i \in \mathbb{Z}^d} |f_i|}_{:= ||f||_1} < \infty \right\}$

Man kann zeigen (Übung): $f, g \in \ell^1 \Rightarrow f * g \in \ell^1$ und $||f * g||_1 \leq ||f||_1 \cdot ||g||_1$. Wobei oft die Gleichheit gilt.

Alles gilt auch in der Kontinuierlichen Version:

$$L^1(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{R} : \underbrace{\int_{\mathbb{R}^d} |f| \, dx}_{:= ||f||_1} < \infty \right\}$$

$$f, g \in L^1(\mathbb{R}^d) : (g * f)(x) = \int_{\mathbb{R}^d} g(x - y) f(y) dy, \ y, x \in \mathbb{R}^d$$

Beispiel für den kontinueirlichen Fall:
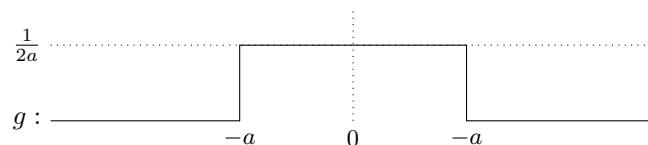
Hierbei gilt $\int_{\mathbb{R}} g(x) dx = 1$
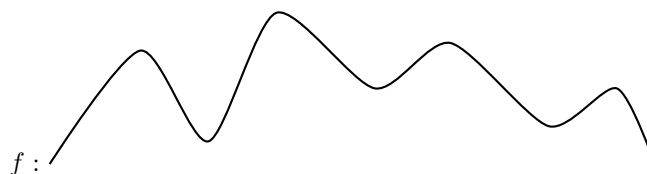
$g \boxplus f = \underline{\text{gleitendes Mittel}}$.

Layout!

Weitere Eigenschaften der Faltung:

Für alle $f, g \in L^1$ or ℓ^1

$$\left.\begin{array}{r} (g_1 + g_2) * f = (g_1 * f) + (g_2 * g) \\ (\alpha g) * f = \alpha(g * f) \end{array}\right\} = \text{Linearität}$$

Somit ist:

$$g \mapsto f * g$$

ein linearer Operator.

Formt ℓ^1 bzw. L^1 eine Algebra mit neutralem Element δ?

ℓ^1?:

$$\delta: \quad \cdots \boxed{\quad 0 \quad | \quad 0 \quad | \quad 1 \quad | \quad 0 \quad | \quad 0 \quad} \cdots$$
$$\underset{\underset{\text{Pos 0}}{\uparrow}}{}$$

Ja!

L^1?: Für ein solches Element muss gelten:

$\forall f \in L^1 : d * f = f$

$\forall x \in \mathbb{R} : \int_{\mathbb{R}^d} \underbrace{\delta(x - y)}_{=0 \forall x \neq y} f(y) dy = f(x)$

Diese Funktion wird <u>Dirac-Impuls</u> gennant ist aber kein Element von L^1.

<u>Nun zu Masken in 2D:</u>

$$u = m \boxplus f \text{ mit } m = \begin{array}{ccc} & \boxed{\alpha} & \\ \boxed{\beta} & \boxed{\gamma} & \boxed{\delta} \\ & \boxed{\epsilon} & \end{array}$$

wobei $\alpha + \beta + \gamma + \delta + \epsilon = 1$

Kurzschreibweise: $u_{ij} := u(x)$ wobei $x = \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2$, analog für f_{ij}.

$$\Rightarrow u_{ij} = \alpha f_{i-1,j} + \beta f_{i,j-i} + \gamma f_{ij} + \delta f_{i,j+1} + \epsilon f_{i+1,j}$$

$$u = m \boxplus f = \tilde{m} * f \text{ mit } \tilde{m} = \begin{array}{ccc} & \boxed{\epsilon} & \\ \boxed{\delta} & \boxed{\gamma} & \boxed{\beta} \\ & \boxed{\alpha} & \end{array}$$

<u>Symmetrischer Fall:</u>

$$\tilde{m} = \begin{array}{ccc} & \boxed{\alpha} & \\ \boxed{\alpha} & \boxed{\gamma} & \boxed{\alpha} \\ & \boxed{\alpha} & \end{array} \text{ mit } \gamma = 1 - 4\alpha$$

$$u_{ij} = (1 - 4\alpha)f_{ij} + \alpha(f_{i-1,j} + f_{i,j-1} + f_{i,j+1} + f_{i+1,j}) \tag{5.6}$$

Erinnerung:

$$\boxed{f_0} \xrightarrow{+ \text{ Rauschen}} \boxed{f} \xrightarrow{\text{Entrauschen}} \boxed{u}$$

Sauberes Bild   Gestörtes Bild   Resultat
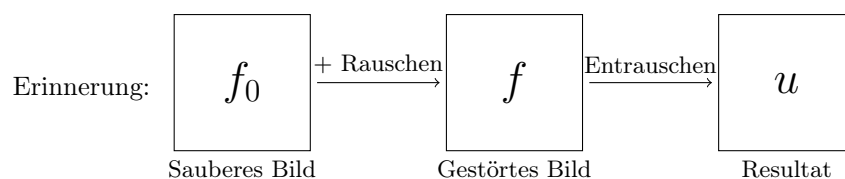
Annahme: $f_{ij} = f_{ij} + r_{ij}$ mit $r_{ij} \sim N(0, \sigma^2)$ iid.

z.z.: $Var(u_{ij}) \leq Var(f_{ij})$

$$Var(f_{ij}) = E(\underbrace{f_{ij} - \overbrace{Ef_{ij}}^{f_{ij}^0}}_{r_{ij}})^2 = \sigma^2$$

Layout!

$$Var(u_{ij}) = E(u_{ij} - Eu_{ij})^2 = E((1 - 4\alpha)(\underbrace{f_{ij} - f_{ij}^0}_{r_{ij}}) + \alpha((\underbrace{f_{i-1,j} - f_{i-1,j}^0}_{r_{i-1,j}}) + ... + (\underbrace{f_{i+1,j} - f_{i+1,j}^0}_{r_{i+1,j}})))^2$$

$$= E((1 - 4\alpha)^2 r_{ij}^2 + \alpha^2(r_{i-1,j}^2 + r_{i,j-1}^2 + r_{i,j+1}^2 + r_{i+1,j}^2) + 2(1 - 4\alpha)\alpha r_{ij}r_{i-1,j}...)$$

$$= (1 - 4\alpha)^2 \underbrace{Er_{i,j}^2}_{\sigma^2} + \alpha^2(Er_{i-1,j}^2 + ... + Er_{i+1,j}^2) + 2(1 - 4\alpha)\alpha \underbrace{E(r_{ij}r_{i-1,j})}_{\underbrace{Er_{ij}Er_{i-1,j}}_{0}} + \underbrace{...}_{0})$$

$$= (1 - 4\alpha)^2 \sigma^2 + \alpha^2 4\sigma^2 = (1 - 8\alpha + 16\alpha^2 + 4\alpha^2)\sigma^2$$

Da $0 \leq \alpha$ und $0 \leq 1 - 4\alpha \Rightarrow 0 \leq \alpha \leq \frac{1}{4}$:

$$(1 - 8\alpha + 16\alpha^2 + 4\alpha^2)\sigma^2 = 1 + \underbrace{\underbrace{20\alpha}_{\geq 0}\underbrace{(\alpha - \frac{2}{5})}_{< 0}}_{\leq 1}$$

$\Rightarrow Var(u_{ij}) \leq Var(f_{ij})$ für $\alpha \in [0, \frac{1}{4}]$

Dabei gilt: $Var(u_{ij}) \xrightarrow{\alpha} d\min \iff 1 - 8\alpha + 20\alpha^2 \xrightarrow{\alpha} \min \iff -8 + 40\alpha = 0 \iff \alpha = \frac{1}{5}$

$$\Rightarrow \text{bester Filter}: \quad \begin{array}{ccc} & \frac{1}{5} & \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ & \frac{1}{5} & \end{array}$$

Kapitel sollte noch fehlergelesen werden. Es könnte noch einiges aus dem Skript übernommen werden. Es braucht etwas Layout

5.3   Frequenzfilter

Ansatz: Rauschen $\approx$ hochfrequente Anteile des Bildes/Signals

$\Rightarrow$ gezieltes entfernen

Wichtiges Instrument: Fouriertransformation (FT)

$$\mathcal{F}: f \mapsto \hat{f} \text{ mit } \hat{f}(z) = \frac{1}{(2\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} dx$$

hier fehlt der rest aus einer Vorlesung
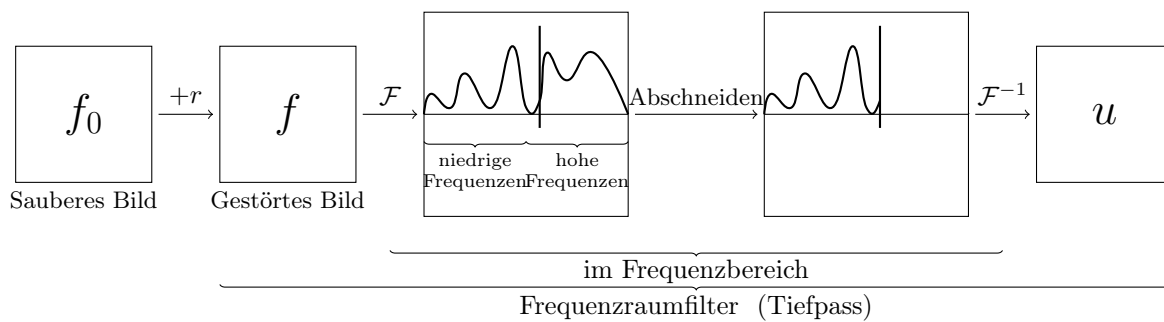
Wobei $z \in \mathbb{R}^d$, $f \in L^1(\mathbb{R}^d)$.

Falls auch $\hat{f} \in L^1(\mathbb{R}^d)$ ist ,dann lässt sich f wie folgt mittels der inversen Fouriertransformation aus $\hat{f}$ rekonstruieren:

$$\mathcal{F}^{-1} : \hat{f} \mapsto f$$

$$\boxed{\hat{f}(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{i\langle z, x\rangle} dx} \tag{5.7}$$

Wobei $x \in \mathbb{R}^d$.

Man hat also $\mathcal{F}^{-1}\mathcal{F}f$, d.h.

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(y) e^{-i\langle z, y\rangle} dy \right) e^{i\langle z, x\rangle} dz$$

Sei nun $e_z(x) := e^{i\langle z, x\rangle}$, $x \in \mathbb{R}^d$ mit Parameter $z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}$.

Also $e_z(x) = e^{i\langle \binom{z_1}{z_2}, \binom{x_1}{x_2}\rangle} = e^{i(z_1 x_1 + z_2 x_2)}$

Beispiele in $2D$:

(Hier stellen die Linien, Punkte mit konstantem wert dar)

$z = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_z(x) = e^{ix_1}$:

$z = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ e_z(x) = e^{i2x_1}$:

$z = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ e_z(x) = e^{ix_2}$:

$z = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ e_z(x) = e^{i2x_2}$:

$z = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ e_z(x) = e^{i(x_1+x_2)}$:

$z = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ e_z(x) = e^{i(-2x_1+x_2)}$:
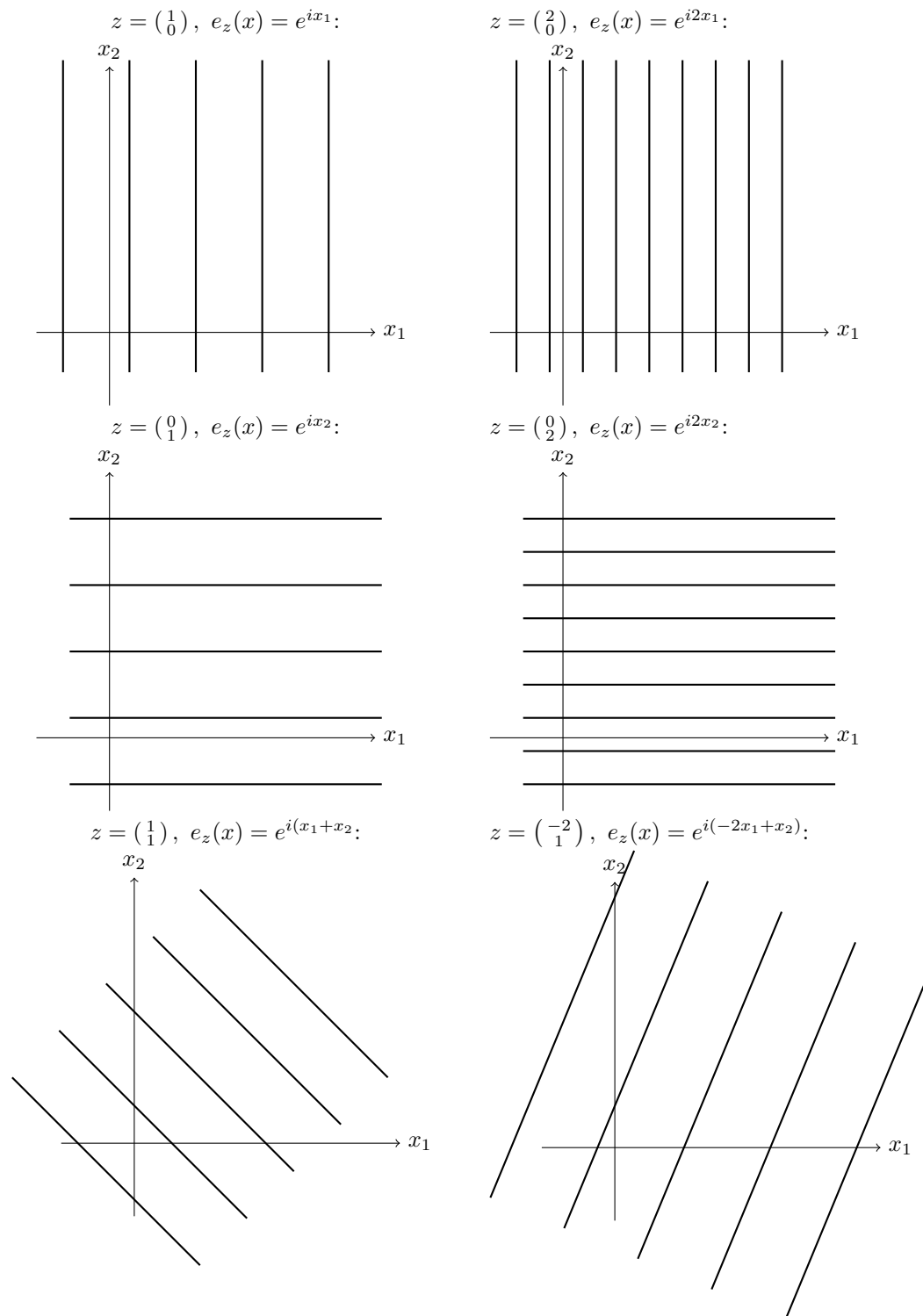
$f \in L^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \,|\, \int_{\mathbb{R}^d} |f|^2\,dx < \infty\}$ ist

- ein normierter Raum mit $+$, $\alpha\cdot$ und $||\cdot||_2 := \sqrt{\int_{\mathbb{R}^d} |f(x)|^2\,dx}$

- ein Skalarproduktraum mit $\langle f, g \rangle := \int_{\mathbb{R}^d} f\bar{g}\,dx$, wobei $||f||_2^2 = \langle f, f \rangle$

- ein vollständiger Raum, also <u>Banachraum</u>

Ein vollständiger normierter Banachraum mit Skalarproduk heißt <u>Hilbertraum</u> .

$\mathcal{F}$ kann auch als Abbildung auf $L^2(\mathbb{R}^d)$ betrachtet werden. Dann gilt:

$$\hat{f} = \mathcal{F}f \in L^2(\mathbb{R}^d)$$

und

$$\left\| \hat{f} \right\|_2 = \|f\|_2 \tag{5.8}$$

und sogar

$$\left\langle \hat{f}, \hat{g} \right\rangle_2 = \langle f, g \rangle_2 \tag{5.9}$$

für alle $f, g \in L^2(\mathbb{R}^d)$.

Weitere Eigenschaften der Fouriertransformation:

- $f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f}$ stetig und $\lim\limits_{|z| \to \infty} \hat{f}(z) = 0$

- $\mathcal{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ ist eine lineare Abbildung

- $\mathcal{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ ist eine beschränkte/stetige Abbildung

- Verschiebung $\overset{\mathcal{F}}{\to}$ Modulation, d.h.

$$g(x) = f(x + a) \Rightarrow \hat{g}(z) = e^{i\langle a, z \rangle} \hat{f}(z)$$

- Modulation $\overset{\mathcal{F}}{\to}$ Verschiebung, d.h.

$$g(x) = e^{i\langle x, a \rangle} f(x) \Rightarrow \hat{g}(z) = \hat{f}(z - a)$$

- Skalierung $\overset{\mathcal{F}}{\to}$ inverse Skalierung, d.h.

$$g(x) = f(cx) \Rightarrow \hat{g}(z) = \frac{1}{|c|} \hat{f}(\frac{z}{|c|})$$

- Konjugation: $g(x) = \overline{f(x)} \Rightarrow \hat{g}(z) = \overline{\hat{f}(-z)}$
  Folglich: f reelwertig $\Rightarrow \hat{f}(z) = \overline{\hat{f}(-z)}$

-

$$\text{Grundmode: } \hat{f}(0) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)dx$$

$$\text{Analog: } f(0) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{f}(x)dx$$

- Differentation $\overset{\mathcal{F}}{\to}$ Multiplikation mit Potenzen von z, d.h.

$$g(x) = \frac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x) \Rightarrow \hat{g}(z) = i^{\alpha_1 + \cdots + \alpha_d} z_1^{\alpha_1} \cdots z_d^{\alpha_d} \hat{f}(z)$$

- Unkehrung des letzten Punktes:

$$g(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d} f(x) \Rightarrow \hat{g}(z) = i^{\alpha_1 + \cdots + \alpha_d} \frac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial x_1^{\alpha_1}} \hat{f}(z)$$

-

$$\text{Faltungssatz: } \mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}} \mathcal{F}(f) \cdot \mathcal{F}(g), \quad \widehat{f * g} = (2\pi)^{\frac{d}{2}} \hat{f} \cdot \hat{g}$$

$$\text{Analog: } \mathcal{F}(f \cdot g) = \frac{1}{(2\pi)^{\frac{d}{2}}} \mathcal{F}(f) * \mathcal{F}(g), \quad \widehat{f \cdot g} = \frac{1}{(2\pi)^{\frac{d}{2}}} \hat{f} * \hat{g}$$

d.h.: Faltung $\overset{\mathcal{F}}{\to}$ Multiplikation und umgekehrt

Zur Erinnerung:

$$f_0 \xrightarrow{\ \mathcal{F}\ } \hat{f} \xrightarrow[\text{abschneiden}]{\text{Hohe Frequenzen}} \hat{u} \xrightarrow{\ \mathcal{F}^{-1}\ } u$$

$$\underbrace{}_{\text{tiefe Frequenzen}} \qquad -r \quad 0 \quad r$$

$$\bullet \quad -r \quad 0 \quad r$$

$\underline{\text{Zeitbereich}}$	$\underline{\text{Frequenzbereich}}$

$$f \xrightarrow[\quad n\mathbb{C}log(n)\quad]{\mathcal{F}} \hat{f}$$

Falltung, $*g$ $\Big| n^2$ $\qquad n \cdot log(n) \qquad n \Big|$ Mult. mit:

$$\hat{g} := \quad \underset{0}{\rule{2cm}{0pt}} \quad \tfrac{1}{(2\pi)^{\frac{d}{2}}}$$

$$f * g = u \xleftarrow[\mathcal{F}^{-1}]{\quad n \cdot log(n)\quad} \begin{array}{l}\hat{u} = \hat{f}\cdot\hat{g}\cdot(2\pi)^{\frac{d}{2}} \\ \quad = \widehat{f*g}\end{array}$$
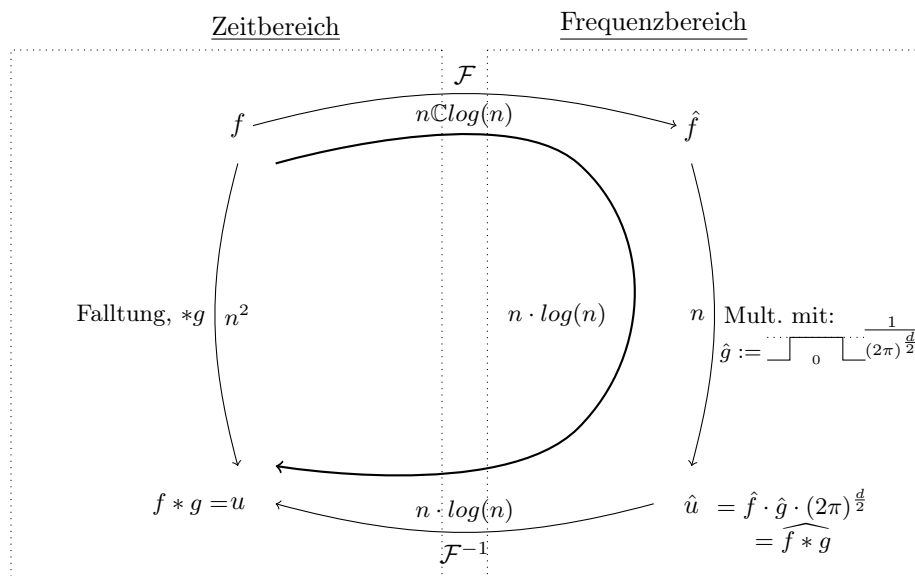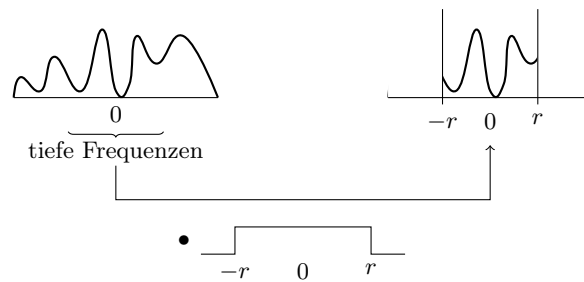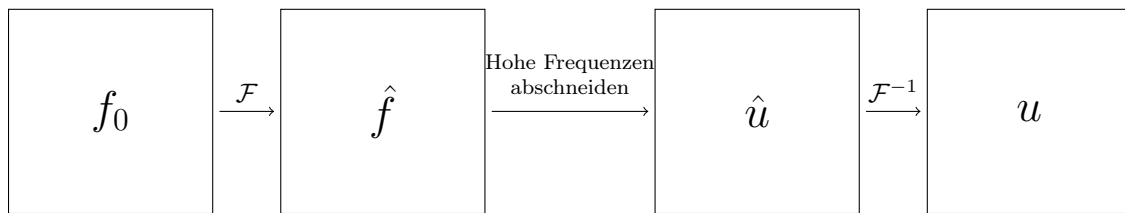
Genauer:

$$\mathcal{F}u = \hat{v} =$$

$$g(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}(\mathcal{F}^{-1}\chi_{[-r,r]^d})(x)$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\chi_{[-r,r]^d}(z)e^{i\langle z,x\rangle}dz$$

$$\overset{1d}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{[-r,r]}e^{izx}dz$$

$$= \frac{1}{2\pi}\int_{-r}^{r}e^{izx}dz = \frac{1}{2\pi}\left.\frac{e^{izx}}{ix}\right|_{z=-r}^{r}$$

$$= \frac{1}{2\pi ix}(e^{irx}-e^{-irx}) = \frac{1}{\pi x}\sin(rx)$$

$$\hat{g}(0) = (\mathcal{F}g)(0) = \frac{1}{2}$$

Es ist zu bemerken, dass g eine Art Tensor Struktur besitzt, was in etwa bedeutet das sich die Funktion in belibigen Dimensionen als Produkt der Funktion in einer Dimensionen darstellen lässt.

<u>Gauß-Kern</u> :

$$G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2}} \Rightarrow G\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-x_1^2 - x_2^2 + \cdots + x_d^2}{2}}$$

$$= \left(\frac{1}{(2\pi)^{\frac{1}{2}}} e^{\frac{-x_1^2}{2}}\right) \cdot \cdots \cdot \left(\frac{1}{(2\pi)^{\frac{1}{2}}} e^{\frac{-x_d^2}{2}}\right) = G(x_1) \cdot \cdots \cdot G(x_d)$$

allerhand noch im Skript und ein Tafelfoto

5.4   Filterbreite und Glättung

klar ist: $\frac{1}{25}$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

'glättet mehr als': $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

Im Kontinuierlichen: Sei $m \in L^1(\mathbb{R}^d)$ und $s > 0$. Setze

$$m_s(x) := \frac{1}{s^d} m(\frac{x}{s}), \quad x \in \mathbb{R}^d$$

Bsp (in $d = 1$):
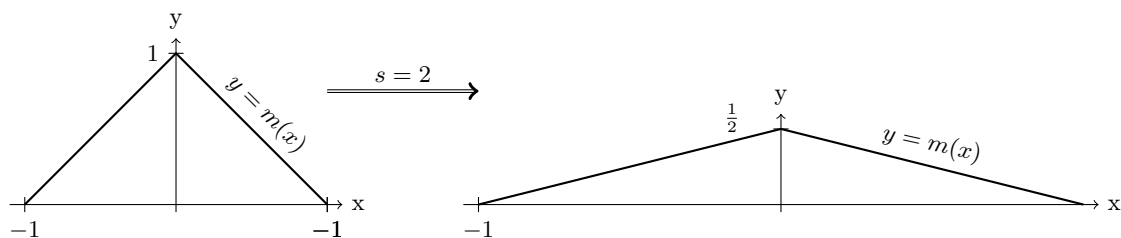
Bsp: Gauß-Kern $G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2}}$

Skalierung mit Fehler $s > 0$

$$\Rightarrow G_s(x) = \frac{1}{s^d} G\left(\frac{x}{s}\right) = \frac{1}{s^d} \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x|}{2}} = \frac{1}{(2\pi s^2)^{\frac{d}{2}}} e^{\frac{-|x|^2}{2s^2}}$$

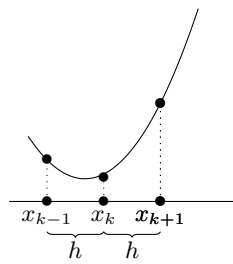Skalierung $s \mathrel{\hat=}$ Standardabweichung σ

hier noch mehr im Skript p. 45

5.4.1   Differenzenfilter

Bisher: Glättung $\mathrel{\hat=}$ Mittelwert bilden $\mathrel{\hat=}$ Summe/Integrale
Jetzt: Schärfen $\mathrel{\hat=}$ Differenzen/Kontraste hervorheben $\mathrel{\hat=}$ Differenzen/Ableitungen

Diskretisierung von Ableitungen durch Differenzenquotienten

(hier bedeutet $f(k) = f(x_k)$)

Vorwärts: $u(h) = \dfrac{f(k+1) - f(k)}{h}$ $\qquad u = \dfrac{1}{h}\boxed{\;0\;|\;\text{-}1\;|\;1\;}\boxtimes f$

Rückwärts: $u(h) = \dfrac{f(k) - f(k-1)}{h}$ $\qquad u = \dfrac{1}{h}\boxed{\;0\;|\;\text{-}1\;|\;1\;}\boxtimes f$

Zentral: $u(h) = \dfrac{f(k+1) - f(k-1)}{2h}$ $\qquad u = \dfrac{1}{2h}\boxed{\;0\;|\;\text{-}1\;|\;1\;}\boxtimes f$

2. Abbleitung:

$$u(h) \approx \frac{f'(k+1) - f'(k)}{h} \text{(vorwärts)}$$

$$\approx \frac{\frac{f(k+1)-f(k)}{h} - \frac{f(k)-f(k-1)}{h}}{h} \text{(rückwärts)}$$

$$= \frac{f(k+1) - 2f(k) + f(k+1)}{h^2}$$

Also folgt $u := \boxed{\;1\;|\;\text{-}2\;|\;1\;}\boxtimes f$ und $\dfrac{1}{h^2}\boxed{\;1\;|\;\text{-}2\;|\;1\;} = \dfrac{1}{h}\boxed{\;0\;|\;\text{-}1\;|\;1\;} * \dfrac{1}{h}\boxed{\;\text{-}1\;|\;1\;|\;0\;}$

Denn:

$$\frac{1}{h}\boxed{\;\text{-}1\;|\;1\;|\;0\;}$$

$$= \frac{1}{h}\boxed{\;0\;|\;1\;|\;\text{-}1\;} * \left(\frac{1}{h}\boxed{\;1\;|\;\text{-}1\;|\;0\;} * f\right)$$

$$= \left(\frac{1}{h}\boxed{\;0\;|\;1\;|\;\text{-}1\;} * \frac{1}{h}\boxed{\;1\;|\;\text{-}1\;|\;0\;}\right) * f$$

$$= \left(\frac{1}{h}\boxed{\;\text{-}1\;|\;1\;|\;0\;} \boxtimes \frac{1}{h}\boxed{\;1\;|\;\text{-}1\;|\;0\;}\right) * f$$

$$= \frac{1}{h^2}\boxed{\;1\;|\;\text{-}2\;|\;1\;} * f$$

$$= \frac{1}{h^2}\boxed{\;1\;|\;\text{-}2\;|\;1\;} \boxtimes f$$

In 2D: $\dfrac{\partial}{\partial x} \widehat{=} \boxed{\;0\;|\;\text{-}1\;|\;1\;}$, $\dfrac{\partial}{\partial y} \widehat{=} \boxed{\begin{array}{c} 0 \\ \text{-}1 \\ 1 \end{array}}$, $\dfrac{\partial^2}{\partial x^2} \widehat{=} \boxed{\;1\;|\;\text{-}2\;|\;1\;}$, $\dfrac{\partial^2}{\partial y^2} \widehat{=} \boxed{\begin{array}{c} 1 \\ \text{-}2 \\ 1 \end{array}}$.

Diskreter Laplace Operator :

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \widehat{=} \boxed{\;1\;|\;\text{-}2\;|\;1\;} + \boxed{\begin{array}{c} 1 \\ \text{-}2 \\ 1 \end{array}} = \boxed{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & \text{-}4 & 1 \\ 0 & 1 & 0 \end{array}}$$

5.4.2 Glättungsfilter und partielle Differentialgleichungen

Wir haben gesehen: $m = \dfrac{1}{5}\boxed{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}}$ ist unter allen 5-Punkt Filtern der am besten glättende.

Idee: Rauschen weiter verringern indem man $m\boxtimes$ wiederholt anwendet $\Rightarrow$ Folge von Bildern:

$$f_{:= u^{(0)}} \xrightarrow{m \boxplus} u^{(1)} \xrightarrow{m \boxplus} u^{(2)} \cdots$$

$$\Rightarrow u^{(n+1)} - u^{(n)} = \text{(Unterschied zwischen 'Zeit' Punkt } n \text{ und } n+1)$$

$$= \underbrace{m \boxplus u^{(n)}}_{u^{n+1}} - \underbrace{\delta \boxplus u^{(n)}}_{u^{(n)}} \text{ mit } \delta = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

$$= (m - \delta) \boxplus u^{(n)}$$

$$= \left(\frac{1}{5} \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} - \frac{1}{5} \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 5 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \right) \boxplus u^{(n)}$$

$$= \frac{1}{5} \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & \text{-4} & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} u^{(n)}$$

Somit gilt insgesamt:

$$\underbrace{u^{(n+1)} - u^{(n)}}_{\hat{=} \frac{\partial u}{\partial t}} = \frac{1}{5} \underbrace{\begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & \text{-4} & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}}_{\hat{=} \Delta u} \tag{5.10}$$

Kontinuierlich: Funktion u

$$u(x,t) \quad x \in \mathbb{R}^2, \ t \text{ Zeit}$$

(5.10) ist eine Diskretisierung (1 Zeitschritt im Eulerverfahren) der partiellen Differentialgleichungen

$$\frac{\partial u}{\partial t} = \Delta u \tag{5.11}$$

Bekannt als Wärmegleichung oder Diffusionsgleichung .
Zum Zeitpunkt $t = 0$ möge die Anfangsbedingung

$$u(x,0) = u^{(0)} = f(x) \tag{5.12}$$

gelten. Vorranschreiten der Zeit t repräsentiert Diffusion.
Für einen stationären Zustand, also keine Änderung $\frac{\partial u}{\partial t}$ dann muss auch $\Delta u = 0$ gelten.
Diese wird unteranderem von konstanten Funktionen oder linearen Funktionen $u(x_1, x_2) = ax_1 + bx_2$ erfüllt.

Es existiert auch einen explizite Formel für die Lösung der Diffusionsgleichung (5.11) mit Anfangsbedingung (5.12):
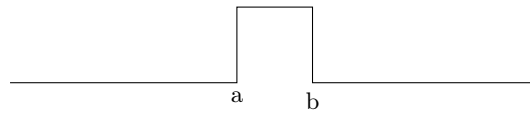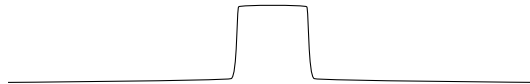
$$u(x,t) = \left(G_{\sqrt{2t}} * u^{(0)} \right)(x)$$

Wobei $\sqrt{2t}$ für eine Skalierung um diesen Wert steht.
Zu zeigen ist: $\frac{\partial u}{\partial t} = \Delta u$

$$\frac{\partial}{\partial t} \left(G_{\sqrt{2t}} * u^{(0)} \right) = \Delta \left(G_{\sqrt{2t}} * u^{(0)} \right)$$

$$\stackrel{\text{mit Satz}}{\Longrightarrow} \left(\frac{\partial}{\partial t} G_{\sqrt{2t}} \right) * u^{(0)} = \left(\Delta G_{\sqrt{2t}} \right) * u^{(0)}$$
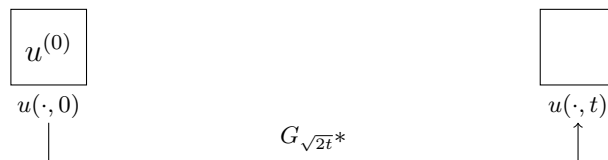
Es bleibt somit z.z.: $\frac{\partial}{\partial t} G_{\sqrt{2t}} = \Delta G_{\sqrt{2t}}$.

$t = 0$:

$t > 0$:

Bemerkenswert ist das, für $t = 0$ die Funktion nicht stetig ist, aber für alle $t > 0$ die Funktion beliebig oft differenzierbar ist.

Insgesamt lässt sich die Idee darstellen als:

kontinueirlich: ─────────────────────────────────────→ t

$u^{(0)}$

$u(\cdot, 0)$ $u(\cdot, t)$

$G_{\sqrt{2t}}*$

diskret:

$u^{(0)} \xrightarrow{m \boxast} u^{(1)} \xrightarrow{m \boxast} \quad \cdots \quad \xrightarrow{m \boxast} u^{(n)}$