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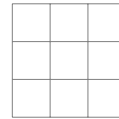
Kapitel 1

1. Overview

- „image society“ (webpages: 1995 text-based, 2005 image based, 2015 video based ...)
 - data transfer rates \uparrow , compression rates \uparrow
 - critical shift: reading \rightarrow watching
- „Photoshop“-ing (remove wrinkles, bumps, ...)
- Images in medicine („medical image processing“), x-ray, CT, MRI, ultrasound, ... („modalities“).
different questions:

1.) Layout!

align bottom measurements $\xrightarrow{?}$ image
 expl: tomography
 \Rightarrow difficult mathematical problems



2.) Image enhancements

- denoising
 - simple pixels/lines: „sandpaper“ interpolation
 - global noise: smoothing
- grayscale
 - histogramm balancing (spreading)
- distortion
 - makes straight lines (in real world) straight (in the images)
- edge detection
 - contour enhancement
- segmentation
 - detect and separate parts of the image
- registration
 - sequence of images of the same object \Rightarrow Wort?, compare Skizze
 - \nearrow object following in a movie

so richtig?

Our Focus:

- mathematical models/methods/ideas
- (algorithms)
- ((implementation))

skipped: Very fast intro: Matlab and images

Kapitel 2

2. What is an image?

2.1 Discrete and continuous images

There are (at least) two different points of view:



Abbildung 2.1: Discrete Image

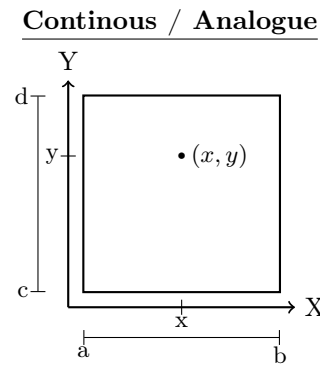


Abbildung 2.2: Continuous Image

object: matrix
tools: linear algebra (SVD, ...)
pros: (finite storage) storage, complexity
cons: limitations: zooming, rotations, ...

function
 analysis (differentiation, integrate, ...)
 freedom, tools, **motions? P.4**
 (e.g. edge discontinuity)
 storage (infinite amount of data)

arguably, one has:

- real life \Rightarrow continuous „images“ (objects)
- digital cameras \Rightarrow discrete images

In general we will say:

Definition 2.1 ((mathematical) image). A (mathematical) *image* is a function

$$u : \Omega \rightarrow F,$$

where: $\Omega \subset \mathbb{Z}^d$ (discrete) or $\Omega \subset \mathbb{R}^d$ (continuous) ... *domain*

$d = 2$ (typical case 2D), $d = 3$ („3D image“ = body or $\underbrace{2D + \text{time}}_{\text{movie}}$)

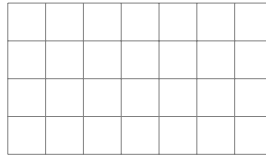
$d = 4$ (3D + time)

$F \dots$ range of colours

$F = \mathbb{R}$ or $[0, \infty]$ or $[0, 1]$ or $\{0, \dots, 255\}$, ... grayscale (light intensity)

$F \subset \mathbb{R}^3 \dots$ RGB image (colored)

$F = \{0, 1\} \dots$ black/white



3 Layers

\Rightarrow colored images:w

Matlab stuff

Large parts of the course: analytical approach (i.e. continuous domain Ω)

Since we want to differentiate, ... the image u .

Still: need to assume that also F is continuous (not as $\{0, 1\}$, $\{0, 1, \dots, 255\}$ or \mathbb{N})

since otherwise the only differentiable (actually, the only continuous) functions $u : \Omega \rightarrow F$ are *constant* functions \Leftrightarrow single-colour images

Also: We usually take F one-dimensional ($F \subset \mathbb{R}$). Think of it as either

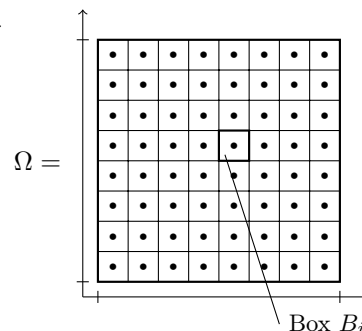
- gray scaled image, or
- treating R,G & B layer separately

2.2 Switching between discrete and continuous images

continuous \rightarrow discrete:

- divide the continuous image in small squared pieces (boxes) (superimpose grid)
- now: represent each box by *one* value
 - strategy 1: take function value $u(x_i)$
for $x_i = \text{midpoint of box } B_i$
 - strategy 2: use mean value

$$\frac{1}{|B_i|} \int_{B_i} u(x) dx$$



\Rightarrow discrete image

strategy 1: simple (and quick) but problematic ($u(x_i)$ might represent $u|_{B_i}$ badly; for $u \in L^p$, single point evaluation not even defined)

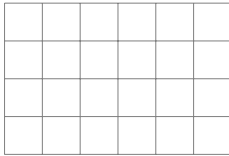
strategy 2: more complex but also more „democratic“ (actually closer to the way how CCD Sensors in digital cameras work)

often the image value of the box B_i gets also digitized, i.e. fitted (by scaling & rounding) into range $\{0, 1, \dots, 255\}$

discrete \rightarrow continuous

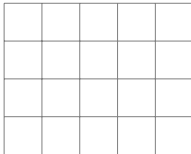
This is of course more tricky ...

- Again: each pixel of the discrete image corresponds to a „box“ of the continuous image (that is still to be constructed)
- Usually: pixel value \mapsto function value at the *midpoint* of the box
- Question: How to get the other function values (in the box)?



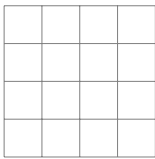
idea 1: just take the function value of the nearest midpoint („nearest neighbour interpolation“)

For each $x \in B_i : u(x) := u(x_j)$ where $|x - x_j| = \min_k |x - x_k|$



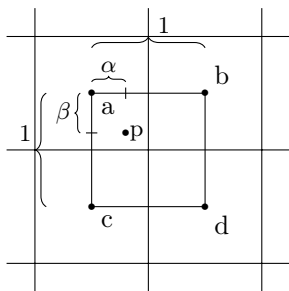
$\Rightarrow u(x) = u(x_i)$ for all $x \in B_i$
 \Rightarrow each box is uni-color
 \Rightarrow the continuous image is essentially still discrete

idea 2: (bi-) linear interpolation



Let $a, b, c, d \dots$ function values at 4 surrounding adjacent midpoints (\nearrow figure)
 $\alpha, \beta, 1 - \alpha, 1 - \beta \dots$ distance to dotted lines (\nearrow figure, w.l.o.g, bob is 1×1)

interpolation (linear) on the dotted line between a and b :



$e := a + \alpha(b - a) = (1 - \alpha)a + \alpha b$
 (1D - interpolation, convex combination)

Similarly: $f = (1 - \alpha)c + \alpha d$

Then: The same 1D-interpolation between e and f
 $\Rightarrow u(x) := (1 - \beta) \cdot e + \beta \cdot f$

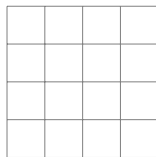
$$= (1 - \beta)[(1 - \alpha)a + \alpha b] + \beta[(1 - \alpha)c + \alpha d]$$

$$= \underbrace{(1 - \alpha)(1 - \beta)a}_{\in [0, 1]} + \underbrace{\alpha(1 - \beta)b}_{\in [0, 1]} + \underbrace{(1 - \alpha)\beta c}_{\in [0, 1]} + \underbrace{\alpha\beta d}_{\in [0, 1]}$$

$\in [0, 1] \wedge \sum = 1$

\Rightarrow convex combination of the function values a, b, c, d at the the surrounding 4 midpoints (on which points is the nearest, instead of taking just a, b, c or d - depending)

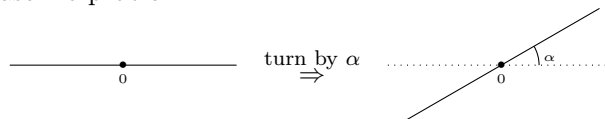
\Rightarrow 2D linear interpolation, *bi-linear interpolation* (can be interpreted as spline interpolation with bilinear **basis** splines).



Beispiel 2.2. Rotate image

by angle $\phi \neq k \cdot \frac{\pi}{2}$

- continuous image case: no problem



$$x = D_\varphi y \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, D_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

2D rotation matrix

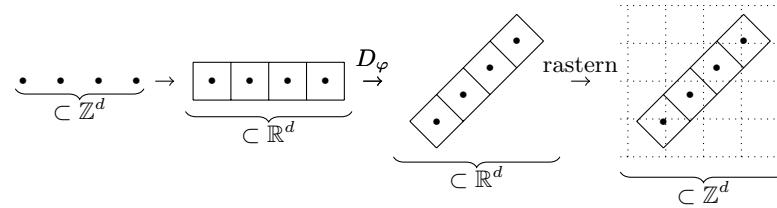
$$y = D_\varphi^{-1} x = D_{-\varphi} x$$

$$\Rightarrow v(x) := u(y) = u(D_{-\varphi} x) \quad \forall x \in \text{domain of the rotated image}$$

- discrete image case: problem !

For $x \in \text{domain of notated image}$, in general $D_{-\varphi} x \notin \text{domain of original image}$ ¹

Way out: $v(x) := \text{interpolation}$ between the $u(\cdot)$ of the 4 surrounding pixels of $D_{-\varphi}$



Something to think about:

What happens in the limit (?) if we, starting with an image (discrete or continuous), repeatedly switch between discrete and continuous, non-stop ... ?

Does the answer depend on the way of switching ? (continuous \rightarrow discrete: midpoint or average, discrete \rightarrow continuous: nearest neighbour or bilinear?)

¹it's not an integer

Kapitel 3

3. Histogramm and first applicatsion

3.1 The histogramm

Definition 3.1 (histogram). Let $\Omega \subset \mathbb{Z}^d$, $F \subset \mathbb{R}$ discrete and $u : \Omega \rightarrow F$ a discrete discrete image. The function

$$H_u : F \rightarrow \mathbb{N}_0 \quad (:= \mathbb{N} \cup \{0\})$$

with

$$H_u(k) := \# \{x \in \Omega : u(x) = k\}, \quad k \in F$$

is called *histogramm* of the image u .

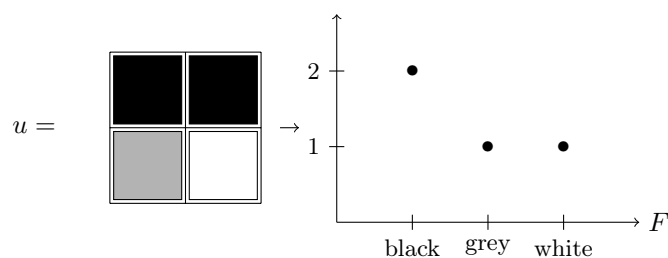
$H_u(k)$ counts how often colour k appears in u .

$$\sum_{k \in F} H_u(k) = |\Omega| = \text{number of pixels in the whole image}$$

or

$$\frac{H_u(k)}{|\Omega|} = \begin{array}{l} \text{relative frequency of colour } k \text{ in image } u \\ \text{(relative H\u00e4ufigkeit)} \end{array}$$

Beispiel 3.2.



If u is a continuous image, H_u can be understood as a measure (generalized function)¹.

Another way to write this:

$$H_u(k) = \sum_{x \in \Omega} \delta_{u(x)}(k), \quad k \in F \qquad H_u(k) = \int_{\Omega} \delta_{u(x)}(k) dx, \quad k \in F$$

hier fehlt noch das Kronecker underarrow

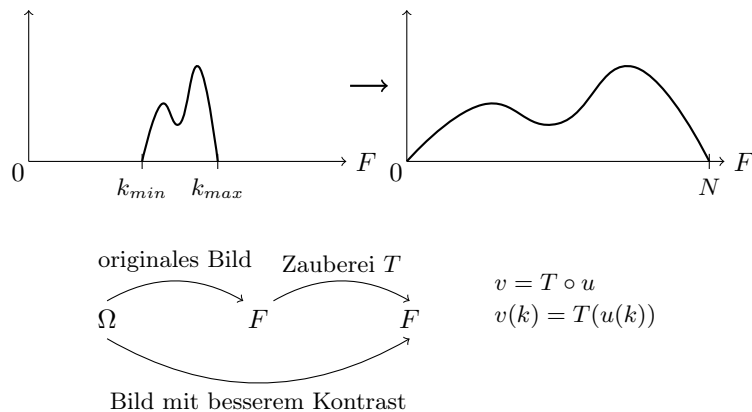
Matlab-Code

¹density of a probability distribution

3.2 Application: contrast enhancement

If the image only uses a small part of the available colour/grayscale „palette“ F , then its contrast can be improved by „spreading“ the histogram over all of F .

Simple idea:



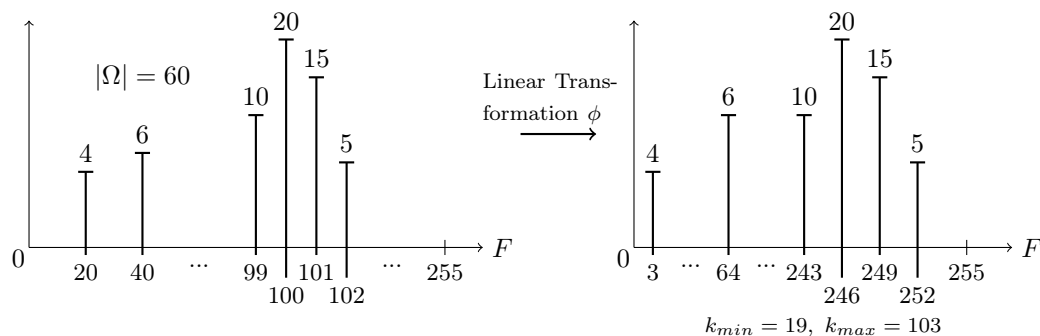
5

The above simple idea („contrast stretching“) corresponds to

$$\begin{aligned}
 \varphi : k_{\min} &\mapsto 0 \\
 k_{\max} &\mapsto N \\
 &\text{and linear in between} \\
 \text{i.e.} \quad \varphi(k) &= \left\lceil \frac{k - k_{\min}}{k_{\max} - k_{\min}} \right\rceil
 \end{aligned}$$

Where $\lceil \cdot \rceil$ means ...rounding to the nearest integer (assuming that $F = \{0, 1, \dots, N\}$).

Example histogram:

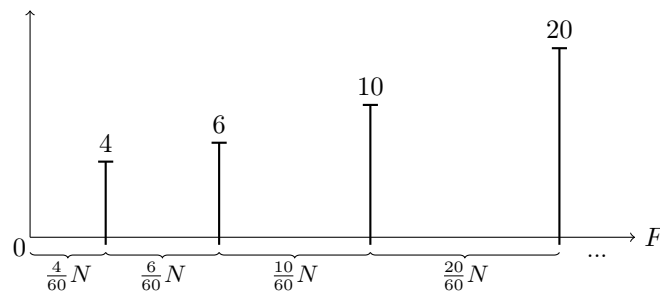


A bit more sophisticated:

$$\begin{aligned}
 \varphi : (k_{\min} &\mapsto 0) \\
 k_{\max} &\mapsto N \\
 &\text{and **non** linear in between}
 \end{aligned}$$

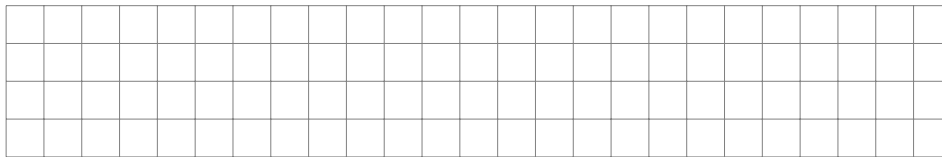
such that colour ranges that occur more frequently in u can occupy a larger range of colours in v .
 (\Rightarrow visibility \uparrow)

Example histogram spread out according to frequency of occurrence:



\Rightarrow „density“ is equalized over $F = \{0, \dots, N\}$

Ideal would be:



Layout S.12 u

Note: The new colours (i.e the location of the bars in the histogram of u) only depend on the frequencies / height of the bars in H_u but not on the colours/location of the bars in H_u

Finally: The formula

$$\varphi(k) = \left\lceil \frac{N}{|\Omega|} \sum_{l=0}^k H_u(l) \right\rceil$$

This process is called „histogramm equalization“

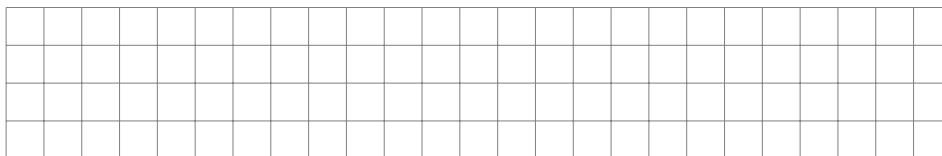
Exercise ?!

3.3 Another application: conversion to b/w

Task: convert grayscale image to black white

- interesting for object detection/*segmentation* ...!

Idea: Find a threshold $t \in T$ s.t. the histogram splits into two „characteristic“ parts



For $t \in F$ put

$$\text{black} := \{k \in F : k \leq t\}$$

$$\text{white} := \{k \in F : k > t\}$$

and

$$\tilde{u} := \begin{cases} 0, & u(x) \in \text{black} \\ 1, & u(x) \in \text{white} \end{cases} \quad \tilde{F} = \{0, 1\}$$

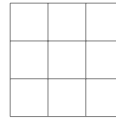
How to find the threshold t :

1.) Shape based methods

If the histogram is „biomodal“

Put $t := \frac{k_{\max_1} + k_{\max_2}}{2}$

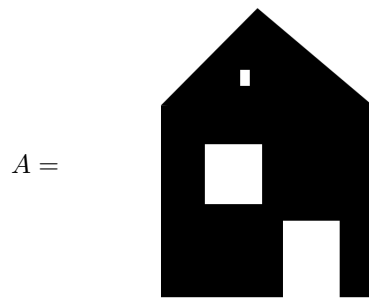
or $t := k_{\min}$



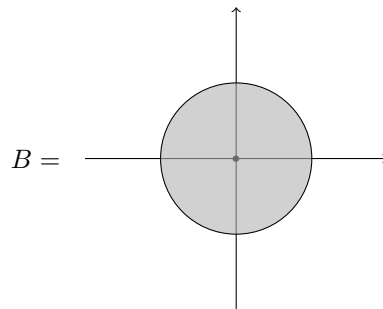
Kapitel 4

4. Basic Morphological Operations

B/W Bild:



Structural element :



4.1 Operations on A and B

$$A + B := \{a + b : a \in A, b \in B\}$$

This is called dilation.

You might imagine that at every dark point in the image A the Structurelement is applied.

$$A + B =$$

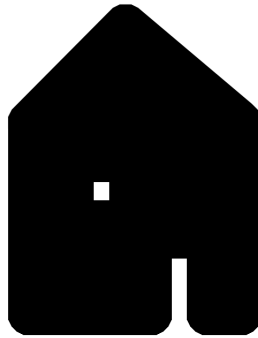


Image created in Matlab through:

```

1 I=imread('Bild1.png');
2 se=strel('disk',40,8);
3 I2=imcomplement(imdilate(imcomplement(I),se));%I am using the complement of the image
   here so that the structural element is applied to the dark parts of the image
4 imshow(I2);

```

$$A - B := \{a : a + B \subset A\}$$

This is called erosion.

You can imagine that you search for the points in which the structural element fits.

$$A - B =$$

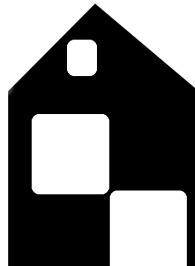


Image created in Matlab thorough:

```

1 I=imread('Bild1.png');
2 se=strel('disk',20,8);
3 I2=imcomplement(imerode(imcomplement(I),se));
4 imshow(I2);

```

One may quickly realize that $A \neq (A + B) - B$, so a new Operation is introduced:

$$A \bullet B := (A + B) - B$$

This is called closing and is used to e.g. remove noise. In the example image you might notice that the upper window is missing.

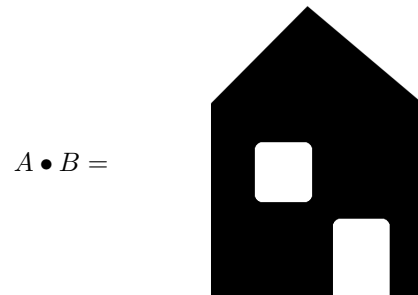


Image created in Matlab thorough:

```

1 I=imread('Bild1.png');
2 se=strel('disk',20,8);
3 I2=imcomplement(imdilate(imcomplement(I),se));
4 I3=imcomplement(imerode(imcomplement(I2),se));
5 imshow(I3);

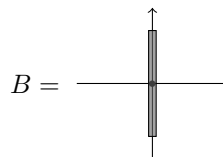
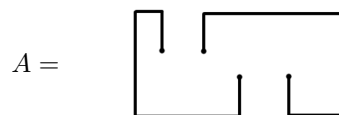
```

The inverse also exists:

$$A \circ B := (A - B) + B$$

This is called opening.

This time with a new example:



$$A \circ B = \begin{array}{cccc} | & | & | & \\ | & & & \\ & & & | \\ & & | & | \\ & & & | \end{array}$$

Image created in Matlab thorough:

```

1 I=imread('Bild2.png');
2 se=strel('line',10,90);
3 I2=imcomplement(imerode(imcomplement(I),se));
4 I3=imcomplement(imerode(imcomplement(I2),se));
5 imshow(I3);

```

Kapitel 5

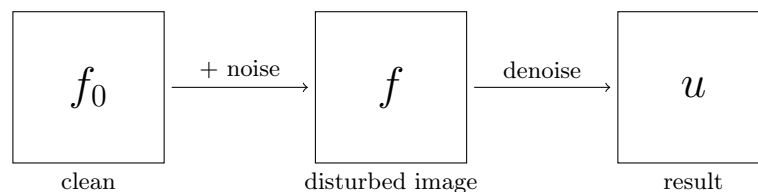
5. Entrauschen: Filter und Co

5.0.1 Noise

Noise : Unwanted disturbances in an image

- point wise
- random
- independent
- additive (for multiplicative noise use \log)

Notation:



The quality of the denoised image u compared to the original image f_0 is described by norms.

$$\begin{aligned}
 & \|f - f_0\|, \text{ noise} \\
 & \|u - f_0\|, \text{ absolute error} \\
 & \frac{\|u - f_0\|}{\|f - f_0\|}, \text{ relative error compared to the noise} \\
 & \frac{\|u - f_0\|}{\|f_0\|}, \text{ relative error compared to the signal}
 \end{aligned}$$

Typically the chosen norm is:

$$\|f\| = \|f\|_2 = \sqrt{\int_{\Omega} |f(x)|^2 dx}$$

or in the discrete:

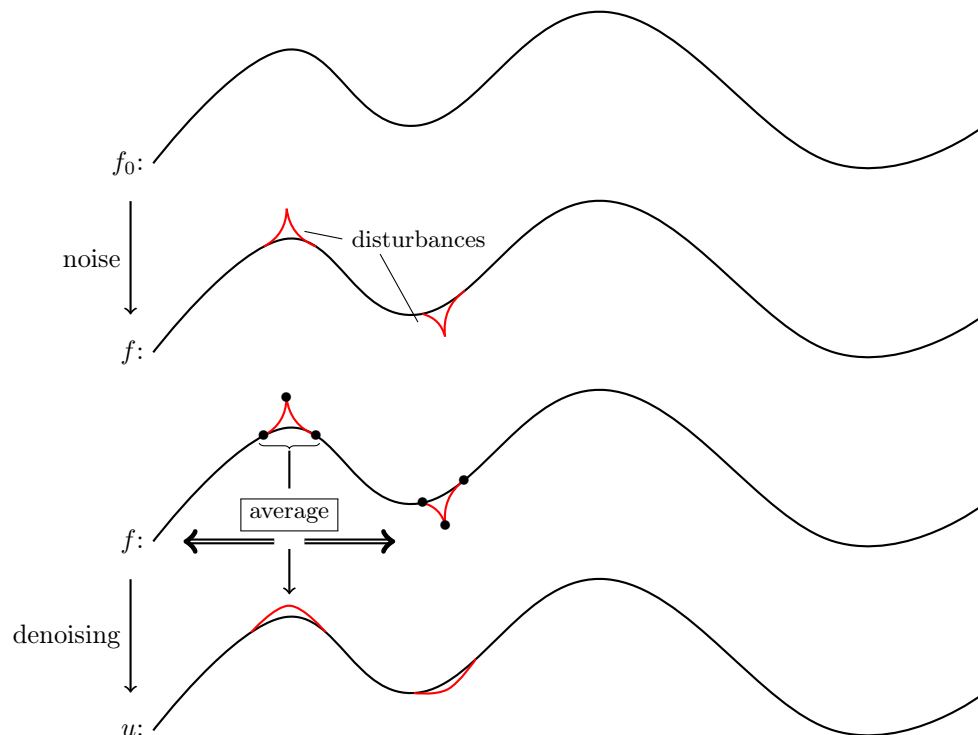
$$\|f\|_2 = \sqrt{\sum_{x \in \Omega} |f(x)|^2}$$

Closely connected is the Signal to noise ratio (SNR):

$$\log\left(\underbrace{\frac{\|f_0\|_2}{\|u - f_0\|_2}}_{\in [1, \infty)}\right) \in [0, +\infty), \text{ where } 0 \text{ is bad and } +\infty \text{ is good.}$$

5.0.2 smoothing filter

Idea: (to simplify in 1D)

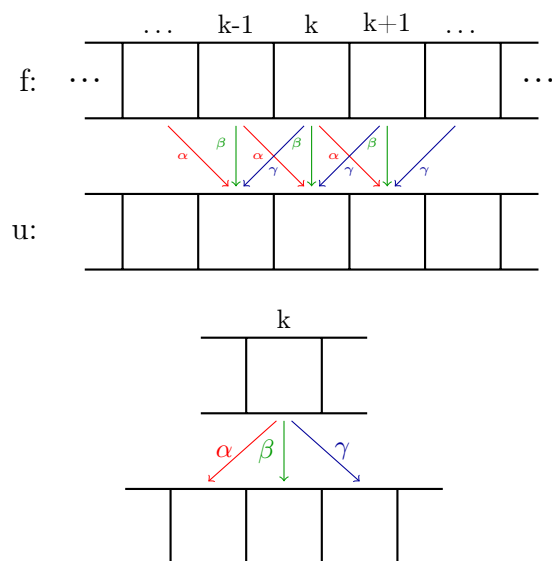


$$u(k) := \alpha \cdot f(k-1) + \beta \cdot f(k) + \gamma \cdot f(k+1) \quad (5.1)$$

where:

$$\alpha + \beta + \gamma = 1 \quad (5.2)$$

More precisely (5.1) means:



With (5.1) there is a mapping $f \mapsto u$, we write

$$u = m \boxtimes f, \text{ this is called } \underline{\text{Correlation}} .$$

where:

$$(m \boxtimes f)(k) = \sum_{i \in \text{supp}(m)} m(i) f(k+i) \quad (5.3)$$

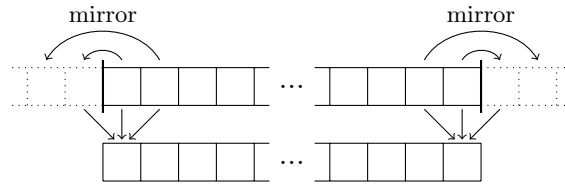
and:

$$m = \begin{array}{ccccccc} & \dots & -1 & 0 & 1 & \dots & \\ & \hline \dots & \alpha & \beta & \gamma & \dots & & \\ & \hline \end{array} \quad \text{called } \underline{\text{mask}}.$$

If you set $j := k + i$ in (5.1), then $i = j - k$, which means:

$$(m \boxtimes f)(k) = \sum_{i \in \text{supp}(m)} m(j-k) f(j) \quad (5.4)$$

To apply the mapping onto the boundary the image is reflected, in 1D:



in 2D:

d	b	d
q	p	q
d	b	d

Formula (5.4) might remind one of the convolution :

$$(g * f)(k) = \sum_{j \in \mathbb{Z}} g(\underbrace{k-j}_{\text{Difference to (5.4)}}) \cdot f(j) \quad (5.5)$$

If you set $g(i) := m(-i) =: \tilde{m}(i)$, which corresponds to a reflection of the Mask, then

$$m \boxtimes f = g * f = \tilde{m} * f$$

Properties of the convolution:

1. $(f * g) * h = f * (g * h)$, Associativity
2. $f * g = g * f$, Commutativity

3. $\tilde{f} * \tilde{g} = \widetilde{f * g}$, Compatibility with reflection

Properties of the correlation:

1. $f \boxtimes (g \boxtimes h) = \tilde{f} * (\tilde{g} * h) \stackrel{\boxed{1}}{=} (\tilde{f} * \tilde{g}) * h \stackrel{\boxed{3}}{=} \widetilde{(f * g) * h} = (f * g) \boxtimes h \neq (f \boxtimes g) \boxtimes h$, not associative!
2. $f \boxtimes g = \tilde{f} * g \stackrel{\boxed{2}}{=} g * \tilde{f} = \tilde{g} * \tilde{f} \stackrel{\boxed{3}}{=} \widetilde{(\tilde{g} * f)} = \widetilde{g \boxtimes f} \neq g \boxtimes f$, not commutative!
3. $\tilde{f} \boxtimes \tilde{g} = \tilde{\tilde{f} * \tilde{g}} \stackrel{\boxed{3}}{=} \widetilde{(\tilde{f} * g)} = \widetilde{f \boxtimes g}$, Compatibility with reflection