Домашня робота

16.03.23р.

$$\begin{cases} \sum_{i=1}^{n} q_i\, a(\varphi_i, \varphi_j) = \langle l, \varphi_j \rangle, \quad j=\overline{1,n} \\ a(\varphi_i,\varphi_j) = \mu \int_0^4 \varphi_i' \varphi_j'\, dx + \beta \int_0^4 \varphi_i' \varphi_j\, dx + \sigma \int_0^4 \varphi_i \varphi_j\, dx \\ \langle l, \varphi_j\rangle = f \int_0^4 \varphi_j\, dx + 2\mu\, \varphi_j(4) \end{cases}$$

Для початку визначимо φ:

$$\varphi_j = \begin{cases} 0, & x \in [0;4] / (x_{j-1}; x_{j+1}); \\ \frac{x - x_{j-1}}{h}, & x \in [x_{j-1}; x_j]; \\ \frac{x_{j+1} - x}{h}, & x \in (x_j; x_{j+1}]. \end{cases}$$

Тоді φ_j':

$$\varphi_j' = \begin{cases} 0, & x \in [0;4] / (x_{j-1}; x_{j+1}); \\ 1/h, & x \in [x_{j-1}; x_j]; \\ -1/h, & x \in [x_j; x_{j+1}]. \end{cases}$$

Знайдемо усі інтеграли:

1) $$\mu \int_0^4 \varphi_i' \varphi_j'\, dx = \begin{cases} 0, & |i-j| > 1; \\ 2\mu/h, & |i-j| \le 1 \wedge j \ne n; \\ \mu/h, & |i-j| \le 1 \wedge j = n. \end{cases}$$

а) коли $|i-j|>1$: $\int_0^4 \varphi_i' \varphi_j'\, dx = 0$.

б) коли $|i-j|=0$ $(i=j)$: $\int_0^4 (\varphi_j')^2\, dx = \int_{x_{j-1}}^{x_{j+1}} \frac{1}{h^2}\, dx = \frac{1}{h^2} \cdot 2h = \frac{2}{h}$

(якщо $j \ne n$)

якщо $j=n$: $\int_0^4 (\varphi_j')^2\, dx = \frac{1}{h}$ (оск. це край, то вдвічі менше)

в) коли $|i-j|=1$: $\int_0^4 \varphi_i' \varphi_j'\, dx = \int_{x_{j-1}}^{x_{j+1}} \frac{1}{h} \cdot \left(-\frac{1}{h}\right) dx = \frac{2}{h}$

аналог. якщо $j=n$: $\int_0^4 \varphi_i' \varphi_j'\, dx = \frac{1}{h}$

2) $\beta \int_0^4 \varphi_i' \varphi_j' \, dx = \begin{cases} 0, \ (|i-j|>1) \cup (i=j \wedge i\neq 0, i\neq n); \\ -1/2\beta \ (1=j-i) \cup (i=j \wedge i=0); \\ 1/2\beta \ (i-j=1) \vee (i=j \wedge i=n). \end{cases}$

а) $|i-j|>1$: $\int_0^4 \varphi_i' \varphi_j' \, dx = 0$

б) $i-j=1$: $(i>j)$

$$\int_0^4 \frac{x_{j+1}-x}{h}\cdot\frac{1}{h}\,dx = \frac{2x_{j+1}x - x^2}{2h^2}\Big|_{x_j}^{x_{j+1}} =$$

$$= \frac{2x_{j+1}x_{j+1} - \cancel{x_{j+1}^2} - 2x_{j+1}x_j + x_j^2}{2h^2} = \frac{\overset{h^2}{(x_{j+1}-x_j)^2}}{2h^2} = 1/2$$

в) $j-i=1$: $(j>i)$

$$\int_0^4 -\frac{1}{h}\cdot\frac{x-x_{j-1}}{h}\,dx = \frac{2x_{j-1}x - x^2}{2h^2}\Big|_{x_{j-1}}^{x_j} =$$

$$= \frac{2x_{j-1}x_j - x_j^2 - 2x_{j-1}^2 + \cancel{x_{j-1}^2}}{2h^2} = -\frac{(x_{j-1}-x_j)^2}{2h^2} = -1/2$$

г) $i=j$; $i\neq 0, i\neq n$

$$\int_0^4 \varphi_j' \varphi_j' \, dx = \int_{x_{j-1}}^{x_j} \varphi_j' \varphi_j' \, dx + \int_{x_j}^{x_{j+1}} \varphi_j' \varphi_j' \, dx = \int_{x_{j-1}}^{x_j} \frac{x-x_{j-1}}{h}\cdot\frac{1}{h}\,dx$$

$$+ \int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h}\cdot\left(-\frac{1}{h}\right)dx = \frac{x^2-2x_{j-1}x}{2h^2}\Big|_{x_{j-1}}^{x_j} +$$

$$+ \frac{x^2-2x_{j+1}x}{2h^2}\Big|_{x_j}^{x_{j+1}} = \frac{x_j^2 - 2x_{j-1}x_j - \cancel{x_{j-1}^2} + \cancel{2}x_{j-1}^2}{2h^2} + \frac{\cancel{x_{j+1}^2} - 2x_{j+1}^2 + x_j^2}{2h^2}$$

$$= \frac{(x_j - x_{j-1})^2}{2h^2} + \frac{-(x_{j+1} - x_j)^2}{2h^2} = \frac{h^2}{2h^2} - \frac{h^2}{2h^2} = 0$$

e) $i = j,\ i = 0$:

$$\int_0^L \varphi_i \varphi_j \,dx = \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \cdot \left(-\frac{1}{h}\right) dx = -\frac{(x_{j+1} - x_j)^2}{2h^2} = -\frac{h^2}{2h^2} = -1/2.$$

g) $i = j,\ i = n$:

$$\int_0^L \varphi_j \varphi_j \,dx = \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \cdot \frac{1}{h} = \left(\frac{x_j - x_{j-1}}{2h^2}\right)^2 = 1/2.$$

3) $$\int_0^L \varphi_i \varphi_j \,dx = \begin{cases} 0, & |i-j| > 1 \\ 4/3 \cdot h, & |i-j| = 1 \\ 2h/3, & |i-j| = 0,\ i \neq 0 \wedge i \neq n \\ h/3 & |i-j| = 0 \quad i = 0 \vee i = n \end{cases}$$

а) $|i-j| > 1$: $\int_0^L \varphi_i \varphi_j \,dx = 0$.

б) $i - j = 1\ (i > j) \vee j - i = 1\ (j > i)$

$$\int_0^L \varphi_i \varphi_j \,dx = \int_{x_{j-1}}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h} \cdot \frac{x - x_{j-1}}{h}\right) dx = \frac{3x_{j+1}x^2 + 3x_{j-1}x^2 -}{6h^2}$$

$$\frac{-6x_{j+1}x_{j-1}x - 2x^3}{6h^2} \Bigg|_{x_{j-1}}^{x_{j+1}} = \frac{3x_{j+1}^3 + 3x_{j-1}x_{j+1}^2 - 6x_{j+1}^2 x_{j-1}}{6h^2}$$

$$\frac{-2x_{j+1}^3 - 3x_{j-1}^3 x_{j+1} - 3x_{j-1}^3 + 6x_{j+1}x_{j-1}^2 + 2x_{j-1}^3}{6h^2} = \frac{x_{j+1}^3 - 3x_{j+1}^2 x_{j-1} +}{6h^2}$$

$$\frac{+3x_{j+1}x_{j-1}^2 - x_{j-1}^3}{6h^2} = \frac{(x_{j+1} - x_{j-1})^3}{6h^2} = \frac{8h^3}{6h^2} = \frac{4}{3}h$$

в) $i=j$, $i\neq 0$ і $i\neq n$

$$\int_0^L \varphi_i \varphi_j \, dx = \int_0^L \varphi_j^2 \, dx = \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{h}\right)^2 dx + \int_{x_{j-1}}^{x_j} \left(\frac{x-x_{j-1}}{h}\right)^2 dx =$$

$$= \frac{3x_{j+1}^2 x - 3x_{j+1} x^2 + x^3}{3h^2}\Big|_{x_j}^{x_{j+1}} + \frac{x^3 - 3x_{j-1}x^2 + 3x_{j-1}^2 x}{3h^2}\Big|_{x_{j-1}}^{x_j} =$$

$$= \frac{\cancel{3x_{j+1}^3} - \cancel{3x_{j+1}^3} + x_{j+1}^3 - 3x_{j+1}^2 x_j + 3x_{j+1} x_j^2 - x_j^3}{3h^2} +$$

$$+ \frac{x_j^3 - 3x_{j-1}x_j^2 + 3x_{j-1}^2 x_j - x_{j-1}^3 + \cancel{3x_{j-1}^3} - \cancel{3x_{j-1}^3}}{3h^2} =$$

$$= \frac{\overbrace{(x_{j+1}-x_j)}^{=h}{}^3}{3h^2} + \frac{\overbrace{(x_j - x_{j-1})}^{=h}{}^3}{3h^2} = \frac{h^3+h^3}{3h^2} = \frac{2h}{3}$$

г) $i=j \wedge (i=0 \vee i=n)$

$$\int_0^L \varphi_i \varphi_j \, dx = \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{h}\right)^2 dx = h/3$$

4) $f_0 \int_0^L \varphi_j \, dx = \begin{cases} fh, & j\neq 0 \text{ і } j\neq n \\ fh/2, & j=0 \text{ або } j=1 \end{cases}$

$j\neq 0$ і $j\neq n$:

а) $$\int_0^L \varphi_j \, dx = \int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h} dx + \int_{x_{j-1}}^{x_j} \frac{x-x_{j-1}}{h} dx =$$

$$= \frac{-x^2 + 2x_{j+1}x}{2h}\Big|_{x_j}^{x_{j+1}} + \frac{x^2 - 2x_{j-1}x}{2h}\Big|_{x_{j-1}}^{x_j} = \frac{-\cancel{x_{j+1}^2} + \cancel{2}x_{j+1}^2 + x_j^2}{2h}$$

$$- \frac{2x_{j+1}x_j}{2h} + \frac{x_j^2 - 2x_{j-1}x_j - \cancel{x_{j-1}^2} + \cancel{2}x_{j-1}^2}{2h} = \frac{\overbrace{(x_{j+1}-x_j)}^{=h}{}^2}{2h} + \frac{\overbrace{(x_j - x_{j-1})}^{=h}{}^2}{2h} =$$

$$= \frac{h^2}{2h} + \frac{h^2}{2h} = \frac{2h^2}{2h} = h.$$

в) $j=0$ або $j=n$: $\int_0^u \varphi_j \, dx = h/2.$

Отже:

$$\mu \int_0^u \varphi_i' \varphi_j' \, dx = \begin{cases} 0, & |i-j|>1 \\ 2\mu/h, & |i-j|\le 1 \quad ij \ne 0 \text{ або } j \ne n \\ \mu/h & |i-j|< \quad j=n \end{cases}$$

$$\beta \int_0^u \varphi_i' \varphi_j \, dx = \begin{cases} 0, & (|i-j|>1) \vee (i=j \wedge i\ne 0, i \ne n) \\ -\beta/2 & (j-i=1) \vee (i=j,\ i=0) \\ \beta/2 & (i-j=1) \vee (i=j,\ i=n) \end{cases}$$

$$\sigma \int_0^u \varphi_i \varphi_j \, dx = \begin{cases} 0, & |i-j|>1 \\ 4\sigma h/3, & |i-j|=1 \\ 2\sigma h/3, & (|i-j|=0) \wedge (i\ne 0 \wedge i \ne n) \\ \sigma h/3, & (|i-j|=0) \wedge (i=0 \vee i=n) \end{cases}$$

$$f \int_0^u \varphi_i \, dx = \begin{cases} fh, & j\ne 0 \wedge j \ne n \\ fh/2, & j=0 \vee j=n. \end{cases}$$

матриця матиме вигляд:

$$\begin{array}{cccccc} & & & & n-1 & n \\ \end{array}$$

$$\begin{pmatrix} \boxed{\mu/h + \beta/2 + \sigma h/3} & \boxed{\mu/h + \beta/2 + \frac{4\sigma h}{3}} & \boxed{0} & \dots & \boxed{0} & \boxed{0} \\ \boxed{\mu/h - \beta/2 + \frac{4\sigma h}{3}} & \boxed{\frac{2\mu}{h} + \frac{2\sigma h}{3}} & \boxed{\frac{\mu}{h} + \frac{\beta}{2} + \frac{4\sigma h}{3}} & \dots & \boxed{0} & \boxed{0} \\ \text{↳ зсувається} & & \dots & & & \\ \boxed{0} & \boxed{0} & \dots & & \boxed{\frac{\mu}{h} - \beta/2 + \frac{4\sigma h}{3}} & \boxed{\frac{\mu}{h} + \frac{\sigma h}{3}} \end{pmatrix}$$

$$\begin{array}{cc} & n-1 \end{array}$$

Вектор вільних членів матиме вигляд:

$$\left(\frac{fh}{2} \quad fh \quad fh \quad \ldots \quad \overset{n-1}{fh} \quad \overset{n}{\frac{fh}{2}+\alpha H} \right)$$