

Goal

Add two bits:

a
 b

A normal (classical) 1-bit addition gives:

a b SUM CARRY

0 0 0 0

0 1 1 0

1 0 1 0

1 1 0 1

2. Qubits

Use 3 qubits:

$q_0 = a$

$q_1 = b$

$q_2 = \text{carry}$ (initially 0)

Initial state:

$|000\rangle$

3. Load inputs (a and b)

Qubits always start at 0.

To put a 1 into a qubit, we flip it using the X gate. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

if $a = 1 \rightarrow \text{flip } q_0$

if $b = 1 \rightarrow \text{flip } q_1$

Example: $a=1, b=1$:

$|000\rangle \rightarrow |100\rangle \rightarrow |110\rangle$

Now the qubits hold the inputs.

4. Compute the CARRY using CCNOT

CCNOT rule:

Flip the third qubit only when the first and second are both 1.

So:

$q_0=1 \text{ AND } q_1=1 \rightarrow \text{flip } q_2$

Example starting from $|110\rangle$: $|110\rangle \rightarrow |111\rangle$

This correctly produces:

CARRY = 1 (since $1 \text{ AND } 1 = 1$)

If a or b is 0, CCNOT does nothing.

5. Compute SUM using CNOT gates

CNOT rule:

Flip the target qubit only if the control qubit is 1.

We want:

SUM = $a \text{ XOR } b$

For XOR, one qubit must flip depending on the other.

Two CNOTs do this:

CNOT($q_0 \rightarrow q_1$)

CNOT($q_1 \rightarrow q_0$)

These two flips create the XOR behavior.

Example ($a=1, b=1$):

Start after CCNOT:

$|111\rangle$

First CNOT ($q_0 \rightarrow q_1$)

$q_0 = 1 \rightarrow \text{flip } q_1$:

Second CNOT ($q_1 \rightarrow q_0$)

$q_1 = 0 \rightarrow \text{no flip}$:

$|101\rangle$ (final)

Meaning:

$$\text{SUM} = q_0 = 0$$

$$\text{CARRY} = q_2 = 1$$

Correct for:

$$1 + 1 = 10 \text{ (binary)}$$

6. Final Output

From the final qubits:

$$\text{Sum} = q_0$$

$$\text{Carry} = q_2$$

Case 1: $a = 0, b = 0$

Start:

$$|0,0,0\rangle$$

CCNOT:

$$c = 0 \oplus (0 \cdot 0) = 0$$

State:

$$|0,0,0\rangle$$

CNOT:

$$s = 0 \oplus 0 = 0$$

Final:

$$|0,0,0\rangle$$

Sum = 0, Carry = 0

Case 2: $a = 0, b = 1$

Start:

$$|0,1,0\rangle$$

CCNOT:

$$c = 0 \oplus (0 \cdot 1) = 0$$

CNOT:

$$s = 0 \oplus 1 = 1$$

Final:

$$|0,1,0\rangle$$

Sum = 1, Carry = 0

Case 3: $a = 1, b = 0$

Start:

$$|1,0,0\rangle$$

CCNOT:

$$c = 0 \oplus (1 \cdot 0) = 0$$

CNOT:

$$s = 1 \oplus 0 = 1$$

Final:

$$|1,1,0\rangle$$

Sum = 1, Carry = 0

Case 4: $a = 1, b = 1$

Start:

$$|1, 1, 0\rangle$$

CCNOT:

$$c = 0 \oplus (1 \cdot 1) = 1$$

State becomes:

$$|1, 1, 1\rangle$$

CNOT:

$$s = 1 \oplus 1 = 0$$

Final:

$$|1, 0, 1\rangle$$

Sum = 0, Carry = 1