

Goal

Add two bits:

*a*  
*b*

A normal (classical) 1-bit addition gives:

a b SUM CARRY

0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

## 2. Qubits

Use 3 qubits:

$q_0 = a$

$q_1 = b$

$q_2 = \text{carry (initially 0)}$

Initial state:

$|000\rangle$

## 3. Load inputs (a and b)

Qubits always start at 0.

To put a 1 into a qubit, we flip it using the X gate.

[0 1  
1 0]

if  $a = 1 \rightarrow$  flip  $q_0$

if  $b = 1 \rightarrow$  flip  $q_1$

Example:  $a=1, b=1$ :

$|000\rangle \rightarrow |100\rangle \rightarrow |110\rangle$

Now the qubits hold the inputs.

## 4. Compute the CARRY using CCNOT

CCNOT rule:

Flip the third qubit only when the first and second are both 1.

So:

$q_0=1 \text{ AND } q_1=1 \rightarrow$  flip  $q_2$

Example starting from  $|110\rangle$ :  $|110\rangle \rightarrow |111\rangle$

This correctly produces:

CARRY = 1 (since 1 AND 1 = 1)

If a or b is 0, CCNOT does nothing.

## 5. Compute SUM using CNOT gates

CNOT rule:

Flip the target qubit only if the control qubit is 1.

We want:

SUM =  $a \oplus b$

For XOR, one qubit must flip depending on the other.

Two CNOTs do this:

$\text{CNOT}(q_0 \rightarrow q_1)$

$\text{CNOT}(q_1 \rightarrow q_0)$

These two flips create the XOR behavior.

Example ( $a=1, b=1$ ):

Start after CCNOT:

$|111\rangle$

First CNOT ( $q_0 \rightarrow q_1$ )

$q_0 = 1 \rightarrow$  flip  $q_1$ :

Second CNOT ( $q_1 \rightarrow q_0$ )

$q_1 = 0 \rightarrow$  no flip:

$|101\rangle$  (final)

Meaning:

$$\text{SUM} = q_0 = 0$$

$$\text{CARRY} = q_2 = 1$$

Correct for:

$$1 + 1 = 10 \text{ (binary)}$$

6. Final Output

From the final qubits:

$$\text{Sum} = q_0$$

$$\text{Carry} = q_2$$

Case 1:  $a = 0, b = 0$

Start:

$$| 0,0,0 \rangle$$

CCNOT:

$$c = 0 \oplus (0 \cdot 0) = 0$$

State:

$$| 0,0,0 \rangle$$

CNOT:

$$s = 0 \oplus 0 = 0$$

Final:

$$| 0,0,0 \rangle$$

Sum = 0, Carry = 0

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Case 2:  $a = 0, b = 1$

Start:

$$| 0,1,0 \rangle$$

CCNOT:

$$c = 0 \oplus (0 \cdot 1) = 0$$

CNOT:

$$s = 0 \oplus 1 = 1$$

Final:

$$| 0,1,0 \rangle$$

Sum = 1, Carry = 0

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Case 3:  $a = 1, b = 0$

Start:

$$| 1,0,0 \rangle$$

CCNOT:

$$c = 0 \oplus (1 \cdot 0) = 0$$

CNOT:

$$s = 1 \oplus 0 = 1$$

Final:

$$| 1,1,0 \rangle$$

Sum = 1, Carry = 0

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Case 4:  $a = 1, b = 1$

Start:

$$|1,1,0\rangle$$

CCNOT:

$$c = 0 \oplus (1 \cdot 1) = 1$$

State becomes:

$$|1,1,1\rangle$$

CNOT:

$$s = 1 \oplus 1 = 0$$

Final:

$$|1,0,1\rangle$$

Sum = 0, Carry = 1