DEUTSCH ALGORITHM

Goal

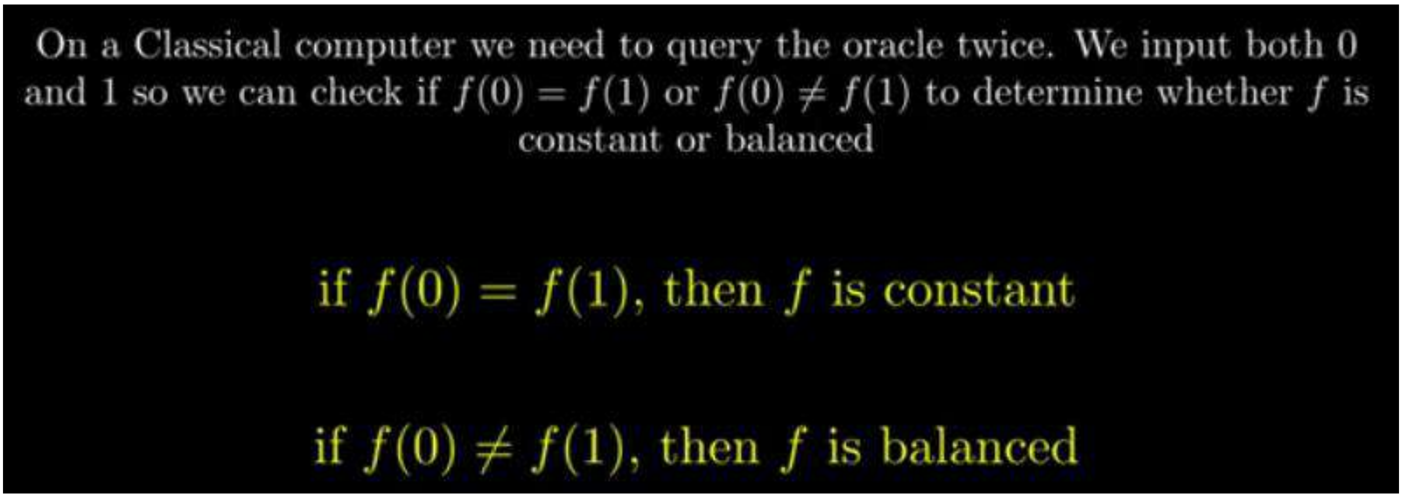
Given a function

$$f: \{0,1\} \rightarrow \{0,1\}$$

determine whether:

- **f is constant** (same for both inputs)
  - f(0) = f(1)
- **f is balanced** (different for inputs)
  - f(0) ≠ f(1)

Classically: need **2 evaluations** (f(0) and f(1))
Quantumly: **1 evaluation (one oracle call)**



On a Classical computer we need to query the oracle twice. We input both 0 and 1 so we can check if $f(0) = f(1)$ or $f(0) \neq f(1)$ to determine whether f is constant or balanced

if $f(0) = f(1)$, then f is constant

if $f(0) \neq f(1)$, then f is balanced

STEP 1: INITIAL STATE
Circuit starts with:

$$|0\rangle\,|1\rangle$$

Mathematically:

$$|\psi_0\rangle = |0\rangle \otimes |1\rangle$$

STEP 2: APPLY H TO BOTH QUBITS

Hadamard on |0>

$$H\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Hadamard on |1>

$$H\,|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Expand:

$$|\psi_1\rangle = \frac{1}{2}(|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle))$$

This simplifies to:

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

This state is **a superposition of both inputs 0 and 1**, ready for single oracle query.

STEP 3: ORACLE OPERATION U_f

Quantum oracle:

$$U_f \mid x, y\rangle = \mid x, y \oplus f(x)\rangle$$

Key identity:

$$U_f(\mid x\rangle \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}) = (-1)^{f(x)} \mid x\rangle \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}$$

Proof

Try both cases:

1. If f(x) = 0:

$$y \oplus 0 = y \Rightarrow \text{oracle does nothing}$$

2. If f(x) = 1:

$$\mid 0\rangle - \mid 1\rangle \mapsto \mid 1\rangle - \mid 0\rangle = -(\mid 0\rangle - \mid 1\rangle)$$

Thus oracle only adds a phase:

$$(-1)^{f(x)}$$

The second qubit disappears from the math.

Thus the state becomes:

$$\mid \psi_2\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)} \mid 0\rangle + (-1)^{f(1)} \mid 1\rangle) \otimes \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}$$

All information about f is now inside the PHASE of the first qubit.

STEP 4: APPLY HADAMARD TO FIRST QUBIT

We apply H to:

$$\mid \phi\rangle = \frac{(-1)^{f(0)} \mid 0\rangle + (-1)^{f(1)} \mid 1\rangle}{\sqrt{2}}$$

Hadamard identities:

$$H \mid 0\rangle = \frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}}$$

$$H \mid 0\rangle = \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}$$

Apply H:

$$H \mid \phi\rangle = \frac{1}{\sqrt{2}}[(-1)^{f(0)} \frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}} + (-1)^{f(1)} \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}]$$

Simplify:

$$H \mid \phi\rangle = \frac{1}{2}[((-1)^{f(0)} + (-1)^{f(1)}) \mid 0\rangle + ((-1)^{f(0)} - (-1)^{f(1)}) \mid 1\rangle]$$

Now we analyze each case.

FINAL OUTCOME: TWO POSSIBILITIES

Case 1: f is constant

Then:

$$f(0) = f(1) \Rightarrow (-1)^{f(0)} = (-1)^{f(1)}$$

So:
Coefficient of |0>:

$$(-1)^{f(0)} + (-1)^{f(1)} = 2(-1)^{f(0)} \neq 0$$

Coefficient of |1>:

$$(-1)^{f(0)} - (-1)^{f(1)} = 0$$

Final state is:

$$|\ 0\rangle$$

Thus measurement gives:

$$0(\text{constant})$$

Case 2: f is balanced
Then:

$$f(0) \neq f(1) \Rightarrow (-1)^{f(0)} = -(-1)^{f(1)}$$

So:
Coefficient of |0>:

$$(-1)^{f(0)} + (-1)^{f(1)} = 0$$

Coefficient of |1>:

$$(-1)^{f(0)} - (-1)^{f(1)} = \pm 2$$

Final state is:

$$|\ 1\rangle$$

Measurement gives:

$$1(\text{balanced})$$

DEUTSCH–JOZSA ALGORITHM

1. Deutsch–Jozsa Algorithm

Goal:

Determine whether a function $f(x): \{0,1\}^n \to \{0,1\}$is **constant** or **balanced** with a single query.

Idea:

Classically, $2^{n-1} + 1$evaluations are needed.

Quantumly, only **one** query.

Math Concept:

- Prepare uniform superposition of all inputs using Hadamard gates.
- Apply the **oracle** $U_f \mid x, y\rangle = \mid x, y \oplus f(x)\rangle$.
- Apply another layer of Hadamards to interfere all paths.

If $f$is constant → output all zeros

If $f$is balanced → output non-zero pattern.

Goal

Given a Boolean function

$$f(x): \{0,1\}^n \to \{0,1\},$$

determine **one thing only**:

✓ Is $f$constant?

(returns same value for all inputs)

✓ Or is $f$balanced?

(returns 0 for half inputs and 1 for half)

CLASSICAL COST

To be sure:

- Worst case: you may need to check **$2^{n-1} + 1$** inputs
  (half + one more to confirm not constant)

For large n, that is exponential.

QUANTUM COST

Using the Deutsch–Jozsa algorithm:

✓ Only ONE evaluation of the oracle U_f

regardless of n.

QUANTUM MATH BEHIND THE ALGORITHM

Let's go step-by-step with **vectors and operations**.

STEP 1 — Initial State

We use:

- $n$input qubits (start in $\mid 0\rangle^{\otimes n}$)
- 1 output qubit (start in $\mid 1\rangle$)

So initial state:

$$\mid \psi_0\rangle = \mid 0\rangle^{\otimes n} \mid 1\rangle$$

STEP 2 — Apply Hadamards

Apply $H$to **all qubits**.

For the input qubits:

Hadamard on $\mid 0\rangle$:

$$H \mid 0\rangle = \frac{1}{\sqrt{2}}(\mid 0\rangle + \mid 1\rangle)$$

After applying to all $n$qubits:

$$H^{\otimes n} \mid 0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \mid x\rangle$$

This creates a **uniform superposition of all inputs**.

For the output (last) qubit:

$$H \mid 1\rangle = \frac{1}{\sqrt{2}}(\mid 0\rangle - \mid 1\rangle)$$

STEP 3 — Apply the oracle

The quantum oracle performs:

$$U_f \mid x\rangle \mid y\rangle = \mid x\rangle \mid y \oplus f(x)\rangle$$

Now plug in $y = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$U_f|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} = (-1)^{f(x)}|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Important effect:

✓ The output qubit never changes

✓ The function value $f(x)$becomes a phase:

$$\mid x\rangle \rightarrow (-1)^{f(x)} \mid x\rangle$$

So the total state becomes:

$$\mid \psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \mid x\rangle \otimes \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}$$

STEP 4 — Apply Hadamard again (interference step)

Apply $H^{\otimes n}$to the **input qubits only**.

Key formula:

$$H^{\otimes n} \mid x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{x \cdot z} \mid z\rangle$$

Where $x \cdot z$is bitwise dot product mod 2.

So the state becomes:

$$\mid \psi_4\rangle = \frac{1}{2^n} \sum_{z=0}^{2^n-1} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)}(-1)^{x \cdot z} \right) \mid z\rangle \otimes \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}}$$

The inner sum determines everything:

$$S(z) = \sum_{x} (-1)^{f(x)}(-1)^{x \cdot z}$$

STEP 5 — Measure

We measure the **first n qubits**.

2. Bernstein–Vazirani Algorithm

Goal:

Find a secret bit string $s \in \{0,1\}^n$in the function

$$f(x) = s \cdot x \text{ (mod 2)}$$

Idea:

The oracle encodes the hidden string $s$via inner product.

Quantum parallelism + interference reveals $s$in one query.

Math Example:

For $s = 101$,

$$f(011) = 1 * 0 \oplus 0 * 1 \oplus 1 * 1 = 1$$

Circuit:

- Apply **Hadamard** to all qubits.
- Query the **oracle** U_f.
- Apply **Hadamard** again.
- Measure → output gives s.

Bernstein–Vazirani Algorithm

Goal

Given a function

$$f(x) = s \cdot x(\text{mod}2)$$

where

- $x \in \{0,1\}^n$
- $s = (s_1, \dots, s_n)$is a *hidden bit string*,
- the dot product is

$$s \cdot x = \bigoplus_{i=1}^{n} (s_i x_i)$$

Find the string $s$using **one** query to the oracle U_f.

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1. Oracle Definition

The oracle behaves as:

$$U_f \mid x\rangle \mid y\rangle =\mid x\rangle \mid y \oplus f(x)\rangle.$$

Because $y$starts in $\mid 1\rangle$, and because

$$\mid 1\rangle \overset{Z}{\rightarrow} (-1)^{f(x)} \mid 1\rangle,$$

Apply Hadamards Again

Now apply $H^{\otimes n}$to the first register.

Key identity:

$$H^{\otimes n} \mid x\rangle = \frac{1}{\sqrt{2^n}} \sum_z \quad (-1)^{x \cdot z} \mid z\rangle.$$

Apply to each term:

$$H^{\otimes n}(\sum_x \quad (-1)^{s \cdot x} \mid x\rangle) = \sum_x \quad (-1)^{s \cdot x}(\frac{1}{\sqrt{2^n}} \sum_z \quad (-1)^{x \cdot z} \mid z\rangle)$$

Swap the sums:

$$\frac{1}{\sqrt{2^n}} \sum_z (\sum_x \quad (-1)^{s \cdot x}(-1)^{x \cdot z}) \mid z\rangle.$$

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$$\frac{1}{\sqrt{2^n}} \sum_z (\sum_x \quad (-1)^{s \cdot x}(-1)^{x \cdot z}) \mid z\rangle.$$

So we have:

$$\frac{1}{\sqrt{2^n}} \sum_z (\sum_x \quad (-1)^{x \cdot (s \oplus z)}) \mid z\rangle.$$

5. Orthogonality Property

Key fact:

$$\sum_x \quad (-1)^{x \cdot (s \oplus z)} = \{\begin{matrix} 2^n & \text{if } z = s \\ 0 & \text{if } z \neq s \end{matrix}$$

This means **all amplitudes vanish except for the one at** $z = s$.

Thus: $\mid s\rangle$

remains after normalization.

Final Output

After measurement, the student **always** sees:

$$\boxed{s}$$

with **probability 1**.