Problem Setup

Imagine you have a list (or database) of Npossible items — say, $N = 2^n$possibilities, encoded in $n$qubits.
Only one of them (or a few) is the “marked” or “correct” solution.
Example:
You want to find the input $x$such that

$$f(x) = 1$$

for a black-box function $f(x)$that outputs 1 only for the correct solution(s).
Classically, you would need to check, on average, $N/2$possibilities.
Grover’s algorithm can find it in only about

$$O(\sqrt{N})$$

steps — a *quadratic speedup*.

Main Idea (Amplitude Amplification)

In quantum computing, you can put the system in a **superposition** of all possible states:

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

Then you use a clever combination of quantum operations to **amplify** the amplitude of the correct state — increasing its probability of being measured.
That’s where **Amplitude Amplification** (the heart of Grover’s algorithm) comes in.

Steps of the Algorithm

Let’s go step-by-step

Step 1 – Initialization

Start with $n$qubits in the state:

$$|0\rangle^{\otimes n}$$

Apply a **Hadamard gate (H)** to each qubit:

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

Now you’re in an **equal superposition** of all possible states.

Step 2 – Oracle (Phase Inversion)

The **Oracle** $O_f$is a quantum subroutine that marks the correct solution by flipping its phase:

$$O_f |x\rangle = \begin{cases} -|x\rangle & \text{if } f(x) = 1 \\ |x\rangle & \text{if } f(x) = 0 \end{cases}$$

In practice: it multiplies the amplitude of the correct state(s) by **−1**.

This is like putting a “flag” on the correct answer, but quantum-mechanically.

Step 3 – Diffusion Operator (Inversion about the Mean)

This operator reflects the amplitudes of all states around their **average amplitude**.

Mathematically, it’s:

$$D = 2 \mid \psi_0\rangle\langle\psi_0 \mid - I$$

Intuitively:

- After the Oracle, the marked state has a *negative amplitude*.
- The Diffusion operator then flips it again — **amplifying** it beyond the average.

This is the **amplitude amplification** step.

Step 4 – Grover Iteration

Each **Grover iteration** = Oracle + Diffusion operator.

Every iteration rotates the state vector closer to the target state in the Hilbert space.

After $k$iterations, the probability of measuring the correct state increases roughly as:

$$\sin^2((2k+1)\theta)$$

where $\sin^2(\theta) = 1/N$.

You should perform about:

$$k \approx \frac{\pi}{4}\sqrt{N}$$

iterations to maximize the success probability.

Step 5 – Measurement

Finally, measure the qubits.

With high probability, you’ll obtain the correct marked state!

Visualization (Amplitude Rotation)

If you think geometrically:

- The quantum state lives in a 2D plane spanned by:
  - $\mid w\rangle$: the marked state(s)
  - $\mid r\rangle$: the rest of the states
- Each iteration rotates the vector by an angle 2θtoward $\mid w\rangle$.

After a few rotations, the vector points almost entirely along $\mid w\rangle$→ high probability of success when measured.

Example (for 2 qubits, N=4)

We want to find $x = 11$such that $f(11) = 1$.

1. Initialize → superposition:

$$\frac{1}{2}(\mid 00\rangle + \mid 01\rangle + \mid 10\rangle + \mid 11\rangle)$$

2. Oracle → flips $\mid 11\rangle$amplitude to negative.
3. Diffusion → amplifies $\mid 11\rangle$.
4. Measure → you get 11 with ~100% probability.