ORFE 526: Probability Theory
Chapter 1

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Exercise 1: Show that an intersection of an arbitrary (countable or uncountable) family of σ-algebras on E is again a σ-algebra on E. What about unions of σ-algebras?

Answer: To be a σ-algebra, it must be closed under complements and under countable union. Indeed let $\mathcal{E}_n$ be a sequence of σ-algebras,

$$A \in \bigcap_n \mathcal{E}_n \implies A \in \mathcal{E}_n \ \forall n$$
$$\implies A^c \in \mathcal{E}_n \ \forall n \implies A^c \in \bigcap_n \mathcal{E}_n$$

Likewise, let A_m be a collection of sets in $\bigcap_n \mathcal{E}_n$,

$$A_m \in \bigcap_n \mathcal{E}_n \ \forall m \implies A_m \in \mathcal{E}_n \ \forall m, n$$
$$\implies \bigcup_m A_m \in \mathcal{E}_n \ \forall n \implies \bigcup_m A_m \in \bigcap_n \mathcal{E}_n$$

Thus σ-algebras are closed under intersection. However, they are not closed under union. Consider $\{\emptyset, A, A^c, E\}$ and $\{\emptyset, B, B^c, E\}$ which are both σ-algebras, but their union $\{\emptyset, A, A^c, B, B^c, E\}$ is not a σ-algebra since it is missing $A \cup B$.

Exercise 2: Show that if $\mathcal{E}$ is a σ-algebra, then

i) $\mathcal{E}$ is a p-system

ii) $\mathcal{E}$ is a d-system if

Answer:

i) $\mathcal{E}$ is a p-system if it is non-empty and it is closed under intersection. Obviously, a σ-algebra is non-empty. It is also closed under intersection due to De Morgan's law that $\bigcap\limits_{n\geq 1} A_i = \left(\bigcup\limits_{n\geq 1} A_i^c\right)^c$. As well, σ-algebras are closed under complementation and union:

$$
\begin{aligned}
(A_n)_n \subset \mathcal{E} &\implies (A_n^c)_n \subset \mathcal{E} && \text{Since closed under complementation}\\
&\implies \bigcup_{n\geq 1} A_i^c \in \mathcal{E} && \text{Since closed under union}\\
&\implies \left(\bigcup_{n\geq 1} A_i^c\right)^c \in \mathcal{E} && \text{Since closed under complementation}\\
&\implies \bigcap_{n\geq 1} A_i \in \mathcal{E} && \text{By De Morgan's}
\end{aligned}
$$

ii) $\mathcal{E}$ is a d-system if

i) $E \in \mathcal{D}$

ii) if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \backslash A \in \mathcal{D}$

iii) if $A_1, A_2, A_3, \ldots \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then $A_n \nearrow A \in \mathcal{D}$

E is in $\mathcal{E}$ since $A \cup A^c \in \mathcal{E}$ and $A \cup A^c = E$. Now, $B\backslash A = B \cap A^c$ which is in $\mathcal{E}$ since σ-algebras are closed under complementation and intersection (since they are d-systems by the first part). We also have $A_n \nearrow A \in \mathcal{E}$ since $A_n \nearrow A = \bigcup_n A_n$ and σ-algebras are closed under union.

Exercise 3: Let $\mathcal{D}$ be a a d-system on E. Fix D in $\mathcal{D}$ and define

$$\widehat{\mathcal{D}} = \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$$

Prove that $\widehat{\mathcal{D}}$ is a d-system.

Answer: To show that something is a d-system, we must show the following 3 properties:

i) $E \in \mathcal{D}$

ii) if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \backslash A \in \mathcal{D}$

iii) if $A_1, A_2, A_3, \ldots \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then $A_n \nearrow A \in \mathcal{D}$

Proof of these properties for $\widehat{\mathcal{D}} = \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$:

i) Let $A = E \in \mathcal{D}$, then $A \cap D = E \cap D = D$ since the intersection of a set and the universe is just the set. Note that $D \in \mathcal{D}$ by definition. So, $E \cap D \in \mathcal{D}$ which implies $E \in \widehat{\mathcal{D}}$

ii) Let $A, B \in \widehat{\mathcal{D}}$ and $A \subseteq B$, then

Since $A, B \in \widehat{\mathcal{D}}$, then $(B \cap D) \in \mathcal{D}$ and $(A \cap D) \in \mathcal{D}$
Since $\mathcal{D}$ is a d-system, then $(B \cap D) \backslash (A \cap D) \in \mathcal{D}$
Note that $(B \cap D) \backslash (A \cap D) = (B \backslash A) \cap D$
So, $(B \backslash A) \cap D \in \mathcal{D}$
Thus, $B \backslash A \in \widehat{\mathcal{D}}$

iii) Let $A_1, A_2, A_3, \ldots \in \widehat{\mathcal{D}}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then

Since $A_1, A_2, A_3, \ldots \in \widehat{\mathcal{D}}$, then $A_n \cap D \in \mathcal{D}, n \geq 1$
Since $\mathcal{D}$ is a d-system, then $\displaystyle\bigcup_{n \geq 1} (A_n \cap D) \in \mathcal{D}$
Distributing the union we have, $(\cup_{n \geq 1} A_n) \bigcap (\cup_{n \geq 1} D) = A \cap D$
Where $\cup_{n \geq 1} A_n = A$ since $\mathcal{D}$ is a d-system
So, $A \cap D \in \mathcal{D}$
Thus, $A \in \widehat{\mathcal{D}}$

All 3 properties hold, thus $\widehat{\mathcal{D}}$ is a d-system.

Exercise 4: Show than an intersection of arbitrary (countable or uncountable) family of d-systems on E is again a d-system on E. What about p-systems?

Answer: To show that something is a d-system, we must show the following 3 properties:

i) $E \in \mathcal{D}$

ii) if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \backslash A \in \mathcal{D}$

iii) if $A_1, A_2, A_3, \ldots \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then $A_n \nearrow A \in \mathcal{D}$

Let $\mathcal{D}_n$ be a collection of d-systems. Then,

i) Since $\mathcal{D}_n$ are all d-systems, we have

$$E \in \mathcal{D}_n \ \forall n \implies E \in \bigcap_n \mathcal{D}_n$$

ii) if $A, B \in \mathcal{D}_n$ and $A \subseteq B$, then $B \backslash A \in \mathcal{D}_n$ $\forall n$ which implies $B \backslash A \in \bigcap_n \mathcal{D}_n$

iii) if $A_1, A_2, A_3, \ldots \in \mathcal{D}_n$ and $A_m \subseteq A_{m+1}$, $m \geq 1$, then $A_m \nearrow A \in \mathcal{D}_n$ $\forall n$ which implies $A_m \nearrow A \in \bigcap_n \mathcal{D}_n$

Thus d-systems are closed under intersection. However, p-system need not be closed under intersection. Consider $\{A, B, A \cap B\}$ and $\{C, D, C \cap D\}$ which are both p-systems. However, their intersection is $\emptyset$ which is not a p-system.

Exercise 5: Let $\mathcal{C}$ be a countable partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from $\mathcal{C}$.

Answer: Since $\mathcal{C}$ is a partition of E. Then, for any $A \subset \mathcal{C}$, A^c is the union of every other element of the partition not in A. That is, all complements are countable unions of the elements of the partition. Also, there are no intersections since this is a partition. By definition, if $A, B \in \mathcal{C}$ then $A \cap B = \emptyset$. Thus, $\sigma\mathcal{C}$ is simply all the countable unions of the partition $\mathcal{C}$.

Exercise 6: Let $\mathcal{C}, \mathcal{D} \subset 2^E$. Show the following:

i) $\mathcal{C} \subset \mathcal{D} \implies \sigma\mathcal{C} \subset \sigma\mathcal{D}$

ii) $\mathcal{C} \subset \sigma\mathcal{D} \implies \sigma\mathcal{C} \subset \sigma\mathcal{D}$

iii) If $\mathcal{C} \subset \sigma\mathcal{D}$ and $\mathcal{D} \subset \sigma\mathcal{C} \implies \sigma\mathcal{C} = \sigma\mathcal{D}$

iv) $\mathcal{C} \subset \mathcal{D}\sigma\mathcal{C} \implies \sigma\mathcal{C} = \sigma\mathcal{D}$

Answer:

i) By assumption and definition,

   $\mathcal{C} \subset \mathcal{D} \subset \sigma\mathcal{D}$

   The definition of $\sigma\mathcal{C}$ is: $\bigcap \mathcal{E} = \sigma\mathcal{C}$ where $\mathcal{E}$ is any σ-algebra containing $\mathcal{C}$

   $\sigma\mathcal{D}$ is one such $\mathcal{E}$. Thus, $\sigma\mathcal{C} \subset \sigma\mathcal{D}$

ii) This follows from the first part. No assumption was made if $\mathcal{D}$ was a σ-algebra or not.

iii) By the previous part, $\mathcal{C} \subset \sigma\mathcal{D} \implies \sigma\mathcal{C} \subset \sigma\mathcal{D}$ and $\mathcal{D} \subset \sigma\mathcal{C} \implies \sigma\mathcal{D} \subset \sigma\mathcal{C}$. So, $\sigma\mathcal{C} = \sigma\mathcal{D}$.

iv) By the first part, $\mathcal{C} \subset \mathcal{D} \implies \sigma\mathcal{C} \subset \sigma\mathcal{D}$. The second part shows that $\mathcal{D} \subset \sigma\mathcal{C} \implies \sigma\mathcal{D} \subset \sigma\mathcal{C}$. So, $\sigma\mathcal{C} = \sigma\mathcal{D}$.

Exercise 7: Show that $\mathcal{B}(\mathbb{R})$ can be generated as

i) $\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, x]; x \in \mathbb{R}\}$

ii) $\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, x); x \in \mathbb{R}\}$

iii) $\mathcal{B}(\mathbb{R}) = \sigma\{(x, y]; x, y \in \mathbb{R}\}$

iv) $\mathcal{B}(\mathbb{R}) = \sigma\{(x, \infty); x \in \mathbb{R}\}$

Answer:

i) All open sets are a countable union of open intervals (due to the fact the the rationals are dense but countable in $\mathbb{R}$). Thus, we just have to show that any open interval (a, b) is the countable union (and complement) of intervals of the form $(-\infty, x]$. Consider $a_n = a + \frac{1}{n}$ and $b_n = b - \frac{1}{n}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} (a_n, b_n] = \bigcup_{n=1}^{\infty} (-\infty, a_n]^c \cap (-\infty, b_n]$$

ii) We follow the same technique as the first part.

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n) = \bigcup_{n=1}^{\infty} (-\infty, a_n)^c \cap (-\infty, b_n)$$

iii) We follow the same technique as the first part.

$$(a, b) = \bigcup_{n=1}^{\infty} (a_n, b_n]$$

iv) Notice that $(x, \infty)^c = (-\infty, x]$ thus the proof is the complement of the first part.

Exercise 8: Let $(E, \mathcal{E})$ be a measurable space. Fix D in E and let

$$D = \mathcal{E} \cap D = \{A \cap D : A \in \mathcal{E}\}$$

Show that $\mathcal{D}$ is a σ-algebra on D.

Answer: Obviously, $\mathcal{D}$ is non-empty since $D \in \mathcal{D}$. We show closed under complement first. Suppose $A \cap D \in \mathcal{D}$. Then

$$(A \cap D)^c = A^c \cup D^c$$
$$\implies (A^c \cup D^c) \cap D = A^c \cap D \bigcup D^c \cap D = A^c \cap D \in \mathcal{D}$$

Where the last equality is true because $A^c \in \mathcal{E}$. Thus, $(A \cap D)^c \in \mathcal{D}$. Now for countable union. Suppose $A_n \cap D \in \mathcal{D}$. Then,

$$\bigcup_n (A_n \cap D) = \cup_n A_n \bigcap D \in \mathcal{D}$$

Since $\cup_n A_n \in \mathcal{E}$.

Exercise 9: Consider a function $f : E \to F$. Show the following:

i) $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(F) = E$

ii) $f^{-1}(B\backslash C) = f^{-1}(B)\backslash f^{-1}(C)$

iii) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$

iv) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$

for any $B_i \in 2^E$

Answer:

i) By definition of a function, it must map an element in E to an element in F. Thus, we cannot have $A \neq \emptyset$ and $f(A) = \emptyset$. Thus, $f^{-1}(\emptyset) = \emptyset$. Now, we have by sequence of logic,

$$\begin{aligned} f^{-1}(F) &= \{A \in E : f(A) \in F\} \\ &= \{A \in E : f(A) \notin \emptyset\} \\ &= \{A \in E : f(A) \in \emptyset\}^c \\ &= \{A \in E : f(A) \in \emptyset\}^c = \emptyset^c = E \end{aligned}$$

ii) Using the same technique as the first part,

$$\begin{aligned} f^{-1}(B\backslash C) &= \{A \in E : f(A) \in B\backslash C\} \\ &= \{A \in E : f(A) \in B \cap C^c\} \\ &= \{A \in E : f(A) \in B\} \cap \{A \in E : f(A) \in C^c\} \\ &= f^{-1}(B) \cap f^{-1}(C^c) = f^{-1}(B)\backslash f^{-1}(C) \end{aligned}$$

iii) Using the same technique as the first part,

$$\begin{aligned} f^{-1}(\cup_i B_i) &= \{A \in E : f(A) \in \cup_i B_i\} \\ &= \cup_i \{A \in E : f(A) \in B_i\} \\ &= \cup_i f^{-1}(B_i) \end{aligned}$$

iv) Using the same technique as the first part,

$$\begin{aligned} f^{-1}(\cap_i B_i) &= \{A \in E : f(A) \in \cap_i B_i\} \\ &= \cap_i \{A \in E : f(A) \in B_i\} \\ &= \cap_i f^{-1}(B_i) \end{aligned}$$

Exercise 10: Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces and consider a function $f : E \to F$. Define $\mathcal{F}_1 = \{B \in \mathcal{F} : f^{-1}(B) \in \mathcal{E}\}$. Show that $\mathcal{F}_1$ is a σ-algebra on $\mathcal{F}$.

Answer: We prove closed under complements first. Let $A \in \mathcal{F}_1$. Then

$$A \in \mathcal{F}_1 = \{A \in \mathcal{F} : f^{-1}(A) \in \mathcal{E}\}$$

But $\mathcal{E}$ is a σ-algebra, so $f^{-1}(A)^c \in \mathcal{E}$. From the previous exercise

$$f^{-1}(A^c) = f^{-1}(E \backslash A) = f^{-1}(E) \backslash f^{-1}(A) = f^{-1}(A)^c$$

So, $f^{-1}(A^c) \in \mathcal{E}$. We also have $A^c \in \mathcal{F}$ since it is also a σ-algebra. Thus, $A^c \in \mathcal{F}_1$.

Now to show that it is closed under countable union. Let $A_n \in \mathcal{F}_1$. Then by the previous exercise,

$$f^{-1}(\cup_n A_n) = \cup_n f^{-1}(A_n)$$

Since $\mathcal{E}$ is a σ-algebra, then $\cup_n f^{-1}(A_n) \in \mathcal{E}$. We also have $\cup_n A_n \in \mathcal{F}$ since it is also a σ-algebra. Thus, $\cup_n A_n \in \mathcal{F}_1$.

Exercise 11: Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a function $f : E \to F$. Define $f^{-1}(\mathcal{F}) = \{f^{-1}(B) : B \in \mathcal{F}\}$. Prove that:

(i) $f^{-1}(\mathcal{F})$ is a σ-algebra.

(ii) $f^{-1}(\mathcal{F})$ is the smallest σ-algebra on E such that f is measurable relative to it and $\mathcal{F}$.

Answer:

(i) First, note that the inverse image preserves complements and union:

Complementation:

$$\begin{aligned} a \in f^{-1}(A^c) &\iff f(a) \in A^c \\ &\iff f(a) \notin A \\ &\iff a \notin f^{-1}(A) \\ &\iff a \in f^{-1}(A)^c \end{aligned}$$

Union:

$$\begin{aligned} a \in f^{-1}(A \cup B) &\iff f(a) \in A \cup B \\ &\iff f(a) \in A \text{ or } f(a) \in B \\ &\iff a \in f^{-1}(A) \text{ or } a \in f^{-1}(B) \\ &\iff a \in f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

We now use these to show $f^{-1}(\mathcal{F})$ is closed under complementation and countable union:

Complementation:

$$\begin{aligned} A \in f^{-1}(\mathcal{F}) &\iff A = f^{-1}(B) \text{ for some } B \\ &\iff A^c = f^{-1}(B)^c \\ &\iff A^c = f^{-1}(B^c) \text{ , note that } B^c \in \mathcal{F} \text{ since it is } \sigma\text{-algebra} \\ &\iff A^c \in f^{-1}(\mathcal{F}) \end{aligned}$$

Union:

$$\begin{aligned}
A_n \in f^{-1}(\mathcal{F}) &\iff A_n = f^{-1}(B_n) \text{ for some } B_n \\
&\iff \bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} f^{-1}(B_n) \\
&\iff \bigcup_{n\geq 1} A_n = f^{-1}\big(\bigcup_{n\geq 1} B_n\big)
\end{aligned}$$

Note that $\bigcup_{n\geq 1} B_n \in \mathcal{F}$ since it is σ-algebra

$$\iff \bigcup_{n\geq 1} A_n \in f^{-1}(\mathcal{F})$$

Also, $E \in f^{-1}(\mathcal{F})$ since $f^{-1}(F) = E$. Thus, $f^{-1}(\mathcal{F})$ is a σ-algebra.

(ii) Assume there is a smaller σ-algebra such that $\mathcal{E} \subset f^{-1}(\mathcal{F})$. But we know that f is measurable so, $f^{-1}(\mathcal{F}) \subset \mathcal{E}$. But this contradicts our assumption, therefore there does not exist a smaller σ-algebra and $f^{-1}(\mathcal{F}) = \mathcal{E}$.

Exercise 12: Let $\mathcal{E}$ be a σ-algebra and consider a sequence $(A_n)_n \subset \mathcal{E}$. Prove that $\bigcap\limits_{n\geq 1} A_n \in \mathcal{E}$.

Answer: We know by De Morgan's law that $\bigcap\limits_{n\geq 1} A_i = \left(\bigcup\limits_{n\geq 1} A_i^c\right)^c$. As well, σ-algebras are closed under complementation and union:

$$\begin{aligned}
(A_n)_n \subset \mathcal{E} &\implies (A_n^c)_n \subset \mathcal{E} && \text{Since closed under complementation}\\
&\implies \bigcup_{n\geq 1} A_i^c \in \mathcal{E} && \text{Since closed under union}\\
&\implies \left(\bigcup_{n\geq 1} A_i^c\right)^c \in \mathcal{E} && \text{Since closed under complementation}\\
&\implies \bigcap_{n\geq 1} A_i \in \mathcal{E} && \text{By De Morgan's}
\end{aligned}$$

Exercise 13: Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be an increasing function. Show that f is Borel measurable.

Answer: A function is Borel measurable if for every $r \in \mathbb{R}$, then $E = \{x : f(x) \leq r\}$ is measurable.

Define $b = \sup f^{-1}\big((-\infty, r]\big)$
If $b = \infty$, then $E = \mathbb{R}$
If $b = -\infty$, then $E = \emptyset$
If $b \in \mathbb{R}$, then $E = (-\infty, b]$ or $E = (-\infty, b)$
because $f(x) \leq r$ for all $x \in E$ since f is increasing.

All of these E's are elements of the Borel set and hence Borel measurable.

Exercise 14: Let $(E, \mathcal{E})$ be a measurable space and $f : E \to \mathbb{R}$ a Borel measurable function.

(i) Show that $|f|$ is measurable

(ii) Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Show that $|f| = f^+ + f^-$

(iii) Use (i) and (ii) to show that f^+ and f^- are measurable

Answer:

(i) We will show that the absolute value does not change measurability

For $r < 0$, $\{x : |f(x)| \leq r\} = \emptyset$
For $r = \infty$, $\{x : |f(x)| \leq \infty\} = E$
For $r \geq 0$, $\{x : |f(x)| \leq r\} = \{x : -r \leq f(x) \leq r\}$
$= \{x : f(x) \leq r\} \bigcap \{x : f(x) \leq -r\}^c$
Note that the final line is the intersection of two measurable sets since f itself is measurable. Note that an intersection of two measurable set is measurable since σ-algebras are closed under countable intersection.

Thus, since for every $r \in \mathbb{R}$, $\{x : |f(x)| \leq r\}$ is measurable, then $|f|$ is measurable.

(ii) Break f into two cases, $f \geq 0$ and $f < 0$:

$$
\begin{aligned}
&f \geq 0 : \text{ Then } \max\{f, 0\} = f \text{ and } -\min\{f, 0\} = 0 \\
&f^+ + f^- = f + 0 = f \\
&f < 0 : \text{ Then } \max\{f, 0\} = 0 \text{ and } -\min\{f, 0\} = -f \\
&f^+ + f^- = 0 - f = -f
\end{aligned}
$$

Thus, we have $f^+ + f^-$ defined piecewise as: $f = \begin{cases} f & f \geq 0 \\ -f & f < 0 \end{cases}$

Which is the exact definition of $|f|$. Hence, $|f| = f^+ + f^-$

(iii) One cleverly notes that we can rewrite f^+ and f^- as follows:

$$f^+ = \frac{|f| + f}{2}$$
$$f^- = \frac{|f| - f}{2}$$

And note from lecture that the sum of two measurable functions is measurable.