Homework 9

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Question 1

Let us consider the Crank-Nicholson equation in an alternate form:

$$v_i^{j+1} - v_i^j = \frac{\alpha \Delta t}{2} \left(\delta^2 v_i^{j+1} + \delta^2 v_i^j \right)$$

First, let us do the left side. We have

$$v_i^{j+1} - v_i^j = \partial_t v_i^j \Delta t + \mathcal{O}(\Delta t^2) \quad (1)$$

Now, let us compute $\delta^2 v_i^j$. We have by the Crank-Nicholson scheme and Taylor expansion

$$\begin{aligned}
\delta^2 v_i^j &= \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{\Delta x^2} \\
&= \frac{v_{i+1}^j - v_i^j + v_{i-1}^j - v_i^j}{\Delta x^2} \\
&= \frac{\partial_x v_i^j \Delta x + \frac{1}{2}\partial_{xx} v_i^j \Delta x^2 + \frac{1}{6}\partial_{xxx} v_i^j \Delta x^3 - \partial_x v_i^j \Delta x + \frac{1}{2}\partial_{xx} v_i^j \Delta x^2 - \frac{1}{6}\partial_{xxx} v_i^j \Delta x^3 + \mathcal{O}(\Delta x^4)}{\Delta x^2} \\
&= \partial_{xx} v_i^j + \mathcal{O}(\Delta x^2) \quad (2)
\end{aligned}$$

Now, for the hard expansion

$$\begin{aligned}
\delta^2 v_i^{j+1} &= \frac{v_{i+1}^{j+1} - 2v_i^{j+1} + v_{i-1}^{j+1}}{\Delta x^2} \\
&= \frac{v_{i+1}^{j+1} - v_i^j - 2(v_i^{j+1} - v_i^j) + v_{i-1}^{j+1} - v_i^j}{\Delta x^2} \\
&= \frac{1}{\Delta x^2} \Bigg(\partial_x v_i^j \Delta x + \partial_t v_i^j \Delta t + \frac{1}{2}\partial_{xx} v_i^j \Delta x^2 + \frac{1}{2} v_i^j \Delta t^2 + \partial_{xt} v_i^j \Delta x \Delta t \\
&+ \frac{1}{6}\partial_{xxx} v_i^j \Delta x^3 + \frac{1}{6}\partial_{ttt} v_i^j \Delta t^3 + \frac{1}{2}\partial_{xxt} v_i^j \Delta x^2 \Delta t + \frac{1}{2}\partial_{ttx} v_i^j \Delta t^2 \Delta x \Bigg) \\
&- \frac{2}{\Delta x^2} \left(\partial_t v_i^j \Delta t + \frac{1}{2} v_i^j \Delta t^2 + \frac{1}{6}\partial_{ttt} v_i^j \Delta t^3 \right) \\
&+ \frac{1}{\Delta x^2} \Bigg(-\partial_x v_i^j \Delta x + \partial_t v_i^j \Delta t + \frac{1}{2}\partial_{xx} v_i^j \Delta x^2 + \frac{1}{2} v_i^j \Delta t^2 - \partial_{xt} v_i^j \Delta x \Delta t \\
&- \frac{1}{6}\partial_{xxx} v_i^j \Delta x^3 + \frac{1}{6}\partial_{ttt} v_i^j \Delta t^3 + \frac{1}{2}\partial_{xxt} v_i^j \Delta x^2 \Delta t - \frac{1}{2}\partial_{ttx} v_i^j \Delta t^2 \Delta x \Bigg) + \mathcal{O}(\Delta t^4) + \mathcal{O}(\Delta x^2) \\
&= \partial_{xx} v_i^j + \partial_{xxt} v_i^j \Delta t + \mathcal{O}(\Delta t^4) + \mathcal{O}(\Delta x^2) \quad (3)
\end{aligned}$$

Putting together equations (1), (2), and (3) yields

$$\partial_t v_i^j \Delta t = \alpha \partial_{xx} v_i^j \Delta t + \frac{1}{2} \alpha \partial_{xxt} v_i^j \Delta t^2 + \mathcal{O}(\Delta t^4) + \mathcal{O}(\Delta x^2)$$

Now, we divide by Δt which yields the result

$$\partial_t v_i^j = \alpha \partial_{xx} v_i^j + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$$