Homework #7, ORF 526

Assigned on November 15; Due November 22, 2016, 5:00 pm

1. *(2 points)* An inhomogeneous Poisson process with intensity function $\lambda(t) > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$P(N_T - N_t = n) = \frac{1}{n!}\Big(\int_t^T \lambda(s)\,ds\Big)^n e^{-\int_t^T \lambda(s)\,ds}.$$

   The intensity $\lambda(t)$ is a non-negative function on time only. Consider the filtration $\mathcal{F}_t$ defined by the process N_t.

   (a) Find $\mathbb{E}[N_{t+s} - N_t|\mathcal{F}_t]$.

   (b) Prove that $M_t = N_t - \int_0^t \lambda(s)\,ds$ is an $\mathcal{F}_t$-martingale.

2. Denote by T_k the time of the kth jump of a Poisson process N_t with rate $\lambda > 0$. Let $\tau_1 = T_1$, $\tau_k = T_k - T_{k-1}$, for $k \geq 1$, be the interarrival times (the time elapsed between two consecutive jumps).

   (a) Show that the random variables τ_k are independent.

   (b) Prove that the random variables τ_k are exponentially distributed.

   (c) Show that $\mathbb{E}[\tau_k] = 1/\lambda$.

   (d) Verify that the probability density of T_k is a gamma distribution. What is its mean and variance?

3. Let $(M_n)_{n\geq 1}$ be a process such that:

   (i) M_n is $\mathcal{F}_n$-martingale;

   (ii) M_n is $\mathcal{F}_n$-predictable.

   (a) Show that M_n is constant, $M_n = M_0$, as.

   (b) What happens if (i) is replaced by the condition "M_n an is $\mathcal{F}_n$-submartingale?"

4. Let X be an integrable random variable. Show that

$$E[|X|] = \int_0^\infty P(|X| > y)\,dy.$$

Please solve each problem on a separate sheet of paper. Staple the sheets and attach a cover page with your name and course number. The Homework will be turned in 526 mailbox in Sherred Hall, cabinet A. You can retrieve your graded Homework from your folder situated in the same room, a week later.