**ELE 435/535 - Machine Learning and Pattern Recognition**

Fall 2018

**HOMEWORK 9: Theory**

ELE 435: do questions 1-3; ELE 535: do all questions.

Q1 Kernels:

(a) Let $\mathcal{A}$ be a finite set and for each subset $\mathcal{U} \subseteq \mathcal{A}$ let $|\mathcal{U}|$ denote the number of elements in $\mathcal{U}$. For $\mathcal{U}, \mathcal{V} \subset \mathcal{A}$, let $k(\mathcal{U}, \mathcal{V}) = |\mathcal{U} \cap \mathcal{V}|$. By finding a suitable feature map, show that $k(\cdot, \cdot)$ is a kernel on the power set $\mathcal{P}(\mathcal{A})$ of all subsets of $\mathcal{A}$.

(b) Show that $k(x, z) = \sum_{i=1}^{n} \cos^2(x_i - z_i)$ is a kernel on $\mathbb{R}^n$.

(b) Let $P \in \mathbb{R}^{n \times n}$ be symmetric PSD. Show that $k(x, z) = e^{-\frac{1}{2}(x-z)^T P(x-z)}$ is a kernel on $\mathbb{R}^n$.

(c) $k(x, y) = h_t(Ax)^T h_t(Ay)$ where $A \in \mathbb{R}^{n \times n}$ and $h_t$ is a thresholding function that maps $z = [z_i]$ to $h_t(z) = [\tilde{z}_i]$ with $\tilde{z}_i = z_i$ if $|z_i| > t$ and 0 otherwise.

Q2 Let $k_j$ be a kernel on $\mathcal{X}$ with feature map $\phi_j \colon \mathcal{X} \to \mathbb{R}^q$, $j = 1, 2$. In each part below, find a simple feature map for the kernel $k$ in terms of feature maps for the kernels $k_j$. By this means, give an interpretation for the new kernel $k$.

(a) $k(x, z) = k_1(x, z) + k_2(x, z)$

(b) $k(x, z) = k_1(x, z) k_2(x, z)$

(c) $k(x, z) = k_1(x, z) / \sqrt{k_1(x, x) k_1(z, z)}$

Q3 A binary labelled set of data in $\mathbb{R}^2$ is used to learn a SVM using the homogeneous quadratic kernel. By writing the equation for the decision boundary in terms of a quadratic form, reason about the types of decision boundaries that are possible in $\mathbb{R}^2$. In each case, give a neat sketch.

Q4 Consider the definite integral[1]

$$\int_0^\infty e^{-(ax^2 + \frac{b}{x^2})} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}. \tag{1}$$

Show that (1) can be rewritten as

$$\frac{\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\frac{s}{t^2}} e^{-(\frac{\alpha t}{2})^2} dt = e^{-\alpha\sqrt{s}}, \quad \alpha > 0.$$

Now use the above results to show that $k(x, z) = e^{-\alpha \|x - z\|_2}$ is a kernel.

Q5 Let $a > 0$ and $L_2[0, a]$ denote the set of real valued square integrable functions on the interval $[0, a]$. $L_2[0, a]$ is a Hilbert space under the inner product $<g, h> = \int_0^a g(s)h(s)ds$. For $f \in L_2[0, a]$, let $g(t) = \int_0^t f^2(s)ds$ and $h(t) = \int_t^a f^2(s)ds$, where $t \in [0, a]$.

Show that for each $a > 0$,

(a) $k(x, z) = \min(x, z)$ is a kernel on $[0, a]$

(b) $k(x, z) = a - \max(x, z)$ is a kernel on $[0, a]$

(c) $k(x, z) = e^{-(\max(x,z) - \min(x,z))}$ is a kernel on $[0, a]$. Plot the function $\max(x, z) - \min(x, z)$ and use this to simplify the result further.

(d) for each $a > 0$, and $\gamma \geq 0$, $k(x, z) = e^{-\gamma|x-z|}$ is a kernel on $[-a, a]$

(e) for each $\gamma \geq 0$, $k(x, z) = e^{-\gamma|x-z|}$ is a kernel on $\mathbb{R}$.

[1] http://www.sosmath.com/tables/integral/integ38/integ38.html