ORFE 526: Probability Theory
Homework 7

Zachary Hervieux-Moore

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Exercise 1: An inhomogeneous Poisson process with intensity function $\lambda(t) > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$P(N_T - N_t = n) = \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n e^{-\int_t^T \lambda(s)ds}$$

The intensity $\lambda(t)$ is a non-negative function on time only. Consider the filtration $\mathcal{F}_t$ defined by the process N_t.

a) Find $\mathbb{E}[N_{t+s} - N_t|\mathcal{F}_t]$

b) Prove that $M_t = N_t - \int_0^t \lambda(s)ds$ is an $\mathcal{F}_t$-martingale.

Answer:

a) Calculating the conditional expectation,

$$\begin{aligned}
\mathbb{E}[N_{t+s} - N_t|\mathcal{F}_t] &= \sum_{n-0}^{\infty} nP(N_{t+s} - N_t = n|\mathcal{F}_t) \\
&= \sum_{n=0}^{\infty} \frac{n}{n!}\left(\int_t^{t+s} \lambda(s)ds\right)^n e^{-\int_t^{t+s} \lambda(s)ds} \\
&= \left(\int_t^{t+s} \lambda(s)ds\right) e^{-\int_t^{t+s} \lambda(s)ds} \sum_{n=0}^{\infty} \frac{1}{(n-1)!}\left(\int_t^{t+s} \lambda(s)ds\right)^{n-1} \\
&= \left(\int_t^{t+s} \lambda(s)ds\right) e^{-\int_t^{t+s} \lambda(s)ds} e^{\int_t^{t+s} \lambda(s)ds} \\
&= \int_t^{t+s} \lambda(s)ds
\end{aligned}$$

Where I used the fact that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ and the fact that the finite integral is well defined.

b) First we show M_t is integrable.

$$\begin{aligned}
\mathbb{E}[|M_t|] &= \mathbb{E}[|N_t - \int_0^t \lambda(s)ds|] \\
&\leq \mathbb{E}[|N_t|] + \mathbb{E}[|\int_0^t \lambda(s)ds|] \\
&= \mathbb{E}[|N_t - N_0||\mathcal{F}_0] + |\int_0^t \lambda(s)ds| \\
&= |\int_0^t \lambda(s)ds| + |\int_0^t \lambda(s)ds| \\
&= 2|\int_0^t \lambda(s)ds| < \infty
\end{aligned}$$

Since it is a finite integral.

Also, M_t is continuous as it is a continuous function of N_t and $\int_0^t \lambda(s)ds$ which are both continuous.

Now, we check the last property of martingales,

$$\begin{aligned}
\mathbb{E}[M_{t+r} - M_t|\mathcal{F}_t] &= \mathbb{E}[N_{t+r} - N_t - \int_0^{t+r} \lambda(s)ds + \int_0^t \lambda(s)ds|\mathcal{F}_t] \\
&= \mathbb{E}[N_{t+r} - N_t|\mathcal{F}_t] - \int_0^{t+r} \lambda(s)ds + \int_0^t \lambda(s)ds \\
&= \int_t^{t+r} \lambda(s)ds - \int_t^{t+1} \lambda(s)ds = 0
\end{aligned}$$

Thus, $\mathbb{E}[M_{t+r} - M_t|\mathcal{F}_t] = 0$ implies that $\mathbb{E}[M_{t+r}|\mathcal{F}_t] = M_t$. Thus, M_t is and $\mathcal{F}_t$-martingale.

Exercise 2: Denote by T_k the time of the k^{th} jump of a Poisson process N_t with rate $\lambda > 0$. Let $\tau_1 = T_1, \tau_k = T_k - T_{k-1}$, for $k \geq 1$, be the interarrival times (the time elapsed between two consecutive jumps).

a) Show that the random variable τ_k are independent.

b) Prove that the random variables τ_k are exponentially distributed.

c) Show that $\mathbb{E}[\tau_k] = 1/\lambda$.

d) Verify that the probability density of T_k is a gamma distribution. What is its mean and variance.

Answer:

a) We have

$$\begin{aligned} P(\tau_k = k, \tau_l = l) &= P(T_k - T_{k-1} = k, T_l - T_{l-1} = l) \\ &= P(N_{T_k} - N_{T_{k-1}} = 0, N_{T_l} - N_{T_{l-1}} = 0) \\ &= P(N_{T_k} - N_{T_{k-1}} = 0) P(N_{T_l} - N_{T_{l-1}} = 0) \end{aligned}$$

Where the last equality holds because increments are independent in Poisson processes.

$$= P(\tau_k = k) P(\tau_l = l)$$

Thus, the τ_k's are independent.

b) We have that,

$$P(T_1 > t) = P(N_t = 0) = e^{-\lambda t}$$

Thus, T_1 is exponential with rate λ. Now, for $s < t$,

$$P(T_2 > t | T_1 = s) = P(N_{T_2} > t - s) = e^{-\lambda(t-s)}$$

Which gives us

$$\begin{gathered} P(T_2 > t) \\ = P(T_2 > t | T_1 = s < t) P(T_1 = s) + P(T_2 > t | T_1 = s \geq t) P(T_1 = s) \\ = e^{-\lambda(t-s)} e^{-\lambda s} + 0 = e^{-\lambda t} \end{gathered}$$

By induction, all $P(T_k)$ are distributed exponentially with rate λ and independent. Finally,

$$
\begin{aligned}
P(\tau_k > t) &= P(T_k - T_{k-1} > t) \\
&= P(T_k > t - s | T_{k-1} = s) P(T_{k-1} = s) \\
&= e^{-\lambda(t-s)} e^{-\lambda s} = e^{-\lambda t}
\end{aligned}
$$

c) We know that τ_k is distributed exponentially with rate λ. So,

$$
\mathbb{E}[\tau_k] = \int_0^\infty t\lambda e^{-\lambda t} dt
$$

Which is readily integrated by parts,

$$
\begin{aligned}
&= \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t} dt \\
&= 0 - \left[\frac{e^{-\lambda t}}{\lambda}\right]_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}
$$

d) First note that $T_k = \tau_k + T_{k-1} = \ldots = \tau_k + \tau_{k+1} + \ldots + \tau_1$. Thus T_k is distributed as a sum of i.i.d. exponentials. We will show that this is a gamma distribution by using MGF's.

$$
\begin{aligned}
\mathbb{E}[e^{t\tau_k}] &= \mathbb{E}[e^{t(\sum T_n)}] \\
&= \mathbb{E}[\prod e^{tT_n}] \\
&= \prod_{n=1}^{k} \mathbb{E}[e^{tT_n}] \\
&= \mathbb{E}[e^{tT_1}]^k
\end{aligned}
$$

Since the T_k's are independent, we can break up the expectation by pieces. The last part is due to the fact that they are identically distributed. The inside term is the MGF of the exponential distribution.

$$\begin{aligned}\mathbb{E}[e^{tT_1}] &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx \\ &= \int_0^\infty \lambda e^{(t-\lambda)x}dx \\ &= \frac{\lambda}{\lambda - t}\end{aligned}$$

Where $t < \lambda$. Thus, the MGF of the sum is $\left(\frac{\lambda}{\lambda-t}\right)^k$ which is precisly the MGF for the gamma distribution with rate λ and shape k.

For the expectation of T_k, we use linearity of expectation to get

$$\begin{aligned}\mathbb{E}[T_k] &= \mathbb{E}[\sum_{n=1}^k \tau_n] \\ &= \sum_{n=1}^k \mathbb{E}[\tau_n] = \frac{k}{\lambda}\end{aligned}$$

Likewise, since they are independent, the variance of T_k is the sum of all the variances of τ_n's. Since they are exponential, the variance of τ_k is $\frac{1}{\lambda^2}$. Thus,

$$\begin{aligned}Var(T_k) = Var(\sum_{n=1}^k \tau_n) &= \sum_{n=1}^k Var(\tau_n) \\ &= \frac{k}{\lambda^2}\end{aligned}$$

Exercise 3: Let $(M_n)_{n\geq 1}$ be a process such that:

i) M_n is $\mathcal{F}_n$-martingale

ii) M_n is $\mathcal{F}_n$-predictable

a) Show that M_n is constant, $M_n = M_0$ a.s.

b) What happens if i) is replaced by the condition "M_n is $\mathcal{F}_n$ submartingale"

Answer:

a) Since we have that M_n is a martingale, we have

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$$

However, we also have that M_n is predictable,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_{n+1}$$

Thus, $X_{n+1} = X_n$. And in particular, $X_1 = X_0$. By induction, $X_n = X_0$ for all n.

b) If we replace the first condition with submartingale,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$$

So, $X_{n+1} \geq X_n$. Thus, X_n is non-decreasing.

Exercise 4: Let X be an integrable random variable. Show that

$$E[|X|] = \int_0^\infty P(|X| > y)dy$$

Answer: Starting with the right hand side,

$$\begin{aligned} &\int_0^\infty P(|X| > y)dy \\ &= \int_0^\infty \int_{-\infty}^\infty 1_{\{|x|>y\}} P(x)dxdy \end{aligned}$$

Since X is integrable, the above integral is finite and so we can exchange the itegrals due to Fubini's theorem,

$$\begin{aligned} &= \int_{-\infty}^\infty \int_0^\infty 1_{\{|x|>y\}} P(x)dydx \\ &= \int_{-\infty}^\infty P(x) \int_0^\infty 1_{\{|x|>y\}} dydx \\ &= \int_{-\infty}^\infty P(x) \int_0^{|x|} dydx \\ &= \int_{-\infty}^\infty P(x)|x|dx \\ &= \mathbb{E}[|X|] \end{aligned}$$