Homework #8, ORF 526

Assigned on November 22; Due November 29, 2016, 5:00 pm

1. *(1.5 points)* Let $(M_n)_n$ be an $\mathcal{F}_n$-adapted process such that $\mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = 0$, $\forall n \geq 0$. Prove that for any $p \geq 1$ we have

$$\mathbb{E}[M_{n+p} - M_n | \mathcal{F}_n] = 0, \quad \forall n \geq 0.$$

2. *(1 point)* Let T be a stopping time and define the process

$$F_n = \begin{cases} 1, & \text{if } \ n \leq T(\omega) \\ 0, & \text{if } \ n > T(\omega). \end{cases}$$

Show that F_n is a predictable process.

3. *(1.5 points)* Let S and T be stopping times with respect to filtration $\mathcal{F}_n$, with $S \leq T$. Define the process

$$X_n(\omega) = 1_{(S,T]}(n, \omega) = \begin{cases} 1, & \text{if } S(\omega) < n \leq T(\omega) \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that X_n is an $\mathcal{F}_n$-predictable process.

(b) If M_n is a martingale, show that $\mathbb{E}(M_{T \wedge n}) = \mathbb{E}(M_{S \wedge n})$.

4. *(2 points)* Assume that $X_1, X_2, X_3, \cdots$ are i.i.d. random variables with the same distribution as X

$$P(X = 1) = p, \qquad P(X = -1) = q,$$

where $0 < p = 1 - q < 1$, and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n = a + X_1 + \cdots + X_n, \qquad T = \inf\{n; S_n = 0 \text{ or } S_n = b\}.$$

Consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$, $n \geq 1$.

(a) Prove that $M_n = \left(\frac{q}{p}\right)^{S_n}$ and $N_n = S_n - n(p - q)$ are $\mathcal{F}$-martingales.

(b) Assuming $\mathbb{E}[T] < \infty$, find the values of $P(S_T = 0)$ and $\mathbb{E}[T]$.

5. *(1 point)* Let M_n be an $\mathcal{F}_n$-martingale and T a bounded stopping time. Show that M_T is integrable.

6. (Optional Stopping Theorem, variant I) *(1.5 points)* Let M_n be a martingale and T be a stopping time. Assume that M_n is bounded and $T < \infty$ a.s. Show that M_T is integrable and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

7. (Optional Stopping Theorem, variant II) *(1.5 points)* Let M_n be a martingale and T be a stopping time. Assume that $\mathbb{E}[T] < \infty$ and

$$|M_n(\omega) - M_{n-1}(\omega)| \leq K, \qquad \forall (n, \omega),$$

with $K > 0$ constant. Show that M_T is integrable and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Please solve each problem on a separate sheet of paper. Staple the sheets and attach a cover page with your name and course number. The Homework will be turned in 526 mailbox in Sherred Hall, cabinet A. You can retrieve your graded Homework from your folder situated in the same room, a week later.