Homework #6, ORF 526

Assigned on November 8; Due November 15, 2016, 5:00 pm

1. *(2 points)* (a) Consider the normal random variable $X \sim N(\mu, \sigma^2)$, with $\mu \neq 0$. Prove that there is a unique $\theta \neq 0$ such that $\mathbb{E}[e^{\theta X}] = 1$.

   (b) Let $(X_i)_{i \geq 0}$ be a sequence of independent random variables identically distributed as $N(\mu, \sigma^2)$, with $\mu \neq 0$. Consider the sum $S_n = \sum_{j=0}^{n} X_j$. Show that $Z_n = e^{\theta S_n}$ is a martingale, with θ defined in part (a).

2. *(2 points)* Two processes X_t and Y_t are called conditionally uncorrelated, given $\mathcal{F}_t$, if

   $$\mathbb{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0, \ a.s. \qquad \forall 0 \leq s < t < \infty.$$

   Let X_t and Y_t be martingales with respect to filtration $\mathcal{F}_t$. Show that the process $Z_t = X_t Y_t$ is an $\mathcal{F}_t$-martingale if and only if X_t and Y_t are conditionally uncorrelated. Assume that X_t, Y_t and Z_t are integrable.

3. *(2 points)* Let W_t and $\widetilde{W}_t$ be two independent Wiener processes and ρ be a constant with $|\rho| \leq 1$.

   (a) Show that the process $X_t = \rho W_t + \sqrt{1-\rho^2}\, \widetilde{W}_t$ is continuous and has the distribution $N(0, t)$.

   (b) Is X_t a Wiener process?

4. *(2 points)* (a) Find $\mathbb{E}[W_t^2|\mathcal{F}_s]$ for $0 < s < t$.

   (b) Show $\mathbb{E}[W_t^3|\mathcal{F}_s] = 3(t-s)W_s + W_s^3$, for $0 < s < t$.

   (c) Show that $\mathbb{E}\Big[\displaystyle\int_s^t W_u\, du|\mathcal{F}_s\Big] = (t-s)W_s$.

   (d) Show that the process

   $$X_t = W_t^3 - 3\int_0^t W_s\, ds$$

   is a martingale with respect to $\mathcal{F}_t = \sigma\{W_s; s \leq t\}$.

5. *(2 points)* The process $X_t = W_t - tW_1$ is called the Brownian bridge pinned at both 0 and 1 (because $X_0 = X_1 = 0$).

   (a) Write X_t as a convex combination of W_t and $W_t - W_1$.

   (b) Show that $X_t \sim N\big(0, t(1-t)\big)$.

Please solve each problem on a separate sheet of paper. Staple the sheets and attach a cover page with your name and course number. The Homework will be turned in 526 mailbox in Sherred Hall, cabinet A. You can retrieve your graded Homework from your folder situated in the same room, a week later.