ORFE 526: Probability Theory Homework 8

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Exercise 1: Let $(M_n)_n$ be an $\mathcal{F}_n$-adapted process such that $\mathbb{E}[M_{n+1}-M_n|\mathcal{F}_n] = 0$, $\forall n \geq 0$. Prove that for any $p \geq 1$ we have

$$\mathbb{E}[M_{n+p} - M_n|\mathcal{F}_n] = 0, \quad \forall n \geq 0$$

Answer: We rewrite the expression as

$$\begin{gathered} \mathbb{E}[M_{n+p} - M_n|\mathcal{F}_n] \\ = \mathbb{E}[M_{n+p} - M_{n+p-1} + M_{n+p-1} - \ldots - M_{n+1} + M_{n+1} - M_n|\mathcal{F}_n] \end{gathered}$$

Then using linearity of conditional expectation

$$\begin{gathered} = \mathbb{E}[M_{n+p} - M_{n+p-1}|\mathcal{F}_n] + \mathbb{E}[M_{n+p-1} - M_{n+p-2}|\mathcal{F}_n] + \ldots \\ +\mathbb{E}[M_{n+2} - M_{n+1}|\mathcal{F}_n] + \mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] \\ = 0 + 0 + \ldots + 0 = 0 \end{gathered}$$

By assumption, all of these terms are 0. Thus, $\mathbb{E}[M_{n+p} - M_n|\mathcal{F}_n] = 0$.

Exercise 2: Let T be a stopping time and define the process

$$F_n = \begin{cases} 1, & \text{if } n \leq T(\omega) \\ 0, & \text{if } n > T(\omega) \end{cases}$$

Show that F_n is a predictable process.

Answer: Since T is a stopping time, we know that $\{T \leq t\} \in \mathcal{F}_t$. Also, since it is discrete time, we have that

$$\{T \leq t\}^c = \{T > t\} = \{T \geq t+1\} \in \mathcal{F}_t$$

Since $\{F_{n+1} = 1\} = \{T \geq t+1\}$ then F_{n+1} is adapted to $\mathcal{F}_n$ and so predictable.

Exercise 3: Let S and T be stopping times with respect to filtration $\mathcal{F}_n$, with $S \leq T$. Define the process

$$X_n(\omega) = 1_{(S,T]}(n, \omega) = \begin{cases} 1, & \text{if } S(\omega) < n \leq T(\omega) \\ 0, & \text{otherwise} \end{cases}$$

a) Show that X_n is an $\mathcal{F}_n$-predictable process

b) If M_n is a martingale, show that $\mathbb{E}[M_{T \wedge n}] = \mathbb{E}[M_{S \wedge n}]$

Answer:

a) Similar to the previous question, due to discrete case

$$\begin{aligned} \{X_n = 1\} &= \{n < S_n\} \bigcap \{n \leq T_n\} \\ &= \{n - 1 \leq S_n\} \bigcap \{n > T_n\}^c \\ &= \{n - 1 \leq S_n\} \bigcap \{n - 1 \geq T_n\}^c \end{aligned}$$

And so X_n is adapted to $\mathcal{F}_{n-1}$ and so predictable.

b) Since M_n is a martingale, then $M_{T \wedge n}$ and $M_{S \wedge n}$ are stopped processes and we have shown that for stopped processes

$$\mathbb{E}[M_{T \wedge n}] = \mathbb{E}[M_0]$$
$$\mathbb{E}[M_{S \wedge n}] = \mathbb{E}[M_0]$$

And so we have $\mathbb{E}[M_{T \wedge n}] = \mathbb{E}[M_{S \wedge n}]$.

Exercise 4: Assume that $X_1, X_2, X_3, \dots$ are i.i.d. random variables with the same distribution as X

$$P(X = 1) = p, \quad P(X = -1) = q$$

where $0 < p = 1 - q < p$, and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n = a + X_1 + \ldots + X_n, \quad T = \inf\{n : S_n = 0 \text{ or } S_n = b\}$$

Consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n \geq 1$.

a) Prove that $M_n = \left(\frac{q}{p}\right)^{S_n}$ and $N_n = S_n - n(p - q)$ are $\mathcal{F}$-martingales.

b) Assuming $\mathbb{E}[T] < \infty$, find the values of $P(S_T = 0)$ and $\mathbb{E}[T]$.

Answer:

a) First we show that M_n is a martingale. We show integrability,

$$\begin{aligned}\mathbb{E}[|M_n|] &= \mathbb{E}[|(q/p)^{S_n}|] \\ &= \mathbb{E}[|(q/p)^{a+X_1+\ldots+X_n}|]\end{aligned}$$

Since the X_i's are independent,

$$= \mathbb{E}[|(q/p)^a|] \cdot \mathbb{E}[|(q/p)^{X_1}|] \cdot \ldots \cdot \mathbb{E}[|(q/p)^{X_n}|]$$

Noting that

$$\begin{aligned}= \mathbb{E}[|(q/p)^X|] &= p(q/p)^1 + q(q/p)^{-1} \\ &= q + p = 1\end{aligned}$$

We have

$$\mathbb{E}[|M_n|] = (q/p)^a \cdot 1 \cdot \ldots \cdot 1 = (q/p)^a < \infty$$

Thus, M_n is integrable.

M_n is also $\mathcal{F}_n$ measurable since it is a function of $X_, \dots, X_n$ which are $\mathcal{F}_n$ measurable since $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

Finally, by independence

$$
\begin{aligned}
\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[(q/p)^a|\mathcal{F}_s] \cdot \mathbb{E}[(q/p)^{X_1}|\mathcal{F}_s] \cdot \ldots \cdot \mathbb{E}[(q/p)^{X_t}|\mathcal{F}_s] \\
= (q/p)^a \cdot (q/p)^{X_1} \cdot \ldots \cdot (q/p)^{X_s} \cdot \mathbb{E}[(q/p)^{X_s+1}|\mathcal{F}_s] \cdot \ldots \cdot \mathbb{E}[(q/p)^{X_t}|\mathcal{F}_s] \\
= (q/p)^{a+X_1+\ldots+X_s} \cdot \mathbb{E}[(q/p)^{X_s+1}] \cdot \ldots \cdot \mathbb{E}[(q/p)^{X_t}] \\
= M_s \cdot 1 \cdot \ldots \cdot 1 = M_s
\end{aligned}
$$

Where the conditioning dropped since they are independent and s ince $\mathbb{E}[(q/p)^{X_s+1}] = 1$ as before. So $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$. And we conclude that M_n is a martingale.

Now for N_n.

$$
\begin{aligned}
\mathbb{E}[|N_n|] = \mathbb{E}[|S_n - n(p-q)|] \\
= \mathbb{E}[|a + X_1 + \ldots + X_n - n(p-q)|] \\
\leq \mathbb{E}[|a - n(p-q)|] + \mathbb{E}[|X_1|] + \ldots + \mathbb{E}[|X_n|]
\end{aligned}
$$

Note that

$$
\mathbb{E}[|X_n|] = p + q = 1
$$

So

$$
\begin{aligned}
= |a - n(p-q)| + 1 + \ldots + 1 \\
= |a - n(p-q)| + n < \infty
\end{aligned}
$$

So N_n is integrable.

N_n is obviously $\mathcal{F}_n$ measurable since it is a function of $X_1, \ldots, X_n$ which are $\mathcal{F}_n$ measurable by construction.

Finally, by linearity

$$
\begin{aligned}
\mathbb{E}[N_t|\mathcal{F}_s] = \mathbb{E}[a + X_1 + \ldots + X_t - t(p-q)|\mathcal{F}_s] \\
= \mathbb{E}[a + X_1 + \ldots + X_s - s(p-q)|\mathcal{F}_s] \\
+\mathbb{E}[X_{s+1} + \ldots + X_t - (t-s)(p-q)|\mathcal{F}_s] \\
= a + X_1 + \ldots + X_s - s(p-q) + \mathbb{E}[X_{s+1} + \ldots + X_t - (t-s)(p-q)] \\
= N_s + \mathbb{E}[X_{s+1}] + \ldots + \mathbb{E}[X_t] - (t-s)(p-q) \\
= N_s + (p-q) + \ldots + (p-q) - (t-s)(p-q) = N_s
\end{aligned}
$$

Since $\mathbb{E}[X_t] = p - q$. Thus, $\mathbb{E}[N_t|\mathcal{F}_s] = N_s$ and so N_n is a martingale.

b) We satisfy the conditions for the Optional Stopping Theorem variant II in question 7. Using $P(S_T = 0) = P(M_T = 1)$,

$$(q/p)^a = \mathbb{E}[M_0] = \mathbb{E}[M_T] = 1 \cdot P(M_T = 1) + (q/p)^b \cdot P(M_T = (q/p)^b)$$
$$(q/p)^a = P(S_T = 0) + (q/p)^b(1 - P(S_T = 0))$$
$$P(S_T = 0) = \frac{(q/p)^a - (q/p)^b}{1 - (q/p)^b}$$

Thus, $P(S_T = 0) = \frac{(q/p)^a - (q/p)^b}{1-(q/p)^b}$.

Now,

$$\mathbb{E}[N_n] = \mathbb{E}[S_n] - \mathbb{E}[n(p - q)]$$
$$\mathbb{E}[N_n] = \mathbb{E}[S_n] - \mathbb{E}[n](p - q)$$

Thus using Optional Sampling Theorem,

$$\mathbb{E}[T] = \frac{\mathbb{E}[S_T] - \mathbb{E}[N_T]}{p - q}$$
$$\mathbb{E}[T] = \frac{0 \cdot P(S_T = 0) + b \cdot P(S_T = b) - \mathbb{E}[N_0]}{p - q}$$
$$\mathbb{E}[T] = \frac{b \cdot (1 - \frac{(q/p)^a - (q/p)^b}{1-(q/p)^b}) - a}{p - q}$$

Exercise 5: Let M_n be an $\mathcal{F}_n$-martingale and T a bounded stopping time. Show that M_T is integrable.

Answer: We have that M_n is an $\mathcal{F}_n$-martingale. Thus,

$$\mathbb{E}[|M_n|] < \infty, \quad \forall t \in \mathbb{T}$$

Note that for $k > T$ we have

$$\begin{aligned} \mathbb{E}[|M_T|] &< \mathbb{E}[\sum_{n=1}^{k} |M_n|] \\ &= \sum_{n=1}^{k} \mathbb{E}[|M_n|] < \infty \end{aligned}$$

Since each M_n is integrable since it is a martingale.

Exercise 6: (Optional Stopping Theorem, variant I) Let M_n be a martingale and T be a stopping time. Assume that M_n is bounded and $T < \infty$ a.s. Show that M_T is integrable and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Answer: Using the same decomposition as in class

$$M_T = M_{T\wedge n} + (M_T - M_n) \cdot 1_{\{T>n\}} \quad \forall n \geq 0$$

Taking expectations, since $M_{T\wedge n}$ is a stopped process, then $\mathbb{E}[M_{T\wedge n}] = \mathbb{E}[M_0]$. Since $T < \infty$ a.s. then there exists N such that $T(\omega) < N$ a.s. Hence, taking $n > N$, then $1_{\{T>n\}} = 0$ a.s. which yields the result

$$\mathbb{E}[M_T] = \mathbb{E}[M_{T\wedge n}] = \mathbb{E}[M_0]$$

Integrability follows from the fact that picking n sufficiently large again then $M_T = M_{T\wedge n}$ a.s. where we note that $M_{T\wedge n}$ is also a martingale. Hence,

$$\mathbb{E}[|M_T|] = \mathbb{E}[|M_{T\wedge n}|] < \infty$$

Exercise 7: (Optional Stopping Theorem, variant II) Let M_n be a martingale and T be a stopping time. Assume that $\mathbb{E}[T] < \infty$ and

$$|M_n(\omega) - M_{n-1}(\omega)| \leq K, \quad \forall (n, \omega)$$

with $K > 0$ constant. Show that M_T is integrable and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Answer: To show integrability, we note the following

$$\begin{aligned} \mathbb{E}[|M_T|] &= \mathbb{E}[|M_T - M_{T-1} + M_{T-1}|] \\ &\leq \mathbb{E}[|M_T - M_{T-1}|] + \mathbb{E}[|M_{T-1}|] \end{aligned}$$

Where the inequality is due to the triangle inequality. Applying the same procedure recursively, this yields

$$\leq \mathbb{E}[|M_T - M_{T-1}|] + \mathbb{E}[|M_{T-1} - M_{T-2}|] + \ldots + \mathbb{E}[|M_1 - M_0|] + \mathbb{E}[|M_0|]$$

Now we use the assumption that $|M_n(\omega) - M_{n-1}(\omega)| \leq K$.

$$\leq T \cdot K + \mathbb{E}[|M_0|]$$

Since M_n is a martingale, then $\mathbb{E}[|M_0|] < \infty$ and so M_T is integrable. Also, since $\mathbb{E}[T] < \infty$ this implies that $P(T > n) \to 0$ as $n \to \infty$. Equivalently, $1_{\{T>n\}} \to 0$ as $n \to \infty$. Thus, picking n sufficiently large and using the decomposition used in the previous question

$$M_T = M_{T \wedge n} + (M_T - M_n) \cdot 1_{\{T>n\}} \quad \forall n \geq 0$$

We get that $M_T = M_{T \wedge n}$ and so $\mathbb{E}[M_T] = \mathbb{E}[M_{T \wedge n}] = \mathbb{E}[M_0]$.