ORFE 523: Conic and Convex Optimization
Homework 3

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Exercise 1:

1) Solve the following problem

$$\min_{a,b,\eta} \|a\| + \gamma\|\eta\|_1$$
$$\text{s.t. } y_i(a^T x_i - b) \geq 1 - \eta_i \ \forall i = 1, \ldots, m$$
$$\eta_i \geq 0 \forall i = 1, \ldots, m$$

For $\gamma = 0.1, 1, 10$, What is the optimal a^* and b^* for each γ?

2) Which γ gives the best success rate in terms of prediction? Take a look at the entries of a^* in this case, what does this suggest about the people who vote for Hillary compared to those who vote for Bernie?

Answer:

1) The code used to generate the optimal values is below. Table 1 shoes the optimal values for each γ.

γ	a^*	b^*
0.1	(0.141, 0.183, -0.723, -0.110, 0.381)	-3.147
1	(0.209, -0.979, -1.620, -0.460, 3.769)	-9.241
10	(0.151, -0.913, -1.524, -0.464, 4.821)	-8.805

2) Table 2 shows the success rate for each γ. When $\gamma = 0.1$, this gives the best success for prediction. Using part 1), this suggest that people who have higher incomes, Hispanics, and those who live in high density areas (cities) vote for Hillary. Conversely, white demographics and those who pocess a Bachelor's degree tended to vote for Bernie.

γ	Rate
0.1	0.952
1	0.9048
10	0.9048

Code Appendix

```
clear;
clc;
X = load('Hillary_vs_Bernie.mat');

cvx_begin
    variable a(5);
    variable b(1);
    variable eta(175);
    % Vary this
    gamma = 0.1;

    minimize(norm(a) + gamma*norm(eta, 1));

    subject to

    for i = 1:length(X.features_train)
      X.labels_train(i)*(a'*X.features_train(i,:)' - b) >= (1
          - eta(i));
        eta(i) >= 0;
    end
cvx_end

correct = 0;

for i = 1:length(X.features_test)
    if(sign(a'*X.features_test(i,:)' - b) == X.labels_test(i))
        ;
        correct = correct + 1;
    end
end

correct/length(X.features_test)
```

Exercise 2: The goal is to optimize radiation treatment. We have that b_j is the power of beam j for $j = 1, \dots, n$. We must have $0 \le b_j \le B^{max}$. B^{max} is the maximum possible beam level. The exposure area is divided into m vocels labeled $i = 1, \dots, m$. Dose d_i delivered to voxel i must be linear, i.e., $d_i = \sum_{j=1}^{n} A_{ij} b_j$. Where $A \in \mathbb{R}_+^{m \times n}$ is a known matirx. Furthermore, D^{target} must be administered to each voxel, i.e., $d_i \ge D^{target}$ for $i \in \mathcal{T}$ where $\mathcal{T}$ is the target regions. Lastly, $d_i \le D^{other}$ for non-target voxels. Generally, this isn't possible to we settle to minimize

$$E = \sum_{i \notin \mathcal{T}} (d_i - D^{other})_+$$

1) Show that the treatment planning problem is a linear program. Th optimization variable is $b \in \mathbb{R}^n$; the problem data are $B^{max}, A, \mathcal{T}, D^{target}$, and D^{other}.

2) Solve the problem for the data in `treatment_planning_data.m`. Make a brief comment on what you see. *Remark:* The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.

Answer:

1) The optimization problem is

$$\begin{aligned} \min \sum_{i \notin \mathcal{T}} & (d_i - D^{other})_+ \\ \text{s.t. } d_i &\ge D^{target} \ \forall i \in \mathcal{T} \\ 0 \le b_i &\le B^{max} \\ d_i &= \sum_{j=1}^{n} A_{ij} b_j \end{aligned}$$

We have that d_i is an linear function of b. Furthermore, $\min(x)_+$ can be turned into the following,

$$\begin{aligned} &\min \alpha \\ &\text{s.t. } \alpha \ge 0 \\ &\alpha \ge x \end{aligned}$$

So, the objective $(\cdot)_+$ can be linearized. Thus, the above problem has a linear objective and linear constraints. The problem is the linear program below

$$\begin{aligned} \min \sum_{i \notin \mathcal{T}} \alpha_i \\ \text{s.t.} \quad \sum_{j=1}^{n} A_{ij} b_j \geq D^{target} \ \forall i \in \mathcal{T} \\ 0 \leq b_i \leq B^{max} \\ 0 \leq \alpha_i \\ \sum_{j=1}^{n} A_{ij} b_j - D^{other} \leq \alpha_i \end{aligned}$$

But $\alpha_i = (\sum_{j=1}^{n} A_{ij} b_j - D^{other})_+$ and so the only decision variable is really b_i.

2) Figure 1 shows the histogram for voxels being targeted. Figure 2 shows the histogram for those not being targeted. The code used to generate the histograms are below. Notice that most of the voxels being targeted are above the required dose by just a little (1 to 1.2). Also, the vast majority of voxels not being targeted are under $D^{other} = 0.25$. This is exactly what we would want with a real patient.

Figure 1: Histogram for D^{target}

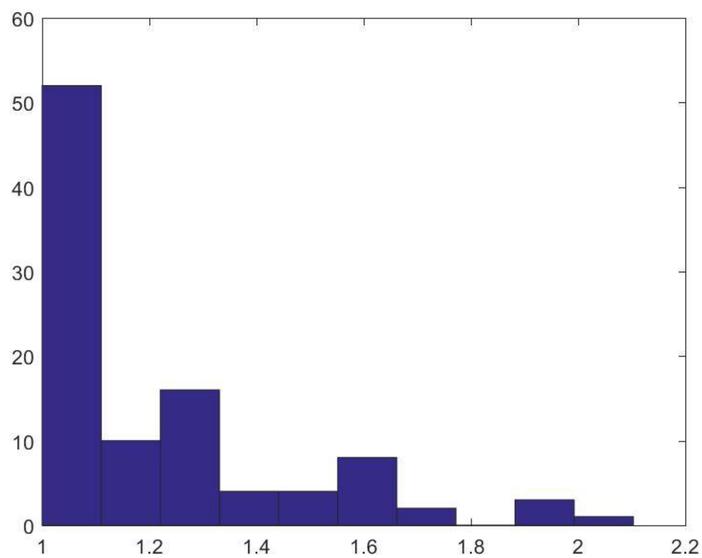
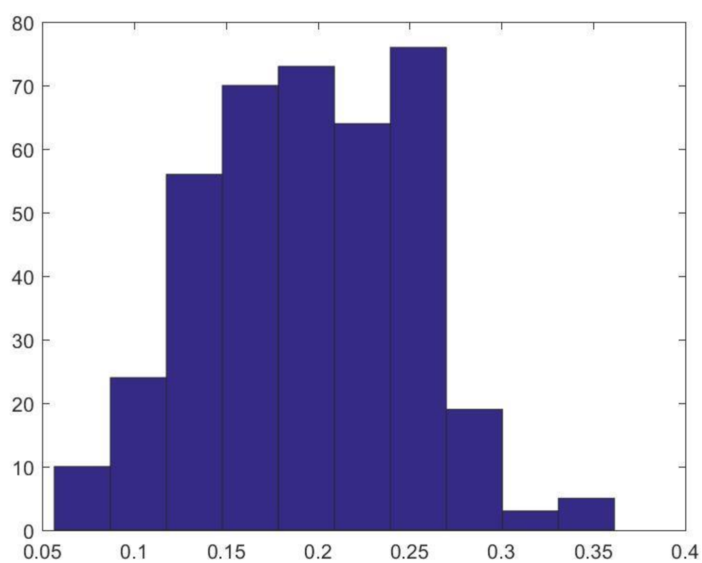


Figure 2: Histogram for D^{other}



Code Appendix

```
clear;
clc;
run('treatment_planning_data.m');

cvx_begin
    variable b(n);
    d_tumor = Atumor*b;
    d_other = Aother*b;
    minimize(sum(max(0,d_other-Dother)));

    subject to
    d_tumor >= Dtarget;
    0 <= b;
    b <= Bmax
cvx_end

figure1 = figure
hist(d_tumor)
saveas(figure1,'dose_tumor.jpg')

figure2 = figure
hist(d_other)
saveas(figure2,'dose_other.jpg')
```

Exercise 3: Suppose $\mathcal{G} = \{Q_1, \ldots, Q_k\} \subseteq \mathbb{R}^{n \times n}$ is a group, i.e., closed under products and inverse. We say that the funciton $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{G}$-invariant, or *symmetrix with respect to* $\mathcal{G}$, if $f(Q_i x) = f(x)$ holds for all x and $i = 1, \ldots, k$. We define $\bar{x} = (1/k) \sum_{i=1}^{k} Q_i x$, which is the average of x over its $\mathcal{G}$-orbit. We define the *fixed subspace* of $\mathcal{G}$ as

$$\mathcal{F} = \{x : Q_i x = x, i = 1, \ldots, k\}$$

a) Show that for any $x \in \mathbb{R}^n$ we have $\bar{x} \in \mathcal{F}$.

b) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\mathcal{G}$-invariant, then $f(\bar{x}) \leq f(x)$.

c) We say the optimization problem

$$\begin{aligned} &\min f_0(x) \\ \text{s.t. } &f_i(x) \leq 0, i = 1, \ldots, m \end{aligned}$$

is $\mathcal{G}$-invariant if the objective f_0 is $\mathcal{G}$-invariant and the feasible set is $\mathcal{G}$-invariant, which means

$$f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \to f_1(Q_i x) \leq 0, \ldots, f_m(Q_i x) \leq 0$$

for $i = 1, \ldots, k$. Show that if the problem is convex and $\mathcal{G}$pinvariant, and there exists an optimal point, then there exists an optimal point in $\mathcal{F}$. In other words, we can adjoin the equality constraints $x \in \mathcal{F}$ to the problem, without loss of generality.

d) As an example, suppose that f is convex and symmetric, i.e., $f(Px) = f(x)$ for every permutation P. Show that if f has a minimizer, then it has a minimizer of the form $\alpha 1$. (This means to minimize f over $x \in \mathbb{R}^n$, we can just as well minimize $f(t1)$ over $t \in \mathbb{R}$).

Answer:

a) We have that

$$\bar{x} = (1/k) \sum_{i=1}^{k} Q_i x$$

Thus,

$$Q_j(\bar{x}) = (1/k) \sum_{i=1}^{k} Q_j Q_i x$$
$$Q_j(\bar{x}) = (1/k) \sum_{i=1}^{k} Q_i x = x$$

Where the second line results since $\mathcal{G}$ is a group and so closure under products implies that multiplying by Q_j simply permutes the original values Q_i.

b) Suppose that f is convex and $\mathcal{G}$-invariant. Thus, by convexity

$$f(\bar{x}) = f((1/k) \sum_{i=1}^{k} Q_i x) \leq (1/k) \sum_{i=1}^{k} f(Q_i x)$$

By $\mathcal{G}$-invariance

$$= (1/k) \sum_{i=1}^{k} f(x) = f(x)$$

So we conclude that $f(\bar{x}) \leq f(x)$.

c) Suppose that x^* satisfies the optimization problem. By part b), we have that $f_0(\bar{x^*}) \leq f_0(x^*)$ since f_0 is $\mathcal{G}$-invariant. From part a), we know $\bar{x} \in \mathcal{F}$. Thus, we only have to show that $\bar{x}$ is feasible. Since the problem is convex, then $f_i(\cdot)$ is convex and so

$$f_i(\bar{x}) \leq (1/k) \sum_{i=1}^{k} f(Q_i x)$$
$$\leq (1/k) \sum_{i=1}^{k} 0 = 0$$

Where the second line if the result of $\mathcal{G}$-invariance of the feasible set. Thus $\bar{x}$ is feasible and we conclude that we can add the constraint $x \in \mathcal{F}$ with no loss in generality.

d) If $f(Px) = x$ for every permutation P. Then we have that

$$\mathcal{F} = \{t\mathbb{1} : t \in \mathbb{R}\}$$

Which is due to the fact that if we permute any elements of x and get back x, then x must have the same element in all of its entries. Thus, by part c), f if convex and P-invariant, if we minimize f over $x \in \mathbb{R}^n$, then we can just minimize over $x \in \mathcal{F}$. That is, we can minimize over $(t\mathbb{1})$ such that $t \in \mathbb{R}$.

Exercise 4: Define M_C as the following function of a convex set C in $\mathbb{R}^n$:

$$M_C(x) = \inf\{t > 0 : \frac{x}{t} \in C\}$$

over the domain

$$dom(M_C) = \{x \in \mathbb{R}^n : \frac{x}{t} \in C \text{ for some } t > 0\}$$

1) Show that M_C is a convex function.

2) Suppose C is also compact, origin symmetric ($x \in C$ if and only if $-x \in C$), and has nonempty interior. Show that M_C is a norm over $\mathbb{R}^n$. What is its unit ball?

3) Show that an even degree homogeneous polynomial is convex if and only if it is quasiconvex. (Hint: use what you proved in the previous parts of this question).

Answer:

1) We have for $\lambda \in (0,1)$ and $x, y \in \mathbb{R}^n$

$$t^* = M_C(\lambda x + (1-\lambda)y) = \inf\{t > 0 : \frac{\lambda x + (1-\lambda)y}{t} \in C\}$$

If we show that $t^* \leq \lambda M_C(x) + (1-\lambda)M_C(y)$, then we are done. So we check that $t = \lambda M_C(x) + (1-\lambda)M_C(y)$ is feasible. Since $\lambda M_C(x) + (1-\lambda)M_C(y) > 0$, we only need to check that $\frac{\lambda x + (1-\lambda)y}{\lambda M_C(x) + (1-\lambda)M_C(y)} \in C$. We have

$$\begin{aligned}
&\frac{\lambda x + (1-\lambda)y}{\lambda M_C(x) + (1-\lambda)M_C(y)} \\
&= \alpha \frac{x}{M_C(x)} + (1-\alpha)\frac{y}{M_C(y)} \\
&\text{where } \alpha = \frac{1}{1 + \frac{M_C(y)}{M_C(x)}\frac{1-\lambda}{\lambda}}
\end{aligned}$$

Clearly, $\alpha \in (0,1)$. WLOG, assume C is closed so that $\frac{x}{M_C(x)}, \frac{y}{M_C(y)} \in C$, otherwise, just work with the closure of C. Since C is convex, we then have that $\alpha \frac{x}{M_C(x)} + (1-\alpha)\frac{y}{M_C(y)} \in C$. Thus,

$$\begin{aligned}
&t^* \leq \lambda M_C(x) + (1-\lambda)M_C(y) \\
\implies &M_C(\lambda x + (1-\lambda)y) \leq \lambda M_C(x) + (1-\lambda)M_C(y)
\end{aligned}$$

So $M_C(\cdot)$ is convex.

2) We show the three properties of a norm

- (positivity) Cleary, $M_C(x) \geq 0$ since $t > 0$. Suppose we have that $M_C(x) = 0$. Since C is compact, then it is bounded. By contradiction, suppose $x \neq 0$, then as $t \to 0$, then $\|\frac{x}{t}\| \to \infty$. But then this means that $M_C(x) \neq 0$ as $\|\frac{x}{t}\|$ leaves C since it is bounded. Therefore, we must have that $M_C(x) = 0$ implies that $x = 0$. Note that $x \in C$ since it is non-empty and since x and $-x$ are in C, this means that 0 must be in C.
- (homogeneity) Suppose $M_C(x) = a$. Then $M_C(\alpha x) = \inf\{t > 0 : \frac{\alpha x}{t} \in C\}$. By origin symmetry of C, if $\alpha < 0$, then $M_C(\alpha x) = \inf\{t > 0 : \frac{-\alpha x}{t} \in C\}$. So we can consider $M_C(\alpha x) = \inf\{t > 0 : \frac{|\alpha| x}{t} \in C\}$. We relabel $t' = \frac{t}{|\alpha|}$ to get $M_C(\alpha x) = \inf\{t' > 0 : \frac{x}{t'} \in C\}$ and so we conclude that $\frac{M_C(\alpha x)}{|\alpha|} = a$ which means that $M_C(\alpha x) = |\alpha| M_C(x)$.
- (triangle inequality) We have by homogeneity of M_C and convexity of M_C that

  $$M_C(x+y) = M_C(2(\frac{1}{2}x + \frac{1}{2}y)) = 2M_C(\frac{1}{2}x + \frac{1}{2}y) \leq M_C(x) + M_C(y)$$

  Thus, M_C is a norm if it is compact and origin symmetric with non-empty interior. Now we show what the unit ball is. Recall that $0 \in C$ since it is non-empty. By definition, $\frac{x}{M_C(x)} \in C$ since C is compact (that is the infimum is achieved). Suppose that $M_C(x) \leq 1$. By convexity of C we have

  $$x = M_C(x)\frac{x}{M_C(x)} + (1 - M_C(x))0$$

  But this is a convex combination of elements in C and so $x \in C$. That is, if $M_C(x) \leq 1$ then $x \in C$. Also, if $x \in C$, then $M_C(x) \leq 1$ since $\frac{x}{1} = x \in C$. Thus, the unit ball is precisely C.

3) Consider the polynomial $p(x)$. Convex implies quasiconvex as shown in class. To show the converse consider the sublevel set S_1. Since it is

a homogeneous polynomial, we also have that $p(tx) = t^d p(x)$. Thus,

$$\begin{aligned} p(x) &= \inf\{t > 0 : \frac{p(x)}{t} \leq 1\} \\ &= \inf\{t > 0 : p(\frac{x}{t^{1/d}}) \leq 1\} \text{ by homogeneity} \\ &= \inf\{t > 0 : \frac{x}{t^{1/d}} \in S_1\} \\ &= M_{S_1}(x)^d \end{aligned}$$

Since M_{S_1} is convex as it is a norm as shown in step 2) and x^d is convex since d is even. Also, M_{S_1} is non-negative and x^d is non-decreasing on $[0, \infty)$ we conclude that $p(x)$ is convex since it is a composition of two convex functions with the preceding properties.

Exercise 5: A matrix A is called *doubly stochastic* if A is nonnegative and each of its rows and columns sum up to 1. A *permutation* matrix is an integral doubly stochastic matrix. Show that every doubly stochastic matrix is a convex combination of permuation matrices. (Hint: relate this to bipartite matching).

Answer: As per the hint, we define the following bipartite graph. Make two columns of nodes from 1 to n, where n is the dimension of the square matrix A. Then create an edge of weight 1 from the node i in column 1 to nodes j column 2 if A_{ij} is non zero. Let X be the incidence matrix of this graph.

By recreating the maximum matching linear program done in class and using the fact that the incidence matrix of a bipartite graph is TUM, then there exists a matrix P that achieves maximal matching where $P_{ij} = 1$ if both nodes i and j are connected in the solution. Since A is doubly stochastic, we have that the maximal matching has precisely a value of n. Since P is a solution to a matching problem, then each column and row has at most one 1. We conclude that P is a permutation matrix since there are n rows and columns and the values must sum up to n.

Now, we create a new matrix $A_1 = A - \gamma P$ where γ is chosen such that $A - \gamma P \geq 0$. Then, by construction, A_1 has at least 1 more zero entry than A. However, $\frac{A_1}{1-\gamma}$ is still doubly stochastic ($A - \gamma P$ has row and column sums of $(1-\gamma)$) and so we can repeat the process above. We finish when we reach an iteration when $\gamma = 1$.

Exercise 6: Let A be an integral matrix. Show that A is totally unimodular if and only if for every integral vector b, the polyhedon $\{x : x \geq 0, Ax \leq b\}$ is integral. (Hint: If A is not totally unimodular, use the inverse of a submatrix which does not have determinant $\{0, -1, +1\}$ to construct an integer vector b that generates a non-integral vertex in the polyhedron).

Answer: We showed in class that if A is TUM, then the polyhedron $P = \{x : x \geq 0, Ax \leq b\}$ is integral. Now, suppose that A is not TUM. Then, this means that there is a submatrix B of A which does not have a determinant in $\{-1, 0, +1\}$. Furthermore, since A is integral, the determinant of B is also integral (since the determinant is an integral function of the entries). However, since $|det(B)| > 1$, then we must have that $det(B^{-1}) = \frac{1}{det(B)}$ is not an integer. This implies that B^{-1} has a non integer entry, because if it had all integer entries, its determinant would be integral.

Suppose that the non integer entry of B is in its j^{th} column. Now, let us define x that satisfies

$$B^{-1}e_j = x$$

Where e_j is the canonical basis vector. Then we have that x is non integral. We correct the case where x has negative entries. Thus, we must add a sufficiently large integral y to x to make it non negative

$$x = B^{-1}e_j + y$$

Now suppose $b = e_j$,

$$\implies Bx = b + By$$

Note that $b + By$ is still integral for suitably chosen y. However, we still must correct for the dimension difference between B and A. Without loss in generality, assume that B is the upper left $k \times k$ block of A. Thus, we pad our x to be

$$x^* = \begin{bmatrix} x \\ 0 \\ \vdots \\ 0 \end{bmatrix}, b^* = \begin{bmatrix} b \\ * \\ \vdots \\ * \end{bmatrix}$$

Where the $*$'s in b are chosen integers that are larger than the $n-k$ bottom rows of Ax. Thus, we have

$$Ax = \begin{bmatrix} b + By \\ * \\ \vdots \\ * \end{bmatrix} \leq b^*$$

This x is indeed a vertex. This is because we have k binding constraints from $Ax \leq b$ since our construction leads the first k rows of Ax to be equal b. We also have $n-k$ binding constraints from the $n-k$ entries of x which are 0. Thus, this x is specified by n constraints and hence this is a vertex. We conclude that b is not integral and thus if A is not TUM then there exists a b such that the polyhedron is not integral.