ORFE 527: Stochastic Calculus
Homework 2

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Exercise 1: (Continuous local martingales) Let M be a continuous local martingale. Show the following statements.

a) If there exists a constant $C_1 > -\infty$ such that $M_t \geq C_1$ for all $t \geq 0$, then M is a supermartingale.

b) If there exists constants $-\infty < C_1 \leq C_2 < \infty$ such that $C_1 \leq M_t \leq C_2$ for all $t \geq 0$, then M is a martingale.

Answer:

a) We wish to apply Fatou's lemma which only works for nonnegative random variables. Thus, we define $M' = M + C_1$ so that $M' \geq 0$. Since M is in $\mathcal{M}^c_{loc}$ then so is M' as it is simply shifted by a constant. Thus, there exists a sequence of stopping times τ_n such that $M_{t\wedge\tau_n}$ are martingales. We now show that M' is integrable. Recall, it is nonnegative so,

$$\mathbb{E}[|M'_t|] = \mathbb{E}[M'_t]$$
$$= \mathbb{E}[\lim_{n\to\infty} \inf M'_{t\wedge\tau_n}]$$

By Fatou's lemma,

$$\leq \lim_{n\to\infty} \inf \mathbb{E}[M'_{t\wedge\tau_n}]$$

Now, since $M_{t\wedge\tau_n}$ is a martingale for all n, then we have by the martingale property

$$= \mathbb{E}[M'_0] < \infty$$

Thus, we have that M' is integrable. Now, to show it has the supermartingale property. Let $A \in \mathcal{F}_s$. Then,

$$\mathbb{E}[M'_t \cdot 1_{\{A\}}] = \mathbb{E}[\lim_{n\to\infty} \inf M'_{t\wedge\tau_n} \cdot 1_{\{A\}}]$$

Again, by Fatou's lemma

$$\leq \lim_{n\to\infty} \inf \mathbb{E}[M'_{t\wedge\tau_n} \cdot 1_{\{A\}}]$$

Now, since $M_{t\wedge\tau_n}$, we apply the martingale property,

$$= \lim_{n\to\infty} \inf M'_{s\wedge\tau_n} = M'_s$$

Thus, we have $\mathbb{E}[M'_t|\mathcal{F}_s] \leq M'_s$ and so we have that M' is a supermartingale. Since $M = M' - C_1$, then we have that M must also be a supermartingale since

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M'_t|\mathcal{F}_s] - C_1 \leq M'_s - C_1 = M_s$$

b) Since $C_1 \leq M_t \leq C_2$ we have that M_t is a supermartingale by part a). Since M_t is a supermartingale, then $-M_t$ is a submartingale. But we also have that $M_t \leq C_2$ is equivalent to $C_2 \leq -M_t$. Thus, we have that $-M_t$ is also a supermartingale. The only way that $-M_t$ is both a supermartingale and a submartingale is if it is a martingale. Since $-M_t$ is a martingale, then so is M_t.

Exercise 2: (Exponential local martingale) For $f \in L^2([0,T])$ and a standard Brownian motion B show from scratch that the process

$$\exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f(s)^2 ds\right),\ t \in [0,T]$$

is a martingale. Conclude that the law of the process $B_t + \int_0^t f(s)ds$, $t \in [0,T]$ is absolutely continuous with respect to the law of the process B_t, $t \in [0,T]$. Is the law of the process B_t, $t \in [0,T]$ absolutely continuous with respect to the law of the process $B_t + \int_0^t f(s)ds$, $t \in [0,T]$ as well?

Answer: First, let us note that $\int_0^t f(s)dB_s$ is a Gaussian process with mean 0 and variance $\int_0^t f(s)^2 ds$. Thus, let us define $M_t = \int_0^t f(s)dB_s$. We have then that $e^{M_t - \frac{1}{2}\int_0^t f(s)^2 ds)}$ is clearly $\mathcal{F}_t$ adapted since it is a continuous function of an $\mathcal{F}_t$ adapted process and a deterministic function. Now, for integrability.

$$\begin{aligned} \mathbb{E}[|e^{M_t - \frac{1}{2}\int_0^t f(s)^2 ds}|] &= \mathbb{E}[e^{M_t - \frac{1}{2}\int_0^t f(s)^2 ds}] \\ &\leq \mathbb{E}[e^{M_t}] = e^{\frac{1}{2}\int_0^t f(s)^2 ds} < \infty \end{aligned}$$

Where the last part comes from the fact that it is the MGF of a Gaussian process. The martingale property follows similarly. Since M_s is $\mathcal{F}_s$ adapted, we can pull any measurable function of M_s out of the conditional expectation,

$$\begin{aligned} &\mathbb{E}[e^{M_t - \frac{1}{2}\int_0^t f(u)^2 du}|\mathcal{F}_s] \\ &= e^{M_s - \frac{1}{2}\int_0^s f(u)^2 du} \cdot \mathbb{E}[e^{M_t - \frac{1}{2}\int_0^t f(u)^2 du} \cdot e^{-M_s + \frac{1}{2}\int_0^s f(u)^2 du}|\mathcal{F}_s] \\ &= e^{M_s - \frac{1}{2}\int_0^s f(u)^2 du} \cdot \mathbb{E}[e^{M_t - M_s - \frac{1}{2}\int_s^t f(u)^2 du}|\mathcal{F}_s] \end{aligned}$$

Thus, if we can show that $\mathbb{E}[e^{M_t - M_s - \frac{1}{2}\int_s^t f(u)^2 du}|\mathcal{F}_s] = 1$, we will have shown the martingale property. This is done by noting that $M_t - M_s$ is again Gaussian with mean 0 and variance $\int_s^t f(u)^2 du$ and that increments are independent. Thus, we have

$$\begin{aligned} &\mathbb{E}[e^{M_t - M_s - \frac{1}{2}\int_s^t f(u)^2 du}|\mathcal{F}_s] \\ &= e^{-\frac{1}{2}\int_s^t f(u)^2 du} \cdot \mathbb{E}[e^{M_t - M_s}] \\ &= e^{-\frac{1}{2}\int_s^t f(u)^2 du} \cdot e^{\frac{1}{2}\int_s^t f(u)^2 du} \\ &= 1 \end{aligned}$$

Where, again, we used the fact that the MGF of a Gaussian is $\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. Thus, we conclude that $\exp(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f(s)^2 ds)$ is a martingale.

We now conclude that $B_t + \int_0^t f(s)ds$, $t \in [0,T]$ is absolutely continuous with respect to the law of the process B_t, $t \in [0,T]$ by applying Girsanov's theorem to $X_t = -f(t)$. Since $Z_t(-f)$ is a martingale (as just shown), we have that the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}_T}{d\mathbb{P}} = Z_T(-f)$ exists between B_t and $B_t + \int_0^t f(s)ds$ which means that $B_t + \int_0^t f(s)ds$, $t \in [0,T]$ is absolutely continuous with respect to the law of the process B_t.

Since we have that $Z_T(-f) > 0$ almost surely, we know that the inverse Radon-Nikodym derivative exists ($\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}_T} = 1/Z_T(-f)$) and so we must also have the converse statement also be true.

Exercise 3: (More martingales) Let $B^{(1)}, B^{(2)}, B^{(3)}$ be independent standard Brownian motions and define

$$X_t = \int_0^t \sin(s) dB_s^{(1)} + \int_0^t \sin(s + 2\pi/3) dB_s^{(2)} + \int_0^t \sin(s + 4\pi/3) dB_s^{(3)}$$

For $t \geq 0$. Find the distribution of $\max_{0 \leq t \leq T} X_t$ for any given $T \geq 0$.

Answer: First, we calculate the quadratic variation of X_t. Note that the variation terms are as follows $\langle B^{(i)}, B^{(i)} \rangle = t$ and $\langle B^{(i)}, B^{(j)} \rangle = 0$ if $i \neq j$. Thus, we have that

$$\langle X \rangle_t = \int_0^t \sin^2(s) + \sin^2(s + 2\pi/3) + \sin^2(s + 4\pi/3) ds$$

Using the trigonometric equality $\sin^2(s) = \frac{1 - \cos(2s)}{2}$, we get

$$= \frac{1}{2} \int_0^t 3 - \cos(2s) - \cos(2s + 4\pi/3) - \cos(2s + 8\pi/3) ds$$
$$= \frac{3}{2} t$$

Thus, by Levy's characterization, X_t is a Brownian motion scaled by 3/2. Now, following the reflection principle, we have

$$\mathbb{P}(\max_{0 \leq t \leq T} X_t \geq x) = 2\mathbb{P}(X_T \geq x) = 2(1 - \mathbb{P}(X_T < x))$$

However, we know the distribution of X_T, it is Gaussian with mean 0 and variance $3/2T$. Thus, we have $\mathbb{P}(X_T < x) = \Phi(\frac{\sqrt{2}x}{\sqrt{3T}})$.

$$= 2 \left[1 - \Phi \left(\frac{\sqrt{2}x}{\sqrt{3T}} \right) \right] = 2 - 2\Phi \left(\frac{\sqrt{2}x}{\sqrt{3T}} \right)$$

Recall that $X_0 = 0$ and so $x \in [0, \infty)$.

Exercise 4: (Changing the variances) Show that for a standard Brownian motion B, real constants $\sigma_1 \neq \sigma_2$, and any $T \in (0, \infty)$ the law of the process $\sigma_2 B_t$, $t \in [0, T]$ is not absolutely continuous with respect to the law of the process $\sigma_1 B_t$, $t \in [0, T]$. In other words, even though one can change the means of the increments of a Brownian motion by a change of measure, the same is not the case for their variances.

Answer: Let $\mathbb{P}_1$ be the law of the process $\sigma_1 B_t$ and let $\mathbb{P}_2$ be the law of the process $\sigma_2 B_t$. Now, define the two events $A = \{\text{ quadratic variation } = \sigma_1^2 t\}$ and $B = \{\text{ quadratic variation } = \sigma_2^2 t\}$. Then we have the following probabilities

$$\begin{aligned} \mathbb{P}_1(A) = 1, &\quad \mathbb{P}_2(A) = 0 \\ \mathbb{P}_1(B) = 0, &\quad \mathbb{P}_2(B) = 1 \end{aligned}$$

Since, by definition, the quadratic variances of $\sigma_1 B_t$ and $\sigma_2 B_t$ are $\sigma_1^2 t$ and $\sigma_2^2 t$ respectively. However, this shows that neither law is absolutely continuous with respect to the other one. Thus, there does not exist a Radon-Nikodym derivative $\frac{d\mathbb{P}_2}{d\mathbb{P}_1}$ as

$$1 = \mathbb{P}_2(B) = \int_B \frac{d\mathbb{P}_2}{d\mathbb{P}_1} d\mathbb{P}_1 = 0$$

Which is a contradiction.