Homework #3, ORF 526

Assigned on October 4; Due October 11, 2016, 5:00 pm

1. *(1.5 points)* Let $X, Y \sim Exp(1)$ be two independent random variables, standard exponential distributed. Find the distribution of $R = \frac{Y}{X}$.

2. *(1.5 points)* Let $X : \Omega \to E$ and $Y : \Omega \to F$ be two random variables, where $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Assume there is a measurable function $f : E \to F$ such that $Y = f(X)$. Prove that $\sigma Y \subset \sigma X$.

3. *(1 point)* Consider the inertia momentum about axis $\{x = a\}$ defined by

$$f(a) = \int_\Omega (X(\omega) - a)^2 d\mathbb{P}(\omega).$$

   (i) Show that f is minimized by $a = \mathbb{E}[X]$.

   (ii) What is the value of its minimum?

4. *(1.5 points)* Let μ be a probability measure. Show that its Fourier transform, $\hat{\mu}$, is a bounded, continuous function satisfying $\hat{\mu}(0) = 1$.

5. *(1.5 points)*Show that if $\displaystyle\int_H X \, d\mathbb{P} \geq 0$ for any $H \in \mathcal{H}$, then $X \geq 0$ a.s.

6. *(1.5 points)* Let $X \geq 0$ and $\mathbb{E}[X] < \infty$, with X random variable. Show that $X < \infty$ a.s.

7. *(1.5 points)* Let $(X_n)_n$ be a sequence of bounded random variables, with $|X_n| < M < \infty$, for all $n \geq 1$.

   (i) Show that $\mathbb{E}[|X_n|] \leq M$, for all $n \geq 1$.

   (ii) Use Markov's inequality to show

$$\mathbb{E}[|X_n| 1_{\{|X_n| > b\}}] < \frac{M^2}{b}, \qquad \forall b > 0.$$

   (iii) Prove that the family $\{X_n\}_n$ is uniformly integrable.

Please solve each problem on a separate sheet of paper. Staple the sheets and attach a cover page with your name and course number. The Homework will be turned in 526 mailbox in Sherred Hall, cabinet A. You can retrieve your graded Homework from your folder situated in the same room, a week later.