ORFE 527: Stochastic Calculus
Homework 6

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Exercise 1: (Elliptic maximum principle) Let $D \subset \mathbb{R}^m$ be a bounded set with piecewise C^1 boundary and suppose that $u \in C^2(\bar{D})$ satisfies

$$\sum_{i=1}^{m} b_i(x)\partial_{x_i}u(x) + \frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{m}\sigma_{ki}(x)\sigma_{kj}(x)\partial_{x_ix_j}u(x) = 0$$

on D for some continuous b, σ. Show that u satisfies the following *maximum principle*:

$$\max_{x\in D} u = \max_{x\in\partial D} u$$

In words: u attains its maximum on the boundary of D.

Answer: As b, σ are continuous and D is bounded, then we have that the solution of the above PDE is unique. Also, we have that u satisfies the Feynman-Kac formula with $h = 0$, $g = 0$. Here, $u = f$ on ∂D is unspecified. Thus, the Feynman-Kac formula gives

$$u(x_0) = \mathbb{E}[f(x_{\tau_D})]$$

But, by the setup of the problem, $f(x_{\tau_D}) = u(x_{\tau_D})$ as $x_{\tau_D} \in \partial D$. So,

$$u(x_0) = \mathbb{E}[u(x_{\tau_D})]$$

Now, suppose that

$$x^* = \arg\max_{x\in\bar{D}} u(x)$$

Which is achieved as f is continuous and $\bar{D}$ is compact. But, by the above Feynman-Kac,

$$u(x^*) = \mathbb{E}[u(x_{\tau_D})]$$

However, the above implies that

$$0 = \mathbb{E}[u(x_{\tau_D}) - u(x^*)]$$

But, by maximality of x^*, we have $u(x_{\tau_D}) - u(x^*) \leq 0$ and so we must have that

$$u(x^*) = u(x_{\tau_D}) \in \partial D$$

We conclude that

$$\max_{x\in D} u(x) = \max_{x\in\partial D} u(x)$$

Exercise 2: (Parabolic PDEs) Use the parabolic Feynman-Kac formula to find *explicitly* the unique classical solutions of the following parabolic PDE problems:

a) $\partial_t u(t,x) + \frac{1}{2}\partial_{xx}u(t,x) = e^{2x}$, $(t,x) \in [0,1] \times \mathbb{R}$ with $u(1,x) = e^x$, $x \in \mathbb{R}$.

b) $\partial_t v(t,x) + x\partial_x v(t,x) + \frac{1}{2}\partial_{xx}v(t,x) = \cos x$, $(t,x) \in [0,1] \times \mathbb{R}$ with $u(1,x) = \sin x$.

Hint: recall problem 2 of homework 1.

Answer:

a) We have that b and σ here are bounded and continuous so we apply the parabolic Feynman-Kac formula. We have $f(x) = e^x$, $g(x) = -e^{2x}$, $h(x) = 0$. Then,

$$u(t,x) = \mathbb{E}_{t,x}[e^{x_1} - \int_t^1 e^{2x_s} ds]$$

Now, we use the fact that $x_1 = x + B_1 - B_t$, which comes from the fact that $x_t = x$, $b = 0$, and $\sigma = 1$. Thus, we get by Fubini's and $\mathbb{E}[e^{\sigma B_t}] = e^{\sigma t}$

$$\begin{aligned}
u(t,x) &= \mathbb{E}_{t,x}[e^{x+B_1-B_t}] - \int_t^1 \mathbb{E}[e^{2x_s}]ds] \\
u(t,x) &= e^x \mathbb{E}_{t,x}[e^{B_{1-t}}] - \int_t^1 \mathbb{E}[e^{2x+2B_{1-s}}]ds] \\
u(t,x) &= e^x e^{\frac{1}{2}(1-t)} - \int_t^1 e^{2x+2(1-s)}ds] \\
u(t,x) &= e^{x+\frac{1}{2}(1-t)} + \frac{1}{2}e^{2x}\left(1 - e^{2(1-t)}\right)
\end{aligned}$$

Where we have the sanity check that $u(1,x) = e^x$. As well, this satisfies the PDE.

b) We do the same thing as the previous part except now with $f(x) = \sin x$, $g(x) = -\cos x$, $h(x) = 0$. We also have that the SDE is satisfied by X_t

$$\mathrm{d}X_s = X_s \mathrm{d}s + \mathrm{d}B_s$$
$$X_t = x$$

From the hint, we have that

$$X_s = xe^{s-t} + \int_t^s e^{s-t} \mathrm{d}B_t$$

solves this SDE. Now, by Feynman-Kac

$$v(t,x) = \mathbb{E}_{t,x}[\sin(x_1) - \int_t^1 \cos(x_s) ds]$$
$$v(t,x) = \text{Real}\left(\mathbb{E}_{t,x}[-ie^{iX_1} - \int_t^1 e^{ix_s} ds]\right)$$
$$v(t,x) = \text{Real}\left(-ie^{ixe^{1-t} - \frac{1}{4}(e^{2(1-t)-1})} - \int_t^1 e^{ie^{s-t} - \frac{1}{4}(e^{2(s-t)-1)})} ds\right)$$

Where the last line comes from the fact that

$$X_s \sim \mathcal{N}(xe^{s-t}, \frac{1}{2}(e^{2(s-t)} - 1))$$

which is what we used for the expectation. Now, we can simplify this to

$$v(t,x) = \sin(xe^{1-t})e^{-\frac{1}{4}(e^{2(1-t)}-1)} - \int_t^1 \cos(xe^{s-t})e^{-\frac{1}{4}(e^{2(s-t)}-1)} ds$$

Finally, a sanity check is that $v(1,x) = \sin(x)$. One can also, with your favourite symbolic solver, verify that this satisfies the PDE.

Exercise 3: (Black-Scholes formula) In the Black-Scholes model, the price of any given stock is modelled (under a suitable probability measure) as the unique strong solution of the SDE

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t\mathrm{d}B_t, \quad X_0 = x_0 > 0$$

where $r \in \mathbb{R}$ is interpreted as the interest rate and σ as the volatility of the stock. Suppose you buy at time 0 a *call option* with strike $K > 0$ and maturity $T > 0$ on the stock, i.e. a contract that pays $\max(X_T - K, 0)$ at time T. In this setting, the market efficiency hypothesis implies that the fair price you should pay for the call option is

$$p := \mathbb{E}[e^{-rT}\max(X_T - K, 0)]$$

a) Use the parabolic Feynman-Kac formula to show $p = u(0, x_0)$, where u is the unique classical solution of the parabolic PDE problem

$$\partial_t u(t,x) + rx\partial_x u(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} u(t,x) - ru(t,x) = 0$$
$$(t,x) \in [0,T] \times (0,\infty), u(T,x) = \max(x-K,0), x \in (0,\infty)$$

b) Define the change of variables $s := T - \frac{\sigma^2}{2}(T-t)$, $y := \log x + (r - \frac{\sigma^2}{2})(T-t)$ and find the parabolic PDE problem satisfied by

$$v(s,y) := e^{-r\left(T - \frac{2(T-s)}{\sigma^2}\right)} u\left(T - \frac{2(T-s)}{\sigma^2}, e^{y - \left(\frac{2r}{\sigma^2} - 1\right)(T-s)}\right)$$

c) Use the parabolic Feynman-Kac formula to find v *explicitly.*

d) Compute the price $u(0, x_0)$ of the call option.

Answer:

a) Again, all the coefficients satisfy the requirements for the parabolic Feynman-Kac. With $f(x) = \max(X_T - K, 0)$, $g(x) = 0$, and $h(x) = r$. By Feynman-Kac

$$u(t,x) = \mathbb{E}_{t,x}[\max(X_T - K, 0)e^{-\int_t^T r ds}]$$

Thus, $u(0, x_0) = \mathbb{E}_{x_0}[\max(X_T - K, 0)e^{-rT}] = p$.

b) We first note that we have

$$v(s,y) = e^{-r\left(T-\frac{2(T-s)}{\sigma^2}\right)} u\left(T - \frac{2(T-s)}{\sigma^2}, e^{y-\left(\frac{2r}{\sigma^2}-1\right)(T-s)}\right) = e^{-rt}u(t,x)$$

Now, we calculate all the derivatives of the left hand side and right hand side. We denote the RHS as $f(t,x)$. Note, for the LHS, you have to do the toal derivative. As in, $\partial_t v(s,y) = \frac{\partial s}{\partial t}\frac{\partial}{\partial s}v(s,y) + \frac{\partial s}{\partial y}\frac{\partial}{\partial y}v(s,y)$.

$$\begin{aligned}
\partial_t v(s,y) &= \frac{\sigma^2}{2}\partial_s v(s,y) + \left(\frac{\sigma^2}{2} - 2\right)\partial_y v(s,y) \\
\partial_t f(t,x) &= -re^{-rt}u(t,x) + e^{-rt}\partial_t u(t,x) \\
\partial_x v(s,y) &= \frac{1}{x}\partial_y v(s,y) \\
\partial_x f(t,x) &= e^{-rt}\partial_x u(t,x) \\
\partial_{xx} v(s,y) &= -\frac{1}{x^2}\partial_y v(s,y) + \frac{1}{x^2}\partial_{yy} v(s,y) \\
\partial_{xx} f(t,x) &= e^{-rt}\partial_{xx} u(t,x)
\end{aligned}$$

Since they are two sides of the same equation, the respective pairs of derivatives are equal. Now, we note that

$$\begin{aligned}
&\partial_t f(t,x) + rx\partial_x f(t,x) + \frac{1}{2}\sigma^2\partial_{xx} f(t,x) \\
&= e^{-rt}\left(\partial_t u(t,x) + rx\partial_x u(t,x) + \frac{1}{2}\sigma^2 x^2\partial_{xx} u(t,x) - ru(t,x)\right) \\
&= 0
\end{aligned}$$

But, by substituting the equivalent derivatives, we get

$$\frac{\sigma^2}{2}\partial_s v(s,y) + \left(\frac{\sigma^2}{2} - 2\right)\partial_y v(s,y) + r\partial_y v(s,y) - \frac{\sigma^2}{2}\partial_y v(s,y) + \frac{\sigma^2}{2}\partial_{yy} v(s,y)$$

$$\implies \partial_s v(s,y) + \partial_{yy} v(s,y) = 0$$

Where the final condition is direct from $v(T,y) = e^{-rT}u(T,e^y) = \max(e^y - K, 0)$

c) The PDE generated in the last part generates the SDE according to Feynmac-Kac

$$\tilde{X}_s = y + \sqrt{2}(\tilde{B}_s - \tilde{B}_t), \; s \geq t$$

Hence we have that $\tilde{X}_T \sim \mathcal{N}(y, 2(T-s))$. Now we apply Feynman-Kac equation

$$
\begin{aligned}
v(s,y) &= \mathbb{E}_{s,y}[e^{-rT}\max(e^{\tilde{X}_T} - K, 0)] \\
&= e^{-rT}\int_{\log K}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2(T-s)}} e^{\tilde{X}_T} e^{-\frac{(\tilde{X}_T - y)^2}{4(T-s)}} \mathrm{d}\tilde{X}_T \\
&\quad - K\int_{\log K}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2(T-s)}} e^{-\frac{(\tilde{X}_T - y)^2}{4(T-s)}} \mathrm{d}\tilde{X}_T \\
&= e^{-rT}\left(e^{-4y(T-s)-4(T-s)^2}\Phi\left(-\frac{\log(K) - (y + 2(T-s))}{\sqrt{2(T-s)}}\right)\right. \\
&\quad \left. - K\Phi\left(-\frac{\log(K) - y}{\sqrt{2(T-s)}}\right)\right) \\
&= e^{-rT}\left(e^{y+(T-s)}\Phi\left(-\frac{\log(K) - (y + 2(T-s))}{\sqrt{2(T-s)}}\right)\right. \\
&\quad \left. - K\Phi\left(-\frac{\log(K) - y}{\sqrt{2(T-s)}}\right)\right)
\end{aligned}
$$

Where the last step is generated by completing the square of the exponential and generating a new Gaussian distribution.

d) We have $u(0, x_0) = v(T - \frac{\sigma^2}{2}T, \log(x_0) + (r - \frac{\sigma^2}{2})T)$ and so we plug this into the equation in the previous part and get the incredibly simple price of the call option

$$
\begin{aligned}
u(0, x_0) = e^{-rT}\Bigg(& e^{\log(x_0) + (r - \frac{\sigma^2}{2})T + (\frac{\sigma^2}{2}T)} \\
\cdot \Phi & \left(-\frac{\log(K) - (\log(x_0) + (r - \frac{\sigma^2}{2})T + 2(\frac{\sigma^2}{2}T))}{\sqrt{2(\frac{\sigma^2}{2}T)}}\right) \\
& - K\Phi\left(-\frac{\log(K) - \log(x_0) + (r - \frac{\sigma^2}{2})T}{\sqrt{2(\frac{\sigma^2}{2}T)}}\right)\Bigg)
\end{aligned}
$$

Which simplifies down to

$$u(0, x_0) = x_0 \Phi\left(\frac{\log(K/x_0) + (r + \frac{\sigma^2}{2})T)}{\sigma\sqrt{T}}\right)$$
$$-e^{-rT} K \Phi\left(\frac{\log(K/x_0) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\Bigg)$$