Numerical integration of stochastic differential equations

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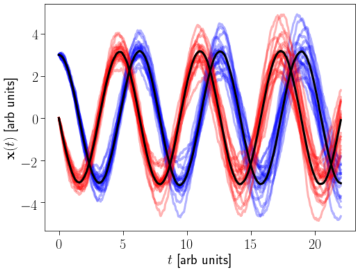
Outline

Motivation

Stochastic differential equations (SDEs)

- physics (quantum to astro), chemistry (molecular dynamics etc), probability theory, finance (Black-Scholes...)
- Anything that evolves continuously and with some non-deterministic component

- Solution very different then for ODEs

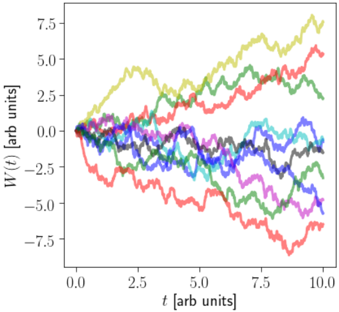


Brownian motion: the Weiner process

- White noise $\zeta(t)$: $\langle \zeta(t) \rangle = 0$, uncorrelated $\langle \zeta(t)\zeta(t') \rangle = \delta(t - t')$
- Weiner process: $W(t) = \int_0^t ds \zeta(s)$

$$p(W, t|0, t_0) = [(2\pi)(t - t_0)]^{-\frac{1}{2}} e^{\frac{-W^2}{2(t-t_0)}}$$

- $\langle W(t) \rangle = 0$, $\langle W(t)^2 \rangle = t - t_0$
- Continuous
- non-differentiable



Stochastic differential equations

- Langavin equation: $\frac{dx}{dt} = A(x,t) + B(x,t)\zeta(t)$
- SDE: $x(t) = x(t_0) + \int_{t_0}^{t} A(x(s),s) + \int_{t_0}^{t} B(x(s),s)dW(s)$
- $dW(t) = W(t+dt) - W(t) = \zeta(t)dt$
- How to evaluate $S = \int f(t')dW(t')$?
- $\lim_{n\to\infty} S_n, \quad S_n = \sum_i^n f(\tau_i)\left(W(t_i) - W(t_{i-1})\right),$

Ito Calculus

- $S_n = \sum_i^n f(\tau_i)\left(W(t_i) - W(t_{i-1})\right)$
- S_n depends on choice of τ_i in interval
- Ito: $\tau = t_{i-1}$
- $\langle \int_{t_0}^{t} f(t')dW(t') \rangle = 0$
- $dW(t)^2 = dt$
- Ito SDE:

$$d\mathbf{x}(t) = \mathbf{A}(\mathbf{x}, t)dt + \mathbf{B}(\mathbf{x}, t)d\mathbf{W}$$

Explicit Euler

- Discretize $x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} A(x(s), s)ds + \int_{t_k}^{t_{k+1}} B(x(s), s)dW(s)$
- $\int_{t_k}^{t_{k+1}} dW(s) = W(t_{k+1}) - W(t_k) = \Delta W_k = \sqrt{\Delta t}\mathcal{N}(0, 1)$

Explicit Euler algorithm

$$x_{k+1} = x_k + A(x_k, t_k)\Delta t + B(x_k, t_k)\Delta W_k$$

Milstein

- Explicit Euler discards a term of order Δt
- $\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dW(s)dW(s') = \frac{1}{2}\left(\Delta W_k^2 - \Delta t\right)$
- Inclusion gives Milstein:

Milstein algorithm

$$
\begin{aligned}
C(x,t) =& \frac{1}{2}B(x,t)\partial_x B(x,t) \\
x_{k+1} =& x_k + A(x_k,t_k)\Delta t + B(x_k,t_k)\Delta W_k + C(x_k,t_k)\left(\Delta W_k^2 - \Delta t\right)
\end{aligned}
$$

Semi-Implicit Euler

- Implicit algorithms have far better stability
- Stratonovich formulation: $S_n = \sum_i^n f(\frac{t_i + t_{i-1}}{2})\,(W(t_i) - W(t_{i-1}))$
- Transform $\mathbf{A}^{\text{strat}}(x, t) = \mathbf{A}^{\text{Ito}}(x, t) - C(x, t)$

Semi-Implicit Euler algorithm

$$x_{k+1} = x_k + A^{\text{Strat}}(\frac{x_{k+1} + x_k}{2}, \frac{t_{k+1} + t_k}{2})\Delta t + B(\frac{x_{k+1} + x_k}{2}, \frac{t_{k+1} + t_k}{2})\Delta W_k$$

Semi-Implicit Euler

Semi-Implicit Euler code

```
x[:, 0]=x0
Weiners = local_state.normal(size=(m, N-1))
for k in range(N-1):
    xtemp = x[:, k]
    for l in range(Niters):
        xtemp =  x[:, k] \
        + Dt*Af(xtemp, tspan[k]+Dt/2)/2   \
        + np. sqrt(Dt)*Bf(xtemp, tspan[k]+Dt/2) \
        @ Weiners[:, k]/2
    x[:, k+1] = 2*xtemp-x[:, k]
```

Weak & Strong Convergence

There are two notions of convergence when it comes to SDE's. This is related to the different notions of convergence for probabilities.

Weak Convergence:

$$\left|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[X_T^{\delta t}\right]\right| \leq O(\delta t)^{\gamma}$$

Strong Convergence:

$$\mathbb{E}\left[|X_T - X_T^{\delta t}|\right] \leq O(\delta t)^{\gamma}$$

Where γ is the rate of convergence of the different types.

Weak & Strong Convergence Differences

Weak Convergence:

- Similar to convergence in distribution
- Statement about the distribution's moments
- Useful for applications where we only care about the state of the system at the end point

Strong Convergence:

- Analogous to convergence in the L^1 norm
- Strong statement about the paths themselves
- Important for applications where path matters such as Exotic Options pricing

Geometric Brownian Motion

Geometric Brownian Motion is a famous model used in finance.

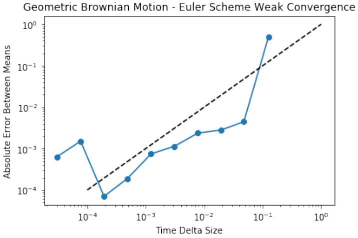
$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

This process has multiplication noise and so it tends to be unstable. Furthermore, we know the solution of this SDE is
Solution:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

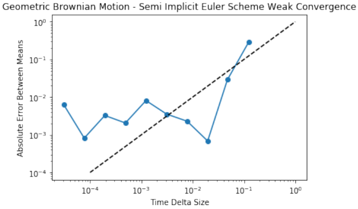
Geometric Brownian Motion - Euler Scheme

Expected Rate of Convergence: 1
Realized Rate of Convergence: 1



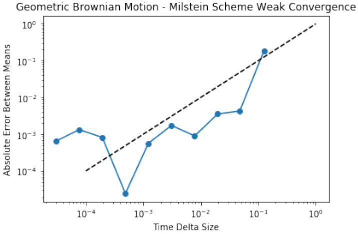
Geometric Brownian Motion - Semi Implicit Euler Scheme

Expected Rate of Convergence: 1
Realized Rate of Convergence: 1



Geometric Brownian Motion - Milstein Scheme

Expected Rate of Convergence: 1
Realized Rate of Convergence: 1



Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is used to model biological systems as it is mean reverting.

$$dX_t = \mu(\theta - X_t)dt + \sigma dW_t$$

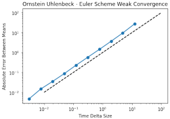
This process differs from GBM in that the noise is only additive. Intuitively, this process tends towards θ as time goes on. The rate at which it tends there is dictated by μ. The solution of the OU process is given by

$$X_t = X_0 e^{-\mu t} + \theta(1 - e^{-\mu T}) + \sqrt{\frac{\sigma}{\mu}(1 - e^{-2\mu t})}W_t$$

Useful for us to consider because $|X_T - X_T^{\delta t}|$ has a closed form.

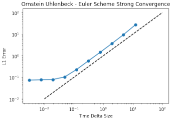
Ornstein-Uhlenbeck - Euler Scheme

Weak Convergence



Theoretical Rate: 1
Realized Rate: 1

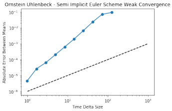
Strong Convergence



Theoretical Rate: 1/2
Realized Rate: 1

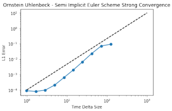
Ornstein-Uhlenbeck - Semi Implicit Euler Scheme

Weak Convergence



Theoretical Rate: 1
Realized Rate: 2

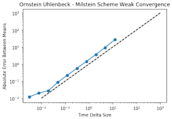
Strong Convergence



Theoretical Rate: 1/2
Realized Rate: 1

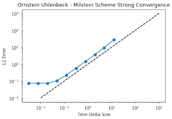
Ornstein-Uhlenbeck - Milstein Scheme

Weak Convergence



Theoretical Rate: 1
Realized Rate: 1

Strong Convergence



Theoretical Rate: 1
Realized Rate: 1