ORFE 526: Probability Theory
Homework 2

Zachary Hervieux-Moore

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Exercise 1: Let $(E, \mathcal{E})$ be a measurable space and $f : E \to \mathbb{R}$ a Borel-measurable function taking finitely many real values. Prove the f is a simple function.

Answer: Let $c_1, c_2, \ldots, c_n$ be the distinct values of f. The let $A_i = \{x : f(x) = c_i\}$ where the A_i's are measurable by assumption and disjoint. Thus, we can write f as:

$$f(x) = \sum_{i=1}^{n} c_i \cdot 1_{\{x \in A_i\}}$$

Which is the exact form of a simple function.

Exercise 2: Let $f, g \in \mathcal{E}_+$. Show that:

a) $f \wedge g, f \vee g \in \mathcal{E}_+$

b) $f + g \in \mathcal{E}_+$

c) $fg \in \mathcal{E}_+$

Answer:

a) Clearly, if $f \geq 0$ and $g \geq 0$, then $f \wedge g \geq 0$ and $f \vee g \geq 0$. We only need to show measurability. Consider the preimages $f^{-1}((-\infty, r])$. Then, for $f \wedge g$,

$$(f \wedge g)^{-1}([r, \infty)) = f^{-1}([r, \infty)) \bigcap g^{-1}([r, \infty))$$

Which is an intersection of two measurable sets by assumptions and hence measurable. Likewise,

$$(f \vee g)^{-1}((-\infty, r]) = f^{-1}((-\infty, r]) \bigcap g^{-1}((-\infty, r])$$

Which is the union of two measurable sets and so measurable.

b) Clearly, if $f \geq 0$ and $g \geq 0$, then $f + g \geq 0$. Again, to show measurability, consider the preimage,

$$(f + g)^{-1}((-\infty, r)) \Leftrightarrow f + g < r \Leftrightarrow f < r - g$$

We know by real analysis, that for every $r \in \mathbb{R}$, we can find a rational $q \in \mathbb{Q}$ s.t.

$$f < q < r - g$$

We then have,

$$f < q \text{ and } q < r - g \Leftrightarrow g < r - q$$

So,

$$(f + g)^{-1}((-\infty, r)) = \bigcup_{q \in \mathcal{Q}} f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, r - q))$$

Which is a countable union of measurable sets, thus it is measurable.

c) Again, if $f \geq 0$ and $g \geq 0$, then $fg \geq 0$. We note the following,

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

We know that adding and subtracting measurable functions does not change measurability. Multiplying by a scalar also doesn't change measurability since for $k \in \mathbb{R}^+$

$$(kf)^{-1}((-\infty, r)) = (f)^{-1}((-\infty, r/k))$$

Which is measurable. For $k \in \mathbb{R}^-$,

$$(kf)^{-1}((-\infty, r)) = (f)^{-1}((r/k, \infty))$$

Finally, for $k = 0$, then $kf = 0$ which is clearly measurable. Thus we only have to show that squaring a function does not change measurability.

$$(f^2)^{-1}((-\infty, r)) = f^{-1}((-\sqrt{r}, \sqrt{r})) = f^{-1}((-\infty, \sqrt{r})) \bigcap f^{-1}((-\sqrt{r}, \infty))$$

Which is the intersection of two measurable sets, thus f^2 is measurable and so fg is measurable.

Exercise 3: Let $\mu_1, \mu_2, \ldots$ be measures on $(E, \mathcal{E})$ and denote $\mu = \sum_{n\geq 1} \mu_n$. Prove that μ is also a measure on $(E, \mathcal{E})$.

Answer: To show something is a measure, we must show $\mu(\emptyset) = 0$, non-negativity, and countable additivity.

$$\mu(\emptyset) = \sum_{n\geq 1} \mu_n(\emptyset) = \sum_{n\geq 1} 0 = 0$$

Non-negativity,

$$\mu(A) = \sum_{n\geq 1} \mu_n(A) \geq \mu_1(A) \geq 0$$

Since μ_n are measures, they satisfy non-negativity. Finally, countable additivity,

$$\begin{aligned}
\mu\left(\bigcup_{i\geq 1} A_i\right) &= \sum_{n\geq 1} \mu_n\left(\bigcup_{i\geq 1} A_i\right) \\
&= \sum_{n\geq 1}\sum_{i\geq 1} \mu_n(A_i)
\end{aligned}$$

Since $\mu_n(A) \geq 0$, Tonelli's theorem states we can interchange the sums,

$$\sum_{n\geq 1}\sum_{i\geq 1} \mu_n(A_i) = \sum_{i\geq 1}\sum_{n\geq 1} \mu_n(A_i) = \sum_{i\geq 1} \mu(A_i)$$

Thus, we have,

$$\mu\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} \mu(A_i)$$

Exercise 4: If δ_{x_0} denotes the Dirac measure sitting at x_0, show that $\delta_{x_0} f = f(x_0)$, for any $f \in \mathcal{E}$.

Answer: We write the definition of Dirac measure,

$$\begin{aligned}\delta_{x_0} f = \int_E f d\delta_{x_0}(x) &= \int_{\{x=x_0\}} f d\delta_{x_0}(x) + \int_{\{x\neq x_0\}} f d\delta_{x_0}(x) \\ &= f(x_0) \int_{\{x=x_0\}} d\delta_{x_0}(x) + 0 = f(x_0)\end{aligned}$$

Where $\int_{\{x\neq x_0\}} f d\delta_{x_0}(x) = 0$ and $\int_{\{x=x_0\}} d\delta_{x_0}(x) = 1$ by definition of the Dirac measure.

Exercise 5: Let $(E, \mathcal{E}, \mu)$ be a measure space, and $p \in \mathcal{E}_+$. Define

$$\nu(A) = \int_A p(x) d\mu(x), \quad \forall A \in \mathcal{E}$$

a) Show that ν is a measure on $(E, \mathcal{E})$

b) Prove that for any $f \in \mathcal{E}_+$ we have

$$\int_E f(x) d\nu(x) = \int_E f(x) p(x) d\mu(x)$$

Answer:

a) We show the three properties of a measure. First,

$$\nu(\emptyset) = \int_\emptyset p(x) d\mu(x) = 0$$

Also, since $p(x) \geq 0 \implies \int_A p(x) d\mu(x) \geq 0$. Now we show countable additivity. First for indicator functions, then simple functions, and finally $p \in \mathcal{E}_+$. Let $p(x) = 1_{\{x \in B\}}$,

$$\nu\left(\bigcup_{i \geq 1} A_i\right) = \int_{\cup_{i\geq 1} A_i} p(x) d\mu(x) = \int_{\cup_{i\geq 1} A_i} 1_{\{x \in B\}} d\mu(x)$$

$$= \mu(B \bigcap \cup_{i\geq 1} A_i) = \sum_{i \geq 1} \mu(B \cap A_i) = \sum_{i\geq 1} \int_{B \cap A_i} d\mu(x) = \sum_{i \geq 1} \int_{A_i} 1_{\{x \in B\}} d\mu(x)$$

$$= \sum_{i \geq 1} \int_{A_i} p(x) d\mu(x) = \sum_{i \geq 1} \nu(A_i)$$

Now let $p(x) = \sum_{j=1}^{n} b_j 1_{\{x \in B_j\}}$,

$$\nu\left(\bigcup_{i\geq 1} A_i\right) = \int_{\cup_{i\geq 1} A_i} p(x) d\mu(x) = \int_{\cup_{i\geq 1} A_i} \sum_{j=1}^{n} b_j 1_{\{x\in B_j\}} d\mu(x)$$
$$= \sum_{j=1}^{n} b_j \int_{\cup_{i\geq 1} A_i} 1_{\{x\in B_j\}} d\mu(x) = \sum_{j=1}^{n} b_j \mu(B_i \bigcap \cup_{i\geq 1} A_i)$$
$$= \sum_{j=1}^{n} b_j \sum_{i\geq 1} \mu(B_j \cap A_i) = \sum_{j=1}^{n} b_j \sum_{i\geq 1} \int_{B_j \cap A_i} d\mu(x)$$
$$= \sum_{j=1}^{n} b_j \sum_{i\geq 1} \int_{A_i} 1_{\{x\in B_j\}} d\mu(x) = \sum_{i\geq 1} \int_{A_i} \sum_{j=1}^{n} b_j 1_{\{x\in B_j\}} d\mu(x)$$
$$= \sum_{i\geq 1} \int_{A_i} p(x) d\mu(x) = \sum_{i\geq 1} \nu(A_i)$$

Where the swapping of sums is justified since the sum is finite. Now suppose $p(x) \in \mathcal{E}_+$, then there exists (p_n) with p_n simple, ≥ 0, and $p_n \nearrow p$.

$$\nu\left(\bigcup_{i\geq 1} A_i\right) = \int_{\cup_{i\geq 1} A_i} p(x) d\mu(x) = \int_{\cup_{i\geq 1} A_i} \lim_{n\to\infty} p_n(x) d\mu(x)$$
$$= \lim_{n\to\infty} \int_{\cup_{i\geq 1} A_i} p_n(x) d\mu(x) \text{ by monotone convergence theorem}$$

Which we proved for simple functions,

$$= \lim_{n\to\infty} \sum_{i\geq 1} \int_{A_i} p_n(x) d\mu(x)$$
$$= \sum_{i\geq 1} \int_{A_i} \lim_{n\to\infty} p_n(x) d\mu(x) \text{ by monotone convergence theorem}$$
$$= \sum_{i\geq 1} \int_{A_i} p(x) d\mu(x) = \sum_{i\geq 1} \nu(A_i)$$

So, $\nu(A)$ satisfies the three properties of a measure, so it is a measure.

b) To prove this, we do it first for indicator functions, then simple functions, and then for $\mathcal{E}_+$. Let f be an indicator,

$$\begin{aligned}\int_E f(x)d\nu(x) &= \int_E 1_{\{x\in A\}}d\nu(x) = \int_A d\nu(x) = \nu(A)\\ &= \int_A p(x)d\mu(x) = \int_E 1_{\{x\in A\}}p(x)d\mu(x) = \int_E f(x)p(x)d\mu(x)\end{aligned}$$

Now suppose f is a simple function,

$$\int_E f(x)d\nu(x) = \int_E \sum_{i=1}^n a_i 1_{\{x\in A_i\}}d\nu(x) = \sum_{i=1}^n a_i \int_A 1_{\{x\in A_i\}}d\nu(x)$$

Where the last equality holds since the sum is finite. Continuing by definition,

$$\begin{aligned}&= \sum_{i=1}^n a_i\nu(A_i) = \sum_{i=1}^n a_i \int_{A_i} p(x)d\mu(x) = \int_{A_i}\sum_{i=1}^n a_i p(x)d\mu(x)\\ &= \int_E \sum_{i=1}^n a_i 1_{\{x\in A_i\}}p(x)d\mu(x) = \int_E f(x)p(x)d\mu(x)\end{aligned}$$

Finally, we consider $f \in \mathcal{E}_+$ where $f_n \nearrow f$, $f_n \geq 0$, and f_n simple.

$$\int_E f(x)d\nu(x) = \int_E \lim_{n\to\infty} f_n(x)d\nu(x) = \lim_{n\to\infty}\int_E f_n(x)d\nu(x)$$

We can exchange the limit and integral due to monotone convergence theorem. Now, since f_n is simple, we can use our previous proof,

$$= \lim_{n\to\infty}\int_E f_n(x)p(x)d\mu(x) = \int_E \lim_{n\to\infty} f_n(x)p(x)d\mu(x) = \int_E f(x)p(x)d\mu(x)$$