Homework 6

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Question 1

First, let us write down the correlation between $Z \sim \mathcal{N}(0,1)$ and Z^2.

$$\rho_{Z,Z^2} = \frac{\text{cov}(Z, Z^2)}{\sigma_Z \sigma_{Z^2}}$$

Now, we compute the covariance term.

$$\begin{aligned}
\text{cov}(Z, Z^2) &= \mathbb{E}[(Z - \mu_Z)(Z^2 - \mu_{Z^2})] \\
&= \mathbb{E}[Z^3 - Z] \\
&= 0
\end{aligned}$$

Where the second line is because $\mu_Z = 0$ by assumption and $\mu_{Z^2} = 1$. The last line is because Z is symmetric with mean 0. Thus, we conclude that $\rho_{Z,Z^2} = 0$ as the denominator is non-zero ($\sigma_Z = 1$ and σ_{Z^2} is well defined).

Clearly, the two variables are not independent. Take for example

$$\mathbb{P}(Z \leq -1, Z^2 \leq 1) = 0 \neq \mathbb{P}(Z \leq -1)\mathbb{P}(Z^2 \leq 1) > 0$$

Thus, we have shown that 0 correlation does not imply independence even though independence implies 0 correlations (if they are independent, the covariance is 0).