ELE 535: Machine Learning and Pattern Recognition
Homework 9

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Exercise 1: Kernels:

a) Let $\mathcal{A}$ be a finite set and for each subset $\mathcal{U} \subseteq \mathcal{A}$ let $|\mathcal{U}|$ denote the number of element in $\mathcal{U}$. For $\mathcal{U}, \mathcal{V} \subset \mathcal{A}$, let $k(\mathcal{U}, \mathcal{V}) = |\mathcal{U} \cap \mathcal{V}|$. By finding a suitable feature map, show that $k(\cdot, \cdot)$ is a kernel on the power set $\mathcal{P}(\mathcal{A})$ of all subsets of $\mathcal{A}$.

b) Show that $k(x, z) = \sum_{i=1}^{n} \cos^2(x_i - z_i)$ is a kernel on $\mathbb{R}^n$.

c) Let $P \in \ltimes \times \ltimes$ be a symmetrix PSD. Show that $k(x, z) = e^{-\frac{1}{2}(x-z)^T P (x-z)}$ is a kernel on $\mathbb{R}^n$.

d) $k(x, y) = h_t(Ax)^T h_t(Ay)$ where $A \in \mathbb{R}^{n \times n}$ and h_t is a thresholding function that maps $z = [z_i]$ to $h_t(z) = [\tilde{z}_i]$ with $\tilde{z}_i = z_i$ if $|z_i| > t$ and 0 otherwise.

Answer:

Exercise 2: Let k_j be a kernel on $\mathcal{X}$ with feature map $\phi_j : \mathcal{X} \to \mathbb{R}^q$, $j = 1, 2$. In each part below, find a simple feature map for the kernel k in terms of feature maps for the kernels k_j. By this means, give an interpretation for the new kernel k.

a) $k(x, z) = k_1(x, z) + k_2(x, z)$

b) $k(x, z) = k_1(x, z)k_2(x, z)$

c) $k(x, z) = k_1(x, z)/\sqrt{k_1(x, x)k_2(z, z)}$

Answer:

Exercise 3: A binary labelled set of data in $\mathbb{R}^2$ is used to learn a SVM using the homogeneous quadratic kernel. By writing the equation for the decision boundary in terms of a quadratic form, reason about the types of decision boundaries that are possible in $\mathbb{R}^2$. In each case, give a neat sketch.

Answer:

Exercise 4: Consider the defintie integral

$$\int_0^\infty e^{-(ax^2+\frac{b}{x^2})}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$$

Show that it can be rewritten as

$$\frac{\alpha}{\sqrt{\pi}}\int_0^\infty e^{-\frac{s}{t^2}}e^{-(\frac{\alpha t}{2})^2}dt = e^{-\alpha\sqrt{s}}, \quad \alpha > 0$$

Now use the above results to show that $k(x,z) = e^{-\alpha\|x-z\|_2}$ is a kernel.

Answer:

Exercise 5: Let $a > 0$ and $L_2[0, a]$ denote the set of real valued square integrable functions on the interval $[0, a]$. $L_2[0, a]$ is a Hilbert space under the inner product $\langle g, h\rangle = \int_0^a g(s)h(s)ds$. For $f \in L_2[0, a]$, let $g(t) = \int_0^t f^2(s)ds$ and $h(t) = \int_t^a f^2)s_d s$, where $t \in [0, a]$. Show that for each $a > 0$,

a) $k(x, z) = \min(x, z)$ is a kernel on $[0, a]$.

b) $k(x, z) = a - \max(x, z)$ is a kernel on $[0, a]$.

c) $k(x, z) = e^{-(\max(x,z)-\min(x,z))}$ is a kernel on $[0, a]$. Plot the function $\max(x, z) - \min(x, z)$ and use this to simplify the result further.

d) for each $a > 0$, and $\gamma \geq 0$, $k(x, z) = e^{-\gamma|x-z|}$ is a kernel on $[-a, a]$.

e) for each $\gamma \geq 0$, $k(x, z) = e^{-\gamma|x-z|}$ is a kernel on $\mathbb{R}$.

Answer: