ORFE 526: Probability Theory
Homework 4

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Exercise 1: Let X be an integrable random variable. Show that any sequence of events H_n such that $P(H_n) \to 0$ as $n \to \infty$, has the property $\lim_{n\to\infty} \mathbb{E}[|X| \cdot 1_{\{H_n\}}] = 0$. Does the converse hold true? (Consider the converse with the conclusion "X is an integrable random variable").

Answer: Consider $|X| \leq b$, then

$$E[|X| \cdot 1_{\{H_n\}}] \leq b \cdot E[1_{\{H_n\}}] = b \cdot P(H_n)$$

Now consider $|X| > b$,

$$E[|X| \cdot 1_{\{H_n\}}] \leq E[|X| \cdot 1_{\{|X|>b\}}]$$

Putting these together,

$$E[|X| \cdot 1_{\{H_n\}}] \leq b \cdot P(H_n) + E[|X| \cdot 1_{\{|X|>b\}}]$$

Since X is integrable, there is a large enough b' such that $E[|X| \cdot 1_{\{|X|>b\}}] < \frac{\epsilon}{2}$,

$$E[|X| \cdot 1_{\{H_n\}}] \leq b' \cdot P(H_n) + \frac{\epsilon}{2}$$

Thus,

$$\lim_{n\to\infty} E[|X| \cdot 1_{\{H_n\}}] \leq \lim_{n\to\infty} b' \cdot P(H_n) + \frac{\epsilon}{2}$$

Since $P(H_n) \to 0$, we can find N sufficiently large such that $P(H_n) < \frac{\epsilon}{2b'}$,

$$\lim_{n\to\infty} E[|X| \cdot 1_{\{H_n\}}] < \frac{\epsilon}{2b'} b' + \frac{\epsilon}{2} < \epsilon, \quad \forall\, \epsilon > 0$$

This proves the direct result.

The converse certainly does hold true. Define H_n to be $\{|X| > n\}$. Then we have that $\lim_{n\to\infty} P(H_n) = 0$ and so we have $\lim_{n\to\infty} \mathbb{E}[|X| \cdot 1_{\{|X|>n\}}] = 0$ by assumption. This is equivalent to integrable as proven in class. Thus, the converse is true. Note that I assumed that X can only take finite values.

Exercise 2: Assume $\sum_{n\geq 1} P\left(|X_n - X| > \frac{1}{n}\right) < \infty$. Show that $X_n \to X$ almost surely.

Answer: We are going to use Borel-Cantelli lemma,

$$\sum_{n\geq 1} P\left(|X_n - X| > \epsilon\right) < \infty \;\forall\; \epsilon > 0 \Longleftrightarrow X_n \to X \text{ a.s.}$$

For every $\epsilon > 0$, there exists N sufficient large such that $\frac{1}{N} < \epsilon$ (namely, $N > \frac{1}{\epsilon}$), then

$$\begin{aligned}\sum_{n\geq 1} P\left(|X_n - X| > \epsilon\right) &= \sum_{n=1}^{N} P\left(|X_n - X| > \epsilon\right) + \sum_{n=N}^{\infty} P\left(|X_n - X| > \epsilon\right) \\ &\leq \sum_{n=1}^{N} P\left(|X_n - X| > \epsilon\right) + \sum_{n=N}^{\infty} P\left(|X_n - X| > \frac{1}{n}\right)\end{aligned}$$

Since $\sum_{n\geq 1} P\left(|X_n - X| > \frac{1}{n}\right) < \infty$ by assumption, then the second sum is finite, and the first sum is clearly finite since it is a finite sum. Thus,

$$\sum_{n\geq 1} P\left(|X_n - X| > \epsilon\right) < \infty \;\forall\; \epsilon > 0$$

Which is equivalent to $X_n \to X$ almost surely by the above Borel-Cantelli lemma.

Exercise 3: Consider the probability space $(\Omega, \mathcal{H}, P)$ with $\Omega = [0,1], \mathcal{H} = \mathcal{B}([0,1])$, and P the Lebesgue measure on $[0,1]$. Define the sequence X_n by

$$X_{2n}(\omega) = \begin{cases} 0, & \text{if } \omega < 1/2 \\ 1, & \text{if } \omega \geq 1/2 \end{cases} \quad X_{2n+1}(\omega) = \begin{cases} 1, & \text{if } \omega < 1/2 \\ 0, & \text{if } \omega \geq 1/2 \end{cases}$$

i) Find the distribution functions of X_{2n} and X_{2n+1}.

ii) Find the characteristic functions of X_{2n} and X_{2n+1}.

iii) Show that X_n converges in distribution.

iv) Show that X_n does not converge in probability to 0.

Answer:

i) We consider the sets $X_{2n}^{-1}((-\infty, x])$ and $X_{2n+1}^{-1}((-\infty, x])$. It is easy to see that the measures are equal,

$$\mu_{2n}((-\infty, x]) = \mu_{2n+1}((-\infty, x]) = \begin{cases} 0, & \text{if } x < 0 \\ 1/2, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x \end{cases}$$

Recognize that this is the distribution of $Bernoulli(1/2)$. That is, $X_n \sim Bernoulli(1/2)$ for all n.

ii) The characteristic functions are,

$$\phi_{X_n}(t) = \mathbb{E}[e^{itX_n}] = e^{it0} \cdot \frac{1}{2} + e^{it} \cdot \frac{1}{2} = \frac{1 + e^{it}}{2}$$

iii) Notice that the μ_n defined in part i) does not depend on n. Thus, $\mu_n = \mu$ and so $\mu_n \to \mu$ and so X_n converges in distribution to $X \sim Bernoulli(1/2)$.

iv) Consider $P(|X_n - 0| > 1/2) = P(|X_n| > 1/2) = 1/2$ for all n. Thus, X_n does not converge in probability to 0.

Exercise 4: Let X_n be a sequence of random variables. Assume there is a constant l such that $\lim_{n\to\infty} \mathbb{E}[\mathbb{X}_\ltimes] = k$ and $\lim_{n\to\infty} Var(X_n) = 0$. Show that $X_n \to k$ in L^2 (i.e. in the mean square).

Answer: We know that,

$$\lim_{n\to\infty} Var(X_n) = 0$$

Rewriting the variancee,

$$\lim_{n\to\infty} \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2 = 0$$

$f(x) = x^2$ is continuous, so we can take the limit inside,

$$\lim_{n\to\infty} \mathbb{E}[X_n^2] - (\lim_{n\to\infty} \mathbb{E}[X_n])^2 = 0 \Longleftrightarrow \lim_{n\to\infty} \mathbb{E}[X_n^2] - k^2 = 0$$
$$\Longleftrightarrow \lim_{n\to\infty} \mathbb{E}[X_n^2] = k^2$$

But $X_n^2 = |X_n|^2$ so,

$$\lim_{n\to\infty} \mathbb{E}[|X_n|^2] = \lim_{n\to\infty} \|X_n\|_2 = k^2$$

We now show this implies convergence in L_2,

$$\lim_{n\to\infty} \mathbb{E}[|X_n - k|^2] = \lim_{n\to\infty} \mathbb{E}[|X_n|^2 - 2kX_n + k^2]$$
$$= \lim_{n\to\infty} \mathbb{E}[|X_n|^2] - 2k\mathbb{E}[X_n] + k^2 = \lim_{n\to\infty} \mathbb{E}[|X_n|^2] - k^2$$

From before, $\lim_{n\to\infty} \mathbb{E}[|X_n|^2] = k^2$,

$$\Longrightarrow \lim_{n\to\infty} \mathbb{E}[|X_n - k|^2] = 0$$

So we conclude $\|X_n - k\|_2 \to 0$ in L^2.

Exercise 5: If $X_n \to X$ and $X_n \to Y$, both in probability, show that $X = Y$ almost surely.

Answer: Since $X_n \to X$ in probability, then there exists a subsequence n_{k_1} such that $X_{n_{k_1}} \to X$ almost surely. Since this is a subsequence of X_n, we also have that $X_{n_{k_1}} \to Y$ in probability. Thus pick another subsequence n_{k2} that converges to Y almost surely. Since this is a subsequence of n_{k_1}, we have that $X_{n_{k_2}} \to X$ almost surely as well. Hence,

$$
\begin{aligned}
P(X - Y = 0) &= P(\lim_{n_{k_2} \to \infty} X_{n_{k_2}} - \lim_{n_{k_2} \to \infty} X_{n_{k_2}} = 0) \\
&= P(\lim_{n_{k_2} \to \infty} X_{n_{k_2}} - X_{n_{k_2}} = 0) = P(0 = 0) = 1
\end{aligned}
$$

So, $X = Y$ almost surely.

Exercise 6: Let $X_n \to 0$ in probability. Show that $X_n^3 \to 0$ in probability.

Answer: We know that if $X_n \to X$ and $Y_n \to Y$ in probability, then $X_n Y_n \to XY$ in probability. So, using X_n above and $Y_n = X_n$, then

$$X_n X_n = X_n^2 \to 0 \text{ in probability}$$

Now let $Y_n = X_n^2$, by the same property,

$$X_n X_n^2 = X_n^3 \to 0 \text{ in probability}$$

Also, $f(x) = x^3$ is continuous, thus if $X_n \to X$ in probability, then $f(X_n) \to f(X)$ in probability. In this case, $X_n^3 \to 0^3 = 0$ in probability.

Exercise 7: Consider a sequence of random variables X_n such that there is another random variable X such that $\sum_n \|X_n - X\|_{L^2}^2 < M < \infty$ a.s. Show that $X_n \to X$ almost surely.

Answer: We have that,

$$M > \sum_n \|X_n - X\|_{L^2}^2 = \sum_n \mathbb{E}[|X_n - X|^2]$$

By Markov's inequality,

$$\sum_n \mathbb{E}[|X_n - X|^2] \geq \sum_n \epsilon^2 P(|X_n - X| > \epsilon)$$

Thus,

$$\sum_n P(|X_n - X| > \epsilon) < \frac{M}{\epsilon^2} < \infty \ \forall \ \epsilon > 0$$

By Borel-Cantelli, then $X_n \to X$ almost surely.