ORFE 527: Stochastic Calculus
Homework 4

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Exercise 1: (Weak solutions) Suppose that the function $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is such that the SDE

$$\mathrm{d}X_t = \sigma(t, X_t)\mathrm{d}B_t$$

has a unique weak solution for any $X_0 \in \mathbb{R}$ (e.g. σ is Lipschitz in the x variable). Suppose further that there exists deterministic constant $c > 0$ such that $\sigma(t, x) \geq c$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$. Show that for any bounded measurable function $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t$$

has a unique weak solution for any $X_0 \in \mathbb{R}$.

Answer: We replicate the steps done in class. Assume there is a standard Brownian motion X on $(\Omega, \mathcal{F}, \mathbb{P})$. Now, define $(\mathcal{F}_t)_{t\geq 0}$ as the filtration generated by X. Now define $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = e^{\int_0^T b(s,X_s)/\sigma(s,X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^T b(s,X_s)^2/\sigma(s,X_s)^2\mathrm{d}s}$$

Since b is bounded and $\sigma \geq c$ then we also have that $b(s, X_s)/\sigma(s, X_s)$ is bounded and so $\mathbb{Q}$ is a probability measure by Novikov's theorem. Thus, by Girsanov's theorem, we have that $B_t := X_t - \int_0^T b(s, X_s)/\sigma(s, X_s)\mathrm{d}s$ is a standard Brownian motion under $\mathbb{Q}$. Thus,

$$\mathrm{d}X_t = b(t, X_t)/\sigma(t, X_t)\mathrm{d}t + \mathrm{d}B_t$$

Now, we can use the theorem from class that

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}B_t$$

has a unique weak solution if $b(t, X_t)$ is bounded. Since, $b(t, X_t)/\sigma(t, X_t) \leq b(t, X_t)/c$ and $b(t, X_t)$ is bounded, we conclude that the original SDE has a unique weak solution.

Exercise 2: (Variation on Yamada-Watanabe Theorem) Suppose that strong existence holds for the initial value problem

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t,\ X_0 = x_0$$

Suppose further that the joint law of (X, B) is uniquely determined for any solution of this initial value problem. Show that any two strong solutions $X, \tilde{X}$ of the initial value problem with respect to the same Brownian motion B are indistinguishable.

Answer: By definition of strong solutions,

$$\mathbb{P}(X_t = x_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s) = 1\ \forall t \geq 0$$
$$\mathbb{P}(\tilde{X}_t = x_0 + \int_0^t b(s, \tilde{X}_s)\mathrm{d}s + \int_0^t \sigma(s, \tilde{X}_s)\mathrm{d}B_s) = 1\ \forall t \geq 0$$

That is, $X_t, \tilde{X}_t \in C([0, \infty), \mathbb{R})$. Also, by uniqueness of the joint law, we have for $A \in C([0, \infty), \mathbb{R})$

$$\mathbb{E}[X_t 1_{B_{[0,t]} \in \{A\}}] = \mathbb{E}[\tilde{X}_t 1_{B_{[0,t]} \in \{A\}}]$$
$$\iff \mathbb{E}[(X_t - \tilde{X}_t) 1_{B_{[0,t]} \in \{A\}}] = 0\ \forall A$$

Therefore $X_t = \tilde{X}_t$ a.s. and we have

$$\mathbb{P}(X_t = \tilde{X}_t\ \forall t \in \mathbb{Q})$$

Since the paths are continuous, we can expand t to all real numbers

$$\mathbb{P}(X_t = \tilde{X}_t\ \forall t \in \mathbb{R})$$

Which implies that X_t and $\tilde{X}_t$ are indistinguishable.

Exercise 3: (Bougerol's identity) Let $B^{(1)}, B^{(2)}$ be independent standard Brownian motions.

a) Find the SDE satisfied by the process

$$X_t := e^{B_t^{(1)}} \int_0^t e^{-B_s^{(1)}} \mathrm{d}B_s^{(2)},\ t \geq 0$$

b) Find the SDE satisfied by the process $Y_t := \sinh B_t^{(1)}$, $t \geq 0$

c) Use (a) and (b) to show the identity in distribution

$$\int_0^t e^{B_s^{(1)}} \mathrm{d}B_s^{(2)} \overset{d}{=} \sinh B_t^{(1)}$$

for any fixed $t \geq 0$. The latter is known as *Bougerol's identity.*

Answer:

a) We apply the multivariate Ito's formula to X_t

$$\begin{aligned}
\mathrm{d}X_t &= \left(e^{B_t^{(1)}} \int_0^t e^{-B_s^{(1)}} \mathrm{d}B_s^{(2)}\right) \mathrm{d}B_t^{(1)} + \left(e^{B_t^{(1)}} e^{-B_t^{(1)}}\right) \mathrm{d}B_t^{(2)} \\
&\quad + \left(e^{B_t^{(1)}} \int_0^t e^{-B_s^{(1)}} \mathrm{d}B_s^{(2)}\right) \mathrm{d}t \\
&= X_t \mathrm{d}B_t^{(1)} + \mathrm{d}B_t^{(2)} + X_t \mathrm{d}t
\end{aligned}$$

b) Apply Ito's formula to Y_t and use hyperbolic trig identity

$$\begin{aligned}
\mathrm{d}Y_t &= \cosh\left(B_t^{(1)}\right) \mathrm{d}B_t^{(1)} + \sinh\left(B_t^{(1)}\right) \mathrm{d}t \\
&= \sqrt{1 + \sinh^2\left(B_t^{(1)}\right)} \mathrm{d}B_t^{(1)} + Y_t \mathrm{d}t \\
&= \sqrt{1 + Y_t^2} \mathrm{d}B_t^{(1)} + Y_t \mathrm{d}t
\end{aligned}$$

c) We define

$$\mathrm{d}Z_t = \frac{\mathrm{d}B_t^{(2)} + X_t \mathrm{d}B_t^{(1)}}{\sqrt{1 + X_t^2}}$$

Note that this is a local martingale. We compute its quadratic variation.

$$\begin{aligned}
\mathrm{d}\langle Z\rangle_t &= \frac{\mathrm{d}t}{1+X_t^2} + \frac{X_t^2\mathrm{d}t}{1+X_t^2} \\
&= \frac{(1+X_t^2)\mathrm{d}t}{1+X_t^2} \\
&= \mathrm{d}t
\end{aligned}$$

Thus, by Levy's characterization, Z_t is a Brownian motion. Furthermore, using part (a),

$$\begin{aligned}
\mathrm{d}X_t &= X_t\mathrm{d}B_t^{(1)} + \mathrm{d}B_t^{(2)} + X_t\mathrm{d}t \\
\implies \mathrm{d}X_t &= \sqrt{1-X_t^2}\mathrm{d}Z_t + X_t\mathrm{d}t
\end{aligned}$$

Thus, X_t and Y_t satisfy the same stochastic differential equation. Thus, $X_t \stackrel{d}{=} Y_t = \sinh B_t^{(1)}$. Now, we show that X_t is the same in distribution as the result we wish.

$$\begin{aligned}
X_t &= e^{B_t^{(1)}} \int_0^t e^{-B_s^{(1)}} \mathrm{d}B_s^{(2)} \\
&= \int_0^t e^{B_t^{(1)} - B_s^{(1)}} \mathrm{d}B_s^{(2)} \\
&= \int_0^t e^{B_{t-s}^{(1)}} \mathrm{d}B_s^{(2)}
\end{aligned}$$

Change of variable $s' = t - s$

$$\begin{aligned}
&= -\int_t^0 e^{B_{s'}^{(1)}} \mathrm{d}B_{s'}^{(2)} \\
&= \int_0^t e^{B_s^{(1)}} \mathrm{d}B_s^{(2)}
\end{aligned}$$

Thus, we have shown precisely that $\int_0^t e^{B_s^{(1)}} \mathrm{d}B_s^{(2)} = X_t \stackrel{d}{=} Y_t = \sinh B_t^{(1)}$