1 List of Problems for Chapter 4

This is a list of problems from Chapter 4 which I left in class as exercises or referred to during proofs. The Homework will be picked from these problems. They are a good practice list for the probability spaces chapter.

$(\Omega, \mathcal{H}, \mathbb{P})$ will denote a probability space.

1. Let $\mathcal{F}$ be a σ-algebra included in $\mathcal{H}$. If X is a $\mathcal{F}$-measurable random variable such that
$$\int_A X\, dP = 0, \qquad \forall A \in \mathcal{F},$$
show that $X = 0$ almost surely.

2. Let $X, Y \geq 0$ are $\mathcal{F}$-measurable such that $\mathbb{E}[fX] \leq \mathbb{E}[fY]$, for all $f \in \mathcal{F}_+$, show that $X \leq Y$ almost surely.

3. (i) Show that $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$
(ii) Let X be an integrable random variable and $Y = \mathbb{E}[X, \mathcal{F}]$. Show that Y is integrable.

4. Show that $\mathbb{E}[X|\mathcal{H}] = X$, where $\mathcal{H}$ is the total information on Ω.

5. Show that if $\mathcal{F} = \{\emptyset, \Omega\}$ then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

6. Let H be an event and let $\mathcal{F} = \sigma H = \{\emptyset, H, H^c, \Omega\}$. Show that $\mathbb{E}[X|\mathcal{F}](\omega) = \mathbb{E}[X|H](\omega)$ for all ω in H.

7. (i) If $X \leq b$ for some constant b, show that $\mathbb{E}[X|\mathcal{F}] \leq b$.
(ii) If $a \leq X \leq b$ for some constants a and b, show that $a \leq \mathbb{E}[X|\mathcal{F}] \leq b$.

8. (i) If $X \in L^1$, show that $\mathbb{E}[X|\mathcal{F}] \in L^1$.
(ii) If $X \in L^2$, show that $\mathbb{E}[X|\mathcal{F}] \in L^2$.
(iii) If $X \in L^\infty$, show that $\mathbb{E}[X|\mathcal{F}] \in L^\infty$.

9. If (X_n) is uniformly integrable and converges to X in probability, then $E[X_n|\mathcal{F}] \longrightarrow E[X_n|\mathcal{F}]$ in L^1.

10. Let X and Y be independent random variables and consider
$$\begin{aligned} S &= X + Y \\ Z &= X - Y. \end{aligned}$$
Compute the conditional expectations $\mathbb{E}[X|S, Z]$, $\mathbb{E}[S|X]$, $\mathbb{E}[Z|Y]$.

11. Toss a fair coin 4 times. Each toss yields either H (heads) or T (tails). Consider the following information sets:
$\mathcal{F}$ = we know the outcomes of the first two tosses
$\mathcal{G}$ = we know the outcomes of the tosses but not the order.
Define the random variables:
X = number of H − number of T
Y = number of T before the first H.
(i) Show that X is $\mathcal{G}$-measurable while Y is not $\mathcal{G}$-measurable.
(ii) Find the expectations $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[X|\mathcal{G}]$.

12. Let X be a random variable which is independent of the σ-algebra $\mathcal{F}$.
(i) Show that $\mathbb{E}[X1_A] = \mathbb{E}[X]P(A)$, for any A in $\mathcal{F}$.
(ii) Prove that $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

13. If Y is $\mathcal{F}$-measurable, show that $\mathbb{E}[XY|\mathcal{F}] = Y\mathbb{E}[X|\mathcal{F}]$.

14. If $\mathcal{F} \subset \mathcal{G}$ prove that:
(i) $$\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{F}].$$
(ii) $$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}].$$

15. Ann and Bob got married at time 0. It is known that they have not divorced until time t. The divorce time is modeled by a positive random variable X with distribution measure μ. Find the expected remainning time they will still be married in the following two cases:
(i) The distribution is exponential with parameter c.
(ii) The distribution is normal with mean m and variance σ^2.

16. Consider X to be a nonegative random variable. Let μ be absolutely continuous and $\mathbb{E}[X|X > t] = t + a$ for some constant $a > 0$ and all $t \geq 0$. Prove that μ is the exponential distribution with parameter $c = 1/a$.

17. Let $X \leq Y$. Show that $\mathbb{E}[X|\mathcal{F}] \leq \mathbb{E}[Y|\mathcal{F}]$.

18. Let c be a constant. Show that $\mathbb{E}[c|\mathcal{F}] = c$.

19. (i) If $X : \Omega \to \mathbb{N}$ is a discrete random variable, show that for any measurable $A \in \mathcal{H}$ we have
$$P(A) = \sum_{n \in \mathbb{N}} P(A|X = n)P(X = n).$$
(ii) What does the relation (i) become in the case when $X : \Omega \to \mathbb{R}$ is a real-valued random variable?

20. (i) If $X : \Omega \to \mathbb{N}$ is a discrete random variable, show that for any integrable random variable Y we have

$$\mathbb{E}(Y) = \sum_{n \in \mathbb{N}} \mathbb{E}(Y|X = n)P(X = n).$$

(ii) Let X be a Poisson distributed random variable with parameter λ. Consider the random variable $Y = \rho^X$, with $\rho > 1$ constant. Find the expectation $\mathbb{E}(Y)$.

21. Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras. Define the commutator of $\mathcal{F}$ and $\mathcal{G}$ acting of a random variable Z as

$$[\mathcal{F}, \mathcal{G}](Z) = \mathbb{E}_{\mathcal{F}} \mathbb{E}_{\mathcal{G}} Z - \mathbb{E}_{\mathcal{G}} \mathbb{E}_{\mathcal{F}} Z.$$

(i) What is $[\mathcal{F}, \mathcal{F}]$?

(ii) If $\mathcal{F} \subset \mathcal{G}$, find $[\mathcal{F}, \mathcal{G}](Z)$;

(iii) What is $[\mathcal{F} \cap \mathcal{G}, \mathcal{G}](Z)$?

(iv) Assume that $\mathcal{F}$ and $\mathcal{G}$ are independent σ-algebras. What is $[\mathcal{F} \cap \mathcal{G}, \mathcal{G}](Z)$ in this case?

22. Prove or disprove: For any random variables X, Y and Z we have

$$\mathbb{E}[Z|X, Y] = \mathbb{E}[\mathbb{E}[Z|X]|Y].$$

Note that the above relation can be written more clearly as $\mathbb{E}_{X,Y} Z = \mathbb{E}_Y \mathbb{E}_X Z$.