ORFE 527: Stochastic Calculus
Homework 1

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Exercise 1: (Quadratic variations)

a) Let M be a continuous martingale and $X \in \mathcal{L}$ with respect to M. Show

$$\left\langle \int_0^t X_s dM_s \right\rangle = \int_0^t X_s^2 d\langle M\rangle_s,\ t \geq 0$$

b) Let $M^{(1)}$, $M^{(2)}$ be continuous martingales with respect to the same filtration, $X^{(1)} \in \mathcal{L}$ with respect to $M^{(1)}$, and $X^{(2)} \in \mathcal{L}$ with respect to $M^{(2)}$. Show that, if $\langle M^{(1)}, M^{(2)}\rangle_t = 0$, $t \geq 0$, then

$$\left\langle \int_0^t X_s^{(1)} dM_s^{(1)}, \int_0^t X_s^{(2)} dM_s^{(2)} \right\rangle = 0,\ t \geq 0$$

c) For an m-dimensional standard Brownian motion $(B^{(1)}, B^{(2)}, \dots, B^{(m)})$ and $R = \sum_{i=1}^m (B^{(i)})^2$ check that

$$\left\langle \sum_{i=1}^m \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)} \right\rangle_t = t,\ t \geq 0$$

Answer:

a) We will first prove Ito's isometry for general $X \in \mathcal{L}$ by extending the result for simple process done in class. That is, we wish to show

$$\mathbb{E}[(I(X)_t - I(X)_s)^2 | \mathcal{F}_s] = \mathbb{E}[\int_s^t X_u^2 d\langle M\rangle_u | \mathcal{F}_s]$$

Recall that $I(X)_t = \int_0^t X_u dM_u$. By definition of conditional expectation, proving the above is the same as proving

$$\mathbb{E}[1_A (I(X)_t - I(X)_s)^2] - \mathbb{E}[1_A \int_s^t X_u^2 d\langle M\rangle_u] = 0$$

For all $A \in \mathcal{F}_s$. Now, using the fact that $\mathcal{L}_0$ is dense in $\mathcal{L}$, pick a sequence of $X^{(n)}$ that converges to X. Then we add and subtract

$(I(X^{(n)})_t - I(X)_s^{(n)})^2$ and $\int_s^t X_u^{(n)} d\langle M\rangle_u$ to the expression above. This yields

$$(1)\mathbb{E}[1_A(I(X)_t - I(X)_s)^2] - \mathbb{E}[1_A(I(X^{(n)})_t - I(X)_s^{(n)})^2]$$
$$(2) + \mathbb{E}[1_A\left((I(X^{(n)})_t - I(X)_s^{(n)})^2 - \int_s^t X_u^{(n)^2} d\langle M\rangle_u\right)]$$
$$(3) + \mathbb{E}[1_A\left(\int_s^t X_u^{(n)^2} d\langle M\rangle_u - \int_s^t X_u^2 d\langle M\rangle_u\right)]$$

Now, we will show that (1) = (2) = (3) = 0 to prove our claim. We have, by the definition of $I(X)_t$ that

$$\|I(X)_t - I(X^{(n)})_t\|_2^2 \to 0 \text{ as } n \to \infty$$

By triangle inequality, this also implies that

$$\|(I(X)_t - I(X)_s) - (I(X^{(n)})_t - I(X^{(n)})_s)\|_2^2 \to 0 \text{ as } n \to \infty$$

Since we have $\|\cdot\|_2^2 \geq \|1_A \cdot\|_2^2$, then this also implies

$$\|1_A(I(X)_t - I(X)_s) - 1_A(I(X^{(n)})_t - I(X^{(n)})_s)\|_2^2 \to 0 \text{ as } n \to \infty$$

Which is implies that

$$\|1_A(I(X)_t - I(X)_s)\|_2^2 \to \|1_A(I(X^{(n)})_t - I(X^{(n)})_s)\|_2^2 \text{ as } n \to \infty$$

Which implies that $(1) \to 0$ as $n \to \infty$.

Since $X^{(n)}$ is in $\mathcal{L}_0$. By applying Ito's isometry for simple processes, $(2) = 0$.

$(3) \to 0$ as $n \to \infty$ also results from the fact that $\|I(X)_t - I(X^{(n)})_t\|_2^2 \to 0$ as $n \to \infty$. (3) is simply this norm defined on a different measure space. Thus, the same argument for (1) holds for (3).

Hence, since we have shown all the summands to be 0, then we have proven Ito's isometry for general $X \in \mathcal{L}$.

We now show that $I(X)_t$ is a martingale. $I(X)_t$ is obviously $\mathcal{F}_t$ adapted since it is a limit of $\mathcal{F}_t$ adapted processes. Integrability follows from Ito's isometry. Since $X \in \mathcal{L}$, we have

$$\mathbb{E}[\int_0^t X_s^2 d\langle M\rangle_s] < \infty$$
$$\implies \mathbb{E}[I(X)_t^2] < \infty$$
$$\implies \mathbb{E}[|I(X)_t|] < \infty$$

To show the martingale property, we wish to show

$$\mathbb{E}[1_A I(X)_t] = \mathbb{E}[1_A I(X)_s] \ \forall A \in \mathcal{F}_s$$

Again, we approximate using a sequence of simple processes $X^{(n)}$

$$\mathbb{E}[1_A I(X^{(n)})_t] = \mathbb{E}[1_A I(X^{(n)})_s] \ \forall A \in \mathcal{F}_s$$

We proved the above in class when we showed $I(X)_t$ was a martingale for simple processes. Since we have convergence of $X^{(n)}$ in L^2, we also have it in L^1. Thus, we conclude that as $n \to \infty$ that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[1_A I(X^{(n)})_t] &= \lim_{n\to\infty} \mathbb{E}[1_A I(X^{(n)})_s] \ \forall A \in \mathcal{F}_s \\ \implies \mathbb{E}[1_A I(X)_t] &= \mathbb{E}[1_A I(X)_s] \ \forall A \in \mathcal{F}_s\end{aligned}$$

Thus, $I(X)_t$ has the martingale property. And so we have proven that $I(X)_t$ is a martingale.

We now use that fact that $I(X)_t$ is a martingale to show that if $I(X)_t^2 - \int_0^t X_u^2 d\langle M\rangle_u$ is a martingale, then $\int_0^t X_u^2 d\langle M\rangle_u$ is precisely the quadratic variation of $I(X)_t$. Integrability results from the argument from before and $\mathcal{F}_t$ adapted results from the definitions of the two terms. Hence, we will just show the martingale property.

$$\begin{aligned}&\mathbb{E}[I(X)_t^2 - \int_0^t X_u^2 d\langle M\rangle_u | \mathcal{F}_s] \\ &= \mathbb{E}[(I(X)_t - I(X)_s)^2 + 2(I(X)_t - I(X)_s)I(X)_s \\ &\quad + I(X)_s^2 - \int_s^t X_u^2 d\langle M\rangle_u - \int_0^s X_u^2 d\langle M\rangle_u | \mathcal{F}_s] \\ &= \mathbb{E}[(I(X)_t - I(X)_s)^2 - \int_s^t X_u^2 d\langle M\rangle_u | \mathcal{F}_s] \\ &\quad + 2I(X)_s \mathbb{E}[I(X)_t - I(X)_s | \mathcal{F}_s] \\ &\quad + I(X)_s^2 - \int_0^s X_u^2 d\langle M\rangle_u\end{aligned}$$

We have that $\mathbb{E}[(I(X)_t - I(X)_s)^2 - \int_s^t X_u^2 d\langle M\rangle_u | \mathcal{F}_s] = 0$ by Ito's isometry. Also, $\mathbb{E}[I(X)_t - I(X)_s | \mathcal{F}_s] = 0$ since $I(X)_t$ is a martingale. Thus we have

$$\mathbb{E}[I(X)_t^2 - \int_0^t X_u^2 d\langle M\rangle_u | \mathcal{F}_s] = I(X)_s^2 - \int_0^s X_u^2 d\langle M\rangle_u$$

This proves that $I(X)_t^2 - \int_0^t X_u^2 d\langle M\rangle_u$ is a martingale and we conclude that

$$\left\langle \int_0^t X_s dM_s \right\rangle = \int_0^t X_s^2 d\langle M\rangle_s$$

b) We first show the result for $X_t^{(1)}, X_t^{(2)} \in \mathcal{L}_0$. Let t_i be the times when either process changes and let $t_{m-1} < s < t_m$ and $t_n < t < t_{n+1}$.

$$\begin{aligned}
&\mathbb{E}[(X_t^{(1)} - X_s^{(1)})(X_t^{(2)} - X_s^{(2)})] \\
&= \mathbb{E}[\alpha_{m-1}^{(1)}\alpha_{m-1}^{(2)}(M_{t_m}^{(1)} - M_s^{(1)})(M_{t_m}^{(2)} - M_s^{(2)}) \\
&+ \sum_{i=m}^{n-1} \alpha_i^{(1)}\alpha_i^{(2)}(M_{t_{i+1}}^{(1)} - M_t^{(1)})(M_{t_{i+1}}^{(2)} - M_t^{(2)}) \\
&+ \alpha_n^{(1)}\alpha_n^{(2)}(M_t^{(1)} - M_{t_n}^{(1)})(M_t^{(2)} - M_{t_n}^{(2)})] \\
&= \mathbb{E}[\alpha_{m-1}^{(1)}\alpha_{m-1}^{(2)}(\langle M^{(1)}, M^{(2)}\rangle_{t_m} - \langle M^{(1)}, M^{(2)}\rangle_s) \\
&+ \sum_{i=m}^{n-1} \alpha_i^{(1)}\alpha_i^{(2)}(\langle M^{(1)}, M^{(2)}\rangle_{t_{i+1}} - \langle M^{(1)}, M^{(2)}\rangle_t) \\
&+ \alpha_n^{(1)}\alpha_n^{(2)}(\langle M^{(1)}, M^{(2)}\rangle_t - \langle M^{(1)}, M^{(2)}\rangle_{t_n})] \\
&= \mathbb{E}[\int_s^t X_u^{(1)} X_u^{(2)} d\langle M^{(1)}, M^{(2)}\rangle_u]
\end{aligned}$$

Thus, if $\langle M^{(1)}, M^{(2)}\rangle_u = 0$, then the measure $d\langle M^{(1)}, M^{(2)}\rangle_u$ is also 0 and so the integral is 0.

Now we want to extend to $X_t^{(1)}, X_t^{(2)} \in \mathcal{L}$. First, we pick the convergent $\mathcal{L}_0$ sequence $X_t^{(1)^{(n)}}$. Then, $\lim_{n\to\infty} \int_0^t (X_u^{(1)^{(n)}} - X_u^{(1)})^2 d\langle M^{(1)}\rangle_u = 0$. Since co-variation is bilinear (partial proof in part c), we have,

$$|\langle I(X^{(1)^{(n)}}) - I(X^{(1)}), X^{(2)}\rangle_t|^2 \leq \langle I(X^{(1)^{(n)}}) - I(X^{(1)})\rangle_t \langle X^{(2)}\rangle_t$$

Using part a), we have

$$= \int_0^t (X_u^{(1)^{(n)}} - X_u^{(1)})^2 d\langle X^{(1)}\rangle_u \langle X^{(2)}\rangle_t$$

Where the first multiplicand goes to 0 as $n \to \infty$. Thus, we have the following conclusion

$$\lim_{n\to\infty} \langle I(X^{(1)^{(n)}}), X^{(2)} \rangle_t = \langle I(X^{(1)}), X^{(2)} \rangle_t$$

Now, we are going to show that
$\langle I(X^{(1)}), X^{(2)} \rangle_t = \int_0^t X^{(1)} d\rangle M^{(1)}, M^{(2)} \rangle_u$. Again, pick our sequence $X^{(1)^{(n)}}$ that converges to $X^{(1)}$. From this sequence, pick the almost sure subsequence. Then we have already shown this for simple processes

$$\langle I(\tilde{X}^{(1)^{(n)}}), X^{(2)} \rangle_t = \int_0^t \tilde{X}_u^{(1)^{(n)}} d\langle M^{(1)}, M^{(2)} \rangle_u$$

Since the convergence is almost sure, we have

$$\lim_{n\to\infty} \langle I(\tilde{X}^{(1)^{(n)}}), X^{(2)} \rangle_t = \lim_{n\to\infty} \int_0^t \tilde{X}_u^{(1)^{(n)}} d\langle M^{(1)}, M^{(2)} \rangle_u$$
$$\langle I(X^{(1)}), X^{(2)} \rangle_t = \int_0^t X^{(1)} d\langle M^{(1)}, M^{(2)} \rangle_u$$

In differential form, we have shown

$$d\langle I(X^{(1)}), X^{(2)} \rangle_t = X^{(1)} d\langle M^{(1)}, M^{(2)} \rangle_u$$

By symmetry, we also have

$$d\langle X^{(1)}, I(X^{(2)}) \rangle_t = X^{(2)} d\langle M^{(1)}, M^{(2)} \rangle_u$$

Thus,

$$\begin{aligned} &\int_0^t X_u^{(1)} X_u^{(2)} d\langle M^{(1)}, M^{(2)} \rangle_u \\ &= \int_0^t X_u^{(1)} d\langle M^{(1)}, I(X^{(2)}) \rangle_u \\ &= \int_0^t d\langle I(X^{(1)}), I(X^{(2)}) \rangle_u \\ &= \langle I(X^{(1)}), I(X^{(2)}) \rangle_t \end{aligned}$$

Thus, we have shown Ito's isometry for co-variation. Now, if
$d\langle M^{(1)}, M^{(2)} \rangle_t = 0$, then the measure is 0 and hence so is the integral. Thus, $\langle I(X^{(1)}), I(X^{(2)}) \rangle_t = 0$.

c) We show that $\langle X+Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$. First, we have that for natural and bounded variation that $\langle X, Y\rangle$ is the bounded variation equivalent to $XY - \langle X, Y\rangle$ being a martingale. We have that $XZ - \langle X, Z\rangle$ and $YZ - \langle Y, Z\rangle$ are martingales by definition. Thus adding them together is also a martingale,

$$XZ + YZ - (\langle X, Z\rangle + \langle Y, Z\rangle)$$
$$(X+Y)Z - (\langle X, Z\rangle + \langle Y, Z\rangle)$$

Thus, we conclude that since this is a martingale, that $\langle X+Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$. Now, we apply this to our problem by letting $X = \sum_{i=1}^m \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)}$ and computing $\langle X, X\rangle$. By additivity of the bilinear covariation,

$$\begin{aligned}\langle X, X\rangle = &\sum_{i=1}^m \left\langle \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)}, \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)} \right\rangle \\ &+ \sum_{i,j,i\neq j}^m \left\langle \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)}, \int_0^t \frac{B_s^{(j)}}{\sqrt{R_s}} dB_s^{(j)} \right\rangle\end{aligned}$$

However, this is a standard Brownian motion vector and so the covariance matrix is some multiple of the identity matrix. That is, the covariances are 0. For a joint Gaussian vector, this means the components are independent. Thus, $\langle B_s^{(i)}, B_s^{(j)}\rangle = 0$. Thus, by part b), the cross terms are all 0. By part a), the diagonal terms become

$$\begin{aligned}\langle X, X\rangle &= \sum_{i=1}^m \left\langle \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)}, \int_0^t \frac{B_s^{(i)}}{\sqrt{R_s}} dB_s^{(i)} \right\rangle \\ &= \sum_{i=1}^m \int_0^t \frac{B_s^{(i)^2}}{R_s} d\langle B^{(i)}\rangle_s = \sum_{i=1}^m \int_0^t \frac{B_s^{(i)^2}}{R_s} ds \\ &= \sum_{i=1}^m \frac{B_s^{(i)^2}}{R_s} \int_0^t ds = \frac{R_s}{R_s} t = t\end{aligned}$$

Which is the result desired.

Exercise 2: (Orstein-Uhlenbeck process) Let $a \neq 0,\ b, \sigma \in \mathbb{R}$. Check that the process

$$X_t := xe^{at} - \frac{b}{a}(1 - e^{at}) + \sigma \int_0^t e^{a(t-s)} dB_s,\ t \geq 0$$

where B is a standard Brownian motion, solves the SDE

$$dX_t = (aX_t + b)dt + \sigma dB_t,\ X_0 = x$$

X is called the *Ornstein-Uhlenbeck (OU) process* with parameters a, b, σ. Find the distribution of X_t for a fixed $t \geq 0$. Do the random variables X_t converge in the limit $t \to \infty$ in distribution and, if yes, to what limit?

Answer: First, it is trivial to check $X_0 = x$. We apply Ito's formula to X_t, with $f(X_t) = X_t$,
$M_t = \sigma \int_0^t e^{a(t-s)} dB_s$, and $\Gamma_t = xe^{at} - \frac{b}{a}(1 - e^{at})$.

$$dX_t = f'(X_t)(dM_t + d\Gamma_t) + \frac{1}{2} f''(X_t) d\langle M \rangle_t$$

We have $f'(X_t) = 1,\ f''(X_t) = 0$. Which simplifies nicely to

$$dX_t = dM_t + d\Gamma_t$$

Now, we compute these derivative. First, $d\Gamma_t$ is deterministic,

$$d\Gamma_t = (xae^{at} + be^{at})dt$$

Now, dM_t can be calculated by noting that $M_t = \sigma e^{at} \int_0^t e^{-as} dB_s$, and applying the multivariate version of Ito's formula to $f(t, Y_t) = e^{at} Y_t$, where $Y_t = \sigma \int_0^t e^{-as} dB_s$ to get

$$dM_t = (ae^{at} \sigma \int_0^t e^{-as} dB_s)dt + e^{at} \sigma e^{-at} dB_t$$

$$dM_t = (a\sigma \int_0^t e^{a(t-s)} dB_s)dt + \sigma dB_t$$

Thus, we get

$$dX_t = (xae^{at} + be^{at} + a\sigma \int_0^t e^{a(t-s)} dB_s)dt + \sigma dB_t$$

which is

$$dX_t = (aX_t + b)dt + \sigma dB_t$$

Now, we show that X_t is Gaussian. First, let us assume $f(t)$ is a simple function. Then we have

$$\int_0^\infty f(s)dB_s = \sum_{k=1}^{n} \alpha_i(B_{t_k} - B_{t_{k-1}})$$

Since increments of Brownian motion is distributed Gaussian, this is a sum of Gaussians and hence is itself a Gaussian. Now, we define f to be the limit of a series of simple functions f_n. Then we have

$$\lim_{n\to\infty} \sqrt{\int_0^\infty (f(t) - f_n(t))^2 dt} = 0$$

Now, we use this fact to show a Cauchy sequence of $\int_0^\infty f_k(t)dB_t$ converges to 0.

$$\begin{aligned}
&\|\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\| \\
&= \sqrt{\mathbb{E}\left[\left(\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right)^2\right]} \\
&= \sqrt{\mathbb{E}\left[\left(\int_0^\infty f_k(t) - f_n(t)dB_t\right)^2\right]} \\
&= \sqrt{\mathbb{E}\left[\int_0^\infty \left(f_k(t) - f_n(t)\right)^2 dt\right]} = \|f_k(t) - f_n(t)\|
\end{aligned}$$

Note that the first norm is the one defined for processes and we used Ito's isometry to change to the L^2 norm. Now we use triangle inequality

$$= \|f_k(t) - f_n(t)\| \leq \|f_k(t) - f(t)\| + \|f(t) - f_n(t)\|$$

But, we have that the right side converges to 0 as $k, n \to \infty$. Since the space of simple functions is dense, we conclude that

$$\int_0^\infty f(t)dB_t = \lim_{n\to\infty} \int_0^\infty f_n(t)dB_t$$

Hence, we have that X_t is a Gaussian added to a deterministic function which is itself Gaussian. Thus, we calculate the mean and variance of X_t to get the parameters.

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\left[xe^{at} - \frac{b}{a}(1-e^{at}) + \sigma\int_0^t e^{a(t-s)}dB_s\right] \\ &= xe^{at} - \frac{b}{a}(1-e^{at}) + \sigma\mathbb{E}\left[\int_0^t e^{a(t-s)}dB_s\right]\end{aligned}$$

Using a similar argument as before, the mean of this integral is 0 since Brownian motion is distributed with mean 0. Thus,

$$\mathbb{E}[X_t] = xe^{at} - \frac{b}{a}(1-e^{at})$$

Now, for the variance, we again use Ito's isometry.

$$\begin{aligned}Var(X_t) &= \mathbb{E}\left[(\sigma\int_0^t e^{a(t-s)}dB_s)^2\right] \\ &= \sigma^2\mathbb{E}\left[\int_0^t e^{2a(t-s)}ds\right] \\ &= \frac{\sigma^2}{2a}(e^{2at}-1)\end{aligned}$$

Thus, we have $X_t \sim \mathcal{N}(xe^{at} - \frac{b}{a}(1-e^{at}), \frac{\sigma^2}{2a}(e^{2at}-1))$. In the limit, we consider two cases. If $a > 0$, then X_t converges to $\mathcal{N}(\infty, \infty)$ which is certainly not a valid distribution. Thus, no convergence when $a > 0$. This can be explained by the fact that our process will drift exponentially larger according to our SDE solution. However, if $a < 0$, then the drift term gets smaller and so X_t converges to $\mathcal{N}(-\frac{b}{a}, -\frac{\sigma^2}{2a})$

Exercise 3: (Kimura diffusion) For any $\sigma \in \mathbb{R}$ find a solution of the SDE

$$dX_t = \sigma X_t(1 - X_t)dB_t - \sigma^2 X_t^2(1 - X_t)dt, \; X_0 = x \in (0, 1)$$

where B is a standard Brownian motion. X is called a *Kimura diffusion* and has been introduced in mathematical biology as a simple model for the relative frequency of a gene in a population.

Hint: start by formally applying Ito's formula to $Y_t := \log \frac{X_t}{1-X_t}, t \geq 0$.

Answer: We apply Ito's formula to $f(X_t) = Y_t$, $\Gamma_t = -\sigma^2 \int_0^t X_s^2(1 - X_s)ds$, and $M_t = \sigma \int_0^t X_s(1 - X_s)dB_s$.

$$dY_t = \left(\frac{1}{X_t} + \frac{1}{1 - X_t}\right)(dM_t + d\Gamma_t) + \frac{1}{2}\left(\frac{1}{(1 - X_t)^2} - \frac{1}{X_t^2}\right) d\langle M \rangle_t$$

Now, we compute dM_t, $d\Gamma_t$, and $d\langle M \rangle_t$.

$$\begin{aligned} dM_t &= \sigma X_t(1 - X_t)dB_t \\ d\Gamma_t &= -\sigma^2 X_t^2(1 - X_t)dt \\ d\langle M \rangle_t &= \sigma^2 X_t^2(1 - X_t)^2 dt \end{aligned}$$

Subbing in these expression,

$$\begin{aligned} dY_t = &\left(\frac{1}{X_t} + \frac{1}{1 - X_t}\right)(\sigma X_t(1 - X_t)dB_t - \sigma^2 X_t^2(1 - X_t)dt) \\ &+\frac{1}{2}\left(\frac{1}{(1 - X_t)^2} - \frac{1}{X_t^2}\right)(\sigma^2 X_t^2(1 - X_t)^2 dt) \\ &= \sigma dB_t - \sigma^2 X_t dt + \frac{\sigma^2}{2}(X_t^2 - (1 - X_t)^2)dt \\ &= \sigma dB_t - \sigma^2 X_t dt + \frac{\sigma^2}{2}(2X_t - 1)dt \\ &= \sigma dB_t - \frac{\sigma^2}{2}dt \end{aligned}$$

Thus, we have that Y_t is a Brownian motion with drift $-\frac{\sigma^2}{2}$ and diffusion σ. Which means that $Y_t \sim \mathcal{N}(-\frac{\sigma^2}{2}t, \sigma^2 t)$. Now, we can write X_t in the following form

$$X_t = \frac{1}{1 + e^{-Y_t}}$$

And so X_t is a process that moves around the logistic function according to the Brownian motion Y_t. That is, for fixed t, a logit performed on a Gaussian RV. Its distribution is given by the logit-normal distribution. In this case, the pdf is

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x(1-x)} e^{-\frac{(\log(\frac{x}{1-x})+\frac{\sigma^2}{2})^2}{2\sigma^2 t}}$$

From here, it is easy to see that the negative causes X_t to converge to 0 as $t \to 0$.