Question 1

We have that the value of a digitial call ($\phi = 1$) or put ($\phi = -1$) is

$$V(t) = e^{-r(T-t)}\mathcal{N}(\phi d_2)$$

with $d_2 = \frac{\ln F/K}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$. Thus, taking the partial derivative with respect to σ yields

$$\frac{\partial V}{\partial \sigma}(t) = e^{-r(T-t)}\mathcal{N}'(\phi d_2) \cdot \phi \left(-\frac{\ln F/K}{\sigma^2\sqrt{T-t}} - \frac{1}{2}\sqrt{T-t} \right)$$

Simplifying and using the fact that the normal distribution is symmetric, we get

$$\frac{\partial V}{\partial \sigma}(t) = -\phi \cdot e^{-r(T-t)}\mathcal{N}'(d_2) \left(\frac{\ln F/K}{\sigma^2\sqrt{T-t}} + \frac{1}{2}\sqrt{T-t} \right)$$

Now, notice that all the terms after the $-\phi$ are all positive. Also, if we are short then we must add a negative sign. Thus, the Vega for a short call or put is

$$\frac{\partial V}{\partial \sigma}(t) = \phi \cdot e^{-r(T-t)}\mathcal{N}'(d_2) \left(\frac{\ln F/K}{\sigma^2\sqrt{T-t}} + \frac{1}{2}\sqrt{T-t} \right)$$

Thus, the portfolio Vega is positive for a short call and negative for a short put.