ORFE 526: Probability Theory
Homework 3

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Exercise 1: Let $X, Y \sim Exp(1)$ be two independent random variables, standard exponential distributed. Find the distribution $R = \frac{Y}{X}$.

Answer: We write the cumulative distribution for R:

$$P(R < r) = P(\frac{Y}{X} < r) = P(Y < rX)$$

Which we condition on X,

$$= \int_0^\infty P(Y < rX | X = x) dP(x)$$

But we note that $P(Y < rX | X = x) = \int_0^{rx} e^{-y} dy$ and $dP(x) = e^{-x} dx$

$$= \int_0^\infty e^{-x} \int_0^{rx} e^{-y} dy dx = \int_0^\infty e^{-x}(1 - e^{-rx}) dx$$
$$= \int_0^\infty e^{-x} - e^{-(1+r)x} dx = 1 - \frac{1}{1+r}$$

Now differentiating to get the distribution,

$$f_R(r) = \frac{d}{dr} 1 - \frac{1}{1+r} = \frac{1}{(1+r)^2}$$

Thus, the distribution for R is $\frac{1}{(1+r)^2}$. For $r \in [0, \infty)$.

Exercise 2: Let $X : \Omega \to E$ and $Y : \Omega \to F$ be two random variables, where $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Assume there is a measurable function $f : E \to F$ such that $Y = f(X)$. Prove that $\sigma Y \subset \sigma X$

Answer: Let $A \in \sigma Y$,

$$
\begin{aligned}
A \in \sigma Y &\implies A \in \{B \subset \Omega : Y(B) \in \mathcal{F}\} \\
&\implies A \in \{B \subset \Omega : f(X(B)) \in \mathcal{F}\} \\
&\implies A \in \{B \subset \Omega : X(B) \in f^{-1}(\mathcal{F})\} \\
&\implies A \in \{B \subset \Omega : X(B) \in \mathcal{E}\} \\
&\implies A \in \sigma X
\end{aligned}
$$

Thus, $\sigma Y \subset \sigma X$

Exercise 3: Consider the inertia momentum about axis $\{x = a\}$ defined by

$$f(a) = \int_\Omega (X(\omega) - a)^2 d\mathbb{P}(\omega)$$

i) Show that f is minimized by $a = \mathbb{E}[X]$

ii) What is the value of its minimum?

Answer: We expand the integral using its linearity,

$$\begin{aligned} f(a) = \int_\Omega (X(\omega) - a)^2 d\mathbb{P}(\omega) &= \int_\Omega (X(\omega))^2 - 2aX(\omega) + a^2 d\mathbb{P}(\omega) \\ &= \int_\Omega (X(\omega))^2 \, d\mathbb{P}(\omega) - 2a \int_\Omega X(\omega) d\mathbb{P}(\omega) + a^2 \int_\Omega d\mathbb{P}(\omega) \end{aligned}$$

By definition,

$$\int_\Omega X(\omega) d\mathbb{P}(\omega) = \mathbb{E}[X] \text{ and } \int_\Omega d\mathbb{P}(\omega) = 1$$

So,

$$f(a) = \int_\Omega (X(\omega))^2 \, d\mathbb{P}(\omega) - 2a\mathbb{E}[X] + a^2$$

Now we minimize over a and we can drop the first time since it has no dependence on a,

$$\min_a f(a) = \min_a -2a\mathbb{E}[X] + a^2$$

This is a convex function and so we can take the derivative and set it to 0 to find the min,

$$\begin{aligned} \frac{d}{da} - 2a\mathbb{E}[X] + a^2 &= -2\mathbb{E}[X] + 2a = 0 \\ &\implies a = \mathbb{E}[X] \end{aligned}$$

Thus, $a = \mathbb{E}[X]$ is the minimizer and $f(\mathbb{E}[X]) = \int_\Omega (X(\omega) - \mathbb{E}[X])^2 d\mathbb{P}(\omega)$ is the precise definition of $Var(X)$

Exercise 4: Let μ be a probability measure. Show that its Fourier transform, $\hat{\mu}$, is a bounded, continuous function satisfying $\hat{\mu}(0) = 1$.

Answer: First, we show that $\hat{\mu}$ is bounded.

$$|\hat{\mu}(r)| = |E[e^{irX}]|$$

By Jensen's inequality since the norm is convex,

$$|E[e^{irX}]| \leq E[|e^{irX}|] = E[1] = 1$$

Since $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$. Thus, we have boundedness. To show continuity, consider the sequence $(r_n)_{n=1}^{\infty}$ with the limit $\lim_{n\to\infty} r_n = r^*$ and consider the sequence of functions $f_n = \hat{\mu}(r_n)$. Note that since it is bounded, let $g(x) = 1$,

$$|f_n| \leq 1 = g(x)$$

and we have that g is integrable since $\int_{\mathbb{R}} g(x) d\mu(x) = \int_{\mathbb{R}} d\mu(x) = 1 < \infty$. Thus, the conditions of dominated convergence theorem are met. So,

$$\lim_{n\to\infty} \hat{\mu}(r_n) = \lim_{n\to\infty} f_n(r) = \lim_{n\to\infty} E[e^{ir_n X}]$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}} e^{ir_n x} d\mu(x) = \int_{\mathbb{R}} \lim_{n\to\infty} e^{ir_n x} d\mu(x) = \int_{\mathbb{R}} e^{ir^* x} d\mu(x) = \hat{\mu}(r^*)$$

Where the limit and the integral can be swapped due to dominated convergence theorem and using the fact that $e^{\alpha r}$ is continuous in r. Thus $\hat{\mu}(r)$ is continuous since for any convergent sequence $(r_n)_{n=1}^{\infty}$, we have $\lim_{n\to\infty} \hat{\mu}(r_n) = \hat{\mu}(r^*)$. Finally, $\hat{\mu}(0) = E[e^0] = E[1] = 1$.

Exercise 5: Show that if $\displaystyle\int_H X d\mathbb{P} \geq 0$ for any $H \in \mathcal{H}$, then $X \geq 0$ a.s.

Answer: Define the sequence of measurable sets $H_n = X^{-1}((-\infty, -\frac{1}{n}))$. Then $H_n \nearrow H = X^{-1}((-\infty, 0))$. Now,

$$\int_{H_n} X d\mathbb{P} = \int_{\mathcal{H}} X \cdot 1_{H_n} d\mathbb{P} \leq \int_{\mathcal{H}} -\frac{1}{n} \cdot 1_{H_n} d\mathbb{P} = -\frac{1}{n} P(H_n)$$

But we have $\int_H X d\mathbb{P} \geq 0$ by assumption,

$$\begin{aligned} 0 \leq \int_{H_n} X d\mathbb{P} &\leq -\frac{1}{n} P(H_n) \\ \implies P(H_n) &= 0 \end{aligned}$$

Thus $P(H_n) = 0$ for all n. So,

$$\lim_{n\to\infty} P(H_n) = P(\lim_{n\to\infty} H_n) = P(H) = \lim_{n\to\infty} 0 = 0$$

The limit can be taken inside the probability by monotone convergence theorem. Hence, $P(X < 0) = 0$ or $P(X \geq 0) = 1$ a.s.

Exercise 6: Let $X \geq 0$ and $\mathbb{E}[X] < \infty$, with X random variable. Show that $X < \infty$ a.s.

Answer: Let $A = X^{-1}(\{\infty\})$. Then,

$$\mathbb{E}[X] = \int X d\mathbb{P} = \int_A X d\mathbb{P} + \int_{A^c} X d\mathbb{P} < \infty$$

However,

$$\int_A X d\mathbb{P} = \infty \cdot P(A) < \infty$$
$$\implies P(A) = 0$$

Thus, $P(A^c) = P(X < \infty) = 1$.

Exercise 7: Let $(X_n)_n$ be a sequence of bounded random variables, with $|X_n| < M < \infty$, for all $n \geq 1$.

i) Show that $\mathbb{E}[|X_n|] \leq M$, for all $n \geq 1$.

ii) Use Markov's inequality to show

$$\mathbb{E}[|X_n| \cdot 1_{\{|X_n|>b\}}] < \frac{M^2}{b}, \quad \forall b > 0$$

iii) Prove that the family $\{X_n\}_n$ is uniformly integrable.

Answer:

i) By definition of the expectation,

$$\mathbb{E}[|X_n|] = \int |X_n| d\mathbb{P} < \int M d\mathbb{P} = M \int d\mathbb{P} = M$$

So, $\mathbb{E}[|X_n|] < M$ for all $n \geq 1$.

ii) We have the following,

$$\mathbb{E}[|X_n| \cdot 1_{\{|X_n|>b\}}] = \int |X_n| \cdot 1_{\{|X_n|>b\}} d\mathbb{P} < M \int 1_{\{|X_n|>b\}} = M \cdot P(|X_n| > b)$$

By Markov's inequality,

$$P(|X_n| > b) \leq \frac{\mathbb{E}[|X_n|]}{b} < \frac{M}{b}$$

Thus,

$$\mathbb{E}[|X_n| \cdot 1_{\{|X_n|>b\}}] < M \cdot P(|X_n| > b) < \frac{M^2}{b} \quad \forall b > 0$$

iii) We have by the previous part $\mathbb{E}[|X_n| \cdot 1_{\{|X_n|>b\}}] < \frac{M^2}{b}$. So,

$$\sup_{X \in \{X_n\}} \mathbb{E}[|X| \cdot 1_{\{|X|>b\}}] \leq \frac{M^2}{b}$$

Taking the limit on both sides,

$$\lim_{b\to\infty} \sup_{X\in\{X_n\}} \mathbb{E}[|X| \cdot 1_{\{|X|>b\}}] \leq \lim_{b\to\infty} \frac{M^2}{b} = 0$$

Thus,

$$\lim_{b\to\infty} \sup_{X\in\{X_n\}} \mathbb{E}[|X| \cdot 1_{\{|X|>b\}}] = 0$$

And hence $\{X_n\}_n$ is uniformly integrable.