ORF 524: Statistical Theory and Methods
Homework 1

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Due: Sep. 30^{th}, 5 p.m.

Exercise 1 (10 points). *Recall the definition of σ-Algebra. Let $(\Omega, \overline{\Sigma})$ be a measurable space, that is, $\overline{\Sigma}$ satisfies the following three properties:*

- $\overline{\Sigma} \neq \emptyset$, $\overline{\Sigma} \subseteq 2^{\Omega}$.
- $A \in \overline{\Sigma}$ *implies that* $A^c \in \overline{\Sigma}$. *Here we use A^c to denote the complement of A.*
- *For any $A_1, A_2, \cdots \in \overline{\Sigma}$, we have $\cap_{i \geq 1} A_i \in \overline{\Sigma}$.*

Based on these properties, solve the following problems.

(1). Show that $\overline{\Sigma}$ is closed under union.

(2). Show that $\overline{\Sigma}$ must contain $\emptyset$ and Ω.

(3). Suppose $A \subseteq \Omega$, what is the smallest σ-algebra containing A?

(4). Show that the set of all rational numbers, denoted by $\mathbb{Q}$, is Borel measurable. That is, $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$.

Exercise 2 (10 points). *Let P be a probability measure on $(\Omega, \overline{\Sigma})$. Only utilizing the definition of probability measure given in the class, solve the following problems.*

(1). Show that for any $A, B \in \overline{\Sigma}$ satisfying $A \subseteq B$, we have $0 \leq P(A) \leq P(B)$.

(2). Show that for any positive integer k, we have

$$P\Big(\bigcup_{i=1}^{k} A_i\Big) \leq \sum_{i=1}^{k} P(A_i). \tag{1}$$

(3). Does inequality (1) still hold when $k = \infty$?

Exercise 3 (10 points). *For any measurable function f, show that*

$$\left|\int f \mathrm{d}P\right| < \infty \ \text{ if and only if } \ \int |f| \mathrm{d}P < \infty.$$

Exercise 4 (10 points). *This exercise, consists of two questions, concerns the σ-finiteness of a measure.*

(1). Show that the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is σ-finite.

(2). Show that the counting measure on $(\Omega, 2^{\Omega})$ is σ-finite if and only if Ω is countable.

Exercise 5 (10 points)**.** *Let $X\colon \Omega \to \mathbb{R}$ be a discrete random variable on probability space $(\Omega, \bar{\Sigma}, P)$ and denote the corresponding induced measure by P_X. We define the support of P_X as*

$$\Omega_X = \{x \in \mathbb{R}\colon P(X = x) > 0\}.$$

Please answer the following two questions.

(1). First assume that $|\Omega_X| < \infty$, that is, Ω_X contains finite number of elements. Show that the probability mass function (pmf)of X, denoted by f, is indeed the density of P_X with respect to the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(2). Show the same thing when $|\Omega_X| = \infty$.