ORFE 527: Stochastic Calculus
Homework 3

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Exercise 1: (Hitting time of drifted Brownian motion) Let $c, \mu \in \mathbb{R}$ and consider the hitting time level c by a Brownian motion with drift μ:

$$\tau_c := \inf\{t \geq 0 : B_t + \mu t = c\}$$

Show that the Laplace transform of τ_c is given by

$$\mathbb{E}[e^{-u\tau_c}] = e^{\mu c - |c|\sqrt{\mu^2+2u}}, u \geq 0$$

Answer: We have a Brownian motion with drift. We wish to apply Girsanov's theorem to work in a space that does not have the drift. Suppose we have a standard Brownian motion B_t. Define our exponential as

$$Z_t(\mu) = e^{-\mu B_t - \frac{1}{2}\mu^2 t}$$

From Homework 2, question 2, this is a martingale. By Girsanov's

$$W_t = B_t + \mu t$$

Where W_t is standard Brownian under $\tilde{\mathbb{P}}$. Note that this is precisely our Brownian motion with drift. Suppose $c > 0$. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[e^{-u\tau_c}] &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-u\tau_c}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}] = \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-u\tau_c}e^{\mu B_{\tau_c} + \frac{1}{2}\mu^2\tau_c}] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-u\tau_c}e^{\mu(c-\mu\tau_c)+\frac{1}{2}\mu^2\tau_c}] = e^{\mu c}\mathbb{E}_{\tilde{\mathbb{P}}}[e^{-\frac{1}{2}(\mu^2+2u)\tau_c}]\end{aligned}$$

We also have that

$$\tilde{\mathbb{P}}(\tau_c < \infty) = 1$$

Because X_t is a standard Brownian motion under $\tilde{\mathbb{P}}$. So, we have an almost surely bounded stopping time so we can apply Optional Sampling Theorem to the martingale $e^{aB_t - \frac{1}{2}a^2 t}$ at times 0 and τ_c,

$$\begin{aligned}1 &= e^{ac}\mathbb{E}_{\tilde{\mathbb{P}}}[e^{-\frac{1}{2}a^2\tau_c}] \\ e^{-ac} &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-\frac{1}{2}a^2\tau_c}]\end{aligned}$$

Now, let $a^2 = (\mu^2 + 2u)$, so we have

$$e^{-c\sqrt{\mu^2+2u}} = \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-\frac{1}{2}(\mu^2+2u)\tau_c}]$$

Putting this into our expression from before,

$$\mathbb{E}_{\mathbb{P}}[e^{-u\tau_c}] = e^{\mu c}e^{-c\sqrt{\mu^2+2u}} = e^{\mu c - c\sqrt{\mu^2+2u}}$$

By symmetry, we must have $\mathbb{E}_{\mathbb{P}}[e^{-u\tau_{-c}}] = \mathbb{E}_{\mathbb{P}}[e^{-u\tau_c}]$ where

$$\tau_{-c} := \inf\{t \geq 0 : -B_t - \mu t = -c\}$$

Notice that we must negate both μ and c, thus we conclude that

$$\mathbb{E}_{\mathbb{P}}[e^{-u\tau_c}] = e^{\mu c}e^{-c\sqrt{\mu^2+2u}} = e^{\mu c - |c|\sqrt{\mu^2+2u}}$$

Exercise 2: (Wald Identity) Let $\Omega := C([0, \infty))$ be the space of continuous functions, $\mathcal{F}$ be the Borel σ-algebra on $C([0, \infty))$ with respect to uniform convergence on compacts, and $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by the projections $\Omega \to \mathbb{R}$, $f \mapsto f(t)$ for $t \geq 0$. Let $\mu \in \mathbb{R}$ and τ be a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$ such that $\tau < \infty$ with probability 1 under the law of a Brownian motion with drift μ. Show that

$$\mathbb{E}\left[e^{\mu B_\tau - \mu^2 \tau/2}\right] = 1$$

for a standard Brownian motion B.

Answer: We have that

$$\mathbb{E}\left[e^{\mu B_t - \mu^2 \tau/2}\right] = \int_\Omega e^{\mu B_t - \mu^2 \tau/2} \mathrm{d}\mathbb{P}(\omega)$$

We know that $e^{\mu B_t - \mu^2 \tau/2}$ is a martingale. Thus we can set $Z_t(\mu) = e^{\mu B_t - \mu^2 t/2}$ and use Girsanov's theorem on a finite time interval $[0, k]$ to get

$$\begin{aligned} \mathbb{E}\left[1_{\{\tau \leq k\}} e^{\mu B_\tau - \mu^2 \tau/2}\right] &= \int_\Omega 1_{\{\tau \leq k\}} e^{\mu B_\tau - \mu^2 \tau/2} \mathrm{d}\mathbb{P}(\omega) \\ &= \int_\Omega 1_{\{\tau \leq k\}} \mathrm{d}\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{P}}(\tau \leq k) \end{aligned}$$

Where $\tilde{\mathbb{P}}$ is the law of a Brownian motion with drift μ. As $k \to \infty$, the RHS is 1 by assumption. That is, $\tilde{\mathbb{P}}(\tau = \infty) = 0$. We also have by Monotone Convergence Theorem that

$$\lim_{k \to \infty} \mathbb{E}\left[1_{\{\tau \leq k\}} e^{\mu B_\tau - \mu^2 \tau/2}\right] = \mathbb{E}\left[e^{\mu B_\tau - \mu^2 \tau/2}\right]$$

So we conclude that

$$\mathbb{E}\left[e^{\mu B_\tau - \mu^2 \tau/2}\right] = 1$$

Exercise 3: (Non-uniqueness) Consider the SDE

$$\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t + 3X_t^{2/3}\mathrm{d}B_t$$

where B is standard Brownian motion. Show that for any initial condition x_0 this SDE has uncountably infinitely many strong solutions.

Answer: We have that the SDE is trivially satisfied by $X_t = 0$ with $x_0 = 0$. Now, consider $X_t = (B_t + x_0^{1/3})^3$, where B_t is standard Brownian. Then we have $X_0 = x_0$ and by Ito's

$$\begin{aligned}
\mathrm{d}X_t &= 3(B_t + x_0^{1/3})^2\mathrm{d}B_t + \frac{1}{2}\cdot 3\cdot 2(B_t + x_0^{1/3})\mathrm{d}t \\
&= 3X_t^{1/3}\mathrm{d}t + 3X_t^{2/3}\mathrm{d}B_t
\end{aligned}$$

Since B_t is Brownian, every state is recurrent and so there are $\tau_1, \tau_2, \ldots$ countably infinite times that $B_t = -x_0^{1/3}$. Now, pick any subset of the natural numbers $I \subseteq \mathbb{N}$ and use these as an indicator set for when you'll switch between the two solutions

$$X_t = \begin{cases} (B_t + x_0^{1/3})^3 & \text{for } t \in [0, \tau_1] \\ (B_t + x_0^{1/3})^3 & \text{for } t \in [\tau_i, \tau_{i+1}] \text{ if } i \in I \\ 0 & \text{for } t \in [\tau_i, \tau_{i+1}] \text{ if } i \notin I \end{cases}$$

We confirm that $X_0 = x_0$ and that the SDE is satisfied by the same steps as before. The number of possible subsets of I is $2^{\mathbb{N}}$ which is uncountably many. We conclude that there are uncountably many strong solutions.

Exercise 4: (Doss-Sussmann method) Let $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $\sigma : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with bounded derivatives. In addition, for any $x \in \mathbb{R}$ consider the solution of the ODE

$$\partial_t h(t,x) = \sigma(h(t,x)), h(0,x) = x$$

Finally, let D_t, $t \geq 0$ be the solution of the random ODE

$$\frac{\mathrm{d}D_t}{\mathrm{d}t} = b(h(B_t, D_t))\exp\left(-\int_0^{B_t} \sigma'(h(s,D_t))\mathrm{d}s\right), D_0 = y$$

for a standard Brownian motion B and $y \in \mathbb{R}$. Find the SDE satisfied by $X_t := h(B_t, D_t)$, $t \geq 0$ and verify that X is its unique strong solution.

Hint: start by finding the ODS satisfied by $g(t,x) := \partial_x h(t,x)$ and writing a formula for its solution.

Answer: As per the hint, we have

$$\begin{aligned} g(t,x) &= \partial_x h(t,x) \\ \implies \partial_t g(t,x) &= \partial_t \partial_x h(t,x) \\ &= \partial_x \partial_t h(t,x) \\ &= \partial_x \sigma(h(t,x)) \\ &= \sigma'(h(t,x))\partial_x h(t,x) \\ &= \sigma'(h(t,x))g(t,x) \end{aligned}$$

Where we can swap the derivatives by Clairaut's theorem. We also have that $g(0,x) = \partial_x h(0,x) = \partial_x x = 1$. Thus, the solution for $g(t,x)$ is an exponential

$$g(t,x) = e^{\int_0^t \sigma'(h(s,x))ds}$$

Now, we write out the SDE for X_t using Ito's formula in multiple variables, and use the fact that D_t has bounded variation (from its definition, $\mathrm{d}D_t$ is bounded since b is Lipschitz and the exponential is also bounded) so that some of the cross variation terms are 0,

$$\begin{aligned}
\mathrm{d}X_t &= \frac{\partial}{\partial t}h(B_t,D_t)\mathrm{d}B_t + \frac{\partial}{\partial x}h(B_t,D_t)\mathrm{d}D_t + \frac{1}{2}\frac{\partial^2}{\partial t^2}h(B_t,D_t)\mathrm{d}t \\
&= \sigma(h(B_t,D_t))\mathrm{d}B_t + g(B_t,D_t)\mathrm{d}D_t + \frac{1}{2}\frac{\partial}{\partial t}\sigma(h(B_t,D_t))\mathrm{d}t \\
&= \sigma(X_t)\mathrm{d}B_t + e^{\int_0^{B_t}\sigma'(h(s,D_t))ds}\cdot b(h(B_t,D_t))e^{-\int_0^{B_t}\sigma'(h(s,D_t))ds}\mathrm{d}t \\
&\quad + \frac{1}{2}\sigma'(h(B_t,D_t))\frac{\partial}{\partial t}h(B_t,D_t)\mathrm{d}t \\
&= \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t + \frac{1}{2}\sigma'(h(B_t,D_t))\sigma(h(B_t,D_t))\mathrm{d}t \\
&= \sigma(X_t)\mathrm{d}B_t + (b(X_t) + \sigma'(X_t)\sigma(X_t))\mathrm{d}t
\end{aligned}$$

Therefore, $X_t = h(B_t, D_t)$ solves the SDE

$$\begin{gathered}
\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + (b(X_t) + \sigma'(X_t)\sigma(X_t))\mathrm{d}t \\
\text{with } X_0 = h(B_0, D_0) = h(0, y) = y
\end{gathered}$$

Now, to show uniqueness of this strong solution, we can show that the coefficients are Lipschitz. By assumption, b is Lipschitz. σ is Lipschitz since it has bounded derivatives (say by M). By the Mean Value Theorem,

$$\begin{gathered}
\sigma(x) - \sigma(y) = \sigma'(c)(x-y) \text{ for some } c \in \mathbb{R} \\
\Leftrightarrow |\sigma(x) - \sigma(y)| = |x-y||\sigma'(c)| \\
\Leftrightarrow |\sigma(x) - \sigma(y)| \le M|x-y|
\end{gathered}$$

Since adding Lipschitz functions is still Lipschitz by the triangle inequality, we just have to show that $\sigma'(\cdot)\sigma(\cdot)$ is locally Lipschitz. We have by triangle inequality

$$\begin{aligned}
|\sigma'(x)\sigma(x) - \sigma'(y)\sigma(y)| &= |\sigma'(x)\sigma(x) - \sigma'(x)\sigma(y) + \sigma'(x)\sigma(y) - \sigma'(y)\sigma(y)| \\
&\le |\sigma'(x)\sigma(x) - \sigma'(x)\sigma(y)| + |\sigma'(x)\sigma(y) - \sigma'(y)\sigma(y)| \\
&\le |\sigma'(x)||\sigma(x) - \sigma(y)| + |\sigma(y)||\sigma'(x) - \sigma'(y)|
\end{aligned}$$

Now, we have that the derivatives are bounded. Therefore, $\sigma(\cdot)$ and $\sigma'(\cdot)$ are Lipschitz as shown previously. By assumption $|\sigma'(x)| \le M$, and $|\sigma(y)| \le N < \infty$ since it is continuous and y is part of some compact set. Therefore, we have

$$\begin{aligned}
&|\sigma'(x)\sigma(x) - \sigma'(y)\sigma(y)| \\
&\leq MK_\sigma|x-y| + NK_{\sigma'}|x-y| \\
&= (MK_\sigma + NK_{\sigma'})|x-y|
\end{aligned}$$

Therefore, $\sigma'(\cdot)\sigma(\cdot)$ is locally Lipschitz. Hence, all the coefficients are locally Lipschitz and we conclude that the strong solution is unique.