ORF 524: Statistical Theory and Methods
Homework 1

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Exercise 1 (10 points)**.** *Recall the definition of σ-Algebra. Let $(\Omega, \bar{\Sigma})$ be a measurable space, that is, $\bar{\Sigma}$ satisfies the following three properties:*

- $\bar{\Sigma} \neq \emptyset$, $\bar{\Sigma} \subseteq 2^{\Omega}$.
- *$A \in \bar{\Sigma}$ implies that $A^c \in \bar{\Sigma}$. Here we use A^c to denote the complement of A.*
- *For any $A_1, A_2, \cdots \in \bar{\Sigma}$, we have $\bigcap_{i \geq 1} A_i \in \bar{\Sigma}$.*

Based on these properties, solve the following problems.

(1). Show that $\bar{\Sigma}$ is closed under union.

(2). Show that $\bar{\Sigma}$ must contain $\emptyset$ and Ω.

(3). Suppose $A \subseteq \Omega$, what is the smallest σ-algebra containing A?

(4). Show that the set of all rational numbers, denoted by $\mathbb{Q}$, is Borel measurable. That is, $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$.

Proof. We first show that $\bar{\Sigma}$ is closed under union. Let $A_1, A_2, \cdots \in \bar{\Sigma}$, by De Morgan's law we have

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c.$$

Since $A_i \in \bar{\Sigma}$ for all $i \geq 1$, it holds that $A_i^c \in \bar{\Sigma}$. Since $\bar{\Sigma}$ is closed under intersection, $\bigcap_i A_i^c$ is in $\bar{\Sigma}$. Thus we have $\bigcup_i A_i \in \bar{\Sigma}$.

For the second problem, we only need to show $\emptyset \in \bar{\Sigma}$, because this implies $\Omega \in \bar{\Sigma}$ by the second property of σ-algebra. Since $\bar{\Sigma} \neq \emptyset$, there exists $A \neq \emptyset$ such that $A \in \bar{\Sigma}$. Thus A^c is also in $\bar{\Sigma}$. Since $\bar{\Sigma}$ is closed under intersection, we have

$$\emptyset = A \bigcap (A^c) \in \bar{\Sigma},$$

which concludes the proof.

For the third problem, for any σ-algebra $\bar{\Sigma}$ containing A, A^c is also in $\bar{\Sigma}$. Thus we have

$$\bar{\Sigma}_0 := \{\emptyset, \Omega, A, A^c\} \subseteq \bar{\Sigma}.$$

Moreover, it is easy to verify that $\bar{\Sigma}_0$ itself is a σ-algebra. Thus $\bar{\Sigma}_0$ is the smallest σ-algebra containing A. Finally, for the last one, since $\mathbb{Q}$ is countable, we can enumerate all its elements by

$$\mathbb{Q} = \{a_1, a_2, \ldots\}.$$

Thus we can write $\mathbb{Q} = \bigcup_{i \geq 1} \{a_i\}$. Since $\{a_i\}^c = (-\infty, a_i) \bigcup (a_i, +\infty)$ is a union of two open sets, $\{a_i\}^c \in \mathcal{B}(\mathbb{R})$. This implies that $\{a_i\} \in \mathcal{B}(\mathbb{R})$. Thus we conclude that $\mathcal{Q} \in \mathcal{B}(\mathbb{R})$. $\square$

Exercise 2 (10 points)**.** *Let P be a probability measure on $(\Omega, \bar{\Sigma})$. Only utilizing the definition of probability measure given in the class, solve the following problems.*

(1). Show that for any $A, B \in \bar{\Sigma}$ satisfying $A \subseteq B$, we have $0 \leq P(A) \leq P(B)$.

(2). Show that for any positive integer k, we have

$$P\Big(\bigcup_{i=1}^{k} A_i\Big) \leq \sum_{i=1}^{k} P(A_i). \tag{1}$$

(3). Does inequality (1) *still hold when $k = \infty$?*

Proof. For the first problem, we write $B = A \cup (B \setminus A)$. Note that A and $B \setminus A$ are disjoint. We set $A_1 = A$, $A_2 = B \setminus A$, and $A_i = \emptyset$ for $i \geq 3$, then these sets are disjoint. Since $P(\emptyset) = 0$, we have

$$P(B) = P\Big(\bigcup_{i} A_i\Big) = \sum_{i} P(A_i) = P(A) + P(B \setminus A) \geq P(A), \tag{2}$$

where the last inequality follows from the non-negativity of probability measure.

For the second problem, we prove by induction. For $k = 1$, the inequality (1) holds trivially. Suppose the argument holds for $k = m$. Now we set $k = m + 1$. Note that we have

$$\bigcup_{i=1}^{m+1} A_i = A_{m+1} \bigcup \Big[\bigcup_{i=1}^{m} (A_i \setminus A_{m+1})\Big]. \tag{3}$$

Similar to (2) we have

$$P\Big(\bigcup_{i=1}^{m+1} A_i\Big) = P(A_{m+1}) + P\Big[\bigcup_{i=1}^{m} (A_i \setminus A_{m+1})\Big]. \tag{4}$$

By the induction assuption, the desired inequality holds for $k = m$, we have

$$P\Big[\bigcup_{i=1}^{m} (A_i \setminus A_{m+1})\Big] \leq \sum_{i=1}^{m} P(A_i \setminus A_{m+1}) \leq \sum_{i=1}^{m} P(A_i), \tag{5}$$

where the last inequality follows from (2). Combining (4) and (5), we have

$$P\Big(\bigcup_{i=1}^{m+1} A_i\Big) \leq \sum_{i=1}^{m+1} P(A_i).$$

Therefore, the desired inequality holds for any integer $k > 0$.

For $k = \infty$, the argument still holds. If $\sum_{i=1}^{\infty} P(A_i) = +\infty$, the inequality holds trivially. Thus we only need to consider the case where $\sum_{i=1}^{\infty} P(A_i) < +\infty$. In this case, note that

$$\bigcup_{i=1}^{\infty} A_i = A_1 \bigcup_{i=1}^{\infty} \Big[A_i \Big\backslash \Big(\bigcup_{j=1}^{i-1} A_j\Big)\Big], \tag{6}$$

where we denote $A_0 = \emptyset$. Since the sets $\{A_i \setminus (\bigcup_{j=1}^{i-1} A_j), i \geq 1\}$ are disjoint, we have

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P\Big[A_i \Big\backslash \Big(\bigcup_{j=1}^{i-1} A_j\Big)\Big] \leq \sum_{i=1}^{\infty} P(A_i).$$

Thus we conclude the proof.

Another proof of the third problem:
As another proof, we show that the continuity of probability measure implies (1) for $k = \infty$.

Theorem 1 (continuity of probability measure)**.** *Let $(\Omega, \mathcal{B}(\Omega), P)$ be a probability space. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of increasing sets such that $B_n \subseteq B_{n+1}$ for $n \geq 1$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then it holds that*

$$\lim_{n\to\infty} P(B_n) = P(B). \tag{7}$$

Now we apply this theorem by letting $B_n = \bigcup_{i=1}^{n} A_i$ in (7) for all $i \geq 1$. Then set B in Theorem 1 is just $\bigcup_{i=1}^{\infty} A_i$. By Theorem 1 we have

$$\lim_{n\to\infty} P\Big(\bigcup_{i=1}^{n} A_i\Big) = P\Big(\bigcup_{i=1}^{\infty} A_i\Big). \tag{8}$$

By the second problem, for each finite n, it holds that

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i). \tag{9}$$

Thus letting n goes to infinity in (9) and combining (8), we finally have

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{\infty} P(A_i),$$

which concludes the proof. $\square$

In what follows we prove Theorem 1.

Proof of Theorem 1. For each $n \geq 1$, we define $C_n = B_n \setminus B_{n-1} = B_n \bigcap B_{n-1}^c$, where we set $B_0 = \emptyset$ Then $\{C_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets. Moreover, we have $B_n = \bigcup_{i=1}^{n} C_i$ and $B = \bigcup_{i=1}^{\infty} C_i$. Thus we have

$$\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(C_i) = \sum_{i=1}^{\infty} P(C_i) = P(B).$$

. Thus we conclude the proof. $\square$

Exercise 3 (10 points)**.** *Show that*

$$\Big|\int f \mathrm{d}P\Big| < \infty \ \textit{ if and only if } \ \int |f| \mathrm{d}P < \infty.$$

Proof. We define $f_+ = f \cdot \mathbb{1}\{f \geq 0\}$ and $f_- = -f \cdot \mathbb{1}\{f \leq 0\}$. By definition, we have

$$\int f \mathrm{d}P = \int f_+ \mathrm{d}P - \int f_- \mathrm{d}P, \tag{10}$$

where at least one of the two terms on the right-hand side of (10) is finite. Thus

$$\left| \int f \mathrm{d}P \right| < \infty \text{ is equivalent to } \int f_+ \mathrm{d}P < \infty \text{ and } \int f_- \mathrm{d}P < \infty.$$

Note that f_+ and f_- are both non-negative. Thus it follows that

$$\left\{ \int f_+ \mathrm{d}P < \infty \text{ and } \int f_- \mathrm{d}P < \infty \right\} \text{ is equivalent to } \int (f_+ + f_-) \mathrm{d}P < \infty.$$

Since $|f| = f_+ + f_-$, we conclude that

$$\left| \int f \mathrm{d}P \right| < \infty \text{ and } \int |f| \mathrm{d}P < \infty$$

are equivalent. □

Exercise 4 (10 points)**.** *This exercise, consists of two questions, concerns the σ-finiteness of a measure.*

(1). Show that the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is σ-finite.

(2). Show that the counting measure on $(\Omega, 2^\Omega)$ is σ-finite if and only if Ω is countable.

Proof. To see that the Lebesgue measure is σ-finite, let $A_i = [i, i+1)$ for $i \in \mathbb{Z}$. Then we have $A_i \in \mathcal{B}(\mathbb{R})$ and

$$\mathbb{R} = \bigcup_{i \in \mathbb{Z}} A_i.$$

Moreover, the Lebesgue measure of A_i is one for each $i \in \mathbb{Z}$. Thus the Lebesgue measure is σ-finite.
Now we consider the second question. Let ν be the counting measure on $(\Omega, 2^\Omega)$. From the definition of σ-finiteness, if ν is σ-finite, there exists $\{A_i, i \geq 1\} \subseteq 2^\Omega$ such that $\nu(A_i) < \infty$ and $\bigcup_{i \geq 1} A_i = \Omega$. Thus each A_i contains finite number of elements in Ω, which implies that

$$\Omega = \bigcup_{i \geq 1} A_i$$

is the countable union of sets each containing finite number of elements. Thus Ω is countable.
On the other hand, if Ω is countable, we could list its elements by

$$\Omega = \{a_1, a_2, \ldots\}.$$

Let $A_i = \{a_i\}$, then we have $\bigcup_{i \geq 1} A_i = \Omega$ and $\nu(A_i) = 1 < \infty$. Thus the counting measure ν is σ-finite. □

Exercise 5 (10 points)**.** *Let $X\colon \Omega \to \mathbb{R}$ be a discrete random variable on probability space $(\Omega, \overline{\Sigma}, P)$ and denote the corresponding induced measure by P_X. We define the support of P_X as*

$$\Omega_X = \{x \in \mathbb{R}\colon P(X = x) > 0\}.$$

Please answer the following two questions.

(1). First assume that $|\Omega_X| < \infty$, that is, Ω_X contains finite number of elements. Show that the probability mass function (pmf) of X, denoted by f, is indeed the density of P_X with respect to the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(2). Show the same thing when $|\Omega_X| = \infty$.

Remark. *In this exercise, the counting measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ is not σ-finite, but this does not contradict Radon-Nikodym theorem. Recall that Radon-Nikodym theorem states that, if a measure μ is absolutely continuous with respect to a σ-finite measure ν, then there exists a function $f\colon \mathcal{X} \to \mathbb{R}_+$ such that, for any measurable set A,*

$$\mu(A) = \int_A f \mathrm{d}\nu. \tag{11}$$

However, even if ν is not σ-finite, as long as (11) *holds for any measurable set A, we might still call f the* density *of μ with respect to ν.*
Therefore, in this exercise, to show that the pmf is the density of P_X with respect to the counting measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, which is denoted by ν, we only need to show that for any $A \in \mathcal{B}(\mathbb{R})$, we have

$$P_X(A) = \int_A f \mathrm{d}\nu. \tag{12}$$

Since X is discrete, in other words, X only takes countable number of values, we have

$$P_X(A) = P_X\big(A \bigcap \Omega_X\big) \;\; \textit{and} \;\; \int_A f \mathrm{d}\nu = \int_{A\cap\Omega_X} f \mathrm{d}\nu.$$

Thus, we only need to show (12) *for any $A \in 2^{\Omega_X}$. In this case, we only need to verify that*

$$\sum_{a\in A} f(a) = \int_A f \mathrm{d}\nu$$

holds for any $A \in 2^{\Omega_X}$.

Proof. **Question (1).** Note that Ω_X is countable since X is discrete. We first assume that Ω_X is a finite set. Let ν be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we show that the pmf of X is the density of P_X with respect to ν. By the definition of density, we need to show that for any $A \subset \mathcal{B}(\mathbb{R})$, it holds that

$$P_X(A) = \int f \cdot \mathbb{1}_A \mathrm{d}\nu. \tag{13}$$

Since X is discrete, we have $P_X(A) = P_X\big(A \bigcap \Omega_X\big)$. In addition, for any $x \in \mathbb{R}$,

$$f(x) = \begin{cases} \mathbb{P}(X = x) & x \in \Omega_X, \\ 0 & x \notin \Omega_X, \end{cases}$$

which implies that $f = f \cdot \mathbb{1}_{\Omega_X}$. Thus we have

$$\int f \cdot \mathbb{1}_A \mathrm{d}\nu = \int f \cdot \mathbb{1}_{\Omega_X} \cdot \mathbb{1}_A \mathrm{d}\nu = \int f \cdot \mathbb{1}_{A \bigcap \Omega_X} \mathrm{d}\nu.$$

Therefore, to show (13) for any $A \in \mathcal{B}(\mathbb{R})$, it suffices to show that (13) holds for any $A \in 2^{\Omega_X}$.
In what follows, for any $A \in 2^{\Omega_X}$, we show that (13) holds. When $|\Omega_X| < \infty$, we can write $A = \{a_1, a_2, \ldots, a_m\}$ where $m \leq |\Omega_X|$.Then we have

$$P_X(A) = P\big(X \in \{a_1, \ldots, a_m\}\big) = \sum_{i=1}^{m} f(a_i) = \sum_{a \in \Omega_X} f(a) \cdot \mathbb{1}_A(a) \cdot \nu(a) = \int f \cdot \mathbb{1}_A \mathrm{d}\nu.$$

Thus we conclude the proof.
Question (2). When $|\Omega_X| = \infty$, the result still holds. Let ν be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Similar to the last question, we only need to show (13) for any $A \in 2^{\Omega_X}$. In what follows, we assume that $|A| = \infty$, since the case where A is finite is already established. Without loss of generality, we can write $A \in 2^{\Omega_X}$ by

$$A = \{a_1, a_2, \ldots\} = \{a_i \colon i \geq 1\} = \bigcup_{i=1}^{\infty} \{a_i\}.$$

Then by the definition of P_X, we have

$$P_X(A) = P(X \in A) = \sum_{i=1}^{\infty} f(a_i) \leq 1, \tag{14}$$

where the last inequality holds because P_X is a probability measure. By the definition of integration, we have

$$\int f \cdot \mathbb{1}_A \mathrm{d}\nu = \sup \int g \mathrm{d}\nu, \tag{15}$$

where the supremum is taken over all simple function g such that $0 \leq g \leq f \cdot \mathbb{1}_A$. Since $f \cdot \mathbb{1}_A$ is supported on A and $0 \leq g \leq f \cdot \mathbb{1}_A$, g is supported on a countable subset of A and for any $i \geq 1$, we have $0 \leq g(a_i) \leq f(a_i)$. Thus it follows that

$$\int g \mathrm{d}\nu \leq \sum_{i=1}^{\infty} g(a_i) \leq \sum_{i=1}^{\infty} f(a_i).$$

Then taking supremum and combining (14), we obtain

$$P_X(A) \geq \int f \cdot \mathbb{1}_A \mathrm{d}\nu.$$

Moreover, by (14), for any $\epsilon > 0$, there exists an integer N such that $\sum_{i > N} f(a_i) < \epsilon$. We define a simple function g_N by

$$g_N = f \cdot \mathbb{1}_{\{a_1, a_2, \ldots, a_N\}}.$$

Then we have

$$\int g_N \mathrm{d}\nu = \sum_{i=1}^{N} f(a_i) \geq P_X(A) - \epsilon. \tag{16}$$

Taking supremum on the left-hand side of (16), we have

$$\int f \cdot \mathbb{1}_A \mathrm{d}\nu \geq P_X(A) - \epsilon,$$

where ϵ can be set arbitrarily small. Therefore, we conclude that

$$P_X(A) = \int f \cdot \mathbb{1}_A \mathrm{d}\nu,$$

which implies that f is the density of P_X with respect to ν. $\square$