1 List of Problems for Chapter 2

This is a list of problems from Chapter 2 which I left in class as exercises or referred to during proofs. The Homework will be picked from these problems. They are a good practice list for the probability spaces chapter.

$(\Omega, \mathcal{H}, \mathbb{P})$ will denote a probability space and X_n a sequence of random variables on it.

1. Let $H_1 \supset H_2 \supset \cdots$ be a descending sequence of events with $\lim_n H_n = \bigcap H_n = H$. Prove that $\mathbb{P}(H_n) \searrow \mathbb{P}(H)$ as $n \to \infty$.

2. Show that if $\displaystyle\int_H X \, d\mathbb{P} \geq 0$ for any $H \in \mathcal{H}$, then $X \geq 0$ a.s.

3. Show that if X_n is a sequence of random variables with $X_n \leq Y$ and Y integrable, such that $X_n \searrow X$, then $\mathbb{E}[X_n] \searrow \mathbb{E}[X]$ as $n \to \infty$.

4. Consider the inertia momentum about axis $\{x = a\}$ defined by
$$f(a) = \int_\Omega (X(\omega) - a)^2 d\mathbb{P}(\omega).$$
   (i) Show that f is minimized by $a = \mathbb{E}[X]$.

   (ii) What is the value of the minimum?

5. Let $X \geq 0$ and $\mathbb{E}[X] < \infty$, with X random variable. Show that $X < \infty$ a.s.

6. Suppose that X is integrable. Show that $Var(a + bX) = b^2 Var(X)$ for all $a, b \in \mathbb{R}$.

7. (*Chebyshev's inequality*) Assume X has finite mean, $\mathbb{E}[X] = \mu$. Show the inequality
$$P\{|X - \mu| > \epsilon\} \leq \frac{1}{\epsilon^2} Var(X).$$

8. (*Markov's inequality*) Assume X is real-valued and $f : \mathbb{R} \to \mathbb{R}_+$ is increasing. Show the inequality
$$P\{X > b\} \leq \frac{1}{f(b)} \mathbb{E}[f(X)].$$

9. Let U have a uniform distribution on $(0, 1)$ and define $X = -\frac{1}{c} \ln U$. Show that X has an exponential distribution with the scale parameter c.

10. Let X and Y be real-valued random variables with finite variances. Define their covariance by
$$Cov(X, Y) = \mathbb{E}\Big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\Big].$$
    Show that:

(i) $Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

(ii) $|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)}$.

(iii) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.

11. The random variable X is called Cauchy distributed if its distribution has the form $\mu(dx) = \dfrac{1}{\pi(1+x^2)}\,dx$. Let $X^+ = X \vee 0$ and $X^- = X \wedge 0$.

    (i) Show that $\mathbb{E}[X^+]$ and $\mathbb{E}[X^-]$ are infinite.

    (ii) Show that the mean and variance of a Cauchy distribution do not exist. What about the other moments?

12. Let X be uniformly distributed on $(-\pi, \pi)$ and let $Y = \tan(X)$. Show that Y is Cauchy distributed.

13. Let X and Y be independent random variables, normally distributed with parameters $\mu = 0$ and $\sigma = 1$.

    (i) Let $Z = X^2 + Y^2$. Show that Z has an exponential distribution, with $\mu(dx) = \frac{1}{2}e^{-z/2}dx$.

    (ii) Let $C = \frac{X}{Y}$. Show that C has a Cauchy distribution.

    (iii) Let $R = \sqrt{X^2+Y^2}$ and $W = \tan^{-1}(\frac{X}{Y})$, with $R \ge 0$ and $-\frac{\pi}{2} < W \le \frac{\pi}{2}$. Show that R and W are independent and that W is uniformly distributed on $(-\pi/2, \pi/2)$. (Hint: Use (ii) and Exercise 12.)

14. Let X be a Poisson distributed random variable.

    (i) Show that the characteristic function of X is $\hat{\mu}(u) = e^{\lambda(e^{iu}-1)}$.

    (ii) Find the generating function of X.

15. Let X be uniformly distributed random variable on $(-a, a)$. Show that the characteristic function of X is $\hat{\mu}(u) = \dfrac{\sin(au)}{au}$.

16. In this exercise you will compute the characteristic function of a normal random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$.

    (i) Using Euler's formula show that $\hat{\mu}(r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos(rx)e^{-x^2/2}\,dx$.

    (ii) Show that $\hat{\mu}(u)$ satisfies the ordinary differential equation

    $$\frac{\hat{\mu}(r)'}{\hat{\mu}(r)} = -r, \qquad \hat{\mu}(0) = 1.$$

    (iii) Solve the previous equation and obtain $\hat{\mu}(r) = e^{-r^2/2}$.

17. Let μ be a probability measure. Show that its Fourier transform, $\hat{\mu}$, is a bounded, continuous function satisfying $\hat{\mu}(0) = 1$. (Hint: for the continuity part use the Bounded Convergence Theorem).

18. Let $X, Y \sim Exp(1)$ be two independent random variables, standard exponential distributed. Find the distribution of $R = \frac{Y}{X}$.

19. Let $\hat{\mu}(r)$ denote the Laplace transform of the random variable X, and assume that $X \geq 0$ and $X < \infty$, a.s.

    (i) Use Fubini theorem for the product measure $\mathbb{P} \times Leb$ to show that

$$\int_r^\infty \mathbb{E}[Xe^{-qX}]dq = \hat{\mu}(r), \qquad r \geq 0.$$

    (ii) Assume $E[X] < \infty$. Use the Dominated Convergence Theorem to show

$$\lim_{r \searrow 0} \frac{d}{dr}\hat{\mu}(r) = -\mathbb{E}[X].$$

20. Show that any family of random variables with bounded second moment is uniformly integrable.

21. Let $X : \Omega \to E$ and $Y : \Omega \to \bar{\mathbb{R}}$ be two random variables, where $(E, \mathcal{E})$ is a measurable space. Prove that $\sigma Y \subset \sigma X$ if and only if there is a measurable function $f : E \to \bar{\mathbb{R}}$ such that $Y = f(X)$.

22. Let X_n be a sequence of integrable random variables such that $\mathbb{E}[|X_n - X|] \to 0$, as $n \to \infty$. Show that $\mathbb{E}[X_n] \to \mathbb{E}[X]$, as $n \to \infty$.

23. Let $(X_n)_n$ be a sequence of bounded random variables, with $|X_n| < M < \infty$, for all $n \geq 1$. Prove that the family $\{X_n\}$ is uniformly integrable.