ORF522 Assignment 2

Due 11:59 PM Oct. 17 2016

1 Problem 1 (20 pts)

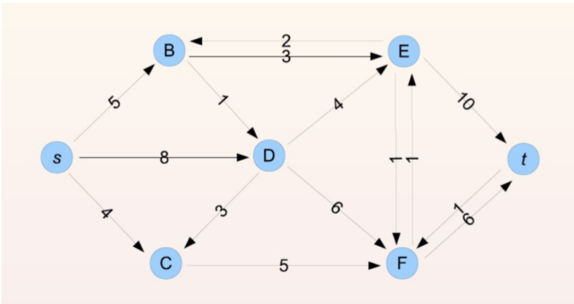
Please show that exactly one of the following systems has a solution

1. $A\mathbf{x} \leq \mathbf{b}$
2. $\mathbf{y}^T A = 0$, $\mathbf{b}^T \mathbf{y} < 0$, $\mathbf{y} \geq 0$

(Hint: using linear duality, you can show that it can't both have solution and if the second one is infeasible, the first one must be feasible.)

2 Problem 2 (15 pts)

Consider max flow problem on the following graph with note s as the source node and node t as the terminal node. A flow of the graph (V, E) is defined as a function $f : E \to \mathbb{R}^+$ subject



to two constraints:

- Capacity constraint: $f(u, v) \leq w(u, v)$ for all $(u, v) \in E$, where $w(u, v)$ is the weight associated with the edge (u, v).
- Conservation of flows: for each $v \in V$ and $v \notin \{s, t\}$, $\sum_{u:(u,v)\in E} f(u, v) = \sum_{u:(v,u)\in E} f(v, u)$.

Max flow problem seeks to maximize the value of the flow $\sum_{v:(s,v)\in E} f(s, v)$.

1. Formulate it as a linear program and solve it by using the simplex method.
2. Formulate the dual of this LP and solve it by using the simplex method.

3 Problem 3 (15 pts)

Use the two-phase simplex method to solve completely the following problem:

$$
\begin{array}{lrrrrrcl}
\text{minimize} & 2x_1 & +x_2 & +3x_3 & +x_4 & -3x_5 & & \\
\text{subject to} & x_1 & +2x_2 & & +4x_4 & -3x_5 & = & 2 \\
 & x_1 & +x_2 & & -3x_4 & +4x_5 & = & 2 \\
 & -x_1 & -3x_2 & +3x_3 & & & = & 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0
\end{array}
$$

For each step, clearly state what is the current basis, the current basic solution, and the corresponding objective values.

4 Problem 4 (30 pts)

A manager of an oil refinery has 8 million barrels of crude oil A and 5 million barrels of crude oil B allocated for production during the coming month. These resources can be used to make gasoline, which sells for 38 per barrel, and home heating oil, which sells for 33 per barrel. There are three production processes available: For example, with the first process, 2 barrels of crude A and 4 barrels of crude B are used to produce 4 barrels of gasoline and 3 barrels of heating oil. The costs in this table refer to variable and allocated overhead costs, and there are no separate cost items for the cost of the crudes.

	Process 1	Process 2	Process 3
Input Crude A	2	1	4
Input Crude B	4	1	2
Output gasoline	4	1	3
Output heating oil	3	1	4
Cost	111	11	100

1. First formulate a LP that would help the manager maximize net revenue over the next month. Then use the simplex method to find an optimal solution. Please write down each iteration.

2. Suppose that the selling price of heating oil is sure to remain fixed over the next month, but the selling price of gasoline may rise. Using the simplex tableau, try to decide how high it can go without causing the optimal solution to change. (Hint: You want to prevent new variable from entering the optimal basis while the price is increasing.)

3. Now the manager has to consider waste disposal as well. Suppose Process 1 generates 4×10^3 tons of waste when it finishes one million production cycle, i.e., produces 4 barrels of gasoline and 3 barrels of heating oil, while Process 2 and 3 generate 3×10^3 tons and 5×10^3 tons of waste respectively when they finish one million production cycle. The refinery can dispose of 14000 tons of waste every month. Will these new constraints change the optimal solution? Why or why not?

4. Let's assume now that the manager wants to maximize his economic utility. His economic utility is actually a non-linear function of net revenue. In fact, it has the following form: $U(r) = \sqrt{r}$ where r is the net revenue. This is a typical utility function because the incremental dollar in net revenue will have declining incremental value to you. How can the problem of maximizing his utility be solved?

5 Problem 5 (20 pts)

Consider an LP problem of standard form

$$\begin{aligned} \text{minimize}_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, A is $m \times n$. Assume that the feasible set, i.e.,

$$P = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\},$$

is nonempty and that A has full rank m. We denote the nullspace of A by

$$\mathbf{N}(A) = \{x \mid Ax = 0\}.$$

Show that there exists an optimal solution to the LP if and only if

$$\{d \mid c^T d < 0\} \cap \mathbf{N}(A) \cap \{x \geq 0\} = \emptyset$$

6 Bonus Problem (20pts)

Consider the linear programming problem in standard form $\min c^\top x$ s.t. $Ax = b, x \geq 0$, where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and matrix $A \in \mathbb{R}^{m \times n}$ $(m < n)$. Suppose x^* is a degenerate BFS with basis B. Now we perturb b with a random variable ϵ uniformly distributed in $[-\sigma, \sigma]$ and obtain new $\widetilde{b} = b + \epsilon$. Prove that there exists $\sigma > 0$ such that the BFS with the same basis B is not degenerate with probability 1.