ELE 535: Machine Learning and Pattern Recognition
Homework 7

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Exercise 1: Sparse Representation in an ON Basis. Let $r \leq n$ and $Q \in \mathbb{R}^{n \times r}$ have orthonormal columns.

a) Find a solution of the following sparse approximation problem and determine if the solution is unique.

$$\min_{w\mathbb{R}^r} \|y - Qw\|_2^2$$
$$\text{s.t. } \|w\|_0 \leq k$$

b) Now let the columns of $X \in \mathbb{R}^{n \times m}$ be a centered set of unlabelled training data and the columns of $Q \in \mathbb{R}^{n \times r}$ be the left singular vectors of a compact SVD of X. In this context, interpret the solution of the above problem.

Answer:

Exercise 2: Let

$$M = \left[\boldsymbol{e}_1 \quad \tfrac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2) \quad \boldsymbol{e}_3 \quad \tfrac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)\right]$$

where $\boldsymbol{e}_i$ denotes the i^{th} standard basis vector in $\mathbb{R}^n$.

a) Show that the columns of M are linearly dependent.

b) Determine spark(M).

c) Determine the mutual coherence $\mu(M)$.

Answer:

Exercise 3: Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = m < n$, and $y \in \mathbb{R}^m$. We seek the sparsest solution of $Ax = y$:

$$\min_{x \in \mathbb{R}^n} \|x\|_0, \text{ s.t. } Ax = y$$

The convex relaxation of this problem is called Basis Pursuit:

$$\min_{x \in \mathbb{R}^n} \|x\|_1, \text{ s.t. } Ax = y$$

Show that Basis Pursuit is equivalent to the linear program:

$$\begin{gathered}\min_{x,z \in \mathbb{R}^n} \mathbf{1}^T z \\ \text{s.t. } Ax = y \\ x - z \leq \mathbf{0} \\ -x - z \leq \mathbf{0}\end{gathered}$$

Answer:

Exercise 4: One way to create a dictionary is to combine known ON bases. Here we explore combining the standard basis with the Haar wavelet basis. The Haar wavelet basis consists of $n = 2^p$ ON vectors in $\mathbb{R}^n$. These can be arranged into the columns of an orthogonal matrix H_p with

$$H_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad H_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{4}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{4}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{4}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{4}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{4}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{4}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{4}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The columns are arranged in groups. The first group consists of the vector $\frac{1}{\sqrt{n}}\mathbf{1}_n$, and second consists of the vector taking the value $\frac{1}{\sqrt{n}}$ in the first half, and $-\frac{1}{\sqrt{n}}$ in the second half. Subsequent group of vectors are derived by subsampling by 2, scaling by $\sqrt{2}$, and translating. This is illustrated above for $p = 1, 2, 3$. Form a dictionary $D \in \mathbb{R}^{n \times 2n}$ by setting D $[I_n, H_p]$ with $n = 2^p$. The matrix H_p is the Haar matrix of size $n = 2^p$. Show that

a) For $p = 1$, $\text{spark}(D) = 3$ and $\mu_(D) = 1/\sqrt{2}$.

b) For all $p > 1$. determine $\text{spark}(D)$ and $\mu(D)$.

c) For a given $y \in \mathbb{R}^n$, we seek the sparsest solution of $y = Dw$. What condition on y is sufficient to ensure the sparsest solution is unique.

Answer: