Homework #4, ORF 526

Assigned on October 11; Due October 18, 2016, 5:00 pm

1. *(1.5 points)* Let X be an integrable random variable. Show that any sequence of events H_n such that $P(H_n) \to 0$ as $n \to \infty$, has the property $\lim_{n\to\infty} \mathbb{E}[|X|1_{H_n}] = 0$. Does the converse hold true? (Consider the converse with the conclusion "X is an integrable random variable").

2. *(1.5 points)* Assume $\sum_{n\geq 1} P\Big(|X_n - X| > \frac{1}{n}\Big) < \infty$. Show that $X_n \longrightarrow X$ almost surely.

3. *(1.5 point)* Consider the probability space $(\Omega, \mathcal{H}, P)$, with $\Omega = [0,1]$, $\mathcal{H} = \mathcal{B}_{[0,1]}$, and P the Leb measure on $[0,1]$. Define the sequence X_n by

$$X_{2n}(\omega) = \begin{cases} 0, & \text{if } \omega < 1/2 \\ 1, & \text{if } \omega \geq 1/2 \end{cases} \qquad X_{2n+1}(\omega) = \begin{cases} 1, & \text{if } \omega < 1/2 \\ 0, & \text{if } \omega \geq 1/2 \end{cases}$$

   (i) Find the distribution functions of X_{2n} and X_{2n+1}.

   (ii) Find the characteristic functions of X_{2n} and X_{2n+1}.

   (iii) Show that X_n converges in distribution.

   (iv) Show that X_n does not converge in probability to 0.

4. *(1.5 points)* Let X_n be a sequence of random variables. Assume there is a constant k such that $\lim_{n\to\infty} \mathbb{E}[X_n] = k$ and $\lim_{n\to\infty} Var(X_n) = 0$. Show that $X_n \longrightarrow k$ in L^2 (i.e. in the mean square).

5. *(1.5 points)* If $X_n \longrightarrow X$ and $X_n \longrightarrow Y$, both in probability, show that $X = Y$ almost surely.

6. *(1.5 points)* Let $X_n \longrightarrow 0$ in probability. Show that $X_n^3 \longrightarrow 0$ in probability.

7. *(1 point)* Consider a sequence of random variables X_n such that there is another random variable X such that $\sum_n \|X_n - X\|_{L^2}^2 < M < \infty$ a.s. Show that $X_n \to X$ almost surely.

Please solve each problem on a separate sheet of paper. Staple the sheets and attach a cover page with your name and course number. The Homework will be turned in 526 mailbox in Sherred Hall, cabinet A. You can retrieve your graded Homework from your folder situated in the same room, a week later.