Homework 2 Solutions

1. Let $\{a_1, \ldots, a_n\}$, $n \geq 1$, be the finite set of *distinct* real values that $f : E \to \mathbb{R}$ can assume. The measurable sets $A_i := f^{-1}(\{a_i\}) = \{x \in E | f(x) = a_i\}$ are disjoint (if $x \in A_i \cap A_j$, then $a_i = f(x) = a_j$, so that necessarily $A_i = A_j$) and exhaust all of E (if $x \in E$, then $f(x) = a_i$ for some $1 \leq i \leq n$, so that $x \in A_i$). Hence, $1_E = \sum_{k=1}^n 1_{A_k}$ and so

$$f(x) = f(x)1_E(x) = \sum_{k=1}^n f(x)1_{A_k}(x) = \sum_{k=1}^n a_k 1_{A_k}.$$

2. First, note that the inverse mapping h^{-1} respects set operations:

$$h^{-1}(A^c) = h^{-1}(A)^c, \quad h^{-1}(\cup_k A_k) = \cup_k h^{-1}(A_k), \quad h^{-1}(\cap_k A_k) = \cap_k h^{-1}(A_k). \tag{1}$$

For (a), note that $(f \wedge g) \geq 0$ since $f, g \geq 0$ and that

$$(f \wedge g)^{-1}([a, \infty)) = \{x \in E | (f \wedge g) \geq a\} = \{x \in E | f(x) \geq a, g(x) \geq a\} = f^{-1}([a, \infty)) \cap g^{-1}([a, \infty)).$$

and since (by the last homework) the collection of Borel sets is generated by the collection of rays $[a, \infty)$ (i.e., $\mathcal{B} = \sigma(\{[a, \infty) | a \in \mathbb{R}\})$), we have by equation (1)

$$(f \wedge g)^{-1}(\mathcal{B}) = \sigma(\{(f \wedge g)^{-1}([a, \infty)) | a \in \mathbb{R}\}) \subset \mathcal{E}.$$

Hence, $(f \wedge g) \in \mathcal{E}_+$. A similar proof holds for $(f \vee g)$ with the complementary generating collection $\{(-\infty, a]\}$.

For (b) and (c), approximate f and g by simple functions f_n, g_n and observe that $f_n + g_n$ and $f_n \cdot g_n$ are both simple functions (why?). Then $f + g = \lim_{n \to \infty}(f_n + g_n)$ and $f \cdot g = \lim_{n \to \infty} f_n \cdot g_n$ are both measurable.

3. If $\mu = \sum_n \mu_n$, then obviously $\mu(\emptyset) = 0$ and $\mu(A) \geq 0$ for any measurable A. Let ν denote the counting measure on the counting number $\mathbb{N} := \{1, 2, 3, \ldots\}$, i.e., for any $E \subset \mathbb{N}$, $\nu(E)$ is the number of elements in E. If $A_1, \ldots, A_n, \ldots$ is a disjoint sequence of measure sets, then

$$\begin{aligned}\mu(\cup_k A_k) &= \sum_n \mu_n(\cup_k A_k) = \sum_n \sum_k \mu_n(A_k) \\ &= \int_{\mathbb{N}} \sum_k \mu_n(A_k) d\nu(n) \overset{MCT}{=} \sum_k \int_{\mathbb{N}} \mu_n(A_k) d\nu(n) = \sum_k \sum_n \mu_n(A_k) = \sum_k \mu(A_k).\end{aligned}$$

where we have used the fact that $\sum_{k=1}^m \mu_n(A_k) \uparrow \sum_{k=1}^\infty \mu_n(A_k)$ as $m \to \infty$ since $\mu_n(A_k) \geq 0$. (Note the interchange of summation does not hold in general!)

4. First, if $f = 1_A$ for some measurable A, then

$$\delta_{x_0} f = \int 1_A(x) d\delta_{x_0}(x) = \delta_{x_0}(A) = 1_A(x_0) = f(x_0).$$

Second, by linearity, the same holds if f is a simple functions (i.e., a linear combination of indicator functions 1_A). Third, if $f \in \mathcal{E}_+$, then there exists an increasing sequence $0 \leq f_n \uparrow f$ and by the monotone convergence theorem

$$\delta_{x_0} f = \delta_{x_0} \lim_n f_n \overset{MCT}{=} \lim_n \delta_{x_0} f_n = \lim_n f_n(x_0) = f(x_0).$$

Fourth and lastly, if $f \in \mathcal{E}$, then $f^+, f^- \in \mathcal{E}_+$ and again by linearity

$$\delta_{x_0} f = \delta_{x_0}(f^+ - f^-) = \delta_{x_0} f^+ - \delta_{x_0} f^- = f^+(x_0) - f^-(x_0) = f(x_0).$$

5. For (a), first assume $p = 1_B$ for some measurable $B \in \mathcal{E}$. Then for any $A \in \mathcal{E}$,

$$\nu(A) := \int_A p(x) d\mu(x) = \int 1_A(x) 1_B(x) d\mu(x) = \int 1_{A \cap B}(x) d\mu(x) = \mu(A \cap B) \geq 0.$$

Then $\nu(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ and for a disjoint sequence $A_1, A_2, \ldots$,

$$\nu(\cup_k A_k) = \mu(A \cap \cup_k A_k) = \mu(\cup_k (A \cap A_k)) = \sum_k \mu(A \cap A_k) = \sum_k \nu(A_k),$$

where we have used the fact that the sequence $A \cap A_1, A \cap A_2, \ldots$ is also disjoint. For (b), the result is easy to check if $f = 1_A$ for some measurable A since

$$\int_E f(x) d\nu(x) = \int_E 1_A(x) d\nu(x) = \mu(A \cap B) = \int_E 1_{A \cap B}(x) d\mu(x) = \int_E f(x) p(x) d\mu(x)$$

This confirms the identities required for (a) and (b) when f and p are indicators. These identities then hold by linearity in the case that p is a simple function, and by the monotone convergence theorem, for any $p \in \mathcal{E}_+$, being a limit of simple functions $0 \leq p_n \uparrow p$. Following the same steps for f (i.e., moving next to the case that f is simple, and then to any $f \in \mathcal{E}_+$ by approximating simple functions and monotone convergence) completes the proof.