ORFE 526: Probability Theory
Homework 9

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Exercise 1: Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables, with $X_n \geq 0$ and $\mathbb{E}[X_n] = 1$, for $n \geq 1$. Let $M_0 = 1$ and define $M_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$.

a) Show that there is an $M_\infty \in L^1$ such that $M_n \to M_\infty$ almost surely, as $n \to \infty$.

b) Assume that the sequence $(X_n)_n$ does not converge to 1 a.s., as $n \to \infty$. Find the limit M_∞ in this case.

Answer:

a) We first note that M_n is a martingale. Integrabilility follows from,

$$\begin{aligned}\mathbb{E}[|M_n|] &= \mathbb{E}[|X_1 \cdot \ldots \cdot X_n|] \\ &= \mathbb{E}[X_1 \cdot \ldots \cdot X_n] \\ &= \mathbb{E}[X_1] \cdot \ldots \cdot \mathbb{E}[X_n] = 1 \cdot \ldots \cdot 1 = 1\end{aligned}$$

M_n is also $\mathcal{F}_n$-measurable since it is a function of X_k, $k \leq n$ which are $\mathcal{F}_n$-measurable.

Finally, the last martingale property,

$$\begin{aligned}\mathbb{E}[M_{n+k}|\mathcal{F}_n] &= \mathbb{E}[X_1 \cdot \ldots \cdot X_{n+k}|\mathcal{F}_n] \\ &= X_1 \cdot \ldots \cdot X_n \cdot \mathbb{E}[X_{n+1} \cdot \ldots \cdot X_{n+k}|\mathcal{F}_n] \\ &= M_n \cdot \mathbb{E}[X_{n+1}|\mathcal{F}_n] \cdot \ldots \cdot \mathcal{E}[X_{n+k}|\mathcal{F}_n] \\ &= M_n \cdot 1 \cdot \ldots \cdot 1 = M_n\end{aligned}$$

Where $\mathbb{E}[X_j|\mathcal{F}_n] = 1$ for $j \neq n$ since they are i.i.d.

Now since M_n is a martingale, it is also a submartingale. We also have that $\sup_n \mathbb{E}[|X_n|] = \mathbb{E}[|X_n|]$ since they are i.i.d. We showed that $\mathbb{E}[|X_n|] < \infty$. Thus, we have by martingale convergence theorem that there is an $M_\infty \in L^1$ such that $M_n \to M_\infty$ almost surely, as $n \to \infty$.

b) We first note that

$$\frac{M_{n+1}}{M_n} = X_{n+1}$$

Thus, if $M_\infty = c > 0$, then we have

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \frac{c}{c} = 1 = \lim_{n\to\infty} X_{n+1}$$

However, this contradicts that X_{n+1} does not converge to 1 a.s. We also note that $P(M_\infty = \infty) = 0$ a.s. Thus, since $X_n \geq 0$, the only remaining possibility is that $M_\infty = 0$ a.s.

Exercise 2: Assume that $X_1, X_2, X_3, \dots$ are i.i.d. random variables with the same distribution as X

$$P(X = 1) = P(X = -1) = 1/2$$

Let $S_0 = 0$, $S_n = X_1 + \dots + X_n$, and consider $T = \inf\{n : S_n = 1\}$. Take the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(S_1, \dots, S_n), n \geq 1$$

For $\theta \in \mathbb{R}$, consider the process $M_n = (\text{sech } \theta)^n e^{\theta S_n}$.

a) Show that T is a stopping time.

b) Find the expectation $\mathbb{E}[e^{\theta X_n}]$.

c) Show that M_n is an $\mathcal{F}_n$-martingale.

d) Verify that $\mathbb{E}[M_{T \wedge n}] = 1$.

e) Show that $\mathbb{E}[(\text{sech } \theta)^T] = e^{-\theta}$, for any $\theta > 0$.

f) Find the moment generating function of T.

Answer:

a) Since we have that $S_n \in \mathcal{F}_n$, then we have that

$$\begin{aligned}\{T \leq n\} = \inf\{k : S_k = 1, k \leq n\} &= \bigcup_{k=0}^{n} \{S_k = 1\} \\ &= \bigcup_{k=0}^{n} S_k^{-1}(\{1\}) \in \mathcal{F}_n\end{aligned}$$

Where the last part uses the fact that $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k \leq n$ and that σ-algebras are closed under union.

b) We have

$$\mathbb{E}[e^{\theta X_n}] = \frac{1}{2}e^{\theta} + \frac{1}{2}e^{-\theta} = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh \theta$$

c) We show integrability

$$\mathbb{E}[|M_n|] = \mathbb{E}[|(\text{sech } \theta)^n e^{\theta S_n}|]$$

However, the inside is the product of two positive numbers so we can drop the absolute values

$$\begin{aligned}
&= \mathbb{E}[(\text{sech } \theta)^n e^{\theta S_n}] \\
&= (\text{sech } \theta)^n \mathbb{E}[e^{\theta(X_1+\ldots+X_n)}] \\
&= (\text{sech } \theta)^n \mathbb{E}[e^{\theta X_1} \cdot \ldots \cdot e^{\theta X_n}]
\end{aligned}$$

Now since they are i.i.d.

$$\begin{aligned}
&= (\text{sech } \theta)^n \mathbb{E}[e^{\theta X_1}] \cdot \ldots \cdot \mathbb{E}[e^{\theta X_n}] \\
&= (\text{sech } \theta)^n \cosh \theta \cdot \ldots \cdot \cosh \theta \\
&= (\text{sech } \theta)^n \cdot (\cosh \theta)^n \\
&= 1 < \infty
\end{aligned}$$

We also have M_n is F_n-measurable since it is a function of S_n and $S_n \in \sigma(S_1, \ldots, S_n) = \mathcal{F}_n$.

Now for the last property of matingales,

$$\begin{aligned}
\mathbb{E}[M_{n+k}|\mathcal{F}_n] &= \mathbb{E}[(\text{sech } \theta)^{n+k} e^{\theta S_{n+k}}|\mathcal{F}_n] \\
&= \mathbb{E}[(\text{sech } \theta)^{n+k} e^{\theta(X_1+\ldots+X_{n+k})}|\mathcal{F}_n]
\end{aligned}$$

Using the fact that the X_i are i.i.d. again,

$$= (\text{sech } \theta)^n \mathbb{E}[(\text{sech } \theta)^k e^{\theta X_1} \cdot \ldots \cdot e^{\theta X_{n+k}}|\mathcal{F}_n]$$

Since we have that X_1 to X_n are $\mathcal{F}_n$-measurable, we can take them out of the expectation,

$$\begin{aligned}
&= (\text{sech } \theta)^n e^{\theta X_1} \cdot \ldots \cdot e^{\theta X_n} \mathbb{E}[(\text{sech } \theta)^k e^{\theta X_{n+1}} \cdot \ldots \cdot e^{\theta X_{n+k}}|\mathcal{F}_n] \\
&= M_n \cdot \mathbb{E}[(\text{sech } \theta)^k e^{\theta X_{n+1}} \cdot \ldots \cdot e^{\theta X_{n+k}}|\mathcal{F}_n] \\
&= M_n \cdot (\text{sech } \theta)^k \mathbb{E}[e^{\theta X_{n+1}}|\mathcal{F}_n] \cdot \ldots \cdot \mathbb{E}[e^{\theta X_{n+k}}|\mathcal{F}_n] \\
&= M_n \cdot (\text{sech } \theta)^k \cdot (\cosh \theta)^k \\
&= M_n
\end{aligned}$$

Where we used that $\mathbb{E}[e^{\theta X_{n+1}}|\mathcal{F}_n] = \cosh \theta$ since the X_k's are i.i.d.

d) Since M_n is a martingale, then $M_{T\wedge n}$ is a stopped process and so

$$\mathbb{E}[M_{T\wedge n}] = \mathbb{E}[M_0] = \mathbb{E}[(\text{sech }\theta)^0 e^{\theta S_0}] = \mathbb{E}[1 \cdot e^{\theta \cdot 0}] = \mathbb{E}[1] = 1$$

e) The last part implies that $\mathbb{E}[M_T] = 1$ since we can choose n large enough that $T \wedge n = T$ because random walks in 1 dimension are recurrent. Then,

$$\begin{aligned} &\mathbb{E}[M_T] = 1 \\ \iff &\mathbb{E}[(\text{sech }\theta)^T e^{\theta S_T}] = 1 \\ \iff &\mathbb{E}[(\text{sech }\theta)^T e^{\theta \cdot 1}] = 1 \\ \iff &e^{\theta}\mathbb{E}[(\text{sech }\theta)^T] = 1 \\ \iff &\mathbb{E}[(\text{sech }\theta)^T] = e^{-\theta} \end{aligned}$$

f) Using the previous part, let $\theta = \text{arcsech } e^t$. Then,

$$\begin{aligned} &\mathbb{E}[(\text{sech }\theta)^T] = e^{-\theta} \\ \iff &\mathbb{E}[(\text{sech arcsech } e^t)^T] = e^{-\text{arcsech } e^t} \\ \iff &\mathbb{E}[e^{tT}] = e^{-\ln\left(\frac{1+\sqrt{1-e^{2t}}}{e^t}\right)} \\ \iff &\mathbb{E}[e^{tT}] = e^{\ln\left(\frac{e^t}{1+\sqrt{1-e^{2t}}}\right)} \\ \iff &\mathbb{E}[e^{tT}] = \frac{e^t}{1+\sqrt{1-e^{2t}}} \end{aligned}$$

Exercise 3: Let X_n be a submartingale and consider its Doob decomposition

$$X_n = X_0 + M_n + A_n$$

Show that if X_n is L^1-bounded, then X_n, M_n, and A_n converge almost surely.

Answer: Since X_n is in L^1, then $\mathbb{E}[|X_n|] < \infty$ for all n. In particular, $\sup_n \mathbb{E}[|X_n|] < \infty$. Since it is also a submartingale we have by convergence theorem that $X_n \to X_\infty$ a.s. as $n \to \infty$. Since X_n is a submartingale then A_n is nonnegative and increasing. We also have that A_n is L^1 since

$$\begin{aligned} \mathbb{E}[|A_n|] &= \mathbb{E}[A_n] = \mathbb{E}[X_n] - \mathbb{E}[X_0] - \mathbb{E}[M_n] \\ &\leq \mathbb{E}[|X_n|] + \mathbb{E}[|X_0|] + 0 < \infty \end{aligned}$$

Thus, A_n is L^1-bounded and increasing. Thus, it has a limit a.s. Finally, we note that $M_n = X_n - X_0 - A_n$ and since A_n and X_n have limits a.s. then a linear combination has a limit a.s.

Exercise 4: Let X_n be i.i.d. random variables and T a stopping time for it.

a) Let $\mu = \mathbb{E}[X_n]$ and assume that X_n are nonnegative. Prove that

$$Y_n = \sum_{j=1}^{n} X_j - n\mu, \quad Y_0 = 0$$

is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

b) Show that

$$\mathbb{E}\left[\sum_{n=1}^{T} X_n\right] = \mathbb{E}[T]\mathbb{E}[X_1],$$

provided $\mathbb{E}[T], \mathbb{E}[X_1] < \infty$.

Answer:

a) First show that Y_n is integrable.

$$\begin{aligned}
\mathbb{E}[|Y_n|] &= \mathbb{E}[|\sum_{j=1}^{n} X_j - n\mu|] \\
&\leq \mathbb{E}[\sum_{j=1}^{n} |X_j| + |n\mu|] \\
&= \sum_{j=1}^{n} \mathbb{E}[|X_j|] + |n\mu|
\end{aligned}$$

Since $X_j \geq 0$ then so is μ, and we can drop all the absolute value signs.

$$\begin{aligned}
&= \sum_{j=1}^{n} \mathbb{E}[X_j] + n\mu \\
&= 2n\mu < \infty
\end{aligned}$$

We also have Y_n is F_n-measurable since it is a function of X_n and $X_n \in \sigma(X_1, \ldots, X_n) = \mathcal{F}_n$.

We now show the last property of martingales.

$$\mathbb{E}[Yn+k|\mathcal{F}_n] = \mathbb{E}[\sum_{j=1}^{n+k} X_j - (n+k)\mu|\mathcal{F}_n]$$

$$= \sum_{j=1}^{n} X_j + \mathbb{E}[\sum_{j=n+1}^{n+k} X_j|\mathcal{F}_n] - (n+k)\mu$$

$$= \sum_{j=1}^{n} X_j + \sum_{j=n+1}^{n+k} \mathbb{E}[X_j|\mathcal{F}_n] - (n+k)\mu$$

$$= \sum_{j=1}^{n} X_j + k\mu - (n+k)\mu$$

$$= \sum_{j=1}^{n} X_j - n\mu = Y_n$$

Where $\mathbb{E}[X_j|\mathcal{F}_n] = \mu$ since the X_j are i.i.d. Thus, Y_n is a martingale.

b) We use the fact that $\mathbb{E}[\mathbb{E}[\sum_{n=1}^{T} X_n|T]] = \mathbb{E}[\sum_{n=1}^{T} X_n]$. So

$$\mathbb{E}[\sum_{n=1}^{T} X_n] = \mathbb{E}[\mathbb{E}[\sum_{n=1}^{T} X_n|T]]$$

$$= \mathbb{E}[\sum_{n=1}^{T} \mathbb{E}[X_n|T]] = \mathbb{E}[\sum_{n=1}^{T} \mu]$$

$$= \mathbb{E}[T\mu] = \mathbb{E}[T]\mu = \mathbb{E}[T]\mathbb{E}[X_1]$$

Alternatively, since $\mathbb{E}[T] < \infty$, then T is a.s. bounded and so we can apply the Optional Stopping Theorem

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$$

$$\iff \mathbb{E}[\sum_{j=1}^{T} X_j - T\mu] = 0$$

$$\iff \mathbb{E}[\sum_{j=1}^{T} X_j] = \mathbb{E}[T\mu] = \mathbb{E}[T]\mu = \mathbb{E}[T]\mathbb{E}[X_1]$$