APC 523: Numerical Algorithms for Scientific Computing
Homework 2

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Exercise 1: "Barycentric" interpoloation formula

Answer: a) Recall the definitions:

$$L_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

$$p_N(x) = \sum_{j=0}^{N} f_j L_j(x)$$

Suppose we are interpolating $f(x) = 1$, then our interpolating function $p_N(x) = 1$ and all $f_j = 1$ and thus we have

$$p_N(x) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} L_j(x) = 1$$

b) Substituting their definitions, we get,

$$\begin{aligned}
\frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x) &= \frac{\prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{1}{x_j - x_k}}{x - x_j} \cdot \prod_{k=0}^{N}(x - x_k) \\
&= \prod_{\substack{k=0 \\ k\neq j}} \frac{1}{x_j - x_k} \cdot \prod_{\substack{k=0 \\ k\neq j}} x - x_k \\
&= \prod_{\substack{k=0 \\ k\neq j}} \frac{x - x_k}{x_j - x_k} = L_j(x)
\end{aligned}$$

Now, taking the definition, we have

$$L_j(x_j) = \prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{x_j - x_k}{x_j - x_k} = 1$$

Thus, $L_j(x_j) = 1$ and we have $L_k(x_j) = 0$ for $k \neq j$ since $L^{(N)}(x_j)/(x - x_j) = 0$. Thus, the logical branch is

$$L_j(x) = \begin{cases} \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x), & \text{if } x \neq x_j \\ 1, & \text{o.w.} \end{cases}$$

c) Now, suppose $x = x_k$ for some k, then we have

$$p_N(x_k) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} f_j L_j(x_k) = f_k$$

Thus, we get that $p_N(x)$ can be rewritten as when substituing b) as

$$p_N(x) = \begin{cases} L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j, & x \neq x_0, \ldots, x_N \\ f_k, \quad x = x_k \end{cases}$$

d) Using our results

$$\sum_{j=0}^{N} L_j(x) = 1$$

$$L_j(x) = \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x)$$

then,

$$\sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x) = 1$$

$$\implies L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} = 1$$

Now, injecting this into the denominator of part c) yields,

$$p_N(x) = \begin{cases} \frac{L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j}{L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j}}, & x \neq x_0, \ldots, x_N \\ f_k, \quad x = x_k \end{cases}$$

$$\implies p_N(x) = \begin{cases} \frac{\sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j}{\sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j}}, & x \neq x_0, \ldots, x_N \\ f_k, \quad x = x_k \end{cases}$$

e) See the implementation in `question1.ipynb`

f) See the implementation in `question1.ipynb`

Exercise 2: Chebyshev nodes of the second kind

Answer:

a) See the implementation in `question2.ipynb`

b) See the implementation in `question2.ipynb`

c) See the implementation in `question2.ipynb`

i) The maximum relative errors are listed below

Nodes	Relative error
4	1.58662e-2
8	1.62741e-6
16	1.03328e-13

ii) The maximum relative errors are listed below

Nodes	Relative error
4	1.25691
8	3.67927
16	51.0769

iii) The maximum relative errors are listed below for the 1^{st} kind

Nodes	Relative error
4	1.21570e-2
8	4.84377e-7
16	8.07804e-16

The maximum relative errors are listed below for the 2^{nd} kind

Nodes	Relative error
4	1.17164e-2
8	5.27579e-7
16	1.02670e-15

iv) The maximum relative errors are listed below for the 1^{st} kind

Nodes	Relative error
4	7.50300e-1
8	3.91740e-1
16	8.31070e-2

The maximum relative errors are listed below for the 2^{nd} kind

Nodes	Relative error
4	8.28912e-1
8	4.59605e-1
16	9.93219e-2

d) See the implementation in `question2.ipynb`.

   i) See the plots below. The slopes match the expected κ as N gets large enough.

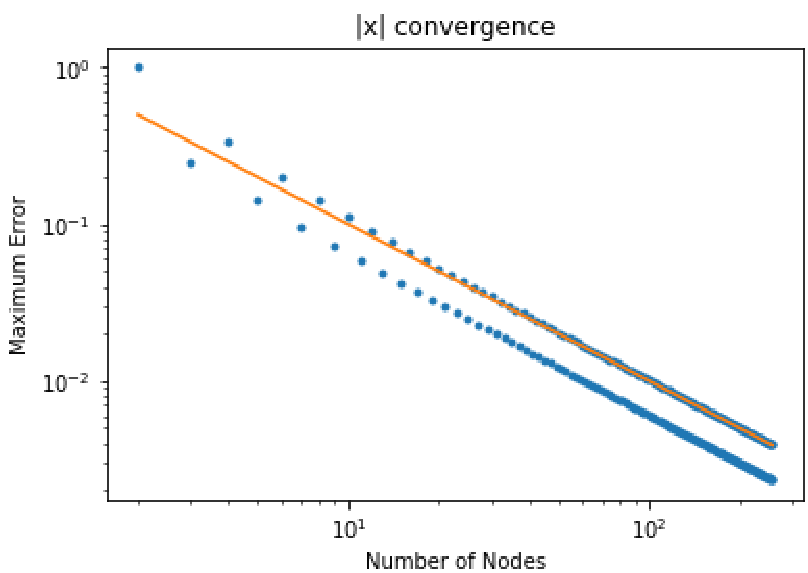

Figure 1: Convergence of $f(x) = |x|$

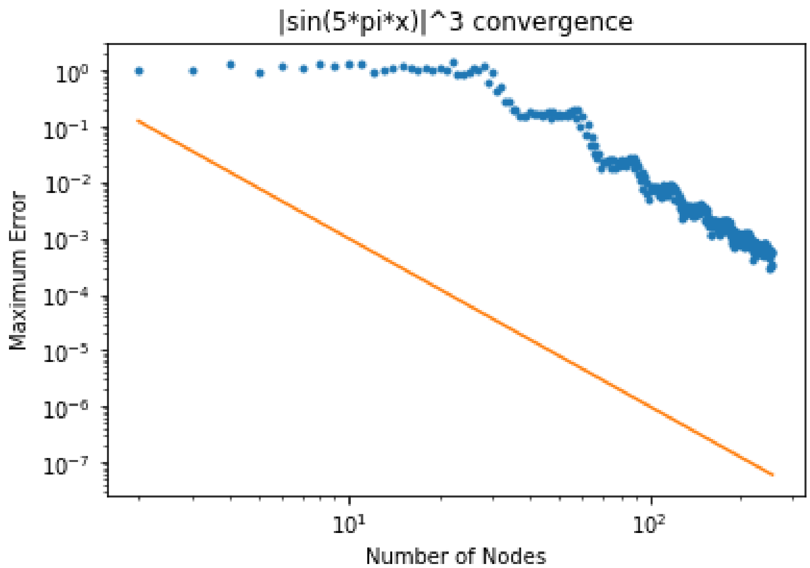

Figure 2: Convergence of $f(x) = |\sin(5\pi x)|^3$

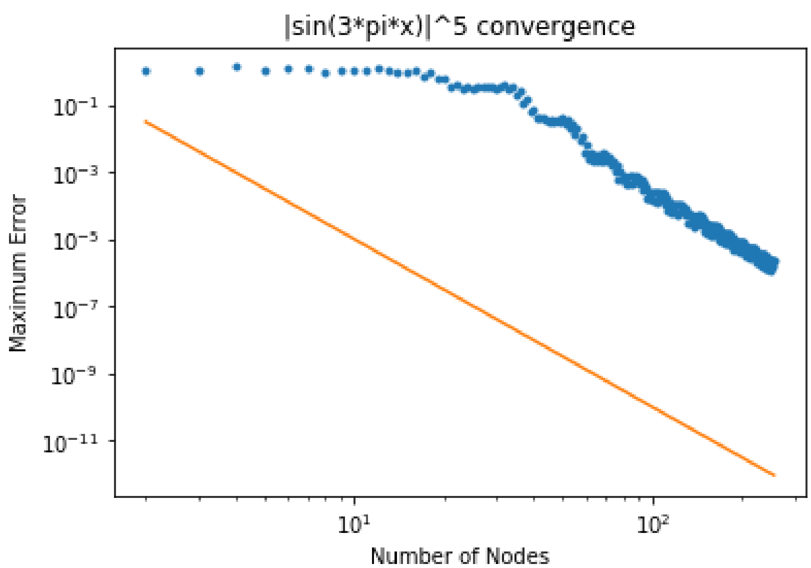

Figure 3: Convergence of $f(x) = |\sin(3\pi x)|^5$

ii) See the plots below. The slopes are linear before rounding errors occur.

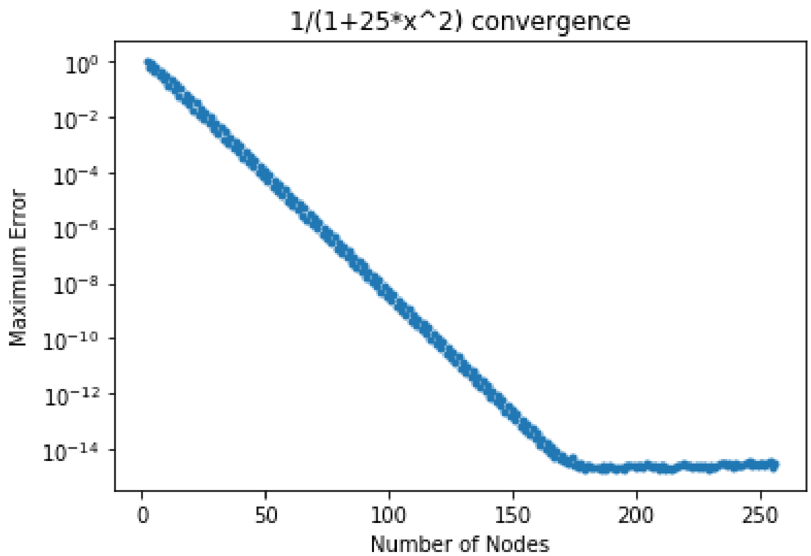

Figure 4: Convergence of $f(x) = \frac{1}{1+25x^2}$

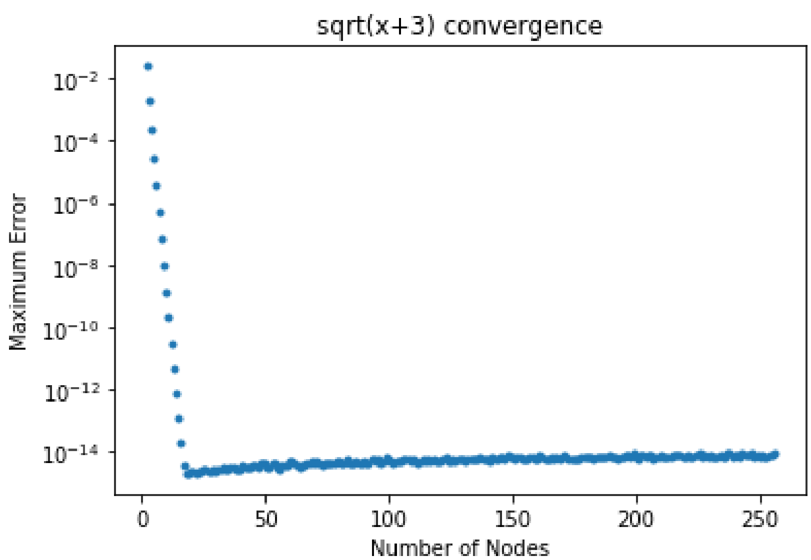

Figure 5: Convergence of $f(x) = \sqrt{x+3}$

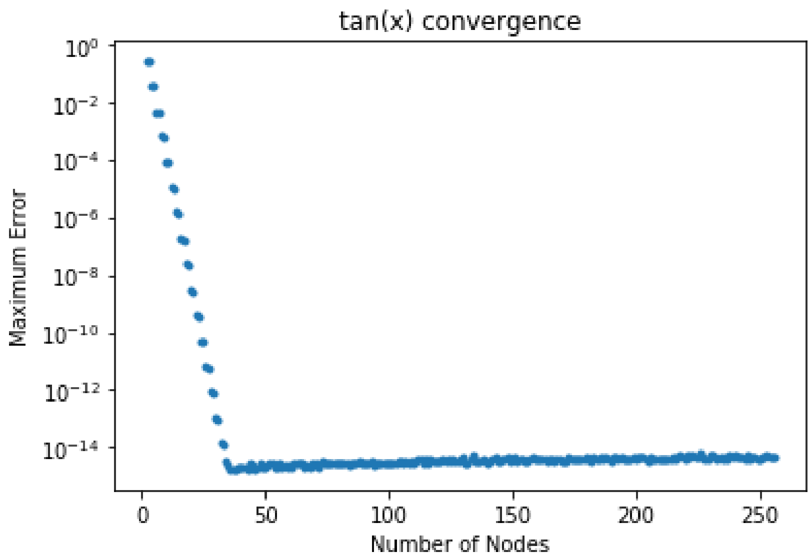

Figure 6: Convergence of $f(x) = \tan x$

iii) See the plots below. The slopes do appear to curve increasingly steep before rounding errors occur for the first two functions. For the last function, $f(x) = \cos(100\pi x)$, the number of zeroes in $[-1, 1]$ are so great that even for $N = 256$, there are not enough points to interpolate accurately.

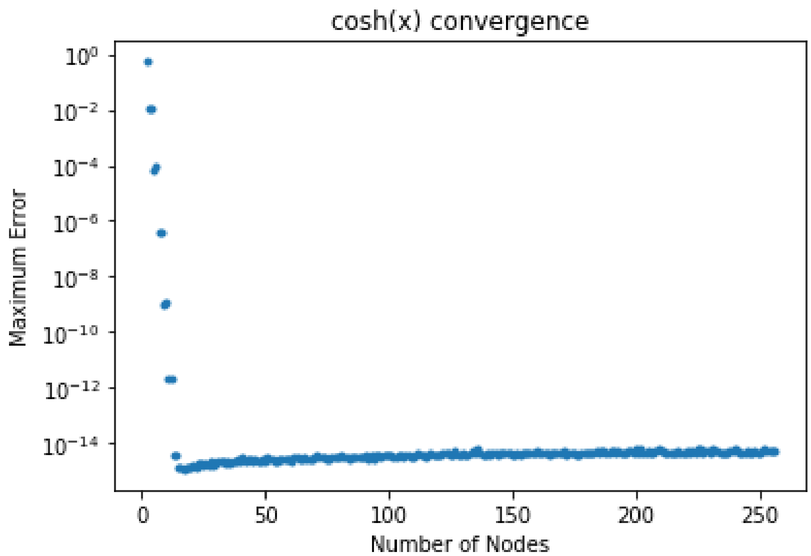

Figure 7: Convergence of $f(x) = \cosh x$

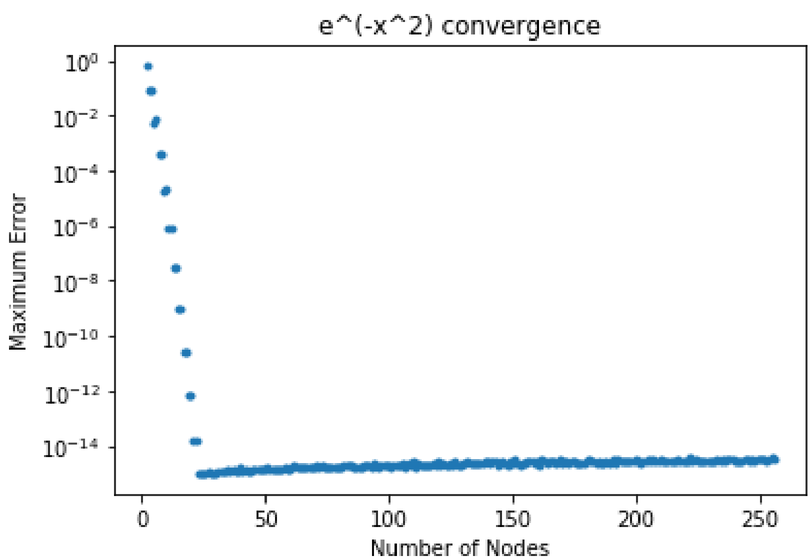

Figure 8: Convergence of $f(x) = e^{-x^2}$

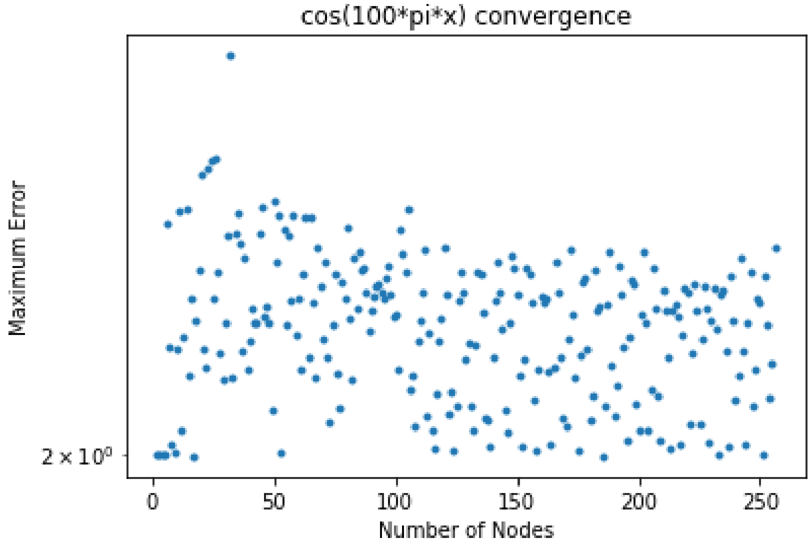

Figure 9: Convergence of $f(x) = \cos(100\pi x)$

Exercise 3: Chebyshev coefficients via the DCT

Answer: a) See the implementation in `question3.ipynb`

b) i) By inspection, the following Chebyshev polynomials satisfy the equation

$$f(x) = T_4(x) + T_3(x) + 5T_2(x) + 4T_1(x) + 5T_0(x)$$

ii) See the implementation in `question3.ipynb`. It matches the above.

c) The running times are in the table below. See the implementation in `question3.ipynb`.

Nodes	Time
2^{10}	330.6 μs
2^{13}	61.03 ms
2^{13}+1	520.0 μs
2^{17}	14.91 s
2^{17}+1	5.792 ms

The plot below shows the computation time for different number of nodes. Notice that the computation time rises somewhere between $O(k)$ (orange line) and $O(k^2)$ (green line).

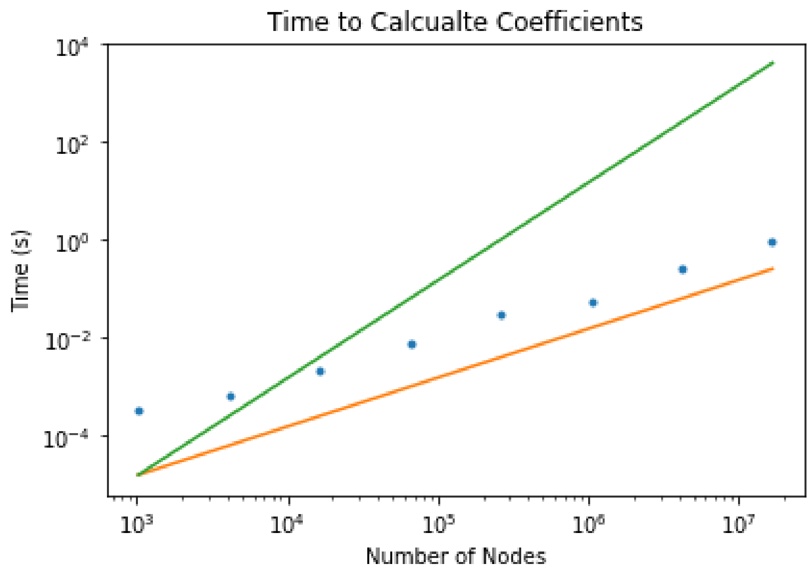

Figure 10: Computation Time vs. Number of Nodes