APC 523: Numerical Algorithms for Scientific Computing
Homework 2

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Exercise 1: "Barycentric" interpoloation formula

Answer: a) Recall the definitions:

$$L_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

$$p_N(x) = \sum_{j=0}^{N} f_j L_j(x)$$

Suppose we are interpolating $f(x) = 1$, then our interpolating function $p_N(x) = 1$ and all $f_j = 1$ and thus we have

$$p_N(x) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} L_j(x) = 1$$

b) Substituting their definitions, we get,

$$\begin{aligned}
\frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x) &= \frac{\prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{1}{x_j - x_k}}{x - x_j} \cdot \prod_{k=0}^{N} (x - x_k) \\
&= \prod_{\substack{k=0 \\ k\neq j}} \frac{1}{x_j - x_k} \cdot \prod_{\substack{k=0 \\ k\neq j}} x - x_k \\
&= \prod_{\substack{k=0 \\ k\neq j}} \frac{x - x_k}{x_j - x_k} = L_j(x)
\end{aligned}$$

Now, taking the definition, we have

$$L_j(x_j) = \prod_{\substack{k=0 \\ k\neq j}}^{N} \frac{x_j - x_k}{x_j - x_k} = 1$$

Thus, $L_j(x_j) = 1$ and we have $L_k(x_j) = 0$ for $k \neq j$ since $L^{(N)}(x_j)/(x - x_j) = 0$. Thus, the logical branch is

$$L_j(x) = \begin{cases} \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x), & \text{if } x \neq x_j \\ 1, & \text{o.w.} \end{cases}$$

c) Now, suppose $x = x_k$ for some k, then we have

$$p_N(x_k) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} f_j L_j(x_k) = f_k$$

Thus, we get that $p_N(x)$ can be rewritten as when substituing b) as

$$p_N(x) = \begin{cases} L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j, & x \neq x_0, \ldots, x_N \\ f_k, & x = x_k \end{cases}$$

d) Using our results

$$\sum_{j=0}^{N} L_j(x) = 1$$

$$L_j(x) = \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x)$$

then,

$$\sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} \cdot L^{(N)}(x) = 1$$

$$\implies L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} = 1$$

Now, injecting this into the denominator of part c) yields,

$$p_N(x) = \begin{cases} \frac{L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j}{L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j}}, & x \neq x_0, \ldots, x_N \\ f_k, & x = x_k \end{cases}$$

$$\implies p_N(x) = \begin{cases} \frac{\sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j} f_j}{\sum_{j=0}^{N} \frac{w_j^{(N)}}{x-x_j}}, & x \neq x_0, \ldots, x_N \\ f_k, & x = x_k \end{cases}$$

e) See the implementation in `question1.ipynb`

f) See the implementation in `question1.ipynb`