# ORFE 526: Probability Theory
# Homework 6

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**Exercise 1:**

a) Consider the normal random variable $X \sim N(\mu, \sigma^2)$, with $\mu \neq 0$. Prove that there is a unique $\theta \neq 0$ such that $\mathbb{E}[e^{\theta X}] = 1$

b) Let $(X_i)_{i \geq 0}$ be a sequence of independent random variables identically distributed as $N(\mu, \sigma^2)$, with $\mu \neq 0$. Consider the sum $S_n = \sum_{j=0}^{n} X_j$. Show that $Z_n = e^{\theta S_n}$ is a martingale, with $\theta$ defined in part a).

**Answer:**

a) We write out $\mathbb{E}[e^{\theta X}]$

$$\mathbb{E}[e^{\theta X}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{\theta x} dx$$

Do a change of variable with $z = \frac{x-\mu}{\sigma}$

$$= e^{\mu\theta} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{z\sigma\theta} dz$$
$$= e^{\mu\theta} e^{\frac{1}{2}\sigma^2\theta^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\theta)^2} e^{z\sigma\theta} dz$$
$$= e^{\mu\theta} e^{\frac{1}{2}\sigma^2\theta^2}$$

Since the inside of the integral is the pdf of a normal distribution $N(\theta, 0)$, it integrates to 1. Setting this result equal to 1 and solving for $\theta$ yields

$$\mu\theta + \frac{1}{2}\sigma^2\theta^2 = 0$$
$$\theta = -\frac{2\mu}{\sigma^2} \text{ (or 0)}$$

We ignore the case $\theta = 0$ by assumption. Thus, $\theta = -\frac{2\mu}{\sigma^2}$ is a unique solution.

b) We must show three things. First, that $Z_t$ is integrable. Since $Z_t \geq 0$ we do not need absolute values.

$$\mathbb{E}[Z_t] = \mathbb{E}[e^{\theta S_t}] = \mathbb{E}[e^{\theta(X_1 + \ldots + X_t)}]$$
$$= \mathbb{E}[e^{\theta X_1}] \cdot \ldots \cdot \mathbb{E}[e^{\theta X_t}] = 1 \cdot \ldots \cdot 1 = 1 < \infty$$

Where we can break up the expectation as they are independent. By choice of $\theta$, all the expectations are 1.

Clearly, $Z_t$ is $\mathcal{F}_t$ measurable as it is a continuous function of $X_1, \ldots, X_t$.

Now for the last property.

$$\begin{aligned}\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}[e^{\theta S_t}|\mathcal{F}_s] = \mathbb{E}[e^{\theta(X_1+\ldots+X_t)}|\mathcal{F}_s] \\ &= e^{\theta(X_1+\ldots+X_s)}\mathbb{E}[e^{\theta(X_{s+1}+\ldots+X_t)}|\mathcal{F}_s]\end{aligned}$$

Since we can pull out all the terms that are $\mathcal{F}_s$-measurable.

$$\begin{aligned}&= e^{\theta S_s}\mathbb{E}[e^{\theta X_{s+1}}] \cdot \ldots \cdot \mathbb{E}[e^{\theta X_t}] \\ &= Z_s 1 \cdot \ldots \cdot 1 = Z_s\end{aligned}$$

Where we can break up the conditional expectation because the $X_t$'s are independent and because for $t > s$ are independent of $\mathcal{F}_s$.

**Exercise 2:** Two processes $X_t$ and $Y_t$ are called conditionally uncorrelated, given $\mathcal{F}_t$, is

$$\mathbb{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0 \text{ a.s.} \quad \forall 0 \le s < t < \infty$$

Let $X_t$ and $Y_t$ be martingales with respect to filtration $\mathcal{F}_t$. Show that the process $Z_t = X_t Y_t$ is an $\mathcal{F}_t$-martingale if and only if $X_t$ and $Y_t$ are conditionally uncorrelated. Assume that $X_t$, $Y_t$, and $Z_t$ are integrable.

**Answer:** By assumption, $Z_t$ is integrable so we have the first property of martingales. Second, $Z_t = X_t Y_t$ so it is clearly $\mathcal{F}_t$-measurable since it is a product of $\mathcal{F}_t$-measurable processes. Thus, we only have to show the last property.

$$\begin{aligned}
&\mathbb{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] \\
&= \mathbb{E}[X_t Y_t - X_t Y_s - X_s Y_t + X_s Y_s|\mathcal{F}_s] \\
&= \mathbb{E}[X_t Y_t|\mathcal{F}_s] - \mathbb{E}[X_t Y_s|\mathcal{F}_s] - \mathbb{E}[X_s Y_t|\mathcal{F}_s] + \mathbb{E}[X_s Y_s|\mathcal{F}_s] \\
&= \mathbb{E}[X_t Y_t|\mathcal{F}_s] - Y_s \mathbb{E}[X_t|\mathcal{F}_s] - X_s \mathbb{E}[Y_t|\mathcal{F}_s] + X_s Y_s \\
&= \mathbb{E}[X_t Y_t|\mathcal{F}_s] - Y_s X_s - X_s Y_s + X_s Y_s \\
&= \mathbb{E}[X_t Y_t|\mathcal{F}_s] - X_s Y_s
\end{aligned}$$

Where the steps taken were expanding the product, then linearity of expectation, then removing $\mathcal{F}_s$-measurable functions from the conditional expectations, then using the fact that $X_t$ and $Y_t$ are martingales. Thus, we have that $X_t$ and $Y_t$ are conditionally uncorrelated iff $\mathbb{E}[X_t Y_t|\mathcal{F}_s] = X_s Y_s$ which is equivalent to $Z_t$ being a martingale. Thus, they are condtionally uncorrelated iff $Z_t$ is an $\mathcal{F}_t$-martingale.

**Exercise 3:** Let $W_t$ and $\widetilde{W}_t$ be two independent Wiener processes and $\rho$ be a constant with $|\rho| \leq 1$.

a) Show that the process $X_t = \rho W_t + \sqrt{1-\rho^2} \cdot \widetilde{W}_t$ is continuous and has the distribution $N(0, t)$.

b) Is $X_t$ a Wiener process?

**Answer:**

a) Notice that we can write $X_t = \rho(W_t - W_0) + \sqrt{1-\rho^2} \cdot (\widetilde{W}_t - \widetilde{W}_0)$. Since $W_t$ and $\widetilde{W}_t$ are martingales, then

$$\begin{aligned} W_t - W_0 &\sim N(0, t) \\ \widetilde{W}_t - \widetilde{W}_0 &\sim N(0, t) \end{aligned}$$

Thus, $\rho(W_t - W_0) \sim N(0, \rho^2 t)$ and $\sqrt{1-\rho^2}(\widetilde{W}_t - \widetilde{W}_0) \sim N(0, (1-\rho^2)t)$. Now, adding the two together yields another normal distribution with their means added and their variaces added. So,

$$\begin{aligned} X_t &= \rho(W_t - W_0) + \sqrt{1-\rho^2} \cdot (\widetilde{W}_t - \widetilde{W}_0) \\ &\sim N(0 + 0, \rho^2 t + (1-\rho^2)t) = N(0, t) \end{aligned}$$

Finally, since $X_t$ is a linear combination of continuous processes, it is itself continuous.

b) $X_t$ is indeed a Wiener process. $X_0 = \rho W_0 + \sqrt{1-\rho^2} \cdot \widetilde{W}_0 = 0$. We showed continuity in the previous part. Also using the previous part, $X_t - X_s \sim N(0, t-s)$. Finally, the increments are independent as $X_t$ is a combination of $W_t$ and $\widetilde{W}_t$ which have independent increments and are independent of each other.

**Exercise 4:**

a) Find $\mathbb{E}[W_t^2|\mathcal{F}_s]$ for $0 < s < t$.

b) Show $\mathbb{E}[W_t^3|\mathcal{F}_s] = 3(t-s)W_s + W_s^3$, for $0 < s < t$.

c) Show that $\mathbb{E}[\int_s^t W_u du|\mathcal{F}_s] = (t-s)W_s$.

d) Show that the process

$$X_t = W_t^3 - 3\int_0^t W_s ds$$

is a martingale with respect to $\mathcal{F}_t = \sigma\{W_s : s \leq t\}$

**Answer:**

a) We will find the equivalent $\mathbb{E}[W_{t+s}^2|\mathcal{F}_t]$

$$\begin{aligned}
&\mathbb{E}[W_{t+s}^2|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t + W_t)^2|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t)^2 - 2W_t(W_{t+s} - W_t) + W_t^2|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t)^2|\mathcal{F}_t] - 2\mathbb{E}[W_t(W_{t+s} - W_t)|\mathcal{F}_t] + \mathbb{E}[W_t^2|\mathcal{F}_t] \\
&= s - 2W_t\mathbb{E}[W_{t+s} - W_t|\mathcal{F}_t] + W_t^2 \\
&= s - 2 \cdot 0 + W_t^2 = W_t^2 + s
\end{aligned}$$

Where $\mathbb{E}[(W_{t+s} - W_t)^2|\mathcal{F}_t] = s$ and $\mathbb{E}[W_{t+s} - W_t|\mathcal{F}_t] = 0$ since $W_{t+s} - W_t \sim N(0, s)$. Thus we conclude that $\mathbb{E}[W_t^2|\mathcal{F}_s] = W_s^2 + t - s$

b) We again work with $\mathbb{E}[W_{t+s}^3|\mathcal{F}_t]$

$$\begin{aligned}
&\mathbb{E}[W_{t+s}^3|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t + W_t)^3|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t)^3 + 3W_t(W_{t+s} - W_t)^2 + 3W_t^2(W_{t+s} - W_t) + W_t^3|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t+s} - W_t)^3|\mathcal{F}_t] + \mathbb{E}[3W_t(W_{t+s} - W_t)^2|\mathcal{F}_t] \\
&\quad\quad + \mathbb{E}[3W_t^2(W_{t+s} - W_t)|\mathcal{F}_t] + \mathbb{E}[W_t^3|\mathcal{F}_t] \\
&= 0 + 3W_t s + 0 + W_t^3 = 3sW_t + W_t^3
\end{aligned}$$

Where we again use the facts $\mathbb{E}[(W_{t+s} - W_t)^2|\mathcal{F}_t] = s$ and $\mathbb{E}[W_{t+s} - W_t|\mathcal{F}_t] = 0$ since $W_{t+s} - W_t \sim N(0, s)$. Also, $\mathbb{E}[(W_{t+s} - W_t)^3|\mathcal{F}_t] = 0$ since the third moment of a normal distribution is $\mu^3 + 3\mu\sigma^2$ which is 0 since $\mu$ is 0. We conclude with $\mathbb{E}[W_t^3|\mathcal{F}_s] = W_s^3 + 3(t-s)W_s$

c) Since the inside integral is finite, we can swap the integral and expectation due to Fubini's theorem. Thus,

$$\mathbb{E}\left[\int_s^t W_u du|\mathcal{F}_s\right] = \int_s^t \mathbb{E}[W_u|\mathcal{F}_s]du$$
$$= \int_s^t W_s du = W_s \int_s^t du = W_s(t-s)$$

d) We show integrability,

$$\mathbb{E}[|X_t|] \leq \mathbb{E}[|W_t^3||] + 3\mathbb{E}\left[|\int_0^t W_s ds|\right]$$
$$\leq \mathbb{E}[|W_t^3||] + 3\mathbb{E}\left[\int_0^t |W_s|ds\right]$$
$$= \mathbb{E}[|(W_t - W_0)^3|] + 3\int_0^t \mathbb{E}[|W_s - W_0|]ds$$

Where we use the fact that the integral is finite to swap with expectation due to Fubini's theorem. We then note that $W_t - W_0 \sim N(0,t)$ and thus all these expectations are finite as the normal distribution has finite absolute moments. Thus, $\mathbb{E}[|X_t|] < \infty$.

Clearly, $X_t$ is a function of $W_s$, where $0 \leq s \leq t$ and those $W_s$ are measurable with respect to $\mathcal{F}_t$. Thus, $X_t$ is measurable with respect to $\mathcal{F}_t$.

Now for the last property of martingales,

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}\left[W_t^3 - 3\int_0^t W_s ds|\mathcal{F}_s\right]$$
$$= \mathbb{E}[W_t^3|\mathcal{F}_s] - 3\mathbb{E}\left[\int_0^t W_s ds|\mathcal{F}_s\right]$$
$$= 3(t-s)W_s + W_s^3 - 3\mathbb{E}\left[\int_0^s W_{s'}ds' + \int_s^t W_{s'}ds'|\mathcal{F}_s\right]$$
$$= 3(t-s)W_s + W_s^3 - 3\mathbb{E}\left[\int_0^s W_{s'}ds'|\mathcal{F}_s\right] - 3\mathbb{E}\left[\int_s^t W_{s'}ds'|\mathcal{F}_s\right]$$
$$= 3(t-s)W_s + W_s^3 - 3\int_0^s W_{s'}ds' - 3(t-s)W_s$$
$$= W_s^3 - 3\int_0^s W_{s'}ds' = X_s$$

**Exercise 5:** The process $X_t = W_t - tW_1$ is called the Brownian bridge pinned at both 0 and 1 (because $X_0 = X_1 = 0$).

a) Write $X_t$ as a convex combination of $W_t$ and $W_t - W_1$

b) Show that $X_t \sim N(0, t(t-1))$

**Answer:**

a) It is easy to verify that

$$X_t = W_t - tW_1 = (1-t)W_t + t(W_t - W_1)$$

b) Since $W_t = (W_t - W_0) \sim N(0, t)$ and $(W_t - W_1) \sim N(0, 1-t)$, we have that

$$\begin{gathered}(1-t)W_t \sim N(0, (1-t)^2 t) \text{ and} \\ t(W_t - W_1) \sim N(0, t^2(1-t))\end{gathered}$$

Thus we conclude that

$$\begin{aligned}X_t = (1-t)W_t + t(W_t - W_1) &\sim N(0, (1-t)^2 t + t^2(1-t)) \\ &= N(0, (1-t)t(1-t+t)) = N(0, (1-t)t)\end{aligned}$$