ORFE 526: Probability Theory
Homework 5

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Exercise 1: Toss a fair coin 4 times. Each toss yields either H (heads) or T (tails). Consider the following set:

$\mathcal{G}$ = we know the outcomes of the tosses but not the order.

Define the random variables:

X = number of $H-$ number of T

Y = number of T before the first H.

i) Show that X is $\mathcal{G}$-measurable while Y is not $\mathcal{G}$-measurable.

ii) Find the expectations $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[X|\mathcal{G}]$.

Answer:

i) X is $\mathcal{G}$-measurable since if we have the outcome, we have both the number of heads and the number of tails. Thus $\sigma(X) \subset \mathcal{G}$. Y is not $\mathcal{G}$-measurable since for a given outcome (say $\{HHTT\}$) we don't know if the order was $\{TTHH\}$ or $\{THHT\}$ which yields a value of $Y = 2$ and $Y = 1$ respectively.

ii) Let G be the number of heads. Then the number of tails is $4 - G$ and so $X = G - 4 + G = 2G - 4$

$$\mathbb{E}[X] = 2\mathbb{E}[G] - 4 = 4 - 4 = 0$$
$$\mathbb{E}[Y] = \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot 2 + \frac{1}{16} \cdot 3 + \frac{1}{16} \cdot 4 = \frac{15}{16}$$

Finally, X is $\mathcal{G}$-measurable so $\mathbb{E}[X|\mathcal{G}] = X$.

Exercise 2: Consider X to be a non-negative random variable with the distribution measure μ. Let μ be absolutely continuous (w.r.t. the Lebesgue measure) and $\mathbb{E}[X|X > t] = t + a$ for some constant $a > 0$ and all $t \geq 0$. Prove that μ is the exponential distribution with parameter $c = 1/a$.

Answer: Since it is absolutely continuous, we know that there exists a density by Radon-Nykodim theorem. So,

$$\mathbb{E}[X|X > t] = \frac{1}{P(X > t)} \int_t^\infty x f(x) dx = \frac{\int_t^\infty x f(x) dx}{\int_t^\infty f(x) dx} = t + a$$

Rearranging,

$$\int_t^\infty x f(x) dx = (t + a) \int_t^\infty f(x) dx$$

By the Lebesgue differentiation theorem, we can differentiate these integrals with respect to t,

$$-t f(t) = -(t + a) f(t) + \int_t^\infty f(x) dx$$
$$a f(t) = \int_t^\infty f(x) dx$$

Which relabelling $F(t) = \int_t^\infty f(x)$ then $F'(t) = -f(t)$,

$$-a F'(t) = F(t)$$

Which by the theory of ODE's has a unique solution. Namely,

$$F(t) = D e^{-\frac{1}{a} t}$$

So, $f(t) = -D \frac{1}{a} e^{-\frac{1}{a} t}$, which integrates to 1 since it is a density,

$$\int_0^\infty -D \frac{1}{a} e^{-\frac{1}{a} t} = D(0 - 1) = 1$$
$$D = -1$$

Which gives us the density $f(t) = \frac{1}{a} e^{-\frac{1}{a} t}$ which is the exponential distribution with parameter $1/a$.

Exercise 3: Prove or disprove: For any real valued random variables X, Y, and Z we have

$$\mathbb{E}[Z|X,Y] = \mathbb{E}[\mathbb{E}[Z|X]|Y]$$

(Note the meaning of the left side as $\mathbb{E}[Z|X,Y] = \mathbb{E}[Z|\sigma(X,Y)]$).

Answer: Consider the case when $\sigma(X) \subset \sigma(Y)$, then by the tower property

$$\mathbb{E}[\mathbb{E}[Z|X]|Y] = \mathbb{E}[Z|X]$$

On the other hand, since $\sigma(X) \subset \sigma(Y)$ then $\sigma(X,Y) = \sigma(Y)$. This yields,

$$\mathbb{E}[Z|X,Y] = \mathbb{E}[Z|Y]$$

In general,

$$\mathbb{E}[Z|X] \neq \mathbb{E}[Z|Y]$$

E.g., X is a constant and Y is non-constant. So $\mathbb{E}[Z|X,Y] = \mathbb{E}[\mathbb{E}[Z|X]|Y]$ does not hold in general.

Exercise 4: Let X be a Poisson distributed random variable with parameter λ. Consider the random variable $Y = \rho^X$, with $\rho > 1$ constant. Find the expectation $\mathbb{E}[Y]$.

Answer: Writting out the expectation,

$$\mathbb{E}[Y] = \mathbb{E}[\rho^X] = \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \rho^i = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda \cdot \rho)^i}{i!}$$

From calculus, this is the Taylor series for e^x, so

$$= e^{-\lambda} \cdot e^{\lambda \cdot \rho} = e^{(\rho-1)\lambda}$$

Exercise 5: Let $\mathcal{F}$ be a σ-algebra included in $\mathcal{H}$. If X is a $\mathcal{F}$-measurable random variable such that

$$\int_A X dP = 0, \quad \forall A \in \mathcal{F},$$

show that $X = 0$ almost surely.

Answer: By definition of conditional expectation,

$$\int_A X dP = 0, \quad \forall A \in \mathcal{F},$$

is equivalent to saying,

$$\mathbb{E}[X|\mathcal{F}] = 0$$

Since X is $\mathcal{F}$-measurable we can pull it out,

$$X \cdot \mathbb{E}[1|\mathcal{F}] = X = 0$$

That is, $X = 0$ almost surely.

Exercise 6: If $\mathcal{F} \subset \mathcal{G}$ prove that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{F}]$$

Answer: We have by the definition of conditional expectation,

$$\int_F \mathbb{E}[X|\mathcal{F}]dP = \int_F XdP \quad \forall F \in \mathcal{F}$$

Since $\mathcal{F} \subset \mathcal{G}$ we also have,

$$\int_F \mathbb{E}[X|\mathcal{G}]dP = \int_F XdP \quad \forall F \in \mathcal{F}$$

Setting these two equations equal,

$$\int_F \mathbb{E}[X|\mathcal{F}]dP = \int_F \mathbb{E}[X|\mathcal{G}]dP \quad \forall F \in \mathcal{F}$$

Which is the definition of conditional expectation on $\mathcal{F}$,

$$\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{F}]$$