Homework 7

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Question 1

First, let us write how the Brownian bridge vector is distributed. We have

$$\begin{pmatrix} W_t \\ W_{t_1} \\ W_{t_2} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t & t_1 & t \\ t_1 & t_1 & t_1 \\ t & t_1 & t_2 \end{pmatrix}\right)$$

Where the covariances are the result of $\mathrm{Cov}(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$ and we have that $t_1 \leq t \leq t_2$. Now, using the notation in the question, we have $X_{[1]} = W_t$, $X_{[2]} = \begin{pmatrix} W_{t_1} \\ W_{t_2} \end{pmatrix}$, and the following parameters

$$\begin{aligned}
\mu_{[1]} &= 0, \\
\mu_{[2]} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
x &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \\
\Sigma_{[11]} &= t, \\
\Sigma_{[12]} = \Sigma_{21}^T &= \begin{pmatrix} t_1 & t \end{pmatrix}, \\
\Sigma_{[22]} &= \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix}
\end{aligned}$$

For completeness, we have that $\Sigma_{[22]}^{-1} = \frac{1}{t_2 - t_1}\begin{pmatrix} t_2/t_1 & -1 \\ -1 & 1 \end{pmatrix}$. Now we are ready to simply apply the conditioning formula for Gaussians, we get that the conditional mean is

$$\begin{aligned}
\mathbb{E}[W_t | W_{t_1} = w_1, W_{t_2} = w_2] &= \mu_{[1]} + \Sigma_{[12]}\Sigma_{[22]}^{-1}(x - \mu_{[2]}) \\
&= 0 + \frac{1}{t_2 - t_1}\begin{pmatrix} t_1 & t \end{pmatrix}\begin{pmatrix} t_2/t_1 & -1 \\ -1 & 1 \end{pmatrix}\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \\
&= \frac{1}{t_2 - t_1}\begin{pmatrix} t_2 - t & -t_1 + t \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= \frac{(t_2 - t)w_1 + (t - t_1)w_2}{t_2 - t_1}
\end{aligned}$$

and the conditial variance is

$$
\begin{aligned}
\mathrm{Var}(W_t | W_{t_1} = w_1, W_{t_2} = w_2) &= \Sigma_{[11]} - \Sigma_{[12]}\Sigma_{[22]}^{-1}\Sigma_{[21]} \\
&= t - \frac{1}{t_2 - t_1}\begin{pmatrix} t_1 & t \end{pmatrix}\begin{pmatrix} t_2/t_1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} t_1 \\ t \end{pmatrix} \\
&= t - \frac{1}{t_2 - t_1}\begin{pmatrix} t_2 - t & -t_1 + t \end{pmatrix}\begin{pmatrix} t_1 \\ t \end{pmatrix} \\
&= t - \frac{t_1 t_2 - t_1 t - t_1 t + t^2}{t_2 - t_1} \\
&= \frac{t t_2 - t t_1 - t_1 t_2 + t_1 t + t_1 t - t^2}{t_2 - t_1} \\
&= \frac{(t_2 - t)(t - t_1)}{t_2 - t_1}
\end{aligned}
$$

Which are precisely the equations we wished to show.