Probability

Homework# 1 Solutions

Question 1.

Let $(A_n)_n \subset \mathcal{E}$. Since by De Morgan's relation

$$\bigcap_n A_n = (\bigcup_n A_n^c)^c,$$

then it suffices to show

$$(\bigcup_n A_n^c)^c \in \mathcal{E}. \tag{$*$}$$

Since $A_n \in \mathcal{E}$ and $\mathcal{E}$ is closed w.r.t. complements, $A_n^c \in \mathcal{E}$. Using that $\mathcal{E}$ is closed w.r.t. countable unions, $\bigcup_n A_n^c \in \mathcal{E}$. Again, using that $\mathcal{E}$ is closed w.r.t. complements, it follows that

$$(\bigcup_n A_n^c)^c \in \mathcal{E},$$

which is (*), and hence it ends the proof.

Question 2.

$$\widehat{\mathcal{D}} = \left\{A \in \mathcal{D}; A \bigcap D \in \mathcal{D}\right\} \tag{*}$$

We verify the axioms of a d-system:
(a) $E \in \widehat{\mathcal{D}}$
We have $E \in \mathcal{D}$ and $E \bigcap D = D \in \mathcal{D}$,

so E satisfies (*) and hence $E \in \widehat{\mathcal{D}}$.

(b) $A, B \in \widehat{\mathcal{D}}$, $A \supset B \implies A \backslash B \in \widehat{\mathcal{D}}$

$A \in \widehat{\mathcal{D}} \Longrightarrow A \in \mathcal{D}$, $\mathcal{A} \bigcap D \in \mathcal{D}$
$B \in \widehat{\mathcal{D}} \Longrightarrow B \in \mathcal{D}$, $\mathcal{B} \bigcap D \in \mathcal{D}$

Since $\mathcal{D}$ is a d-system, $A \backslash B \in \mathcal{D}$.
Then $(A\backslash B) \bigcap D = (A \bigcap D) \backslash (B \bigcap D) \in \mathcal{D}$ since $A \bigcap D \in \mathcal{D}$, $B \bigcap D \in \mathcal{D}$, $A \bigcap D \supset B \bigcap D$ and $\mathcal{D}$ is a d-system.

(c) $(A_n) \subset \widehat{\mathcal{D}}$, $A_n \uparrow A \Rightarrow A \in \widehat{\mathcal{D}}$

Since $A_n \uparrow A$,then $A = \bigcup_n A_n \in \mathcal{D}$,because $A_n \in \mathcal{D}$.

Note that $A_n \in \widehat{\mathcal{D}} \Rightarrow A_n \bigcap D \in \mathcal{D}$.
Then,

$A \bigcap D = (\bigcup_n A_n) \bigcap \mathcal{D} = \bigcup_n \left(\underbrace{A_n \bigcap D}_{\in \mathcal{D}} \right) = \bigcup_n B_n = B \in \mathcal{D}$, since $B_n \uparrow B$.

Hence, $A \in \widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}}$ becomes a d-system.

Question 3.

(i) Show $A \in f^{-1}(\mathcal{F}) \Rightarrow A^c \in f^{-1}(\mathcal{F})$
If $A \in f^{-1}(\mathcal{F})$,there is $B \in \mathcal{F}$ such that $A = f^{-1}(B)$.
Then, $A^c = E \backslash A = f^{-1}(F) \backslash f^{-1}(B) = f^{-1}(F \backslash B) = f^{-1}(B^c)$, since $B^c \in \mathcal{F}$, then $A^c \in f^{-1}(\mathcal{F})$.
Show $A_n \in f^{-1}(\mathcal{F}) \Longrightarrow \bigcup_n A_n \in f^{-1}(\mathcal{F})$
If $A_n \in f^{-1}(\mathcal{F})$, then $A_n = f^{-1}(B_n)$,with $B_n \in \mathcal{F}$.

Then,

$$\bigcup_n A_n = \bigcup_n f^{-1}(B_n) = f^{-1}(\bigcup_n B_n) \in f^{-1}(\mathcal{F}), \text{since} \bigcup_n B_n \in \mathcal{F}.$$

(ii) Almost obvious. Assume that $f^{-1}(\mathcal{F})$ is not the smallest σ-algebra with that property. Then there is a smaller σ-algebra, $\widetilde{\mathcal{E}}$, included strictly in $f^{-1}(\mathcal{F})$:

$$\widetilde{\mathcal{E}} \underset{\neq}{\subset} f^{-1}(\mathcal{F})$$

such that f is $(\widetilde{\mathcal{E}}, \mathcal{F})$–measurable.This means $f^{-1}(\mathcal{F}) \subset \widetilde{\mathcal{E}}$ which leads to the contradiction $f^{-1}(\mathcal{F}) \subset \widetilde{\mathcal{E}} \underset{\neq}{\subset} f^{-1}(\mathcal{F})$.

Question 4.

Let $\mathcal{F}_0 = \{(-\infty, r], r \in \mathbb{R}\}$.
Since $\sigma\mathcal{F}_0 = \mathcal{B}_{\mathbb{R}}$, it suffices to show $f^{-1}(\mathcal{F}_0) \subset \mathcal{B}_{\mathbb{R}}$.
Let $L = \sup f = \lim_{x \to \infty} f(x)$
If $r \geq L$, then

$$f^{-1}\left((-\infty, r]\right) = \{x : f(x) \leq r\} = \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \ .$$

If $r < L$, then

$$f^{-1}\left((-\infty, r]\right) = \{x : f(x) \leq r\} = \{x \leq f^{-1}(r)\} = (-\infty,\ f^{-1}(r)] = (-\infty, p] \in \mathcal{B}_{\mathbb{R}}$$

Hence, $f^{-1}(\mathcal{F}_0) \subset \mathcal{B}_{\mathbb{R}}$ and f is Borel measurable.

Question 5.

$\sigma\mathcal{C} = \bigcap\limits_{\mathcal{C}\subset\mathcal{E}_\alpha} \mathcal{E}_\alpha$ and $\sigma\mathcal{D} = \bigcap\limits_{\mathcal{D}\subset\mathcal{E}_\beta} \mathcal{E}_\beta$

Since $\mathcal{C} \subset \mathcal{D}$, then the family of σ-algebras $\{\mathcal{E}_\alpha\}_\alpha$ is more rich than the family $\{\mathcal{E}_\beta\}_\beta$. Hence, the intersection of the family $\{\mathcal{E}_\alpha\}_\alpha$ is included in the intersection of the family $\{\mathcal{E}_\beta\}_\beta$,i.e.

$$\bigcap_{\mathcal{C}\subset\mathcal{E}_\alpha} \mathcal{E}_\alpha \subset \bigcap_{\mathcal{D}\subset\mathcal{E}_\beta} \mathcal{E}_\beta$$

and hence $\sigma\mathcal{C} \subset\sigma\mathcal{D}$.

Question 6.

(i) Let $g : \mathbb{R} \to \mathbb{R}_+$, $g(x) = |x|$.
Since g is continuous, it is measurable. Then, $|f| = g \circ f$ is measurable, as composition of measurable functions.
(ii) Write explicitly:

$$f^+ = \begin{cases} f, & f \geq 0 \\ 0, & f < 0 \end{cases}$$

and

$$f^- = \begin{cases} 0, & f \geq 0 \\ -f, & f < 0 \end{cases}$$

Then,

$$\begin{aligned} f^+ + f^- &= \begin{cases} f, & f \geq 0 \\ 0 & f < 0 \end{cases} + \begin{cases} 0, & f \geq 0 \\ -f & f < 0 \end{cases} \\ &= \begin{cases} f, & f \geq 0 \\ -f & f < 0 \end{cases} = |f| \end{aligned}$$

(iii) Using, $f = f^+ - f^-$, and $|f| = f^+ + f^-$, solve for f^+ and f^- :

$$f^+ = \frac{f + |f|}{2} \text{ and } f^- = \frac{|f| - f}{2},$$

which are measurable as algebraic combinations of f and $|f|$, which are measurable.