Homework 10

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Question 1 Part 1

First, let us calculate the expected return from λ_{MKT}

$$\begin{aligned}
\mu_P &= r_f + h\lambda \\
\mu_{MKT} &= r_f + h\lambda_{MKT} \\
&= r_f + (\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota}) \cdot \frac{1}{\boldsymbol{\iota}^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})} \\
&= r_f + \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}{\boldsymbol{\iota}^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}
\end{aligned}$$

Now, the expected return of the frontier using the weights is

$$\begin{aligned}
\mu_P &= \boldsymbol{w}_P^T\boldsymbol{\mu} \\
\mu_{MKT} &= \boldsymbol{w}_{MKT}^T\boldsymbol{\mu} \\
&= \lambda_{MKT}\big(\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})\big)^T\boldsymbol{\mu}
\end{aligned}$$

Now, we add and subtract $r_f\boldsymbol{\iota}$ and we note that Σ is symmetric and thus so is its inverse.

$$\begin{aligned}
&= \lambda_{MKT}(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota} + r_f\boldsymbol{\iota}) \\
&= \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}{\boldsymbol{\iota}^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})} + \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(r_f\boldsymbol{\iota})}{\boldsymbol{\iota}^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})} \\
&= \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}{\boldsymbol{\iota}^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})} + r_f
\end{aligned}$$

Where the second term reduce to just r_f because the numerator and denominator are scalars and hence they are equal up to a multiple of r_f as they are transposes of each other. Thus, the expected returns are the same. Now for the variance. Again, start with calculating using λ_{MKT}.

$$\begin{aligned}
\sigma_P^2 &= h\lambda^2 \\
\sigma_{MKT}^2 &= \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}{(\iota^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota}))^T\iota^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})} \\
&= \frac{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}{(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}\iota\iota^T\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota})}
\end{aligned}$$

Now using the weights.

$$\begin{aligned}
\sigma_P^2 &= \boldsymbol{w}^T\Sigma\boldsymbol{w} \\
\sigma_{MKT}^2 &= \lambda_{MKT}^2(\boldsymbol{\mu}^T - r_f\boldsymbol{\iota}^T)\Sigma^{-1}\Sigma\Sigma^{-1}(\boldsymbol{\mu} - r_f\boldsymbol{\iota}) \\
&= \lambda_{MKT}^2 h
\end{aligned}$$

Which is now the same series of steps to get the final answer as before.

Question 1 Part 2

The slope of the CML is $\sqrt{h}$ as given by the equation

$$\mu_P = r_f + \sigma_P \sqrt{h}$$

Now, we have the following derivatives in mean-standard deviation space from (36) in the notes

$$\frac{d\mu_P}{d\lambda} = h$$
$$\frac{d\sigma_P}{d\lambda} = \frac{h\lambda}{\sigma_P}$$

Putting these together yields

$$\frac{d\mu_P}{d\sigma_P} = \frac{\sigma_P}{\lambda}$$

Now, we plug in the proper σ_{MKT} (which we calculated in part 1) and λ_{MKT}

$$\frac{d\mu_{MKT}}{d\sigma_{MKT}} = \frac{\sqrt{(\boldsymbol{\mu}^T - r_f \boldsymbol{\iota}^T)\Sigma^{-1}(\boldsymbol{\mu} - r_f \boldsymbol{\iota})}}{\boldsymbol{\iota}^T \Sigma^{-1}(\boldsymbol{\mu} - r_f \boldsymbol{\iota})} \cdot \boldsymbol{\iota}^T \Sigma^{-1}(\boldsymbol{\mu} - r_f \boldsymbol{\iota}) = \sqrt{h}$$