1 List of Problems for Chapter 3

This is a list of problems from Chapter 3 which I left in class as exercises or referred to during proofs. The Homework will be picked from these problems. They are a good practice list for the probability spaces chapter.

$(\Omega, \mathcal{H}, \mathbb{P})$ will denote a probability space and X_n a sequence of random variables on it.

1. (i) If X is integrable and $\{X_n\}_{n\geq 1}$ is uniformly integrable, show that $\{X_n - X\}_{n\geq 1}$ is uniformly integrable.
   (ii) Show that the result in part (i) holds true if X is replaced by a sequence Y_n of uniformly integrable functions.

2. If $X_n \longrightarrow X$ and $X_n \longrightarrow Y$, both in probability, show that $X = Y$ almost surely.

3. Let $X_n \longrightarrow X$ in probability. Show that $X_n^3 \longrightarrow X^3$ in probability.

4. Let $\mathcal{K}$ be a uniformly integrable family of random variables. Show the following:
   (i) $E[|X|] \leq b + k(b)$;
   (ii) $\mathcal{K} \subset L^1$;
   (iii) $\sup\limits_{X\in\mathcal{K}} \mathbb{E}[|X|] < \infty$.

5. Assume $\sum\limits_{n\geq 1} P\Big(|X_n - X| > \dfrac{1}{n}\Big) < \infty$. Show that $X_n \longrightarrow X$ almost surely.

6. Let X be a random variable and f and g be two functions, both increasing (or both decreasing). Show that
$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

7. Let X_n be a sequence of random variables. Assume there is a constant k such that $\lim\limits_{n\to\infty} \mathbb{E}[X_n] = k$ and $\lim\limits_{n\to\infty} Var(X_n) = 0$. Show that $X_n \longrightarrow k$ in L^2 (i.e. in the mean square).

8. Let $X_n \longrightarrow X$ in L^2, with $\mathbb{E}[X_n^2] < \infty$. Show the following:
   (i) $\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X]$;
   (ii) $\mathbb{E}[X_n^2] \longrightarrow \mathbb{E}[X^2]$;
   $(iii) Var[X_n] \longrightarrow Var[X]$;
   (iv) $Cov(X_n, X) \longrightarrow Var(X)$.

9. Let X_n be a sequence such that $\mathbb{E}[|X_n|] \longrightarrow 0$. Show that $X_n \longrightarrow 0$ in probability.

10. Let $\Omega = (0,1]$, $\mathcal{H} = \mathcal{B}_{(0,1]}$ and $\mathbb{P}$ be the Leb measure. Consider the sequence X_n defined as $X_1 = \mathbf{1}_{(0,1]}$, $X_2 = \mathbf{1}_{(0,1/2]}$, $X_3 = \mathbf{1}_{(1/2,1]}$, $X_4 = \mathbf{1}_{(0,1/3]}$, $X_5 = \mathbf{1}_{(1/3,2/3]}$, $X_5 = \mathbf{1}_{(2/3,1]}$, ...

    (i) Fix $\epsilon \in (0,1)$. Compute the probabilities $\mathbb{P}(X_n > \epsilon)$, for $n = 1, 2, 3....$

    (ii) Show that $X_n \longrightarrow 0$ in probability.

    (iii) Does X_n also converge almost surely?

11. Consider the probability space $(\Omega, \mathcal{H}, P)$, with $\Omega = [0,1]$, $\mathcal{H} = \mathcal{B}_{[0,1]}$, and P the Leb measure on $[0,1]$. Define the sequence X_n by

$$X_{2n}(\omega) = \begin{cases} 0, & \text{if } \omega < 1/2 \\ 1, & \text{if } \omega \geq 1/2 \end{cases} \qquad\qquad X_{2n+1}(\omega) = \begin{cases} 1, & \text{if } \omega < 1/2 \\ 0, & \text{if } \omega \geq 1/2 \end{cases}$$

    (i) Show that X_n converges in distribution.

    (ii) Show that X_n does not converge in probability.

12. Show that:

    (i) If $X_n \longrightarrow 0$ in L^2 and $Y_n \longrightarrow 0$ in L^2, then $X_n + Y_n \longrightarrow 0$ in L^2;

    (ii) If $X_n \longrightarrow X$ in L^2 and $Y_n \longrightarrow Y$ in L^2, then $X_n + Y_n \longrightarrow Y + Y$ in L^2.

13. Let X be a random variable with the distribution function

$$\mu(dx) = \frac{5}{\Gamma(1/5)\Gamma(4/5)} \frac{1}{x^5 + 1} dx, \qquad x \geq 0.$$

    (i) Show $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[X^4] = \infty$;

    (ii) Construct the sequences of random variables $X_n = Y_n = \frac{1}{n} X$.
    Show that: $X_n \longrightarrow 0$ in L^2, $Y_n \longrightarrow 0$ in L^2, and $X_n Y_n \longrightarrow \infty$ in L^2. What do you notice?

14. Let $B_1, B_2, \cdots$ be independent Bernoulli variables with success probability p. Consider $S_n = X_1 + \cdots + X_n$ be the number of successes in the first n trials. Then $\overline{B}_n = \frac{S_n}{n}$ is the success frequency in the first n trials. Show that $\overline{B}_n \to p$ almost surely. Find an approximation of the distribution of $\overline{B}_n$ using a normal distribution for n large.

15. Let $X_1, X_2, \cdots$ be sequence of independent, identically distributed random variables taking values in the measurable space $(E, \mathcal{E})$. Let μ be their common distribution. Define

$$F_n(A) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_A \circ X_k, \qquad A \in \mathcal{E}.$$

    (i) Show that for each $\omega \in \Omega$, the mapping $A \longmapsto F_n(\omega, A)$ is a probability measure on $(E, \mathcal{E})$.

    (ii) Let $A \in \mathcal{E}$. Show that $F_n(A) \longrightarrow \mu(A)$ almost surely.

(*iii*) Let $f \in \mathcal{E}_+$. Show that

$$\frac{1}{n}\sum_{k=1}^{n} f \circ X_k \longrightarrow \mu f, \text{ almost surely.}$$

16. If X is a random variable with $E[|X|] = 0$, show that $X = 0$ a.s.

17. Consider a sequence of random variables X_n such that there is another random variable X such that $\sum_n \|X_n - X\|^2 < \infty$ a.s. Show that $X_n \to X$ almost surely.

18. Let X be an integrable random variable. Show that if H_n is a sequence of events such that $P(H_n) \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \mathbb{E}[|X|1_{H_n}] = 0$. Does the converse hold true?

19. Let $X_n \xrightarrow{d} X$. Show that $c_n(x) \to c(x)$, for all $x \in \mathbb{R}$. (Recall that $c(x)$ denotes the distribution function of X).