ORFE 526: Probability Theory
Homework 1

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Exercise 1: Let $\mathcal{E}$ be a σ-algebra and consider a sequence $(A_n)_n \subset \mathcal{E}$. Prove that $\bigcap\limits_{n\geq 1} A_n \in \mathcal{E}$.

Answer: We know by De Morgan's law that $\bigcap\limits_{n\geq 1} A_i = \left(\bigcup\limits_{n\geq 1} A_i^c\right)^c$. As well, σ-algebras are closed under complementation and union:

$$
\begin{aligned}
(A_n)_n \subset \mathcal{E} &\implies (A_n^c)_n \subset \mathcal{E} && \text{Since closed under complementation} \\
&\implies \bigcup_{n\geq 1} A_i^c \in \mathcal{E} && \text{Since closed under union} \\
&\implies \left(\bigcup_{n\geq 1} A_i^c\right)^c \in \mathcal{E} && \text{Since closed under complementation} \\
&\implies \bigcap_{n\geq 1} A_i \in \mathcal{E} && \text{By De Morgan's}
\end{aligned}
$$

Exercise 2: Let $\mathcal{D}$ be a a d-system on E. Fix D in $\mathcal{D}$ and define

$$\widehat{\mathcal{D}} = \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$$

Prove that $\widehat{\mathcal{D}}$ is a d-system.

Answer: To show that something is a d-system, we must show the following 3 properties:

1. $E \in \mathcal{D}$
2. if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \backslash A \in \mathcal{D}$
3. if $A_1, A_2, A_3, \ldots \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then $A_n \nearrow A \in \mathcal{D}$

Proof of these properties for $\widehat{\mathcal{D}} = \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$:

1. Let $A = E \in \mathcal{D}$, then $A \cap D = E \cap D = D$ since the intersection of a set and the universe is just the set. Note that $D \in \mathcal{D}$ by definition. So, $E \cap D \in \mathcal{D}$ which implies $E \in \widehat{\mathcal{D}}$
2. Let $A, B \in \widehat{\mathcal{D}}$ and $A \subseteq B$, then

   Since $A, B \in \widehat{\mathcal{D}}$, then $(B \cap D) \in \mathcal{D}$ and $(A \cap D) \in \mathcal{D}$
   Since $\mathcal{D}$ is a d-system, then $(B \cap D) \backslash (A \cap D) \in \mathcal{D}$
   Note that $(B \cap D) \backslash (A \cap D) = (B \backslash A) \cap D$
   So, $(B \backslash A) \cap D \in \mathcal{D}$
   Thus, $B \backslash A \in \widehat{\mathcal{D}}$

3. Let $A_1, A_2, A_3, \ldots \in \widehat{\mathcal{D}}$ and $A_n \subseteq A_{n+1}$, $n \geq 1$, then

   Since $A_1, A_2, A_3, \ldots \in \widehat{\mathcal{D}}$, then $A_n \cap D \in \mathcal{D}, n \geq 1$
   Since $\mathcal{D}$ is a d-system, then $\displaystyle\bigcup_{n \geq 1} (A_n \cap D) \in \mathcal{D}$
   Distributing the union we have, $(\cup_{n \geq 1} A_n) \bigcap (\cup_{n \geq 1} D) = A \cap D$
   Where $\cup_{n \geq 1} A_n = A$ since $\mathcal{D}$ is a d-system
   So, $A \cap D \in \mathcal{D}$
   Thus, $A \in \widehat{\mathcal{D}}$

All 3 properties hold, thus $\widehat{\mathcal{D}}$ is a d-system.

Exercise 3: Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a function $f : E \to F$. Define $f^{-1}(\mathcal{F}) = \{f^{-1}(B) : B \in \mathcal{F}\}$. Prove that:

(i) $f^{-1}(\mathcal{F})$ is a σ-algebra.

(ii) $f^{-1}(\mathcal{F})$ is the smallest σ-algebra on E such that f is measurable relative to it and $\mathcal{F}$.

Answer:

(i) First, note that the inverse image preserves complements and union:

Complementation:

$$\begin{aligned} a \in f^{-1}(A^c) &\iff f(a) \in A^c \\ &\iff f(a) \notin A \\ &\iff a \notin f^{-1}(A) \\ &\iff a \in f^{-1}(A)^c \end{aligned}$$

Union:

$$\begin{aligned} a \in f^{-1}(A \cup B) &\iff f(a) \in A \cup B \\ &\iff f(a) \in A \text{ or } f(a) \in B \\ &\iff a \in f^{-1}(A) \text{ or } a \in f^{-1}(B) \\ &\iff a \in f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

We now use these to show $f^{-1}(\mathcal{F})$ is closed under complementation and countable union:

Complementation:

$$\begin{aligned} A \in f^{-1}(\mathcal{F}) &\iff A = f^{-1}(B) \text{ for some } B \\ &\iff A^c = f^{-1}(B)^c \\ &\iff A^c = f^{-1}(B^c) \text{ , note that } B^c \in \mathcal{F} \text{ since it is } \sigma\text{-algebra} \\ &\iff A^c \in f^{-1}(\mathcal{F}) \end{aligned}$$

Union:

$$\begin{aligned}
A_n \in f^{-1}(\mathcal{F}) &\iff A_n = f^{-1}(B_n) \text{ for some } B_n \\
&\iff \bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} f^{-1}(B_n) \\
&\iff \bigcup_{n\geq 1} A_n = f^{-1}\big(\bigcup_{n\geq 1} B_n\big)
\end{aligned}$$

Note that $\bigcup_{n\geq 1} B_n \in \mathcal{F}$ since it is σ-algebra

$$\iff \bigcup_{n\geq 1} A_n \in f^{-1}(\mathcal{F})$$

Also, $E \in f^{-1}(\mathcal{F})$ since $f^{-1}(F) = E$. Thus, $f^{-1}(\mathcal{F})$ is a σ-algebra.

(ii) Assume there is a smaller σ-algebra such that $\mathcal{E} \subset f^{-1}(\mathcal{F})$. But we know that f is measurable so, $f^{-1}(\mathcal{F}) \subset \mathcal{E}$. But this contradicts our assumption, therefore there does not exist a smaller σ-algebra and $f^{-1}(\mathcal{F}) = \mathcal{E}$.

Exercise 4: Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be an increasing function. Show that f is Borel measurable.

Answer: A function is Borel measurable if for every $r \in \mathbb{R}$, then $E = \{x : f(x) \leq r\}$ is measurable.

Define $b = \sup f^{-1}\big((-\infty, r]\big)$
If $b = \infty$, then $E = \mathbb{R}$
If $b = -\infty$, then $E = \emptyset$
If $b \in \mathbb{R}$, then $E = (-\infty, b]$ or $E = (-\infty, b)$
because $f(x) \leq r$ for all $x \in E$ since f is increasing.

All of these E's are elements of the Borel set and hence Borel measurable.

Exercise 5: Let $\mathcal{C}, \mathcal{D} \subset 2^E$. Show that $\mathcal{C} \subset \mathcal{D} \implies \sigma\mathcal{C} \subset \sigma\mathcal{D}$

Answer:

By assumption and definition, $\mathcal{C} \subset \mathcal{D} \subset \sigma\mathcal{D}$
The definition of $\sigma\mathcal{C}$ is: $\bigcap \mathcal{E} = \sigma\mathcal{C}$ where $\mathcal{E}$ is any σ-algebra containing $\mathcal{C}$
$\sigma\mathcal{D}$ is one such $\mathcal{E}$. Thus, $\sigma\mathcal{C} \subset \sigma\mathcal{D}$

Exercise 6: Let $(E, \mathcal{E})$ be a measurable space and $f : E \to \mathbb{R}$ a Borel measurable function.

(i) Show that $|f|$ is measurable

(ii) Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Show that $|f| = f^+ + f^-$

(iii) Use (i) and (ii) to show that f^+ and f^- are measurable

Answer:

(i) We will show that the absolute value does not change measurability

For $r < 0$, $\{x : |f(x)| \leq r\} = \emptyset$
For $r = \infty$, $\{x : |f(x)| \leq \infty\} = E$
For $r \geq 0$, $\{x : |f(x)| \leq r\} = \{x : -r \leq f(x) \leq r\}$
$= \{x : f(x) \leq r\} \bigcap \{x : f(x) \leq -r\}^c$
Note that the final line is the intersection of two measurable sets since f itself is measurable. Note that an intersection of two measurable set is measurable since σ-algebras are closed under countable intersection.

Thus, since for every $r \in \mathbb{R}$, $\{x : |f(x)| \leq r\}$ is measurable, then $|f|$ is measurable.

(ii) Break f into two cases, $f \geq 0$ and $f < 0$:

$$\begin{aligned}
&f \geq 0 : \text{ Then } \max\{f, 0\} = f \text{ and } -\min\{f, 0\} = 0 \\
&f^+ + f^- = f + 0 = f \\
&f < 0 : \text{ Then } \max\{f, 0\} = 0 \text{ and } -\min\{f, 0\} = -f \\
&f^+ + f^- = 0 - f = -f
\end{aligned}$$

Thus, we have $f^+ + f^-$ defined piecewise as: $f = \begin{cases} f & f \geq 0 \\ -f & f < 0 \end{cases}$

Which is the exact definition of $|f|$. Hence, $|f| = f^+ + f^-$

(iii) One cleverly notes that we can rewrite f^+ and f^- as follows:

$$f^+ = \frac{|f| + f}{2}$$
$$f^- = \frac{|f| - f}{2}$$

And note from lecture that the sum of two measurable functions is measurable.