ORFE 527: Stochastic Calculus
Homework 7

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Exercise 1: (Scaling property of local times) Let B be a standard Brownian motion and $\ell(t,a)$, $t \geq 0$, $a \in \mathbb{R}$ be the associated process of local times. Prove that, for any $c > 0$, the process $(B_{ct}/\sqrt{c}, \ell(ct, \sqrt{c}a/\sqrt{c}))$, $t \geq 0$, $a \in \mathbb{R}$ has the same law as $(B_t, \ell(t,a))$, $t \geq 0$, $a \in \mathbb{R}$.

Answer: First, we show that $B_t \overset{d}{=} B_{ct}/\sqrt{c}$. We note that $B_{ct} - B_{cs}$ will be distributed normal with mean zero and variance $c(t-s)$ by the properties of Brownian motion. Taking s to be 0, we have that

$$B_{ct}/\sqrt{c} \sim \mathcal{N}(0,t)$$

and so we conclude that $B_t \overset{d}{=} B_{ct}/\sqrt{c}$. Now we use the fact that $X \overset{d}{=} Y$ and $f(X)|X \overset{d}{=} f(Y)|Y$ iff $(X, f(X)) \overset{d}{=} (Y, f(Y))$. This comes from the fact that

$$\begin{aligned}
&(X, f(X)) \overset{d}{=} (Y, f(Y)) \\
&\iff \mathbb{P}(X, f(X)) = \mathbb{P}(Y, f(Y)) \\
&\iff \mathbb{P}(f(X)|X)\mathbb{P}(X) = \mathbb{P}(f(Y)|Y)\mathbb{P}(Y)
\end{aligned}$$

As we have shown $X \overset{d}{=} Y$, all we have to show is that $f(X)|X$ and $f(Y)|Y$ have the same functional form. By definition, we have

$$\ell(t,a) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|B_s - a| \leq \epsilon\}} \mathrm{d}s$$

Now, we aim to retrieve this from the other conditional law.

$$\ell(ct, \sqrt{c}a)|B_{ct}/\sqrt{c} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^{ct} 1_{\{|B_s - \sqrt{c}a| \leq \epsilon\}} \mathrm{d}s$$

Now, doing a change of variable of $s' = s/c$, we get

$$\begin{aligned}
\ell(ct, \sqrt{c}a) &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t c 1_{\{|B_{cs} - \sqrt{c}a| \leq \epsilon\}} \mathrm{d}s \\
\ell(ct, \sqrt{c}a) &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t c 1_{\{|B_{cs}/\sqrt{c} - a| \leq \epsilon/\sqrt{c}\}} \mathrm{d}s
\end{aligned}$$

Again, we do a change of variable from ϵ to $\epsilon/\sqrt{c}$ which doesn't change the limit

$$\begin{aligned}
\ell(ct, \sqrt{c}a) &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \sqrt{c} 1_{\{|B_{cs}/\sqrt{c} - a| \leq \epsilon\}} \mathrm{d}s \\
\iff \ell(ct, \sqrt{c}a)/\sqrt{c} &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|B_{cs}/\sqrt{c} - a| \leq \epsilon\}} \mathrm{d}s
\end{aligned}$$

Now conditioning the above on $B_{cs}/\sqrt{c}$ and conditioning $\ell(t,a)$ on B_t gives the same functional form. Thus, we have that $\ell(ct,\sqrt{c}a)/\sqrt{c}|B_{cs}/\sqrt{c} \stackrel{d}{=} \ell(t,a)|B_t$ and we conclude that

$$(\ell(t,a),B_t) \stackrel{d}{=} (\ell(ct,\sqrt{c}a)/\sqrt{c},B_{cs}/\sqrt{c})$$

Exercise 2: (Local time at zero) Let B be a standard Brownian motion and $\ell(t,0)$, $t \geq 0$ be the local time it accumulates at 0.

a) Show that the event $\{\ell(t,0) > 0 \text{ for all } t > 0\}$ has probability one. Conclude that for any stopping time τ with $\mathbb{P}(0 < \tau < \infty) = 1$ it holds $\ell(\tau,0) > 0$ with probability one.

b) Find the distribution of the random vector $(B_t, \ell(t,0))$ for any given $t \geq 0$.

Answer: a) First, we know from class that

$$\ell(t,0) \overset{d}{=} \max_{0 \leq s \leq t} B_s$$

From the reflection principle, we also have that

$$\begin{aligned} \mathbb{P}(\max_{0 \leq s \leq t} B_s \geq a) &= 2\mathbb{P}(B_t \geq a) \\ &= \mathbb{P}(|B_t| \geq a) \end{aligned}$$

Thus, we have $|B_t| \overset{d}{=} \ell(t,0)$. Since $\mathbb{P}(|B_t| = 0) = 0$ for all $t > 0$, we have

$$\mathbb{P}(\ell(t,0) > 0) = 1, \ \forall \ t > 0$$

We wish to bring the universal quantifier inside the probability. Take a sequence $t_n > 0$ with $t_n \searrow 0$ as $n \to \infty$. Now, as $\ell(t,0)$ is non-decreasing in t, $\ell(t_n,0) > 0$ implies $\ell(t,0) > 0$ for all $t_n > t$. So we have the following equivalence of events

$$\{\ell(t,0) > 0, \ \forall \ t > 0\} = \bigcap_n \{\ell(t_n,0) > 0\}$$

As all the events in the intersection have probability 1, then the intersection occurs with probability 1. Hence,

$$\mathbb{P}(\ell(t,0) > 0, \ \forall \ t > 0) = 1$$

Now, since we have

$$\{\ell(t,0) > 0, \ \forall \ t > 0\} \cap \{0 < \tau < \infty\} \subset \{\ell(\tau,0) > 0\}$$

The left side is an intersection of events that occur with probability 1, so we conclude

$$\mathbb{P}(\ell(\tau,0) > 0) = 1$$

b) First, let us find the distribution of (M_t, B_t) where $M_t = \max_{0\le s\le t} B_s$. Working through the various cases and applying the reflection principle yields,

$$\begin{aligned} F = \mathbb{P}(M_t \ge x, B_t \ge y) &= 1_{\{x<0 \text{ or } y\ge x\}}\mathbb{P}(B_t \ge y) \\ &\quad + 1_{\{x\ge 0 \text{ and } y<x\}}(\mathbb{P}(B_t \ge x) + \mathbb{P}(B_t \in [x, 2x-y])) \\ &= 1_{\{x<0 \text{ or } y\ge x\}}(1-\Phi(\frac{y}{\sqrt{t}})) \\ &\quad + 1_{\{x\ge 0 \text{ and } y<x\}}(1-2\Phi(\frac{x}{\sqrt{t}}) + \Phi(\frac{2x-y}{\sqrt{t}})) \end{aligned}$$

Now, to find the pdf, we take the partial derivatives with respect to y and x. First, take the derivative with respect to x,

$$\frac{\partial F}{\partial x} = 1_{\{x\ge 0 \text{ and } y<x\}}(-\frac{2}{\sqrt{t}}\phi(\frac{x}{\sqrt{t}}) + \frac{2}{\sqrt{t}}\phi(\frac{2x-y}{\sqrt{t}}))$$

Now the with respect to y,

$$\frac{\partial^2 F}{\partial x \partial y} = -1_{\{x\ge 0\}}1_{\{y<x\}}\frac{2}{t}\phi'(\frac{2x-y}{\sqrt{t}})$$

So we have that

$$\mathbb{P}(M_t \ge x, B_t \ge y) = \int_y^\infty \int_x^\infty 1_{\{u\ge 0\}}1_{\{v<u\}}\frac{-2}{t}\phi'(\frac{2u-v}{\sqrt{t}})dudv$$

From class, we have shown that $(M_t - B_t, M_t) \overset{d}{=} (|B_t|, \ell(t,0))$. Let us relate these two by finding the pdf of $(|B_t|, \ell(t,0))$. By definition of being equal in distribution, we have for all h postive measurable function

$$\mathbb{E}[h(M_t - B_t, M_t)] = \mathbb{E}[h(|B_t|, \ell(t,0))]$$

We calculate both sides of this equation, first the left side

$$\mathbb{E}[h(M_t - B_t, M_t)] = \int_{-\infty}^\infty \int_{-\infty}^\infty h(x-y, x)1_{\{x\ge 0\}}1_{\{y<x\}}\frac{-2}{t}\phi'(\frac{2x-y}{\sqrt{t}})dxdy$$

Now, we want to extract another $f(\cdot,\cdot)$ so we do the change of variable $z = x-y$ which yields

$$\begin{aligned} &\mathbb{E}[h(M_t - B_t, M_t)] \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty h(z, x)1_{\{x\ge 0\}}1_{\{z>0\}}\frac{-2}{t}\phi'(\frac{x+z}{\sqrt{t}})dxdz \end{aligned}$$

Since this equals $\mathbb{E}[h(|B_t|, \ell(t,0))]$, we conclude that the pdf of $(|B_t|, \ell(t,0))$ is

$$f(z,x) = 1_{\{x\geq 0\}} 1_{\{z>0\}} \frac{-2}{t} \phi'(\frac{x+z}{\sqrt{t}})$$

Now, by Tanaka's formula, we have the following equivalence

$$\ell(t,0) = |B_t| - \int_0^t \operatorname{sign}(B_s) \mathrm{d}B_s \quad = |-B_t| - \int_0^t \operatorname{sign}(-B_s) \mathrm{d}(-B_s)$$

Thus we have

$$\begin{aligned} &(-B_t, \ell(t,0)) \stackrel{d}{=} (B_t, \ell(t,0)) \\ \Longleftrightarrow\ &\mathbb{P}(-B_t \leq x, \ell(t,0) \leq y) = \mathbb{P}(B_t \leq x, \ell(t,0) \leq y) \\ \Longleftrightarrow\ &\mathbb{P}(B_t \geq -x, \ell(t,0) \leq y) = \mathbb{P}(B_t \leq x, \ell(t,0) \leq y) \end{aligned}$$

Now, we use this fact to get

$$\begin{aligned} &\mathbb{P}(|B_t| \leq |x|, \ell(t,0) \leq y) \\ &= \mathbb{P}(B_t \leq x, \ell(t,0) \leq y) + \mathbb{P}(B_t \geq -x, \ell(t,0) \leq y) \\ &= 2\mathbb{P}(B_t \leq x, \ell(t,0) \leq y) \end{aligned}$$

Thus, to get the pdf of $(B_t, \ell(t,0))$, we differentiate $\frac{1}{2}\mathbb{P}(|B_t| \leq |x|, \ell(t,0) \leq y)$ with respect to x and y. This gets us

$$g(x,y) = \frac{1}{2} f(|x|, y) = -\frac{1}{t} 1_{\{y\geq 0\}} \phi' \left(\frac{|x| + y}{\sqrt{t}} \right)$$

Plugging in the defition of $\phi'(\cdot)$ yields the resulting pdf of $(B_t, \ell(t,0))$

$$g(x,y) = \frac{1}{t} \frac{1}{\sqrt{2\pi}} \frac{|x|+y}{\sqrt{t}} e^{-\frac{1}{2}(\frac{|x|+y}{\sqrt{t}})^2} 1_{\{y\geq 0\}}$$

Exercise 3: (Yet another definition of local time) Let B be a standard Brownian motion and $\ell(t,0)$, $t \geq 0$ be the local time it accumulates at 0. Prove that

$$\ell(t,0) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{E}[|B_{s+\epsilon}||B_s] - |B_s| \mathrm{d}s$$

almost surely.

Hint: apply the occupation time formula to the integral on the right-hand side.

Answer: To begin, we calculate what $\mathbb{E}[|B_{s+\epsilon}||B_s]$ is. We note that

$$|B_{s+\epsilon}||B_s = |B_\epsilon + x|$$

Where we condition on $B_s = x$. We know that $B_\epsilon + x \sim \mathcal{N}(x, \epsilon)$ and so, taking the absolute values, we have that $|B_\epsilon + x|$ is a folded normal distribution. Thus, the expectation becomes

$$\mathbb{E}[|B_\epsilon + x|] = \sqrt{\frac{2\epsilon}{\pi}} e^{-\frac{x^2}{2\epsilon}} + x - 2x\Phi\left(\frac{-x}{\sqrt{\epsilon}}\right)$$

Using the fact that $\Phi(-x) = 1 - \Phi(x)$, we write the above equivalently as

$$\mathbb{E}[|B_\epsilon + x|] = \sqrt{\frac{2\epsilon}{\pi}} e^{-\frac{x^2}{2\epsilon}} + |x| - 2|x|\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)$$

Now we apply the occupation time formula to get

$$\begin{aligned}
&\int_0^t \mathbb{E}[|B_{s+\epsilon}||B_s] - |B_s| \mathrm{d}s \\
&= \int_{-\infty}^{\infty} \left(\mathbb{E}[|B_{s+\epsilon}||B_s = x] - |x|\right) \ell(t,x) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\sqrt{\frac{2\epsilon}{\pi}} e^{-\frac{x^2}{2\epsilon}} - 2|x|\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t,x) \mathrm{d}x
\end{aligned}$$

Dividing by ϵ yields

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t,x) \mathrm{d}x$$

Now, we note that

$$\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \mathrm{d}x = 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \mathrm{d}x = 2$$

We also have that

$$\begin{aligned}
&\int_{-\infty}^{\infty} -2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\mathrm{d}x \\
&= -2\int_{-\infty}^{\infty} \frac{|x|}{\epsilon} \int_{-\infty}^{-|x|/\sqrt{\epsilon}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y\mathrm{d}x \\
&= -1
\end{aligned}$$

Where the last line is reached using your favourite symbolic solver (or Fubini's to exchange the order of integration). Thus, we have the nice result that

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \mathrm{d}x = 1$$

With these in hand, let us try to compute the integral with the local time. Let us note that $\ell(t, x)$ is integrable as

$$\int_{-\infty}^{\infty} \ell(t, x) = \int_0^t 1\mathrm{d}s = t$$

Now, for all $c > 0$, define

$$\begin{aligned}
(\star) \equiv &\int_c^{\infty} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t, x)\mathrm{d}x \\
+ &\int_{-\infty}^{-c} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t, x)\mathrm{d}x
\end{aligned}$$

Then

$$\begin{aligned}
|(\star)| \leq &\int_{-\infty}^{\infty} \ell(t, x)\mathrm{d}x \cdot \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \right) \\
+ &\int_c^{\infty} \left(2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t, x)\mathrm{d}x \\
+ &\int_{-\infty}^{-c} \left(2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right) \ell(t, x)\mathrm{d}x
\end{aligned}$$

The first term yields

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \ell(t,x)\mathrm{d}x \cdot \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \right) \\
&= \lim_{\epsilon \downarrow 0} t \sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \\
&= 0
\end{aligned}$$

Likewise, we have that

$$2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right) = O(e^{-\frac{|x|}{\sqrt{\epsilon}}}) \leq De^{-\frac{|x|}{\sqrt{\epsilon}}}$$

For some constant D. Then, the remaining two terms yield (using the appropriate bounds of integration)

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \int_{c}^{\infty} \left(2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right) \right) \ell(t,x)\mathrm{d}x \\
&\leq \lim_{\epsilon \downarrow 0} De^{-\frac{|x|}{\sqrt{\epsilon}}} \int_{c}^{\infty} \ell(t,x)\mathrm{d}x \\
&\leq Dte^{-\frac{|x|}{\sqrt{\epsilon}}} \\
&= 0
\end{aligned}$$

Thus, we have that $(\star) \to 0$ as $\epsilon \to 0$. Now, by continuity, we have that there is a $c > 0$ such that $|\ell(t,x) - \ell(t,0)| \leq \delta$ for all $x \in [-c, c]$. We now show that the limit in question approaches $\ell(t,0)$.

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \left| \ell(t,0) - \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right) \right) \ell(t,x)\mathrm{d}x \right| \\
&\leq \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right) \right) |\ell(t,0) - \ell(t,x)|\mathrm{d}x
\end{aligned}$$

Where this results from the convexity of $|\cdot|$ and that the large term integrates to 1. Now, we break up the integrand into four parts

$$\begin{aligned}
&\leq \lim_{\epsilon\downarrow 0}(\star) + \int_{-\infty}^{-c}\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right)\mathrm{d}x\cdot|\ell(t,0)| \\
&\quad + \int_{c}^{\infty}\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right)\mathrm{d}x\cdot|\ell(t,0)| \\
&\quad + \int_{-c}^{c}\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right)\mathrm{d}x\cdot\delta
\end{aligned}$$

Now, we take the limit as $\epsilon \to 0$. We know that $(\star) \to 0$. For the same reason, we have that the next two integrals also go to 0. Now, the last terms is less than δ since the integrand is bounded by 1. Thus, we conclude that

$$\lim_{\epsilon\downarrow 0}\left|\ell(t,0) - \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}} - 2\frac{|x|}{\epsilon}\Phi\left(\frac{-|x|}{\sqrt{\epsilon}}\right)\right)\ell(t,x)\mathrm{d}x\right| \leq \delta$$

And so we have that

$$\ell(t,0) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t \mathbb{E}[|B_{s+\epsilon}||B_s] - |B_s|\mathrm{d}s$$

Exercise 4: (Two-sided Skorokhod problem) Given a continuous function $f : [0, \infty) \to \mathbb{R}$ with $f(0) \in (0, 1)$, show that there is a unique pair of function $g, h : [0, \infty) \to \mathbb{R}$ with the following three properties:

i) $r(t) := f(t) + g(t) - h(t) \in [0, 1], t \geq 0$

ii) $g(0) = h(0) = 0$ and g, h are non-decreasing

iii) $\int_0^\infty 1_{\{r(t)>0\}} \mathrm{d}g(t) = 0$, $\int_0^\infty 1_{\{r(t)<1\}} \mathrm{d}h(t) = 0$

The mapping $\Gamma : f \to r$ is called the *two-sided Skorokhod map* and the image of a standard Brownian motion under Γ is called a *reflected Brownian motion on* $[0, 1]$.

Answer: By the guidance of God (and a paper by Shreve et al. in 2005) let's suppose the functions are

$$g(t) = \max_{s \in [0,t]} (h(s) - f(s))_+$$
$$h(t) = \max_{s \in [0,t]} (f(s) + g(s) - 1)_+$$

Let's show that these satisfy the three properties.

i) $r(t) = f(t) + g(t) - h(t)$. We have that $g(t) \geq h(t) - f(t)$ so

$$r(t) \geq f(t) + h(t) - f(t) - h(t) = 0$$

Likewise, $-h(t) \leq -f(t) - g(t) + 1$ which implies

$$r(t) \leq f(t) + g(t) - f(t) - g(t) + 1 = 1$$

Thus, we have $r(t) \in [0, 1]$ for all $t \geq 0$.

ii) We have that $g(0) = (h(0) - f(0))_+$ and $h(0) = (f(0) + g(0) - 1)_+$. From this, the only way these are simultaneously satisfied is if

$$g(0) = h(0) = 0$$

This comes from the fact that if $g(0) > 0$, then $h(0) > f(0)$. But if $h(0) > f(0)$ then $g(0) > 1$ which is impossible by the previous part. Thus, $g(0) = 0$. As $f(0) \in (0, 1)$ and $g(0) = 0$, then $h(0) = 0$. Also, since these are running maximums, then the functions are non-decreasing.

iii) We use the same tactic from class and use Fatou's lemma

$$\int_0^\infty 1_{\{r(t)>0\}} \mathrm{d}g(t) \leq \liminf_{\epsilon \downarrow 0} \int_0^\infty 1_{\{r(t)>\epsilon\}} \mathrm{d}g(t)$$

Now, we have that $\{t : r(t) > \epsilon\} = \bigcap_{i=1}^\infty (s_i, t_i)$. So,

$$\forall i \int_{s_i}^{t_i} 1_{\{r(t)>\epsilon\}} \mathrm{d}g(t) = g(t_i) - g(s_i)$$

We now have that $r(t) > \epsilon$ if and only if $h(t) - f(t) < g(t) - \epsilon$ which implies,

$$\begin{aligned} g(t_i) &= \max(g(s_i), \max_{s \in [s_i, t_i]} h(s) - f(s)) \\ &\leq \max(g(s_i), \max_{s \in [s_i, t_i]} g(s) - \epsilon) \end{aligned}$$

Now, we cannot have $g(t_i) \leq g(t_i) - \epsilon$ and so we have that $g(t_i) = g(s_i)$ and we conclude that

$$\forall i \int_{s_i}^{t_i} 1_{\{r(t)>\epsilon\}} \mathrm{d}g(t) = g(t_i) - g(s_i) = 0$$

And so

$$\int_0^\infty 1_{\{r(t)>0\}} \mathrm{d}g(t) = 0$$

We now repeat the steps for

$$\int_0^\infty 1_{\{r(t)<1\}} \mathrm{d}h(t) \leq \liminf_{\epsilon \downarrow 0} \int_0^\infty 1_{\{r(t)<1-\epsilon\}} \mathrm{d}h(t)$$

We again have for $\{t : r(t) > \epsilon\} = \bigcap_{i=1}^\infty (s_i, t_i)$,

$$\forall i \int_{s_i}^{t_i} 1_{\{r(t)<1-\epsilon\}} \mathrm{d}h(t) = h(t_i) - h(s_i)$$

We now have that $r(t) < 1 - \epsilon$ if and only if $f(t) + g(t) - 1 < h(t) - \epsilon$ which implies,

$$\begin{aligned} h(t_i) &= \max(h(s_i), \max_{s \in [s_i, t_i]} f(s) + g(s) - 1) \\ &\leq \max(g(s_i), \max_{s \in [s_i, t_i]} h(s) - \epsilon) \end{aligned}$$

Now, we cannot have $h(t_i) \leq h(t_i) - \epsilon$ and so we have that $h(t_i) = h(s_i)$ and we conclude that

$$\forall i \int_{s_i}^{t_i} 1_{\{r(t)<1-\epsilon\}} \mathrm{d}h(t) = h(t_i) - h(s_i) = 0$$

And so

$$\int_0^\infty 1_{\{r(t)<1\}} \mathrm{d}h(t) = 0$$

This proves existence. Now for uniqueness. Consider two possible pairs (g, h) and (g', h'). Following the steps done in class, we prove that $r = r'$. By contradiction, assume that $r \neq r'$. Then, wlog, there is a T such that $r(T) > r'(T)$. Now define

$$\tau = \max(t < T : r(t) = r'(t))$$

Thus, we have that $r(t) > r'(t)$ on $t \in (\tau, T]$. Hence we have

$$0 \leq r'(t) < r(t) \leq 1 \ \forall \ t \in (\tau, T]$$

So $r'(t) < 1$ and $r(t) > 0$ for $t \in (\tau, T]$ and by property iii)

$$\begin{aligned} h'(\tau) &= h'(T) \\ g(\tau) &= g(T) \end{aligned}$$

Now, putting this all together,

$$\begin{aligned} 0 < r(T) - r'(T) &= g(T) - h(T) - g'(T) + h'(T) \\ &= g(\tau) - g'(T) + h'(\tau) - h(T) \\ &\leq g(\tau) - g'(\tau) + h'(\tau) - h(\tau) \\ &= r(\tau) - r'(\tau) = 0 \end{aligned}$$

This is a contradiction and so $r = r'$. Note that $r = r'$ implies that $g - h = g' - h'$. Now we show that $h = h'$. Again, wlog, assume by contradiction that there exists T such that $h(T) > h'(T)$. Define

$$\tau = \max(t < T : h(t) = h'(t))$$

Then we have $h(t) > h'(t)$ for $t \in (\tau, T]$. Using this,

$$\begin{aligned}
&h(T) - h(\tau) - h'(T) - h'(\tau) \\
&= h(T) - h'(T) \\
&= \int_\tau^T \mathrm{d}(h(t) - h'(t)) \\
&= \underbrace{\int_\tau^T 1_{\{r(t)<1\}} \mathrm{d}(h(t) - h'(t))}_{=0} + \int_\tau^T 1_{\{r(t)=1\}} \mathrm{d}(h(t) - h'(t)) \\
&= \int_\tau^T 1_{\{r(t)=1\}} \mathrm{d}(h(t) - h'(t)) > 0
\end{aligned}$$

However, as this is positive, then there exists some interval $[a, b] \subset (\tau, T]$ where $r(t) = 1$ and so

$$h(a) - h'(a) < h(b) - h'(b)$$

But this would imply also that

$$g(a) - g'(a) < g(b) - g'(b)$$

But, by property iii), and $r(t) > 0$ on $[a, b]$ we have that

$$\begin{aligned}
&\int_a^b \mathrm{d}(g(t) - g'(t)) = 0 \\
\implies\ & g(b) - g(a) = g'(b) - g'(a)
\end{aligned}$$

Which is a contradiction and so we conclude $h = h'$. As $r = r'$ and $h = h'$, then $g = g'$. We conclude that the pair (g, h) is unique.