ORFE 524: Statistical Theory and Methods
Homework 1

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Exercise 1: Recall the definition of σ-Algebra. Let (Ω, Σ) be a measurable space, that is, Σ satisfies the following three properties:

1. $\Sigma \neq \emptyset$, $\Sigma \subseteq 2^{\Omega}$.
2. $A \in \Sigma$ implies that $A^c \in \Sigma$. Here we use A^c to denote the complement of A.
3. $\forall A_1, A_2, \ldots \in \Sigma$, we have $\cap_{i \geq 1} A_i \in \Sigma$.

Based on these properties, solve the following problems:

1) Show that Σ is closed under union.
2) Show that Σ must contain $\emptyset$ and Ω.
3) Suppose $A \subseteq \Omega$, what is the smallest σ-algebra containing A?
4) Show that the set of all rational numbers, denoted by $\mathcal{Q}$, is Borel measurable. That is, $\mathcal{Q} \in \mathcal{B}(\mathcal{R})$.

Answer:

1) We know by De Morgan's law that $A \cup B = (A^c \cap B^c)^c$. Let $A, B \in \Sigma$:

$$\begin{aligned} A, B \in \Sigma &\implies A^c, B^c \in \Sigma && \text{By property 2} \\ &\implies A^c \cap B^c \in \Sigma && \text{By property 3} \\ &\implies (A^c \cap B^c)^c \in \Sigma && \text{By property 2} \\ &\implies A \cup B \in \Sigma && \text{By De Morgan's} \end{aligned}$$

2) Let $A \in \Sigma$ then $A^c \in \Sigma$ by property 2. Now,

$$A \cap A^c = \emptyset \in \Sigma \text{ by property 3}$$
$$\emptyset^c = \Omega \in \Sigma \text{ by property 2}$$

Thus, $\emptyset, \Omega \in \Sigma$.

3) If $A \in \Sigma$ then $A^c \in \Sigma$ necessarily. Also, by 2), $\emptyset, \Omega \in \Sigma$, so the smallest possible set is:

$$\Sigma = \{\emptyset, A, A^c, \Omega\}$$

This is clearly closed under complementation. It is also very quick to verify that this set is closed under intersection:

$$X \cap \Omega = X \in \Sigma, \text{ for all } X \in \Sigma$$
$$X \cap \emptyset = \emptyset \in \Sigma, \text{ for all } X \in \Sigma$$
$$\text{Lastly, } A \cap A^c = \emptyset \in \Sigma$$

4) Recall that $\mathcal{B}(\mathbb{R})$ is the σ-algebra formed by the open sets of $\mathbb{R}$. Equivalently, it is also formed by the closed sets. Also recall the facts from real analysis that singletons are closed and the rational numbers are countable. Using 1) that $\mathcal{B}(\mathbb{R})$ is closed under countable union,

$$\mathbb{Q} = \Big(\bigcup_{q \text{ rational}} q \Big) \in \mathcal{B}(\mathbb{R})$$

That is, $\mathbb{Q}$ is the union of a countable number of closed sets, therefore, $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$

Exercise 2: let P be a probability measure on (Ω, Σ). Only utilizing the definition of probability measure given in the class, solve the following problem.

1) Show that for any $A, B \in \Sigma$ satisfying $A \subseteq B$, we have $0 \leq P(A) \leq P(B)$.

2) Show that for any positive k, we have

$$P\big(\bigcup_{i=1}^{k} A_i\big) \leq \sum_{i=1}^{k} P(A_i)$$

3) Does the previous inequality still hold when $k = \infty$?

Answer:

1) In general, we can write $B = B \backslash A \bigcup B \cap A$. But, $B \cap A = A$ since $A \subseteq B$. So, $B = B \backslash A \cup A$. By definition, $B \backslash A$ and A are disjoint. So we can use the property of probability measures:

$$P(B) = P(B \backslash A \cup A) = P(B \backslash A) + P(A)$$
$$P(B) \geq P(A) \text{ since } P(B \backslash A) \geq 0$$

Also, $P(A) \geq 0$ since it is a probability measure. Thus, $0 \leq P(A) \leq P(B)$.

2) We use a similar idea as in 1). We construct a new sequence that is a partition. Define:

$$B_1 = A_1$$
$$B_i = A_i \backslash (A_{i-1} \cup A_{i-2} \cup \ldots \cup A_1)$$

By construction, $B_i \subseteq A_i$ and $\bigcup\limits_{i=1}^{k} A_i = \bigcup\limits_{i=1}^{k} B_i$. So,

$$P\big(\bigcup_{i=1}^{k} A_i\big) = P\big(\bigcup_{i=1}^{k} B_i\big) = \sum_{i=1}^{k} P(B_i)$$

Since $B_i \subseteq A_i$, by the previous part,

$$\sum_{i=1}^{k} P(B_i) \leq \sum_{i=1}^{k} P(A_i)$$

Which gives us the result $P\big(\bigcup\limits_{i=1}^{k} A_i\big) \leq \sum\limits_{i=1}^{k} P(A_i)$.

3) Yes, the inequality still holds if $k = \infty$. Let $\lim_{i\to\infty} A_i = A$. Then by the continuity of the probability measures,

$$\lim_{i\to\infty} P\big(\bigcup_{i=1}^{k} A_i\big) = P\big(\lim_{i\to\infty} \bigcup_{i=1}^{k} A_i\big) = P(A)$$

Now taking the limit on the right side of the inequality,

$$\lim_{i\to\infty} \sum_{i=1}^{k} P(A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Thus, since the left hand side converges,

$$\lim_{i\to\infty} P\big(\bigcup_{i=1}^{k} A_i\big) \leq \lim_{i\to\infty} \sum_{i=1}^{k} P(A_i)$$
$$\implies P\big(\bigcup_{i=1}^{\infty} A_i\big) = P(A) \leq \sum_{i=1}^{\infty} P(A_i)$$

Exercise 3: For any measurable function f, show that

$$\left|\int f dP\right| < \infty \text{ if and only if } \int |f| dP < \infty$$

Answer: Suppose that $\left|\int f dP\right| < \infty$,

$$\left|\int f dP\right| = \left|\int f^+ dP - \int f^- dP\right| < \infty$$

Then f is integrable by definition since this implies $\int f^+ dP < \infty$ and $\int f^- dP < \infty$. Note that $|f| = f^+ + f^-$. So,

$$\int |f| dP = \int f^+ + f^- dP = \int f^+ dP + \int f^- dP$$

But we have shown that both of these integrals are finite. So,

$$\int |f| dP < \infty$$

Now suppose $\int |f| dP < \infty$. Note that,

$$\int f^+ dP \leq \int |f| dP < \infty \text{ and}$$
$$\int f^- dP \leq \int |f| dP < \infty$$

Which implies $\left|\int f dP\right| = \left|\int f^+ dP - \int f^- dP\right| < \infty$.

Exercise 4: This exercise consists of two questions, concerns the σ-finiteness of a measure.

1) Show that the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is σ-finite.
2) Show that the counting measure on $(\Omega, 2^\Omega)$ is σ-finite if and only if Ω is countable.

Answer:

1) Define the sequence of A_i's as follows,

$$A_i = (i, i+1]$$

Then we have $\mu(A_i) = 1$ for $i \in \mathbb{Z}$ and $\bigcup_{i\in\mathbb{Z}} A_i = \mathbb{R}$. Thus the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is σ-finite.

2) Suppose Ω is countable. That is, without loss of generality, $\Omega = \mathbb{N}$. Define the sequence of $A_i = i$. Then, $P(A_i) = 1$ for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} A_i = \mathbb{N}$. Thus, if Ω is countable, the counting measure is σ-finite.

   Now suppose that the counting measure is σ-finite. Thus, there exists $\{A_i\}$ measurable with $\bigcup_{i\in\mathbb{N}} A_i = \Omega$ and $P(A_i) < \infty$ by definition. However, this is a countable union of sets with a finite number of elements. Thus, $\bigcup_{i\in\mathbb{N}} A_i \neq \mathbb{R}$. Therefore, Ω is countable.

Exercise 5: Let $X : \Omega \to \mathbb{R}$ be a discrete random variable on probability space (Ω, Σ, P) and denote the corresponding induced measure by P_X. We define the support of P_X as

$$\Omega_X = \{x \in \mathbb{R} : P(X = x) > 0\}$$

Please answer the following two questions.

1) First assume that $|\Omega_X| < \infty$, that is, Ω_X contains finite numbers of elements. Show that the probability mass function (pmf) of X, denoted f, is indeed the density of P_X with respect to the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

2) Show the same thing when $|\Omega_X| = \infty$.

Answer:

1) Let # denote the counting measure. Then $P_X << \#$ since,

$$\#(A) = 0 \implies P_X(A) = P(X^{-1}(A)) = P(X^{-1}(\emptyset)) = 0$$

Also, we proved that the counting measure # is σ-finite if Ω_X countable in the previous question. Since it is finite, it is countable. By Radon-Nikodym theorem, if we restrict ourselves to Ω_X, then we know a density exists. In this specific case, we do not need σ-finiteness. We only have to show that $P_X(A) = \int_A f d\#$ to show that f is the density. Since Ω_X is finite, then $f : \Omega \to \mathbb{R}$ is simple and has the form,

$$f = \sum_{x \in A} f(x) 1_{\{x\}}$$

Putting this into the integral above,

$$\int_A f d\# = \int_A \sum_{x \in A} f(x) 1_{\{x\}} d\# = \sum_{x \in A} f(x)$$

Which is exactly what we desire,

$$P_X(A) = P(X^{-1}(A)) = P(\omega : X(\omega) \in A) = \sum_{x \in A} f(x)$$

Where the last equality is the definition of the pmf

Thus, $P_X(A) = \int_A f d\#$ iff f is the pmf.

2) If $|\Omega_X| = \infty$ then we must construct our f different. Since Ω_X is σ-finite, we can construct an increasing sequence of sets (A_i) such that $\cup_i A_i = \Omega_X$ and $A_{i-1} \subset A_i$. Then define f_n as,

$$f_n = \sum_{x \in A_n} f(x) 1_{\{x\}}$$

Since (f_n) is an increasing sequence of simple functions with $f_n \to f$. Then,

$$\begin{aligned}\int_{\Omega_X} f d\# &= \lim_{n\to\infty} \int_{\Omega_X} f_n d\# = \lim_{n\to\infty} \int_{\Omega_X} \sum_{x\in A_n} f(x) 1_{\{x\}} d\# \\ &= \lim_{n\to\infty} \sum_{x\in A_n} f(x) = \sum_{x\in\Omega_X} f(x)\end{aligned}$$

We can exchange the limit and the integral due to the monotone convergence theorem. Now, as before,

$$P_X(\Omega_X) = P(X^{-1}(\Omega_X)) = P(\omega : X(\omega) \in \Omega_X) = \sum_{x\in\Omega_X} f(x)$$

Thus, $P_X(A) = \int_A f d\#$ iff f is the pmf.