1 List of Problems for Chapter 1

This is a list of problems from Chapter 1 which I left in class as exercises or referred to during proofs. The Homework will be picked from these problems. They are a good practice list for the measure theory part.

1. Show that an intersection of an arbitrary (countable or uncountable) family of σ-algebras on E is again a σ-algebra on E. What about unions of σ-algebras?

2. Show that if $\mathcal{E}$ is a σ-algebra, then
   (i) $\mathcal{E}$ is a p-system;

   (ii) $\mathcal{E}$ is a d-system.

3. Let $\mathcal{D}$ be a d-system on E. Fix D in $\mathcal{D}$ and define

   $$\widehat{\mathcal{D}} = \{A \in \mathcal{D}; A \cap D \in \mathcal{D}\}.$$

   Prove that $\widehat{\mathcal{D}}$ is a d-system.

4. Show that an intersection of an arbitrary (countable or uncountable) family of d-systems on E is again a d-system on E. What about p-systems?

5. Let $\mathcal{C}$ be a countable partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from $\mathcal{C}$.

6. Let $\mathcal{C}, \mathcal{D} \subset 2^E$. Show the following:

   (i) $\mathcal{C} \subset \mathcal{D} \Rightarrow \sigma\mathcal{C} \subset \sigma\mathcal{D}$;

   (ii) $\mathcal{C} \subset \sigma\mathcal{D} \Rightarrow \sigma\mathcal{C} \subset \sigma\mathcal{D}$;

   (iii) If $\mathcal{C} \subset \sigma\mathcal{D}$ and $\mathcal{D} \subset \sigma\mathcal{C}$, then $\sigma\mathcal{C} = \sigma\mathcal{D}$;

   (iv) $\mathcal{C} \subset \mathcal{D} \subset \sigma\mathcal{C} \Rightarrow \sigma\mathcal{C} = \sigma\mathcal{D}$.

7. Show that $\mathcal{B}_{\mathbb{R}}$ can be generated as

   (i) $\mathcal{B}_{\mathbb{R}} = \sigma\{(-\infty, x]; x \in \mathbb{R}\}$;

   (ii) $\mathcal{B}_{\mathbb{R}} = \sigma\{(-\infty, x); x \in \mathbb{R}\}$;

   (iii) $\mathcal{B}_{\mathbb{R}} = \sigma\{(x, y]; x, y \in \mathbb{R}\}$;

   (iv) $\mathcal{B}_{\mathbb{R}} = \sigma\{(x, \infty); x \in \mathbb{R}\}$.

8. Let $(E, \mathcal{E})$ be a measurable space. Fix D in E and let

   $$\mathcal{D} = \mathcal{E} \cap D = \{A \cap D; A \in \mathcal{E}\}.$$

   Show that $\mathcal{D}$ is a σ-algebra on D.

9. Consider a function $f : E \to F$. Show the following:

   (i) $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(F) = E$;

   (ii)$f^{-1}(B \backslash C) = f^{-1}(B) \backslash f^{-1}(C)$;

   (iii) $f^{-1}\Big(\bigcup_i B_i\Big) = \bigcup_i f^{-1}(B_i)$;

   (iv) $f^{-1}\Big(\bigcap_i B_i\Big) = \bigcap_i f^{-1}(B_i)$, for any $B_i \in 2^E$.

10. Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces and consider a function $f : E \to F$. Define $\mathcal{F}_1 = \{B \in \mathcal{F}; f^{-1}(B) \in \mathcal{E}\}$. Show that $\mathcal{F}_1$ is a σ-algebra on F.

11. Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a function $f : E \to F$. Define $f^{-1}(\mathcal{F}) = \{f^{-1}(B); B \in \mathcal{F}\}$. Prove that:

    (i) $f^{-1}(\mathcal{F})$ is a σ-algebra.

    (ii) $f^{-1}(\mathcal{F})$ is the smallest σ-algebra on E such that f is measurable relative to it and $\mathcal{F}$.

12. Let $\mathcal{E}$ be a σ-algebra and consider the sequence $(A_n)_n \subset \mathcal{E}$. Prove that $\displaystyle\bigcap_{n \geq 1} A_n \in \mathcal{E}$.

13. Let $\mathcal{D}_0$ be the smallest d-system that contain a p-system $\mathcal{C}$. Prove that $\mathcal{D}_0$ is a σ-algebra.

14. $f : E \to \mathbb{R}$ is measurable if and only if both f^+ and f^- are measurable.

15. Let f_n be a sequence of measurable functions. Show that $\inf f_n$ is measurable.

16. Let $f : E \to \mathbb{R}$ be a simple function. Show that there is a measurable partition of E such that

    $$f = \sum_{k=1}^{m} b_k \mathbf{1}_{B_k}, \quad b_k \in \mathbb{R}, B_k \in \mathcal{E},$$

    with $(B_k)_{1 \leq k \leq m}$ partition of E.

17. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be an increasing function. Show that f is Borel measurable.

18. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a continuous function. Show that f is Borel measurable.

19. If $\phi(x)$ is a convex function of x from $\mathbb{R}$ to $\mathbb{R}$ and $f(\omega)$ and $\phi(f(\omega))$ are integrable, show that

    $$\int_E \phi(f(\omega))\, dP \geq \phi\Big(\int_E f(\omega)\, dP\Big),$$

    where P is a probability measure on $(E, \mathcal{E})$.

20. Let $(\Omega, \mathcal{H})$, $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Consider the function $f : \Omega \to E \times F$ with $f = (f_1, f_2)$ such that

    (i) f_1 is $(\mathcal{H}, \mathcal{E})$-measurable;

(ii) f_2 is $(\mathcal{H}, \mathcal{F})$-measurable;

Prove that f is measurable with respect to $\mathcal{H}$ and $\mathcal{E} \otimes \mathcal{F}$. Formulate and prove a converse.

21. Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces and assume there is a subset $\mathcal{F}_0$ of $\mathcal{F}$ with $\sigma\mathcal{F}_0 = \mathcal{F}$. If $f : E \to F$ is a function such that

$$f^{-1}(B) \in \mathcal{E}, \qquad \forall B \in \mathcal{F}_0,$$

prove that f is $(\mathcal{E}, \mathcal{F})$-measurable.

22. Prove that composition of measurable functions is measurable.

23. Let $\mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$ Prove that $\mathbf{1}_A$ is measurable if and only if $A \in \mathcal{E}$.

24. Let $f : E \to \mathbb{R}$ be a simple function. Show that there is a partition $\{B_j\}_{j=1,m}$ of E such that

$$f(x) = \sum_{k=1}^{m} b_k \mathbf{1}_{B_k}(x), \qquad b_k \in \mathbb{R}, B_k \in \mathcal{E}.$$

(This is called the canonical form of a simple function).

25. Let $f : E \to \mathbb{R}$ be a $\mathcal{E}$-measurable function taking finitely many real values. Prove that f is a simple function.

26. Let f and g be simple functions. Show that $f \wedge g$, $f \vee g$, $f \pm g$, fg and f/g (when it makes sense) are simple functions.

27. Let f and g be $\mathcal{E}$-measurable. Show that $f \wedge g$, $f \vee g$, $f \pm g$, fg are $\mathcal{E}$-measurable.

28. For each $n \geq 1$ consider the function

$$d_n(x) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(x) + n\mathbf{1}_{[n,\infty)}(x), \qquad x \in \overline{\mathbb{R}}.$$

Prove that $d_n(x)$ is an increasing, right continuous, simple function on $\overline{\mathbb{R}}$, and $d_n(x) \nearrow x$ as $n \to \infty$.

29. Let $\mathcal{M}_+$ be a collection of positive functions on E. Suppose that

(a) $1 \in \mathcal{M}_+$

(b) $f, g \in \mathcal{M}_+$ and $a, b \in \mathbb{R}$ and $af + bg \geq 0 \Rightarrow af + bg \in \mathcal{M}_+$

(c) $(f_n) \subset \mathcal{M}_+$, $f_n \nearrow f \Rightarrow f \in \mathcal{M}_+$.

Suppose that for some p-system $\mathcal{C}$ generating $\mathcal{E}$ we have $\mathbf{1}_A \in \mathcal{M}_+$ for each $A \in \mathcal{C}$. Prove that $\mathcal{M}_+$ includes every positive $\mathcal{E}$-measurable function. (Hint: Read first the proof of Theorem 2.19, page 10).

30. If $\mu_1, \mu_2, \cdots$ are measures, prove that $\sum_{n\geq 1} \mu_n$ is also a measure.

31. If μ and λ are measures, is $\mu - \lambda$ also a measure? What about $|\mu - \lambda|$?

32. Prove that every σ-finite measure is Σ-finite (hint: see ex. 3.13 page 18).

33. Show that $|f| = f^+ + f^-$.

34. Provide a function that is Lebesgue integrable but it is not Riemannian integrable.

35. Let $f \in \mathcal{E}$. If $A \cup B = C$, $A \cap B = \emptyset$, show that

$$\int_C f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

(hint: see page 22).

36. (i) If ν is finite, show that $\nu \circ h^{-1}$ is finite.

(ii) If ν is $\sum$-finite, show that $\nu \circ h^{-1}$ is $\sum$-finite.

37. Let $(E, \mathcal{E}, \mu)$ be a measure space, and $p \in \mathcal{E}_+$. Define

$$\nu(A) = \int_A p(x)\, d\mu(x), \qquad \forall A \in \mathcal{E}.$$

(i) Show that ν is a measure on $(E, \mathcal{E})$;

(ii) Prove that for any $f \in \mathcal{E}_+$ we have

$$\int_E f(x)\, d\nu(x) = \int_E f(x)p(x)\, d\mu(x).$$

38. Let μ be a Σ-finite measure on $(E, \mathcal{E})$. Show that $\mu = \lambda + \nu$, with λ diffuse measure and ν purely atomic. (Hint: see ex. 3.15, page 18).

39. If $(f_n) \subset \mathcal{E}$, $f_n \leq 0$, show that $\mu(\limsup f_n) \geq \limsup \mu f_n$.

40. If δ_x denotes the Dirac measure sitting at x, show that $\delta_x f = f(x)$, for any $f \in \mathcal{E}$.