# Lecture 21

## 10.2

## Conditional Jointly Continuous Distributions

**Def:**

Let X, Y be 2 jointly continuous RV.
The conditional probability density function of $X|Y=y$, $f_{X|Y}(x|y)$ is:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{for } y \text{ s.t. } f_Y(y) \neq 0$$

① $f_{X|Y}(x|y) \geq 0$

② $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1$

**Prf**

② $$\int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx} = 1$$

**Def**

Let $X, Y$ be 2 jointly continuous RV.
Let $y$ be s.t. $P(Y=y)>0$.

- Then the conditional prob. that $X \in A$ given $Y=y$ is:

$$P(X \in A \mid Y=y) = \int_A f_{x|y}(x)\,dx$$

- The conditional expectation of $X$ given $Y=y$ is:

$$E(X \mid Y=y) = \int_x x\, f_{x|y}(x|y)\,dx$$

- Let $g$ be a real function:

$$E(g(x) \mid Y=y) = \int_x g(x)\, f_{x|y}(x|y)\,dx$$

Ex)

Let $f_{X,Y}(x,y) = \dfrac{e^{-x/2y}\, e^{-y}}{2y}, \quad x, y \geq 0$

a) Find $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx = \int_0^\infty \frac{e^{-x/2y}}{2y} e^{-y}\,dx$$

$$= e^{-y} \underbrace{\int_0^\infty \frac{e^{-x/2y}}{2y}\,dx}_{\exp\left(\frac{1}{2y}\right)}$$

$$= \underbrace{e^{-y} \text{ for } y \geq 0}_{Y \sim \exp(1)}$$

So $f_{X|Y}(x|y) = \dfrac{\frac{e^{-\frac{x}{2y}}}{2y} e^{-y}}{e^{-y}} = \dfrac{e^{-\frac{x}{2y}}}{2y} I(x \geq 0)$

$X|Y=y \sim \exp\left(\frac{1}{2y}\right)$

b) What is $E(X|Y)$?

$$E(X|Y) = \left(\frac{1}{(1/2y)}\right) = 2Y$$

## Lemma

Let $X, Y$ be 2 jointly continuous RV and $g: \mathbb{R} \to \mathbb{R}$

- $f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot f_Y(y)\,dy$

- $E(g(X)) = \int_{-\infty}^{\infty} E(g(X) \mid Y=y) \cdot f_Y(y)\,dy = E\big(\underbrace{E(X|Y)}_{h(Y)}\big)$

function of y

## Prf

1) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} f_{X|Y}(x) \cdot f_Y(y)\,dy$

2) $E(g(X)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) f_{X|Y}(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \left( \int_{-\infty}^{\infty} g(x) f_{X|Y}\,dx \right) dy$$

$$= \int_{-\infty}^{\infty} E(X \mid Y=y) f_Y(y)\,dy$$

$$= E(E(X|Y))$$

Ex)

Back to our example, $Y \sim exp(1)$

and $E(X|Y) = 2Y$.

What is $E(X)$?

$$E(2Y) = 2 \cdot \frac{1}{1} = 2.$$

# Conditional Distributions and Independence

- $P_{X,Y}(x,y) = P_X(x)P_Y(y) \iff X \perp\!\!\!\perp Y$
- $f_{X,Y}(x,y) = f_X(x)f_Y(y) \iff X \perp\!\!\!\perp Y$

We saw:

$$f_{X,Y}(x,y) = f_{X|Y}(x)\cdot f_Y(y)$$
$$= f_{Y|X}(y)\cdot f_X(x)$$

Which is true $\forall$ jointly continuous RV

$\Rightarrow$ **Lemma**

$$X \perp\!\!\!\perp Y$$
$$\iff$$
$$P_{X|Y}(x|y) = P_X(x) \ \forall \ y \ \text{s.t.} \ P_Y(y) > 0$$

or

$$f_{X|Y}(x|y) = f_X(x) \ \forall \ y \ \text{s.t.} \ P_Y(y) > 0$$

So if $f_{X|Y}(x|y)$ contains $y$, then $X \not\perp\!\!\!\perp Y$

Ex)

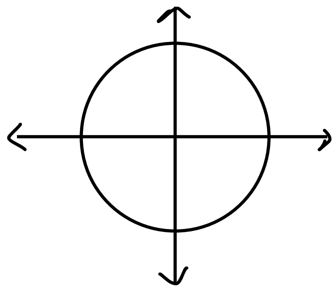
Let $f_{X,Y}(x,y) = \dfrac{e^{-x/2y}\, e^{-y}}{2y}$, $x,y \geq 0$

We saw $f_{X|Y}(x) = \dfrac{e^{-x/2y}}{2y}$ so $X \not\perp\!\!\!\perp Y$.

Ex)

Let $(X,Y)$ be uniformly distributed on a disk $D$, centered at $(0,0)$.

$$f_{X,Y}(x,y) = \frac{1}{\pi r_0^2} \text{ if } (x,y) \in D$$



① What are $f_X(x)$ and $f_Y(y)$?

$$f_X(x) = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\,dy = \frac{2\sqrt{r^2-x^2}}{\pi r^2} I(|x| \leq r)$$

$$f_Y(y) = \frac{2\sqrt{r^2-y^2}}{\pi r^2} I(|y| \leq r) \text{ by symmetry}$$

② What is $f_{X|Y}(x)$?

$$f_{X|Y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{\pi r^2}}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}} = \frac{1}{2\sqrt{r^2-y^2}} \text{ for}$$

$$-\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}$$

(3) Are X and Y independent?

No, $f_{X|Y}(x) \neq f_X(x)$

Ex)

Let $Y \sim \text{Unif}[0,1]$

Let $X|Y \sim \text{Unif}[0, \frac{y}{2}]$.

- What is $f_{X,Y}(x,y)$?

$$f_{X,Y}(x,y) = \frac{2}{y} \text{ for } 0 < x < \frac{y}{2}, \ 0 < y < 1$$

- What is $f_X(x)$?

$$f_X(x) = \int_{2x}^{1} f_{X|Y}(x)\,dy = \int_{2x}^{1} \frac{2}{y}\,dy$$

$x < \frac{y}{2}$

$\Rightarrow y > 2x$

$$= 2\ln|y| \Big|_{2x}^{1}$$

$$= -2\ln(2x)\,I(x \in (0, \tfrac{1}{2}))$$