Lecture 20

10.1

Conditional Distributions

- Conditional Probability Review:

$\Rightarrow$ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow$ Law of Total Probability:

Let $B_1, \ldots, B_n$ be events s.t.
$P(B_1) + \ldots + P(B_n) = 1$ and $P(B_i \cap B_j) = 0$
for $i \neq j$

$\Rightarrow$ B_i "partitions" the probability space
Ω

$\Rightarrow$ Then $\forall A$:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$$

- **Conditioning Discrete RV on Events:**

1) Let X be a discrete RV and B be an event s.t. $P(B) > 0$

Then the PMF of X given B is:

$$P_{X|B}(k) = P(X = k | B) = \frac{P(X = k \cap B)}{P(B)}$$

$\forall$ values k in X's outcome space

$\Rightarrow \quad P_{X|B}(k) \geq 0 \quad \forall \ k$

$\Rightarrow \quad \sum_{k} P_{X|B}(k) = 1$

2) Same conditions for X

Let $B_1, \dots, B_n$ be a partition of Ω

$\left(\begin{array}{l} \sum P(B_i) = 1 \\ P(B_i \cap B_j) = 0, \ i \neq j \end{array} \right)$

$$P_X(k) = P(X = k) = \sum_{i=1}^{n} P(X = k | B_i) P(B_i)$$

$$= \sum_{i=1}^{n} P_{X|B_i}(X | B_i) \cdot P(B_i)$$

Ex)

If it rains the # of customers, $X \sim \text{Pois}(\lambda)$
If not, $X \sim \text{Pois}(\mu)$

$P(\text{rain}) = .4$

What is $P_X(k)$?

$$\Rightarrow P_X(k) = P_{X|\text{rain}}(k|\text{rain}) \cdot P(\text{rain}) + P_{X|\text{no rain}}(k|\text{no rain}) \cdot P(\text{no rain})$$

$$= .4 \cdot \frac{\exp(-\lambda)\lambda^k}{k!} + .6 \frac{\exp(-\mu)\mu^k}{k!}$$

Expectation of Discrete RV

- Let X be a RV, and B be an event w/ $P(B)>0$

$$\underbrace{E(X|B)}_{\text{Conditional expectation}} = \sum_k k \cdot P_{X|B}(k) = \sum_k k \cdot P(X=k|B)$$

- Let X be a RV
  Let $B_1, \dots, B_n$ be a partition of Ω
  s.t. $P(B_i)>0$

$$\Rightarrow E(X) = \sum_{i=1}^{n} E(X|B_i) \cdot P(B_i)$$

Prf

$$E(X) = \sum_k k\, P_X(k) = \sum_k k\left(\sum_{i=1}^{n} P_{X|B_i}(k) P(B_i)\right)$$

$$= \sum_k \sum_i k\, P_{X|B_i}(k) P(B_i)$$

$$\sum_k k P_{X|B_i}(k) = E(X|B_i)$$

$$= \sum_i E(X|B_i) P(B_i)$$

Discrete RV Conditioned on Discrete RV

Let X, Y be 2 discrete RV

The conditional dist of X given Y is:

- $p_{X|Y}(x|y) = P(X = x \mid Y = y) = \dfrac{P(X = x, Y = y)}{p_Y(y)} = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

- $E(X \mid Y = y) = \sum_x x \, p_{X|Y}(x|y)$