Lecture 17

9.3

Convergence in Distribution

- A sequence of RV $\{X_n\}_{n=1}^{\infty}$ converges in dist to a RV X if $\forall\ t \in \mathbb{R}$:

$$\lim_{n \to \infty} F_n(t) = F(t) \Rightarrow X_n \xrightarrow{d} X$$

- **Theorem:**

  Assume the MGF's of the RV in $\{X_n\}_{n=1}^{\infty}$ are finite on $(-\delta, \delta)$ for some $\delta > 0$

  Let X be a RV with MGF that is also finite on $(-\delta, \delta)$.

  Then if $\lim_{n \to \infty} M_{X_n}(t) = M_X(t)\ \forall\ t \in (-\delta, \delta)$

  then: $X_n \xrightarrow{d} X$

Central Limit Theorem

Let $(X_n)_{n=1}^{\infty}$ be a sequence of iid RV with finite $E(X) = \mu$

$Var(X) = \sigma^2$

Then the RV $Z_n = \dfrac{\bar{X}_n - \mu}{(\sigma/\sqrt{n})} \xrightarrow{d} N(0,1)$

i.e.

$\forall \; -\infty \leq a \leq b \leq \infty$:

$$\lim_{n\to\infty} \left(a \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq b \right) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_a^b e^{-z^2/2}\, dz}_{P(a \leq Z \leq b)}$$

for $Z \sim N(0,1)$

Prf

See lecture recording/Textbook prf in section 9.3

(Practical) CLT

Thm

Let $X_1, \dots, X_n$ be iid RV with finite mean and variance.

Let $\overline{X}_n = \dfrac{\sum X_i}{n}$, $\hat{\sigma}_n^2 = \dfrac{\sum (X_i - \overline{X}_n)^2}{n-1}$

Then $$Z_n = \frac{\overline{X}_n - \mu}{\hat{\sigma}_n / \sqrt{n}} \xrightarrow{d} N(0,1)$$