Lecture 9

$\Rightarrow$ hint for Hw #2:

3) The fundamental thm of Calculus could be useful, as well as considering the symmetry of the normal density

(think about $P(X^2 < y)$ and ways to rewrite it)

Consider referencing section 2.1 from Casella + Berger (link on Canvas)

8.1

Linearity of Expection

- Review: $E(X) = \begin{cases} \sum_{k \in \mathbb{Z}} x_k \, p_X(x_k) & \text{discrete} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & \text{continuous} \end{cases}$

Let $X_1, \ldots, X_n$ be RV defined on the same sample space:

$$E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n)$$

Let $g_1, \ldots, g_n$ be functions $(\mathbb{R} \to \mathbb{R})$

then:

$$E\left(\sum_{i=1}^{n} g_i(x_i)\right) = \sum_{i=1}^{n} E(g_i(x_i))$$

Ex) Let $\overline{X}_n = \dfrac{\sum_{i=1}^{n} X_i}{n}$ where $X_1, \ldots, X_n$ are RV's on the same sample space, and we have $E(X_i) = \mu \;\; \forall \; i = 1, \ldots, n$

① What is $E(\overline{X}_n)$?

$$E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n} E(\sum X_i) = \frac{1}{n} \sum E(X_i) = \frac{n \cdot \mu}{n} = \mu$$

Indicator Variables

$$I(X \in A) = \begin{cases} 1 & X \in A \\ 0 & \text{else} \end{cases}$$

If X is a RV then

$$E(I(X \in A)) = 1 \cdot P(X \in A) + 0 \cdot P(X \notin A)$$

$$= P(X \in A)$$

- Can usually decompose discrete RV's as indicator variables

Ex)

Let $X \sim Bin(n, p)$

What is $E(X)$?

We know the sum of n i.i.d. $Ber(p)$ RV's is distributed $Bin(n, p)$.

Let $Z_1, \ldots, Z_n \sim Ber(p)$

So $\sum_{i=1}^{n} Z_i \sim Bin(n, p)$

$$E\left(\sum_{i=1}^{n} Z_i\right) = n \cdot E(Z_i) = np$$

Ex)

You are dealt 5 cards randomly from a standard 52-card deck.

① What is the average sum value of your 5 cards? (Ace = 1, Jack = 11, Queen = 12, King = 13)

Let $X_1, \ldots, X_5$ be the value of the i'th card given to you.

$$E\left(\sum^{5} X_i\right) = \sum E(X_i) = \sum_{i=1}^{5} \left(\frac{\sum_{k=1}^{13} k}{13} \right) = 5 \cdot \frac{(\cancel{13})(14)}{2 \cdot \cancel{13}}$$

$$= 35$$

② How many face cards are you expected to draw (Jack, Queen, King).

Let $X_i = \begin{cases} 1 & i^{th} \text{ card is a face card} \\ 0 & \text{else} \end{cases}$

So $E\left(\sum_{i=1}^{5} X_i\right) = 5 \cdot P(X_i) = 5 \cdot \frac{4}{13} = \frac{20}{13}$

Ex) You are at a party of n people. The probability that any given person knows another person is $\frac{1}{3}$. What is the expected number of groups of size 3 such that everyone knows eachother in that group.

There are $\binom{n}{3}$ possible groupings.

In any given group of 3

$$P(\text{everyone knows eachother}) = \left(\frac{1}{3}\right)^3$$

So

$$E\left(\begin{array}{c}\text{\# of groups where}\\ \text{everyone knows}\\ \text{eachother}\end{array}\right) = \binom{n}{3}\left(\frac{1}{3}\right)^3$$