Lecture 19

9.4

Monte-Carlo Integration

$\Rightarrow$ Goal:

Compute $\int_a^b g(x)\,dx$ but $g(x)$ is really difficult/impossible

Idea:

Let $U \sim Unif[a,b]$

and compute: $E(g(U)) = \int_a^b \frac{g(x)}{b-a}\,dx$

Ex)

Find the area of a circle using MC-integration

$$\iint_{x^2+y^2\le 1} 1\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx$$

$\Rightarrow$ So we can use MC-integration to find $4\int_0^1 \sqrt{1-x^2}\,dx$

- Let $U \sim Unif[0,1]$
- Let $g(U) = 4\sqrt{1-U^2}$
- Simulate a large number of U_i and thus $g(U_i)$
- $E(g(U)) = \int_0^1 4\sqrt{1-x^2}\,dx \approx \dfrac{\sum_{i=1}^{n} g(U_i)}{n}$

- Confidence Intervals for these estimates:

Use CLT

$\Rightarrow$ a $1-\alpha\%$ CI for $\int_0^1 4\sqrt{1-x^2}\,dx$

is $\left[\overline{g(U)}_n \pm \frac{z_\alpha \hat{\sigma}_n}{\sqrt{n}} \right]$

where: $\overline{g(U)}_n = \frac{\sum g(U_i)}{n}$

$\hat{\sigma}_n^2 = \frac{\sum (g(U_i) - \overline{g(U)}_n)^2}{n-1}$

Using R

True Value = $\pi \approx 3.14159$

	n	est	lower	upper
1	1e+01	3.36708609329674	3.04036317848635	3.69380900810713
2	1e+02	3.11323556572598	2.94216249608167	3.28430863537029
3	1e+03	3.13668985591063	3.07985494259809	3.19352476922317
4	1e+04	3.14240504459825	3.12492932120477	3.15988076799173
5	2e+04	3.13924012863554	3.12680226757001	3.15167798970107
6	1e+05	3.14632719083568	3.14080997363652	3.15184440803485
7	1e+06	3.140886311956	3.13913659807691	3.14263602583508

Generic Montecarlo

$$\int_a^b g(x)\,dx$$

Let $f_X(x)$ be the PDF of a RV X over $x \in [a, b]$.

So we can calculate:

$$E\left(\frac{g(x)}{f(x)}\right) = \int_a^b \left(\frac{g(x)}{f(x)}\right) f(x)\,dx$$

$$= \int_a^b g(x)\,dx$$

$\Rightarrow$ Sample $X_1, \dots, X_n$ w/ pdf $f_X(X)$

So $E\left(\frac{g(x)}{f(x)}\right) \approx \frac{1}{n} \sum_{i=1}^{n} \frac{g(X_i)}{f(X_i)}$