# Lecture 5

## Recap

- Joint Distributions + Independence

① Discrete Case: $X_1, \dots, X_n$ discrete RV Independent

$$\iff$$

$$p(k_1, \dots, k_n) = P_{X_1}(k_1) \cdots P_{X_n}(k_n)$$

② Continuous Case: $X_1, \dots, X_n$ RV, jointly contin.

independent

$$\iff$$

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

- If $X_1, \dots, X_n$ are continuous, independent RV then they are jointly continuous with joint PDF $f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

6.4

## Joint CDF

The joint cumulative distribution function of RV $X_1, \ldots, X_n$ is defined as:

$$F(t_1, \ldots, t_n) = P\left(\bigcap_{i=1}^{n} \{X_i \le t_i\}\right)$$

## Lemma

① If $(X, Y)$ are jointly continuous with joint pdf $f(x,y)$ then:

$$\boxed{F(t,s) = \int_{-\infty}^{t} \int_{-\infty}^{s} f(x,y)\,dy\,dx}$$

② If $(X,Y)$ are jointly continuous ($\exists$ a joint pdf) with joint CDF $F$ then:

$$\boxed{\frac{\partial^2}{\partial s \partial t} F(t,s)\Big|_{s=x,\, t=y} = f(x,y)}$$

Ex) Let $(X,Y)$ be a Uniformly random point on a disk centered at $(0,0)$ with radius $r_0$. Let $(R,\Theta)$ be the polar coordinates of the point $(X,Y)$.

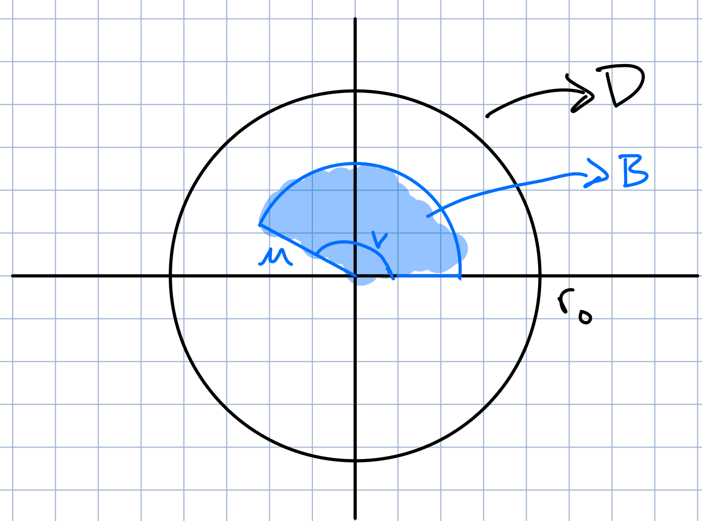
① Find the joint CDF of $(R,\Theta)$

② Find the joint PDF of $(R,\Theta)$

③ Find the marginals for $R$ and $\Theta$

④ Are $R$ and $\Theta$ independent?

① $F_{R,\Theta}(u,v) = P(R \le u,\ \Theta \le v) = P((X,Y) \in B) = \dfrac{\text{Area}(B)}{\text{Area}(D)}$



$\underline{\text{Area of } B:}$ $\pi u^2 = \text{total area}$

So $\text{Area}(B) = \underbrace{\frac{v}{2\pi}}_{\%\text{ of angle}} \cdot (\pi u^2) = \frac{1}{2} v u^2$

So $$\boxed{F_{R,\theta}(u,v) = \frac{\frac{1}{2} v u^2}{\pi r_0^2} = \frac{v u^2}{2\pi r_0^2}}$$

(2) $$\frac{\partial^2}{\partial u \partial v} = \frac{2u}{2\pi r_0^2}\Big|_{u=r,v=\theta} = \boxed{\frac{r}{\pi r_0^2} = f(r,\theta) \quad \text{for } r \in [0, r_0],\ \theta \in [0, 2\pi)}$$

(3) $$f_r(r) = \int_0^{2\pi} \frac{r}{\pi r_0^2}\, d\theta = \frac{2r}{r_0^2}, \quad r \in [0, r_0]$$

$$f_\theta(\theta) = \int_0^{r_0} \frac{r}{\pi r_0^2}\, dr = \frac{r^2}{2\pi r_0^2}\Big|_0^{r_0} = \frac{1}{2\pi}$$

$\theta \in [0, 2\pi)$

(4) Yes since $f_{r,\theta}(r,\theta) = f_r(r) \cdot f_\theta(\theta)$

# Bivariate Normal Distribution

Let $X, Y$ be distributed as BV Normal

with parameter $\rho \in (-1,1)$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2-2\rho xy}{(1-\rho^2)}\right)\right)$$

① Find $f_X(x)$ and $f_Y(y)$ (hint: use symmetry)

② For what values of $\rho$ are $X$ and $Y$ independent?

① $$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2-2\rho xy}{(1-\rho^2)}\right)\right) dy$$

$$= \underbrace{\frac{1}{2\pi\sqrt{1-\rho^2}}\left(\exp\left(-\frac{1}{2}\left(\frac{x^2}{(1-\rho^2)}\right)\right)\right)}_{C} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{y^2-2\rho xy}{(1-\rho^2)}\right)\right) dy$$

- $y^2 - 2\rho xy = (y-\rho x)^2 - \rho^2 x^2$

$$= C \cdot \exp\left( \frac{1}{2} \cdot \frac{\rho^2 x^2}{(1-\rho)^2} \right) \cdot \underbrace{\int_{-A}^{A} \exp\left( -\frac{1}{2} \left( \frac{(y-\rho x)^2}{(1-\rho^2)} \right) \right)}_{Z \sim N(\rho x,\, 1-\rho)} dy$$

$$f_z(z) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot \exp\left( \frac{-1}{2(1-\rho^2)} (y-\rho x)^2 \right)$$

$$= C \cdot \exp\left( \frac{1}{2} \frac{\rho^2 x^2}{(1-\rho^2)} \right) \cdot \sqrt{2\pi(1-\rho^2)}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \exp\left( -\frac{1}{2} \left( \frac{x^2}{(1-\rho^2)} - \frac{\rho^2 x^2}{(1-\rho^2)} \right) \right) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left( \frac{-x^2}{2} \right) \quad \text{So } X \sim N(0,1)$$

So $f_x(x) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right)$

$f_y(y) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{y^2}{2} \right)$

(2) If $\rho = 0$, X and Y are independent.