Lecture 15

9.1/9.2

Concentration Inequalities

Thm: Monotonicity of Expectation

If 2 RV, X and Y, are defined on the same prob. space and

$P(X \geq Y) = 1$ then $E(X) \geq E(Y)$

Prf

Discrete: in Textbook sec 9.1

Continuous:

Let Z be a cont. RV s.t. $P(Z \geq 0) = 1$

Then $E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,dz = \int_{0}^{\infty} z f_Z(z)\,dz \geq 0$

Let $Z = X - Y$ so $P(Z \geq 0) = P(X - Y \geq 0) = 1$

So $E(Z) = E(X-Y) = E(X) - E(Y) \geq 0$

$\Rightarrow E(X) \geq E(Y)$.

Markov's Inequality

Let X be a non-negative RV s.t. $E(X) < \infty$.

Then $\forall$ $c > 0$:

$$P(X \geq c) \leq \frac{E(X)}{c}$$

Proof

Define $I(X \geq c)$

So $X \geq c\, I(X \geq c)$

if $I(X \geq c) = 1$ then $X \geq c$ clearly

If $I(X \geq c) = 0$ then $X \geq 0$ and X non-neg

So $E(X) \geq c\, E(I(X \geq c))$

$$\Rightarrow \frac{E(X)}{c} \geq P(X \geq c)$$

Ex)

Starbucks sells on average 1000 cups of coffee a day. Define this RV as S.

Find an upper bound for the probability Starbucks sells over 1500 cups in a day.

$$\Rightarrow P(X \geq 1500) \leq \frac{E(X)}{c} = \frac{1000}{1500} = \frac{2}{3}$$

Chebyshev's Inequality

Let X be a RV s.t. $E(X) = \mu < \infty$, and $Var(X) = \sigma^2 < \infty$

(or equivalently $E(X), E(X^2) < \infty$)

Then $\forall\ c > 0$:

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

Prf

Let $Z = (X-\mu)^2$. So Z is a RV s.t. $Z \geq 0$

Using Markov's:

$$P((X-\mu)^2 \geq c^2) = P(|X-\mu| \geq c) \leq \frac{E((X-\mu)^2)}{c^2}$$

$$= \frac{\sigma^2}{c^2}$$

Ex)

Starbucks sells S cups of coffee a day.

Let $E(S) = 1000$

$Var(S) = 200$

Find bounds for:

1. Prob. that they will sell between 950 and 1050 coffee tomorrow

2. At least 1400 coffee will be sold.

① $P(950 \leq S \leq 1050)$

$= P(|S-\mu| \leq 50) = 1 - P(|S-\mu| \geq 50)$

We have by Chebyshev:

$$P(|S-\mu| \geq 50) \leq \frac{200}{50^2} = \frac{200}{2500} = \frac{2}{25} = .08$$

So $P(|S-\mu| \leq 50) = 1 - P(|S-\mu| \geq 50) \geq 1 - .08 = .92$

(2) $P(X \geq 1400) \leq \frac{1000}{1400} = \frac{1}{14}$ by Markov

or

$P(X-1000 \geq 400) \leq P(|X-1000| \geq 400) \leq \frac{200}{400^2} = \frac{1}{800}$

by Cheby.

So Chebyshev provides a tigter bound in this case.

Generalizations of Markov

Lemma

Let X be a RV and let $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(X) \geq 0$, $f(X) < \infty$, $E(f(X)) < \infty$, and $f(x)$ is strictly increasing ($f(x) > f(y)$ $\forall$ $x > y$)

Then:

$$P(X \geq c) = P(f(X) \geq f(c)) \leq \frac{E(f(X))}{f(c)}$$

Chernoff's Bound

Let X be a RV s.t. $M_X(t) < \infty$ for $t \in (0, \theta]$ then:

$$P(X \geq c) \leq e^{-tc} M_X(t) \quad \forall \; t \in (0, \theta]$$

Ex) Let $X \sim N(0,1)$.

$M_X(t) = e^{t^2/2}$.

① Find the Chernoff bound for $P(X \geq c)$.

② Find the best possible Chernoff bound