# Lecture 6

## 6.4

## Functions of Random Variables

**Thm**

Let $(X,Y)$ be jointly continuous with PDF $f_{X,Y}(x,y)$.

Denote $S = \{ (x,y) \text{ s.t. } f_{X,Y}(x,y) > 0 \}$

Let $g: R^2 \to R^2$ such that

1. $g$ is invertible on $S$ with inverse

$$\gamma(u,v) = (\alpha(u,v), \beta(u,v))$$

$\alpha: \mathbb{R}^2 \to \mathbb{R}$ $\quad \beta: \mathbb{R}^2 \to \mathbb{R}$

2. $\gamma$ is continuously differentiable on $g(S)$

3. The determinant of the Jacobian, $J_\gamma(u,v)$ does not vanish on $g(S)$

where $J_\gamma(u,v) = \begin{pmatrix} \frac{\partial \alpha}{\partial u} & \frac{\partial \alpha}{\partial v} \\ \frac{\partial \beta}{\partial u} & \frac{\partial \beta}{\partial v} \end{pmatrix}$

Then $(U,V)$ is jointly Continuous with joint pdf

$$f_{U,V}(u,v) = f_{X,Y}(\gamma(u,v)) \cdot \left|\det(J_\gamma(u,v))\right| \cdot I((u,v) \in g(S))$$

Ex) Let $(X,Y)$ be a Uniformly random point on a disk centered at $(0,0)$ with radius $r_0$. Let $(R,\Theta)$ be the polar coordinates of the point $(X,Y)$.

- $S = \{(x,y) \text{ s.t. } x^2 + y^2 \leq r_0^2\}$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$

  $g : (X,Y) \longrightarrow (R,\Theta)$

  $$g(x,y) = \begin{pmatrix} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{pmatrix} \quad \text{s.t.} \quad g(S) = [0, r_0] \times [0, 2\pi]$$

- $g$ is invertible on $S$

  $$g^{-1}(r,\theta) = \gamma(r,\theta) = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

- $\gamma$ is differentiable:

$$J_\gamma(r,\theta) = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb \quad \text{for } 2\times 2 \text{ matrix}}$$

- $\det(J_\gamma(r,\theta)) = r\sin^2\theta + r\cos^2\theta = r(\sin^2\theta + \cos^2\theta) = r$

so $\boxed{f_{U,V}(U,V) = \frac{1}{\pi r_0^2} \cdot r \cdot I(r \in [0, r_0])\ I(\theta \in [0, 2\pi))}$

Ex)

Let $X, Y$ be 2 independent RV

s.t. $X \sim \exp(\lambda)$

$Y \sim \exp(\lambda)$

Let $U = X + Y$, $V = \dfrac{X}{X+Y}$

What is $f_{U,V}$? Are $U$ and $V$ independent?

hints:

- Find $S$, $g(S)$
- Find inverse function, $\gamma$ s.t. $\gamma(U,V) = (X,Y)$
  $\mathbb{R}^2 \to \mathbb{R}^2$
- Find $J_\gamma(U,V)$

$\Rightarrow S = (0,\infty) \times (0,\infty)$

$g(S) = (0,\infty) \times (0,1)$

$f_{X,Y}(X,Y) = \lambda^2 \exp(-\lambda(x+y)) \, I((x,y) \in (0,\infty)\times(0,\infty))$

$$g^{-1}(U,V) = \gamma(U,V) = \begin{pmatrix} UV \\ U - UV \end{pmatrix}$$

$$J_y(u,v) = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} \Rightarrow -uv - u(1-v)$$
$$= -u$$

$$f_{U,V}(u,v) = \lambda^2 u \exp(-\lambda(uv + u - uv))\, I(u \in (0,\infty)) \cdot I(v \in (0,1))$$
$$= \lambda^2 u \exp(-\lambda(u))\, I(u \in (0,\infty))$$

$Z \sim \text{gamma}(\text{shape} = \alpha,\ \text{rate} = \beta)$

$$f_Z(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} \exp(-\beta z)$$

$\Rightarrow U \sim \text{gamma}(2, \lambda)$

Fact:

If $X_1, \ldots, X_n$ are independent exponential($\lambda$) RV then $\sum_{i=1} X_i \sim \text{gamma}(n, \lambda)$