Lecture 20

10.1

Conditional Distributions

- Conditional Probability Review:

  ⇒ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

  ⇒ **Law of Total Probability:**

  Let $B_1, \ldots, B_n$ be events s.t. $P(B_1) + \ldots + P(B_n) = 1$ and $P(B_i \cap B_j) = 0$ for $i \neq j$

  ⇒ B_i "partitions" the probability space Ω

  ⇒ Then $\forall A$:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$$

Conditioning Discrete RV on Events:

1) Let X be a discrete RV and B be an event s.t. $P(B) > 0$

- Then the PMF of X given B is:

$$P_{X|B}(k) = P(X=k|B) = \frac{P(X=k \cap B)}{P(B)}$$

$\forall$ values k in X's outcome space

$\Rightarrow P_{X|B}(k) \geq 0 \quad \forall k$

$\Rightarrow \sum_k P_{X|B}(k) = 1$

2) Same conditions for X

Let $B_1, \ldots, B_n$ be a partition of Ω

$$\left(\begin{array}{l} \sum P(B_i) = 1 \\ P(B_i \cap B_j) = 0,\ i \neq j \end{array} \right)$$

$$P_X(k) = P(X=k) = \sum_{i=1}^{n} P(X=k|B_i)P(B_i)$$

$$= \sum_{i=1}^{n} P_{X|B_i}(X|B_i) \cdot P(B_i)$$

Ex)

If it rains the # of customers, $X \sim Pois(\lambda)$

If not, $X \sim Pois(\mu)$

$P(rain) = .4$

What is $P_X(k)$?

$$\Rightarrow P_X(k) = P_{X|rain}(k|rain) \cdot P(rain) + P_{X|no\ rain}(k|no\ rain) \cdot P(no\ rain)$$

$$= .4 \cdot \frac{\exp(-\lambda)\lambda^k}{k!} + .6 \frac{\exp(-\mu)\mu^k}{k!}$$

Expectation of Discrete RV

- Let X be a RV, and B be an event w/ $P(B)>0$

$$\underbrace{E(X|B)}_{\text{Conditional expectation}} = \sum_k k \cdot P_{X|B}(k) = \sum_k k \cdot P(X=k|B)$$

- Let X be a RV
  Let $B_1, \dots, B_n$ be a partition of Ω
  s.t. $P(B_i)>0$

$$\Rightarrow E(X) = \sum_{i=1}^{n} E(X|B_i) \cdot P(B_i)$$

Prf

$$E(X) = \sum_k k\, p_X(k) = \sum_k k\left(\sum_{i=1}^{n} p_{X|B_i}(k) P(B_i)\right)$$

$$= \sum_k \sum_i k\, p_{X|B_i}(k) P(B_i)$$

$$= \sum_i P(B_i) \underbrace{\sum_k k\, p_{X|B_i}(k)}_{=E(X|B_i)}$$

$$= \sum_i E(X|B_i) P(B_i)$$

Ex)

If it rains the # of customers, $X \sim Pois(\lambda)$

If not, $X \sim Pois(\mu)$

$P(rain) = .4$

What is $E(X)$?

$$E(X) = \sum_i E(X \mid B_i) P(B_i)$$

$$= E(X \mid rain) P(rain) + E(X \mid no\ rain) P(no\ rain)$$

$$\boxed{= \lambda(.4) + \mu(.6)}$$

Discrete RV Conditioned on Discrete RV

Let X, Y be 2 discrete RV

The conditional dist of X given Y is:

- $P_{X|Y}(x|y) = P(X = x \mid Y = y) = \dfrac{P(X = x, Y = y)}{P_Y(y)} = \dfrac{P_{X,Y}(x,y)}{P_Y(y)}$

- $E(X|Y = y) = \sum_x x \, P_{X|Y}(x|y)$

- $P_{X,Y}(x,y) = \underbrace{P_X(x) \cdot P_{Y|X}(y) = P_Y(y) \cdot P_{Y|X}(x)}$

think about this
and how it thus
relates to independence

Ex)

The number of people who enter a store, X, is distributed $Pois(\lambda)$.

Each customer recieves one of 3 coupons with probabilities = p_1, p_2, p_3

Let X_i = # of customers who got coupon i.

① What is the density of (X_1, X_2, X_3) given $X = n$?

$$f_{X_1,X_2,X_3|X=n}(k_1,k_2,k_3) = \binom{n}{k_1\ k_2\ k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3}$$

② What is $f_{X, X_1, X_2, X_3}(X, X_1, X_2, X_3)$

$$f_{X,(X_1,X_2,X_3)} = f_{(X_1,X_2,X_3)|X} \cdot f_X$$

$$= \binom{n}{k_1\ k_2\ k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdot \left(\frac{e^{-\lambda}\lambda^n}{n!}\right)$$

③ What is the $E(X_i | X = n)$?

$$E(X_i | X = n) = n p_i$$

Conditional to Marginal Distributions

Lemma

Let X, Y be 2 discrete RV.

- $P_X(x) = \sum_y P_{X|Y}(x|y) \cdot p(y)$
- $E(X) = \sum_y E(X|Y=y) \cdot P_Y(y) = E_Y(E(X|Y))$

Prf

Y can be thought of as a partition and we thus may use previous definitions.

Lemma

Let X, Y be 2 discrete RV and g a real function.

- $E(g(X)|Y=y) = \sum_x g(x) \cdot P_{X|Y}(x)$
- $E(g(X)) = \sum_y E(g(X)|Y=y) \cdot P_Y(y) = E_Y(E(g(X)|Y))$

Ex)

Going back to our previous example,

$X \sim Pois(\lambda)$, and we found $E(X_i \mid X = n) = np_i$

What is $E(X_i)$?

$$E(X_i) = E(E(X_i / X)) = E(X p_i) = \lambda p_i .$$