Lecture 8

7.1

Sums of Random Variables

Lemma:

Let X, Y be 2 discrete independent RV with PMF's P_X and P_Y.

Then $\forall n \in \mathbb{N}$:

$$P(X+Y=n) = P_{X+Y}(n) = \sum_{k\in\mathbb{Z}} P_X(k)\, P_Y(n-k)$$

$$= \sum_{k\in\mathbb{Z}} P_X(n-k)\, P_Y(k)$$

Proof

$$P(X+Y=n) = \sum_{k\in\mathbb{Z}} P(X=k, Y=n-k)$$

$$= \sum_{k\in\mathbb{Z}} P(X=k)\, P(Y=n-k)$$

Convolution

- Discrete Convolution:

$$h * g(n) := \sum_{k \in \mathbb{Z}} h(k)\, g(n-k) = \sum_{k \in \mathbb{Z}} h(n-k)\, g(k)$$

$$P_{X+Y}(n) = P_X * P_Y(n) \quad \text{for } X, Y \text{ discrete, independ.}$$

Ex)

$X \sim \text{pois}(\lambda)$, $f_X(k) = \dfrac{\exp(-\lambda)\lambda^k}{k!}$, $k = 0, 1, \ldots$

$Y \sim \text{pois}(\mu)$

Let X and Y be independent.

Find $P_Z(n)$ where $Z = X + Y$.

$$P_Z(n) = \sum_{k=0}^{n} P_X(n)\, P_Y(n-k)$$

$$= \sum_{k=0}^{n} \frac{\exp(-\lambda)\lambda^k}{k!} \cdot \frac{\exp(-\mu)\mu^{n-k}}{(n-k)!}$$

$$= \exp(-(\lambda+\mu)) \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \lambda^k \mu^{n-k}$$

$$= \frac{\exp(-(\lambda+\mu))}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k}$$

Fact: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

(Binomial Thm)

$$= \frac{(\mu + \lambda)^n \exp(-(\lambda + \mu))}{n!}$$

So $Z \sim \mathrm{Pois}(\lambda + \mu)$.

Sums of Continuous RV

Let X, Y be 2 continuous independent RV with PDF's f_X and f_Y.

Then $\forall\ z \in \mathbb{R}$:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

Convolution

$$h * g(z) := \int_{-\infty}^{\infty} h(z-x) g(x)\,dx$$

Ex)

Cars drivepast Main Ave at a rate of $\frac{1}{\lambda}$ cars per minute.

Let T be the time till 2 cars have passed. Find $f_T(t)$.

$\Rightarrow$ Let X_i be the time you wait to see the i'th car.

So $X_i \sim \exp(\lambda)$.

So $T = X_1 + X_2$

$$f_T(t) = \int_0^t f_{X_1}(x_1)\, f_{X_2}(t - x_1)\, dx_1$$

$$= \int_0^t \lambda \exp(-\lambda x_1)\, \lambda \exp(-\lambda(t - x_1))\, dx_1$$

$$= \lambda^2 \int_0^t \exp(-\lambda(x_1 + t - x_1))\, dx_1$$

$$= \lambda^2 \exp(-\lambda t) \cdot \int_0^t dx_1$$

$$= \underbrace{\lambda^2\, t \exp(-\lambda t)}_{\text{gamma}(2, \lambda)}$$

* Can show with induction if $X_1, \ldots, X_n \overset{iid}{\sim} \exp(\lambda) \Rightarrow \sum X_i \sim \text{gamma}(n, \lambda)$

Gamma Distribution

- $Z \sim \text{gamma}(\text{shape} = \alpha, \text{rate} = \beta)$

- $f_Z(z) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} \exp(-\beta z)$, $z \in (0, \infty)$

$\Rightarrow$ Gamma is the "sum" of wait times for α events with rate β (if $\alpha \in \mathbb{N}$)

$\Rightarrow$ If $\alpha \in \mathbb{N}$:

$$\Gamma(\alpha) = (\alpha - 1)!$$

(can show by solving $\Gamma(\alpha) = \int_0^\infty \beta^\alpha z^{\alpha-1} \exp(-\beta z)\, dz$ and induction)