Lecture 16

Recap:

MGF's

- $M_X(t) = E(e^{tX})$
- MGF's are unique to distributions
- $E(X^{(n)}) = M^{(n)}(0)$
- $M_{\Sigma X_i}(t) = \prod M_{X_i}(t)$

Markov's Inequality:

For a RV $X \geq 0$, $E(X) < \infty$:

$$\forall\ c > 0 \quad P(X \geq c) \leq \frac{E(X)}{c}$$

Chebyshev's Inequality

For a RV X, $E(X), E(X^2) < \infty$:

$$\forall c > 0,\ P(|X - E(X)| \geq c) \leq \frac{Var(X)}{c^2}$$

Chernoff Bound

For a RV X s.t. $E(e^{tX}) < \infty$ on $t \in (0, \theta]$

then $\forall\, c > 0$ $P(X \geq c) \leq \dfrac{E(e^{tX})}{e^{tc}}$

9.2

Convergence in Probability

A sequence of RV $\{X_n\}_{n=1}^{\infty}$ converges in probability to a RV X if:

$$\forall \varepsilon > 0 \quad \lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.$$

Denoted: $X_n \xrightarrow{P} X$

(Weak) Law of Large Numbers

Let $X_1, \dots, X_n$ be iid RV with $E(X_i) = \mu < \infty$
$Var(X_i) = \sigma^2 < \infty$

Let $\overline{X}_n = \dfrac{\sum_{i=1}^{n} X_i}{n}$

Then $\forall \varepsilon > 0 \quad \lim_{n \to \infty} P(|\overline{X}_n - \mu| < \varepsilon) = 1$

Prf

(hint: Use Chebyshev: $P(|X-E(X)|>c) \le \frac{Var(X)}{c^2}$)

- $\lim_{n\to\infty} P(|\overline{X}_n-\mu|<\varepsilon)=1 \quad \forall\ \varepsilon>0$

1) $E(\overline{X}_n)= \frac{E(\Sigma X_i)}{n} = \frac{\Sigma E(X_i)}{n} = \frac{\Sigma \mu}{n} = \frac{n\mu}{n} = \mu$

$Var(\overline{X}_n)= Var\left(\frac{\Sigma X_i}{n}\right) = \frac{1}{n^2} Var(\Sigma X_i)$

$= \frac{1}{n^2} \Sigma Var(X_i)$ by iid

$= \frac{1}{n^2} \Sigma \sigma^2 = \frac{\sigma^2}{n}$

$\Rightarrow \quad P(|\overline{X}_n-\mu|>c) \le \frac{Var(\overline{X}_n)}{c^2} = \frac{\sigma^2}{nc^2}$

So $\quad P(|\overline{X}_n-\mu|<\varepsilon) \ge 1-\frac{\sigma^2}{n\varepsilon^2} \quad \forall\ \varepsilon>0$

$\Rightarrow \lim_{n\to\infty} P(|\overline{X}_n-\mu|<\varepsilon) \ge \lim_{n\to\infty} 1-\frac{\sigma^2}{n\varepsilon^2}$

$\Rightarrow \lim_{n\to\infty} P(|\overline{X}_n-\mu|<\varepsilon) \ge 1$

$\Rightarrow \lim_{n\to\infty} P(|\overline{X}_n-\mu|<\varepsilon) = 1$

Strong Law of Large Numbers

(will not test)

- **Almost Sure Convergence**

A sequence of RV $\{X_n\}_{n=1}^{\infty}$ converges almost surely to X, $X_n \xrightarrow{a.s.} X$, if:

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1$$

Fact: $X_n \xrightarrow{a.s} X$ implies $X_n \xrightarrow{P} X$

- **SLLN:**

Let $X_1, \dots, X_n$ be iid RV with $E(X_i) = \mu$

then $\bar{X}_n \xrightarrow{a.s.}$, i.e. $P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$