Lecture 4

6.3

Joint Distributions and Independence

- **Independence for Probability:**

RV $X_1, \ldots, X_n$ are independent if for any Borel subsets (can be written as open sets) $B_1, \ldots, B_n \subset \mathbb{R}$

$$P(X_1 \in B_1, \ldots, X_n \in B_n) = P(X_1 \in B_1) \cdots P(X_n \in B_n)$$

- **Joint PMF Case:**

Let $p(k_1, \ldots, k_n)$ be the joint PMF for discrete RV $X_1, \ldots, X_n$. Let $P_{X_j}(k) = P(X_j = k)$ be the marginal PMF of X_j.
Then:

$X_1, \ldots, X_n$ are independent

if and only if

$$p(k_1, \ldots, k_n) = P_{X_1}(k_1) \cdots P_{X_n}(k_n)$$

Ex)

Roll a fair 6-sided die twice.
Record roll 1 as X_1, and roll 2 as X_2.
Let $S = X_1 + X_2$

Are S and X_1 independent?

$P(X_1 = 1, S = 12) = 0$

but $P(X=1)P(S=12) = \frac{1}{6} \cdot \frac{1}{6}^2 \neq 0$

So X_1 is not independent of S.

Ex) Let $X_1, \ldots, X_n$ be independent RV's with $X_i \sim geom(p_i)$, and $P_{X_i}(k) = (1-p_i)^{k-1}(p_i)$

Let $Y = \min(X_1, \ldots, X_n)$.

Find $P_Y(y)$.

$$P(Y>k) = P(X_1>k, X_2>k, \ldots, X_n>k)$$

$$= \prod_{i=1}^{n} P(X_i>k)$$

$$= \prod_{i=1}^{n} (1-p_i)^k$$

$P(X_i<k) = 1-(1-p_i)^k$

$$= \left(\prod_{i=1}^{n}(1-p_i)\right)^k$$

So $P(Y<k) = 1-\left(\prod_{i=1}^{n}(1-p_i)\right)^k$

So $Y \sim geom\left(p_Y = 1-\prod_{i=1}^{n}(1-p_i)\right)$

Thus $P_Y(k) = (1-p_Y)^{k-1}(p_Y)$

- Joint PDF Case:

* Let $X_1, \dots, X_n$ be n jointly continuous Random Variables.

  Then $X_1, \dots, X_n$ are independent

  if and only if

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

  where $f_{X_i}(x_i)$ represents the marginal pdf of X_i.

* If $X_1, \dots, X_n$ are n independent continuous random variables with pdf f_{X_i} for all X_i then they are jointly continuous with joint pdf

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

- Independence of Subsets

Let $X_1, \dots, X_{m+n}$ be independent RV's.

Let $f: \mathbb{R}^m \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$

Then $Y = f(X_1, \dots, X_m)$ and $Z = g(X_{m+1}, \dots X_{m+n})$ are independent random variables.

Ex) $f(x,y) \propto \exp\left(-\frac{1}{2}(y^2-2y+4x)\right)$ $\quad Y \sim N(1,1)$

$x \in (0, \infty)$ $\quad X \sim \exp(2)$

$y \in (-\infty, \infty)$

① Find c

② Find $f_X(x)$

③ Find $f_Y(y)$

④ Are x and y independent?

① $$\int_{-\infty}^{\infty}\int_{0}^{\infty} \exp\left(-\frac{1}{2}(y^2-2y)-2x\right) dx\, dy$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(y^2-2y)\right) dy \underbrace{\int_{0}^{\infty} \exp(-2x)\, dx}$$

expon(2)

$f_x(x) = 2\exp(-2x)$

So $\int_{0}^{\infty} \exp(-2x)\, dx = \frac{1}{2}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(y^2-2y)\right)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}((y-1)^2-1\right) dy$$

$$= \frac{\exp(\frac{1}{2})}{2} \int_{-\infty}^{\infty} \underbrace{\exp\left(-\frac{1}{2}(y-1)^2\right)}_{\substack{Y \sim N(1,1) \\ f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-1)^2\right)}} dy$$

$$= \frac{\sqrt{2\pi}}{2\exp(\frac{1}{2})}$$

So $\boxed{c = \frac{2\exp(-\frac{1}{2})}{\sqrt{2\pi}}}$

(2) $f_X(x) = 2\exp(-2x)$

(3) $f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-1)^2\right)$

(4) Yes, since $f(x,y) = f_X(x) \cdot f_Y(y)$

Ex) Let X denote the time between calls from your parents, and Y the time between calls from your grandparents

Suppose $X \sim \exp(\lambda)$, $Y \sim \exp(\mu)$, and $X \perp Y$

① Find $f(x,y)$

② Find the probability your parents call before your grandparents

③ Let $T = \min(X,Y)$. Find $P_T(t)$.

① X and Y are independent so

$$f(x,y) = f_x(x) \cdot f_y(y)$$

$$= \lambda \mu \exp(-\lambda x - \mu y)\, I(x \geq 0, y \geq 0)$$

② $P(X < Y)$

$$\int_0^\infty \int_0^y \lambda\mu \exp(-\lambda x)\exp(-\mu y)\, dx\, dy$$

$$= \int_0^\infty \mu \exp(-\mu y)\, dy \cdot \left(-\exp(-\lambda x) \Big|_0^y \right)$$

$$= \int_0^\infty \mu \exp(-\mu y) \cdot (-\exp(-\lambda y) + 1)\, dy$$

$$= \int_0^\infty \underbrace{-\mu \exp(-y(\mu+\lambda))}_{\text{exponential}(\mu+\lambda)} + \underbrace{\mu \exp(-\mu y)}_{\text{integrates to } 1} \, dy$$

$$= \frac{-\mu}{\mu+\lambda} + 1 = \boxed{\frac{\lambda}{\mu+\lambda}}$$