Lecture 3

6.2

Joint Probability Density Functions

Let $X_1, \dots, X_n$ have a joint pdf $f: \mathbb{R}^n \mapsto \mathbb{R}$

Let $B \subset \mathbb{R}^n$

Then $$\mathbb{P}((X_1, \dots, X_n) \in B) = \underbrace{\int \cdots \int_B \cdots \int f(x_1, \dots, x_n)\, dx_1 \dots dx_n}_{\text{integrate over the space } B}$$

Properties:

① $f(x_1, \dots, x_n) \geq 0$

② $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_n)\, dx_1 \dots dx_n = 1$

$f(k_1, \dots, k_n) \neq P(X_1 = k_1, \dots, X_n = k_n) = 0$

Excersise:

$f(x,y) = c e^{-(x+2y)\lambda}$ for some $\lambda > 0$ and $x, y \in [0, \infty)$
Find c s.t. this is a pdf.

(1) $$\int_0^\infty \int_0^\infty c e^{-x\lambda} e^{-2y\lambda} \, dx\, dy = 1$$

$$\int_0^\infty e^{-x\lambda} dx = \frac{-1}{\lambda} e^{-x\lambda} \Big|_0^\infty = \frac{1}{\lambda}$$

$$\int_0^\infty e^{-2y\lambda} dy = -\frac{1}{2\lambda} e^{-y2\lambda} \Big|_0^\infty = \frac{1}{2\lambda}$$

So $c \cdot \frac{1}{\lambda} \cdot \frac{1}{2\lambda} = 1 \Rightarrow \boxed{c = 2\lambda^2}$

(2) Form Matching:

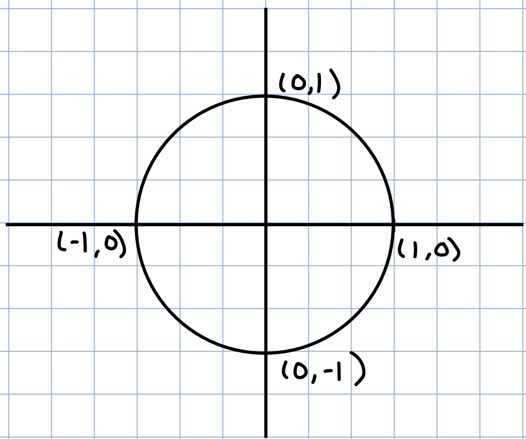
If $X \sim \exp(\lambda) \Rightarrow f(x) = \lambda e^{-x\lambda}$

So $\int_0^\infty \lambda e^{-x\lambda} dx = 1 \Rightarrow \int_0^\infty e^{-x\lambda} dx = \frac{1}{\lambda}$

So $\int_0^\infty e^{-2y\lambda} dy = \frac{1}{2\lambda}$

$\Rightarrow c = 2\lambda^2$

Ex) Take the unit circle

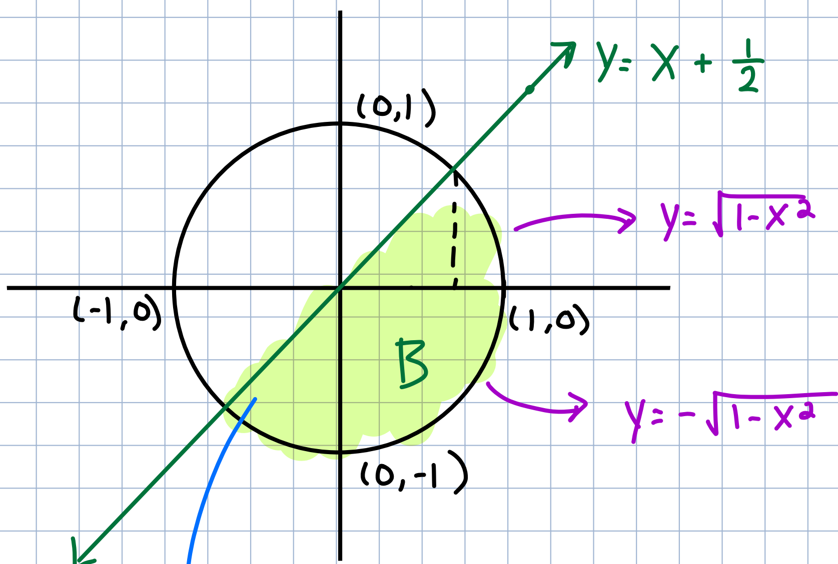


Let (X, Y) represent cartesian coordinates for points uniformly distributed in this circle

(think about why)

We have $f(X,Y) = \frac{1}{\pi}$

What is $P(Y < X)$?



$$\iint_B \frac{1}{\pi}\,dy\,dx = P(X<Y) = \frac{1}{2}$$

Expectation

Let $f(x_1, \dots, x_n)$ be a joint pdf and $g(x_1, \dots, x_n)$ a function $\mathbb{R}^n \mapsto \mathbb{R}$

Then $$E(g(x_1, \dots, x_n)) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) \cdot f(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

Marginal PDF's

$$f_{X_j}(x) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1 \text{ integrals}} f(x_1, \dots, x, \dots, x_n)\, dx_1 \dots dx_{j-1}\, dx_{j+1} \dots dx_n$$

$\Rightarrow$ integrate out the other variables

Uniform Distributions

Let $(X_1, \ldots, X_n) \in \mathbb{R}^n$ be uniformly distributed over a space $A \subset \mathbb{R}^n$.

Then $f(x_1, \ldots, x_n) = \frac{1}{C}$ where

$$C = \int \cdots \int_A dx_1 \ldots dx_n$$

Ex) $\mathbb{R}^1$: $f(x) = \frac{1}{\text{length}} = \frac{1}{b-a}$ for $\text{Unif}(a, b)$

$\mathbb{R}^2$: $f(x) = \frac{1}{\text{area}}$

$\mathbb{R}^3$: $f(x) = \frac{1}{\text{Vol}}$