Lecture 13

Recap:

- $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - E(X)E(Y)$
  1) $Cov(X,Y) = Cov(Y,X)$
  2) $Cov(X,X) = Var(X)$
  3) $Cov(aX+b, Y) = a\,Cov(X,Y)$
  4) Bilinearity:

$$Cov\left(\sum_{j=1}^{m} a_j X_j, \sum_{i=1}^{n} b_i Y_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_j b_i \, Cov(X_j, Y_i)$$

- Variance of a Sum:

$$Var\left(\sum X_i\right) = \sum Var(X_i) + 2\sum_{1 \le i < j \le n} Cov(X_i, X_j)$$

- Correlation:
  - $$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$
  - $Corr(X,Y) \in [-1, 1]$
  - $Corr(aX+b, Y) = \frac{a}{|a|} Corr(X,Y)$

- $\mathrm{Corr} = 1 \iff \exists\, a > 0, b \in \mathbb{R}$ s.t. $Y = aX + b$
- $\mathrm{Corr} = -1 \iff \exists\, a < 0, b \in \mathbb{R}$ s.t. $Y = aX + b$

8.5

Multivariate Normal

Mean Vector

- A multivariate random vector, $\vec{X}$, is a vector $\vec{X} = (X_1, \ldots, X_n)^T = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ whose components are RV on the same prob space.

- $$\mu_X := E(\vec{X}) = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{pmatrix}$$

Ex) Let $(X_1, \ldots, X_r) \sim \text{Multinom}(n, r, p_1, \ldots, p_r)$

What is μ_X?

$$\mu_X = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_r) \end{pmatrix} = \begin{pmatrix} np_1 \\ \vdots \\ np_r \end{pmatrix}$$

Linearity of Expectation (MV case)

Let $\vec{X} = (X_1, \dots, X_n)^T$

$A \in \mathbb{R}^{p \times n}$

$b \in \mathbb{R}^{p}$

then $E(AX + b) = AE(\vec{X}) + b \in \mathbb{R}^{p}$

Covariance Matrix

Let $(X_1, \dots, X_n)^T$ be a Random Vector

The Covariance Matrix of X, S_x is:

$$S_x = \begin{pmatrix} Cov(X_1,X_1) & \dots & Cov(X_1,X_n) \\ \vdots & & \vdots \\ Cov(X_n,X_1) & \dots & Cov(X_n,X_n) \end{pmatrix}$$

So each (i,j)'th entry is $Cov(X_i, X_j)$

Properties

① S_x is Symmetric

② The diagonal $S_{x,ii} = Var(X_i)$

2×2 Case

$$S_{x,y} = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{pmatrix}$$

Random Matrix

A random matrix, $M \in \mathbb{R}^{p \times n}$ has entries which are RV on the same prob space

- We say $E(M) = \left(E(M_{ij})_{\substack{i=1,\dots,p \\ j=1,\dots,n}} \right) \in \mathbb{R}^{p \times n}$

- Linearity:

  $E(AM + B) = A\,E(M) + B$ provided A and B are appropriate sized matrices

Covariance Matrix

1 Var Case:

$$Var(X) = E((X - \mu_x)^2)$$

MV Case:

$$S_x = E((\vec{X} - \mu_x)(\vec{X} - \mu_x)^T)$$

- Properties:

  1) If $Y = AX + b \in \mathbb{R}^p$

  $\Rightarrow S_Y = A S_x A^T \in \mathbb{R}^{p \times p}$

Multivariate Normal Vector

A random vector $\vec{X} = (X_1, \dots, X_n)^T$ is a standard normal vector if:

$X_1, \dots, X_n \overset{iid}{\sim} N(0,1)$ s.t.

- $f_X(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\left(\vec{x}\,\vec{x}^T\right)\right)$

$= x_1^2 + x_2^2 + \dots$

- We have $\mu_X = \vec{0}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$

- We have $S_X = I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix}$

$\Rightarrow X \sim MV\text{-}Norm(\vec{0}_n, I_n)$

MV Norm

$\vec{X} = (X_1, \ldots, X_n)^T \sim MV\text{-}Norm(\mu_x, S_x)$ s.t.

S_x is invertible $\Rightarrow$ has joint PDF:

$$f_x(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \cdot \det(S_x)} \exp\left(\underbrace{-\frac{1}{2}(\vec{X} - \mu_x)^T S_x^{-1} (\vec{X} - \mu_x)}_{-\frac{1}{2} \cdot \frac{(X-\mu)^2}{\sigma^2}}\right)$$

Lemma

1. If X and Y are independent
   $\Rightarrow Cov(X, Y) = 0$

2. If $(X_1, \ldots, X_n)^T \sim MV\text{-}Norm(\mu_x, S_x)$
   $Cov(X_i, X_j) = 0, \; i \neq j$
   $\Longleftrightarrow$
   X_i and X_j are independent

Ex) $(X_1, X_2) \sim \text{MV-Norm}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \right)$

$\Rightarrow$ What is $\underbrace{X_1 - X_2}$ distributed as.

$$\underbrace{\underbrace{\begin{pmatrix} 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}} = X_1 - X_2 = A\vec{X}$$

Def

A Random Vector $\vec{X}$ is a normal random vector if $\exists$ $\mu \in \mathbb{R}^n$
$A \in \mathbb{R}^{p \times n}$
$Z \sim \text{MV-Norm}(\vec{0}_n, I_n)$

s.t. $\vec{X} = AZ + \mu$