Lecture 7

Recap:

- Joint CDF
  - $P(X \leq s, Y \leq t) = F(s,t)$
  - $f_{X,Y}(x,y) = \frac{\partial^2}{\partial s \partial t} F(s,t) \Big|_{\substack{s=x \\ t=y}}$
  - Joint PDF/PMF may not exist if X is discrete and y is continuous, but joint CDF may still exist

- Transformations of Random Variables:
  1. Joint CDF Method (find $P(T(x) < t)$)
  2. Jacobian Method

- Bivariate Normal Distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)\right)$$

6.3

Minimums and Maximums

Lemma:

If $X_1, \dots, X_n$ are n independent RV and $Y = \max\{X_1, \dots, X_n\}$ and $Z = \min\{X_1, \dots, X_n\}$ then:

① $F_Y(t) = \prod_{i=1}^{n} F_{X_i}(t)$ $\Big\}$ $P(\max \le t) = \prod_{i=1}^{n} P(X_i \le t)$

② $1 - F_Z(t) = \prod_{i=1}^{n} (1 - F_{X_i}(t))$ $\Big\}$ $P(\min \ge t) = \prod_{i=1}^{n} (P(X_i \ge t))$

Ex) I can take bus A or bus B to school.
Bus A on average comes around every 10 min
Bus B on average comes around every 4 min

1. What probability distribution can I use to model the bus wait times?

2. If I take the first bus that comes along, what distribution does my wait time follow.

1. Wait times or gaps between events are modeled well by exponential RV.

So let $A \sim \exp(\lambda)$, $B \sim \exp(\mu)$

$$A \sim \exp(\lambda) \rightarrow E(A) = \frac{1}{\lambda} \Rightarrow \hat{\lambda} = \frac{1}{10}.$$
$$B \sim \exp(\mu) \rightarrow E(B) = \frac{1}{\mu} \Rightarrow \hat{\mu} = \frac{1}{4}.$$

2. My wait time, T, is $\min(A, B)$.

So
$$\begin{aligned} P(T \geq t) &= P(A \geq t, B \geq t) \\ &= P(A \geq t)P(B \geq t) \quad \text{by independence} \\ &= \exp(-\lambda t)\exp(-\mu t) \\ &= \exp(-t(\lambda + \mu)) \end{aligned}$$

So $P(T \leq t) = 1 - \exp(-t(\lambda + \mu))$

so $f_T(t) = \frac{d}{dt} P(T \leq t) = (\lambda+\mu) \exp(-t(\lambda+\mu))$

so $T \sim \exp(\lambda+\mu)$.