Lecture 14

Recap:

- **Covariance Matrix:**

  $\vec{X} = (X_1, \dots, X_n)^T$

  - $S_x = E[(\vec{X} - E(\vec{X}))(\vec{X} - E(\vec{X}))^T]$
  - $(S_x)_{i,j} = Cov(X_i, X_j)$

- **MV-Norm:**

  $\vec{X} \sim \text{MV-Norm}(\mu_x, S_x)$

$$f_{\vec{X}}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(S_x)}} \exp\left(-\frac{1}{2}(\vec{x} - \mu_x)^T S_x^{-1} (\vec{x} - \mu_x)\right)$$

$\Rightarrow$ If $\vec{X} \sim \text{MV-Norm}(\mu_x, S_x)$

$\Rightarrow A\vec{X} + b \sim \text{MV-Norm}(A\mu_x + b, A S_x A^T)$

5.1/8.3

Moment Generating Functions

The MGF of X is defined as:

$$M_X(t) = E(e^{tX})$$

Ex)

Let X be a discrete RV s.t.

$$P(X=1) = 1/3$$

$$P(X=-1) = 1/4$$

$$P(X=2) = 5/12$$

- What is $M_X(t)$?

$$E(e^{Xt}) = \frac{1}{3}e^{t} + \frac{1}{4}e^{-t} + \frac{5}{12}e^{2t}$$

Ex) Let $X \sim Pois(\lambda)$, $X = 0, 1, 2, \ldots$

Find $M_X(t)$.

(hint: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and more generally
$e^{f(x)} = \sum_{n=0}^{\infty} \frac{(f(x))^n}{n!}$)

$$p_X(k) = \frac{\exp(-\lambda)\,\lambda^k}{k!}$$

$$M_X(t) = E(e^{tk}) = \sum_{k=0}^{\infty} \frac{e^{tk} e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{e^{tk} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

MGF Properties

- $E(e^{tx}) = E\left(\sum_{n=0}^{\infty} \frac{(tx)^n}{n!}\right)$

$$= E\left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots\right)$$

- What is $\frac{d}{dt} E(e^{xt})\Big|_{t=0} = \frac{d}{dt} M_x(t)\Big|_{t=0}$?

$$\Rightarrow \frac{d}{dt}\left(1 + tE(x) + \frac{t^2}{2!}E(x^2) + \dots\right)$$

$$= 0 + E(X) + tE(X^2) + \dots$$

$t = 0$

$$\Rightarrow = E(X)$$

So $\frac{d}{dt}(M_x(t))\Big|_{t=0}$ or $M_x'(0) = E(X)$

Theorem

Let X be a RV. If $\exists\ \delta > 0$ s.t.

$\forall\ t \in (-\delta, \delta)$, $M_X(t) < \infty$ } $M_X(t)$ is bounded for some interval around zero

then:

$\forall\ n \in \mathbb{N} \setminus \{0\}$, i.e. $n = 1, 2, \ldots$

$$E[X^{(n)}] = \underbrace{\frac{d^{(n)}}{dt^{(n)}} M_X(t) \Big|_{t=0} = M_X^{(n)}(0)}_{\text{n'th derivative of } M_X(t) \text{ evaluated at } t=0.}$$

Ex)

Let $X \sim \exp(\lambda)$

① Find $M_X(t)$.

② Find an expression for $E(X^n)$.

① $f_X(x) = \lambda \exp(-\lambda x)$

$$E(e^{Xt}) = \int_0^\infty \exp(xt)\, \lambda \exp(-\lambda x)\, dx$$

$$= \lambda \int_0^\infty \exp(xt - \lambda x)\, dx$$

$$= \lambda \int_0^\infty \underbrace{\exp(-x(\lambda - t))\, dx}_{\text{if } t<\lambda \text{ this is an } \exp(\lambda - t)}$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if } t < \lambda$$

If $t > \lambda$ then $-x(\lambda - t) > 0$

So $E(e^{Xt}) = \infty$.

$$\Rightarrow M_x(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & t < \lambda \\ \infty & \text{else} \end{cases}$$

② $E(X^n)$:

$$M_x'(t) = \frac{\lambda}{(\lambda - t)^2} \xRightarrow{t=0} \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = E(X)$$

$$M_x''(t) = \frac{2\lambda}{(\lambda - t)^3} \xRightarrow{t=0} \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = E(X^2)$$

$$M_x'''(t) = \frac{3 \cdot 2(\lambda)}{(\lambda - t)^4} \xRightarrow{t=0} \frac{(3 \cdot 2)\lambda}{\lambda^4} = \frac{(3 \cdot 2)}{\lambda^3}$$

$$\Rightarrow \boxed{M_x^{(n)}(0) = \frac{n!}{\lambda^n} = E(X^n)}$$

Equality in Distribution

- Two RV, X and Y are equal in distribution, i.e., $X \overset{d}{=} Y$, if:

$$P(X \in B) = P(Y \in B) \quad \forall \; B \subset \mathbb{R}$$

Theorem

Let X and Y be two R.V.

If $\exists\, \delta > 0$ s.t. $\forall\, t \in (-\delta, \delta)$

$M_X(t), M_Y(t) < \infty$ and $M_X(t) = M_Y(t)$

then: $X \overset{d}{=} Y$

MGF's uniquely define probability distributions

Ex)

$$M_X(t) = \frac{1}{5}e^{-17t} + \frac{1}{4} + \frac{11}{20}e^{2t}.$$

① Is X discrete or continuous?

② Find the distribution of X.

① $M_X(t)$ is a finite sum so X is discrete

② $$P_X(k) = \begin{cases} \frac{1}{5} & k = -17 \\ \frac{1}{4} & k = 0 \\ \frac{11}{20} & k = 2 \end{cases}$$

Sums

So far:

① Jacobian (if we have an invertible transform)

② Convolution (if independent)

Ex) Let X have MGF $M_X(t)$.

Let Y have MGF $M_Y(t)$.

Assume X and Y are independent.

What is $M_{X+Y}(t)$?

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY})$$

$$= \underbrace{E(e^{tX})E(e^{tY})}_{\text{independence}} = M_X(t) \cdot M_Y(t)$$

Lemma

Let $X_1, \dots, X_n$ be independent RV

Then $M_{X_1 + \dots + X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$

Ex) Let $X \sim Pois(\lambda)$
$Y \sim Pois(\mu)$ s.t. X, Y are independent

What is the dist of $X+Y$? Use MGF's.

(Hint: $M_X(t) = e^{\lambda(e^t-1)}$ if $X \sim Pois(\lambda)$)

① $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$
$= e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$
$= e^{(\lambda+\mu)(e^t-1)}$] MGF of $Pois(\lambda+\mu)$

Since MGF's are unique to distributions

$\Rightarrow X+Y \sim Pois(\lambda+\mu)$