Lecture 1: Stat/Math 395

6.1

Joint Distributions

⇒ If we flip 1 coin, with probability p of being heads what distribution does this follow?

Denote X as the R.V. s.t. $X \sim Ber(p)$

$p(x) = p^x (1-p)^{1-x}$ } probability mass function

↳ $p(0) = 1-p$, $p(1) = p$

⇒ What if we flip two coins?

Let X_1 be coin 1

Let X_2 be coin 2

What is $p(x_1, x_2)$? } joint probability mass function

Joint Probability Mass Function

Let $X_1, X_2, \ldots, X_n$ be discrete RV's defined on the same sample space, Ω, then the Joint PMF is defined as:

$$p(x_1, x_2, \ldots, x_n) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

- Properties of the joint PMF:

① $p(x_1, \ldots, x_n) \geq 0 \ \forall \ x_1, x_2, \ldots, x_n \in \Omega$

② $\sum_{x_1, x_2, \ldots, x_n} p(x_1, \ldots, x_n) = 1$

Ex) Lets flip 2 coins with probability p of getting a heads as before

Let X_1 be the result of coin 1 (0 for T, 1 for H)
" " X_2 " " coin 2

$p(X_1, X_2) =$

X_1 \ X_2	0	1
0	$(1-p)^2$	$p(1-p)$
1	$p(1-p)$	p^2

$$= p^{X_1 + X_2}(1-p)^{2-(X_1+X_2)}$$

Expectation of Joint PMF's

$$E(g(X_1,\ldots,X_n)) = \sum_{x_1 \ldots x_n} g(x_1,\ldots,x_n)\, p(x_1,\ldots,x_n)$$

Ex) Consider a 3 sided die which you roll twice, recording each roll as X_1 and X_2

Let $Y \in \{0,1\}$ s.t.

$$Y = \underbrace{I}_{\text{indicator variable}}(X_1 \geq X_2) = \begin{cases} 1 & X_1 \geq X_2 \\ 0 & X_1 < X_2 \end{cases}$$

What is $p(X_1, Y)$?

X_1	Y	$p(X_1, Y)$
1	1	$\frac{1}{3} \cdot (P(X_2 \leq 1)) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
2	1	$\frac{1}{3} \cdot (P(X_2 \leq 2)) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
3	1	$\frac{1}{3} \cdot P(X_2 \leq 3) = \frac{1}{3} \cdot 1 = \frac{1}{3}$
1	0	$\frac{1}{3} \cdot P(X_2 > 1) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
2	0	$\frac{1}{3} \cdot P(X_2 > 2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
3	0	$\frac{1}{3} \cdot P(X_2 > 3) = \frac{1}{3} \cdot 0 = 0$

Let $g(X_1, Y) = (X_1)Y$

What is $E(g(X_1, Y))$?

$$E(g(X_1,Y)) = \sum_{X_1,Y} X_1 Y \cdot p(X_1,Y)$$

$$= 1\cdot\left(\frac{1}{9}\right) + 2\cdot\left(\frac{2}{9}\right) + 3\cdot\left(\frac{1}{3}\right)$$

$$= \frac{14}{9}$$

Marginal Joint PMF

If we have $p(X_1, X_2, \ldots, x_n)$, how do we evaluate $p(X_i = x_i)$?

$\Rightarrow$ **Marginal PMF of X_i :**

$$P_{X_i}(k) = \underbrace{\sum_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n} p(x_1, \ldots, x_{i-1}, k, x_{i+1}, \ldots, x_n)}_{\text{hold } X_i \text{ constant and sum all other possibilities}}$$

Lemma

The pmf of X_i is the marginal pmf of X_i