1 Classification of PDEs

In this lecture we will discuss some classifications of partial differential equations.

1.1 Notation

We let $u(x_1, \dots, x_d)$ be a scalar function, that is, $u(x_1, \dots, x_d) \in \mathbf{R}$ for all $(x_1, \dots, x_d) \in \mathbf{R}^d$. Let us define the partial derivative of u:

$$\partial_{x_i} u(x_1, \dots, x_d) := \lim_{h \to 0} \frac{u(x_1, \dots, x_i + h, \dots, x_d) - u(x_1, \dots, x_i, \dots, x_d)}{h} \tag{1}$$

or equivalently with the notation $x = (x_1, \dots, x_d)$

$$\partial_{x_i} u(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \tag{2}$$

where $e_i = (0, \dots, 1, \dots, 0)$ (1 at ith entry). Recall that $\partial_{x_i} u(x) = \nabla u \cdot e_i$.

Other notations for partial derivative are (and which we will henceforth use)

$$u_{,i}, \quad u_{x_i}, \quad \frac{\partial}{\partial x_i}, \quad D_{x_i}. \tag{3}$$

Similarly higher order partial derivatives are denoted as follows:

$$u_{,ij}, \quad u_{x_i x_j}, \quad \frac{\partial^2}{\partial x_i \partial x_j}, \quad D_{x_i x_j}. \tag{4}$$

We denote by ∇u the gradient of u and by $\nabla^2 u$ the transposed of the Jacobian of ∇u, that is, $\nabla^2 u := D(\nabla u)^\top$.

1.2 Linear elliptic equations

Under a second order linear partial differential operator in divergence form we understand:

$$Lu(x) := -\operatorname{div}(A(x)\nabla u(x)) + b(x) \cdot \nabla u(x) + c(x)u(x) \quad \text{for } x \in \Omega, \tag{5}$$

where $A : \overline{\Omega} \to \mathbf{R}^{d \times d}$ is a matrix-valued function, $b : \overline{\Omega} \to \mathbf{R}^d$ and $c : \overline{\Omega} \to \mathbf{R}$ are given functions. For given $f : \overline{\Omega} \to \mathbf{R}$ one obtains the associated differential equation:

$$Lu(x) = f(x) \quad \text{for } x \in \Omega. \tag{6}$$

We note that (5) can be equivalently written as:

$$Lu(x) = -\sum_{i,j=1}^{d} (a_{ij}u_{x_j})_{x_i} + \sum_{j=1}^{d} b_j(x)u_{x_j}(x) + c(x)u. \tag{7}$$

Here $a_{ij}(x)$ are the entries of the matrix $A(x)$ and $b(x) = (b_1(x), \dots, b_d(x))$.

Definition 1.1. We say the differential operator L is elliptic if A is uniformly positive definite on $\overline{\Omega}$, that is, there is $c > 0$, such that

$$A(x)v \cdot v \geqslant c\|v\|^2 \quad \text{for all } v \in \mathbf{R}^d \tag{8}$$

or equivalently

$$\sum_{i,j=1}^{d} a_{ij}(x)v_i v_j \geqslant c\|v\|^2. \tag{9}$$

Here $\|v\| := \sqrt{\sum_{i=1}^{d} v_i^2}$ denotes the Euclidean norm of $v = (v_1, \dots, v_d) \in \mathbf{R}^d$.

Example 1.1. As an example we consider $A(x) = I$, $b = 0$ and $c = 0$, where $I \in \mathbf{R}^{d\times d}$ denotes the identity matrix. This yields the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega. \tag{10}$$

In order for (10) to have a unique solution we have to impose boundary conditions. For instance

- $u = 0$ on $\partial\Omega$ (homogeneous Dirichlet boundary conditions)
- $\nabla u \cdot n = 0$ on $\partial\Omega$ (homogeneous Neumann boundary conditions)

where n denotes the outward pointing normal vector field along $\partial\Omega$

Example 1.2. A second example is $A(x) = \alpha_1\chi_\omega(x) + \alpha_2\chi_{\Omega\setminus\omega}(x)$, where χ_ω denotes the characteristic function of the set ω, which is 1 if $x \in \omega$ and 0 otherwise. With this

$$\begin{aligned} -\alpha_1\Delta u &= f \quad \text{in } \Omega^+ := \omega \\ -\alpha_2\Delta u &= f \quad \text{in } \Omega^- := \Omega \setminus \overline{\omega}. \end{aligned} \tag{11}$$

In this case a solution u to (11) must satisfy the following transmission conditions:

$$\begin{aligned} u^+ &= u^- \quad \text{on } \partial\Omega^+ \cap \partial\Omega^- \\ \alpha_1\nabla u^+ \cdot n &= \alpha_2\nabla u^- \cdot n \quad \text{on } \partial\Omega^+ \cap \partial\Omega^- \end{aligned} \tag{12}$$

where $u^+ := u|_{\Omega^+}$ and $u^- := u|_{\Omega^-}$ denote the restrictions of u to Ω^+ and Ω^-, respectively.

Notice that if the a_{ij} are differentiable, then one can write by the product rule

$$Lu(x) = -\sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} \tilde{b}_i(x) u_{x_i}(x) + c(x)u. \tag{13}$$

where $\tilde{b}_i(x) := b_i(x) - \sum_{j=1}^{d} (a_{ij})_{x_i} u_{x_j}$. One also says in this case that the operator L is in non-divergence form.

For the sake of completeness we associate a bilinear form with an elliptic operator.

Definition 1.2. We define the bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ by

$$B(u, v) := \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x) + b(x) \cdot u(x)v(x) + c(x)u(x)v(x). \tag{14}$$

With this bilinear form one can formulate for instance (6) when imposing Dirichlet boundary conditions on $\partial\Omega$ as follows: find $u \in H_0^1(\Omega)$, such that

$$B(u, v) = \int_\Omega fv \, dx \quad \text{for all } v \in H_0^1(\Omega). \tag{15}$$

1.3 Linear parabolic equations

Under a linear parabolic equation we understand an equation

$$u_t + Lu(x) = f(x) \quad \text{in } (0, T) \times \Omega, \tag{16}$$

where $u(t, x_1, \ldots, x_d)$ is the unknown, $f : \overline{\Omega} \to \mathbf{R}$ is given and L is a second order differential operator of divergence form

$$Lu(x) = -\sum_{i,j=1}^{d} (a_{ij}(t, x)u_{x_j})_{x_i} + \sum_{j=1}^{d} b_j(t, x)u_{x_j}(x) + c(t, x)u \tag{17}$$

or non-divergence form

$$Lu(x) = -\sum_{i,j=1}^{d} a_{ij}(t, x)u_{x_j x_i} + \sum_{j=1}^{d} b_j(x)u_{x_j}(x) + c(x)u \tag{18}$$

Example 1.3. For $Lu = \Delta u$ we obtain the heat equation

$$u_t - \Delta u = f \quad \text{in } (0, T) \times \Omega. \tag{19}$$

In order for (16) to have a unique solution one has to impose and initial "temperature" and boundary conditions:

- $u(0,x) = g(x) \quad$ for $x \in \Omega$
- $u(t,x) = 0 \quad$ for $(t,x) \in (0,T] \times \partial\Omega$

where $g : \overline{\Omega} \to \mathbf{R}$ is given.

Definition 1.3. We say that the operator $\frac{\partial}{\partial t} + L$ is uniformly parabolic if

$$\sum_{i,j=1}^{d} a_{ij}(t,x)v_i v_j \geqslant c\|v\|^2, \tag{20}$$

for all $v \in \mathbf{R}^d,\ (t,x) \in [0,T] \times \overline{\Omega}$.

1.4 Linear hyperbolic equations

Under a linear hyperbolic equation we understand an equation

$$u_{tt} + Lu(x) = f(x) \quad \text{in } (0,T) \times \Omega, \tag{21}$$

where $u(t, x_1, \dots, x_d)$ is the unknown, $f : \overline{\Omega} \to \mathbf{R}$ is given and L is a second order differential operator of divergence form

$$Lu(x) = -\sum_{i,j=1}^{d} (a_{ij}(t,x)u_{x_j})_{x_i} + \sum_{j=1} b_j(t,x)u_{x_j}(x) + c(t,x)u \tag{22}$$

or non-divergence form

$$Lu(x) = -\sum_{i,j=1}^{d} (a_{ij}(t,x)u_{x_j})_{x_i} + \sum_{j=1} b_j(t,x)u_{x_j}(x) + c(t,x)u \tag{23}$$

Example 1.4. The choice $L = -\Delta$ (that is $A = I$, $b = 0$ and $c = 0$) give the wave equation

$$u_{tt} - \Delta u = f.$$

Definition 1.4. We say that the operator $\frac{\partial^2}{\partial^2 t} + L$ is uniformly hyperbolic if

$$\sum_{i,j=1}^{d} a_{ij}(t,x)v_i v_j \geqslant c\|v\|^2, \tag{24}$$

for all $v \in \mathbf{R}^d,\ (t,x) \in [0,T] \times \overline{\Omega}$.

Similarly to the parabolic equation we need to prescribe initial data and boundary data:

- $u(0,x) = g_1(x) \quad$ for $x \in \Omega$
- $u_t(0,x) = g_2(x) \quad$ for $x \in \Omega$
- $u(t,x) = 0 \quad$ for $(t,x) \in [0,T] \times \partial\Omega$

where $g_1, g_1 : \overline{\Omega} \to \mathbf{R}$ are given.

1.5 First order fully nonlinear equations

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain and let $F : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$. Under a fully nonlinear second order partial differential equation we understand:

$$F(x_1, \dots, x_d, u(x_1, \dots, x_d), \nabla u(x_1, \dots, x_d)) = 0 \tag{25}$$

for all $(x_1, \dots, x_d) \in \Omega$, or shorter with the notation $x := (x_1, \dots, x_d)$

$$F(x, \nabla u(x)) = 0 \quad \text{for all } x \in \Omega. \tag{26}$$

Example 1.5 (Eikonal equation)**.** The Eikonal equation is a first order equation: find $u : \overline{\Omega} \to \mathbf{R}$, such that,

$$\begin{aligned} |\nabla u| &= 1 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{27}$$

The interpretation of $u(x)$ at a point $x \in \Omega$ is the distance to the boundary $\partial\Omega$, that means, $u(x) = \min_{y \in \partial\Omega} \|x - y\|$.

1.6 Second order fully nonlinear equations

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain and let $F : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^{d \times d} \to \mathbf{R}$. Under a fully nonlinear second order partial differential equation we understand:

$$F(x_1, \dots, x_d, u(x_1, \dots, x_d), \nabla u(x_1, \dots, x_d), \nabla^2 u(x_1, \dots, x_d)) = 0 \tag{28}$$

for all $(x_1, \dots, x_d) \in \Omega$, or shorter with the notation $x := (x_1, \dots, x_d)$

$$F(x, \nabla u(x), \nabla^2 u(x)) = 0 \quad \text{for all } x \in \Omega. \tag{29}$$

Example 1.6 (Monge-Ampère)**.** An example of a fully nonlinear PDE is the Monge-Ampère equation:

$$u_{xx}(x,y) u_{yy}(x,y) - (u_{xy}(x,y))^2 = 0 \quad \text{for } (x,y) \in \mathbf{R}^2. \tag{30}$$

A generalisation of this equation to arbitrary space dimension is

$$\det(\nabla^2 u) = \psi \quad \text{in } \mathbf{R}^d, \tag{31}$$

where $\psi : \mathbf{R}^d \to \mathbf{R}$ is some given function.

Example 1.7 (Nonlinear heat equation)**.** Find $u : (0, \infty) \times \mathbf{R} \to \mathbf{R}$, such that

$$u_t = \frac{u_{xx}}{u_x^2}. \tag{32}$$

Assume that u is a solution to (32) and $u_x > 0$. For fixed t we denote the inverse of $x \mapsto u(t,x)$ by $v(t,x) = u^{-1}(t,x)$. The v solves the linear equation:

$$v_t = v_{xx} + v_x. \tag{33}$$

1.7 Semilinear equations

Roughly speaking semilinear equation are equations where the nonlinearity does not appear in the leading part of the operator.

Definition 1.5. We call

$$Lu(x) + \rho(x, u(x)) = f(x) \quad x \in \Omega \tag{34}$$

second order semilinear equation, where $\rho = \rho(x,y) : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a given function and L is a second order differential operator as in Definition 5 or 7. We say the operator $L + \rho$ is monotone provided

- L is uniformly elliptic
- $\partial_y \rho(x,y) \geqslant 0$ for all $(x,y) \in \overline{\Omega} \times \mathbf{R}$

Example 1.8. A typical example is $L = -\Delta$ (that is $A = I$, $b = 0$ and $c = 0$) and $\rho(x,y) = y^3$. Then (34) becomes:

$$-\Delta u + u^3 = f \quad \text{ in } \Omega. \tag{35}$$

Of course also in this case we have to prescribe boundary conditions to get unique solvability. For instance homogeneous Neumann boundary conditions $u = 0$ on $\partial\Omega$.

1.8 Quasilinear equations

Finally let us turn our attention to quasilinear equations.

Definition 1.6. Under a second order quasilinear equation we understand

$$Qu(x) := -\sum_{i,j=1}^{d} a_{ij}(x, u(x), \nabla u(x)) u_{x_i x_j} + b(x, u(x), \nabla u(x)) = f(x) \quad \text{ in } \Omega, \tag{36}$$

where $a_{ij}, b : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ are given functions. The operator Q is called *elliptic* if there is a constant $c >$, such that

$$\sum_{i,j=1}^{d} a_{ij}(x, z, p)\nu_i\nu_j \geqslant c\|\nu\|^2, \tag{37}$$

where $\nu = (\nu_1, \ldots, \nu_d)$.

Similarly to linear second order PDE on says the operator Q is in divergence form if we can write it as

$$Qu(x) = \sum_{i,j=1}^{d} (a_{ij}(x, u(x), \nabla u(x))u_{x_j})_{x_i} + b(x, u(x), \nabla u(x)) \tag{38}$$

or with $A(x, u(x), \nabla u(x)) := (a_{ij})$ as

$$Qu(x) = -\operatorname{div}(A(x, u(x), \nabla u(x))\nabla u(x)) + b(x, u(x), \nabla u(x)). \tag{39}$$

Example 1.9. As an example we consider

$$Qu = -\Delta u + \sum_{i,j=1}^{d} \frac{u_{x_i}u_{x_j}}{(1 + |\nabla u|^2)} u_{x_i x_j}. \tag{40}$$

and $b(x, u(x), \nabla u(x)) = f(x)$.

Example 1.10 (Equation of Capillarity)**.** We consider the equilibrium shape of a liquid surface with constant surface tension. The liquid is supposed to be in a homogeneous gravity field. Then the surface can be described by the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \kappa u. \tag{41}$$

Here u denotes the perturbation of a reference shape $S \subset \mathbf{R}^2$. That means every point $x \in S$ is moved to the equilibrium point $(x, u(x))$.

Example 1.11. Another example is the p-Laplacian

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega. \tag{42}$$

This equation is the Euler-Lagrange equation of the energy

$$E(\varphi) := \frac{1}{p}\int_\Omega |\nabla\varphi|^p - \int_\Omega f\varphi \, dx. \tag{43}$$

Literature

- L. Evans - Partial Differential Equations
- D. Gilbarg, N.S. Trudinger - Elliptic Partial Differential Equations of Second Order