lecture notes

NSSC–II
The Finite Element Method

Some Basics in the Context of Solid Mechanics

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1 Introduction

These lecture notes, and the accompanying classes, aim at a very short introduction into the Finite Element Method. This will be done in the context of solid mechanics, highlighting the intricate material behavior in linear elasticity based on 4^{th} order tensors.
The subject is presented from the perspective of an (mechanical) engineer rather than from the point of view of mathematics. Nevertheless, the mathematical foundation shall become visible as well as the mechanical basics.

The user of Finite Element Method simulation software, and even more the developer, has to have knowledge on both the mathematics of the method and the physics of the modeled problem. This is required to set up appropriate models, to correctly interpret the simulation results, to know the limitations of the approach, and — if necessary — to improve the modeling.

Besides these lecture notes, an introduction to the Finite Element Method and to solid mechanics can be found in the first few chapters of [1]. Fundamental treatment is given e.g. in [2, 3]. The tutorial example for this class is based on [4].
All mentioned literature is available on-line for students of the TU Wien via the TU Wien library (with established VPN connection).

2 Continuum Solid Mechanics

The basics of continuum solid mechanics will be introduced and presented in matrix notation which is advantageous for software implementation. The equations pertain to linearized theory of elasticity for static problems.

2.1 Balance Equation

The local balance equation (condition of equilibrium) in differential form and index notation read,

$$\sigma_{ji,j} + q_i = 0, \quad \sigma_{ij} = \sigma_{ji} \qquad \forall\, i,j = 1,2,3\,(x,y,z)$$

(Einstein summation applies)

σ_{ji} are the components of the symmetric 2$^{\text{nd}}$ order stress tensor and $_{,j}$ denotes the partial derivative. q_i are the vector components of the volume force density, $\vec{q}$, i.e. $\vec{q}^{\,\mathrm{T}} = (q_x, q_y, q_z)$.

Fully expanded for *plane stress* state, the balance equation read,

$$\begin{aligned} \sigma_{zz} &= 0, & \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} &= -q_x \\ \sigma_{zx} &= 0, & \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= -q_y \\ \sigma_{zy} &= 0, & \sigma_{xy} &= \sigma_{yx} \end{aligned}$$

Applying Voigt–Nye matrix notation, a differential operator matrix can be defined as

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = - \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

For general tri-axial stress states the balance equation in Voigt–Nye notation read,

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{pmatrix} = - \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$$

in short,

$$\boxed{\underset{\approx}{\mathbf{d}}^{\mathrm{T}}\underset{\sim}{\sigma} = -\vec{q}}$$

with the corresponding operator matrix $\underset{\approx}{\mathbf{d}}^{\mathrm{T}}$.

The vector of the stress components, $\underset{\sim}{\sigma}$, is a vector in the sense of linear algebra, composed by a particular arrangement of the tensor components; it is not a physical entity. In contrast, the traction vector, $\vec{\sigma}_n$, is a physical entity with describes the force density (per area) with respect to the plane n.

2.2 Displacement—Strain Relation

The linearized displacement—strain relation reads in index notation,

$$\boxed{\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)}$$

with $\vec{u}^{\,\mathrm{T}} = (u_1, u_2, u_3) = (u, v, w)$ being the local displacement vector,

For example,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad , \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\gamma_{xy}}{2}$$

Applying Voigt–Nye notation, the tri-axial relation reads,

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

in short,

$$\boxed{\underset{\sim}{\varepsilon} = \underset{\approx}{\mathbf{d}}\vec{u}}$$

Note 1: The shear contributions in $\underset{\sim}{\varepsilon}$ are expressed by the shear angles $\gamma_{ij} = 2\varepsilon_{ij}$ $(i \neq j)$ (which give rise to certain advantages).

Note 2: The differential operator matrix, $\underset{\approx}{\mathbf{d}}$, is indeed the transposed of the matrix in the balance equations, $\underset{\approx}{\mathbf{d}}^{\mathrm{T}}$, provided the above definition of the vector of strain components, $\underset{\sim}{\varepsilon}$ is used.

2.3 Constitutive Material Law

The relation between stresses and strains is given by the constitutive law. In linear elasticity it is called Hooke's Law, and reads in index notation,

$$\sigma_{ij} = E_{ijkl}\,\varepsilon_{kl}$$

Note, that the elasticity tensor, E_{ijkl}, is a 4^{th} order tensor.

Applying Voigt–Nye matrix notation and assuming isotropic (i.e. direction independent) behavior, the constitutive law can be given as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{pmatrix} = \underbrace{\frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & \underset{\approx}{\mathbf{0}} & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & \underset{\approx}{\mathbf{0}} & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{pmatrix}}_{\underset{\approx}{\mathbf{E}}} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}$$

with E being the Young's modulus and ν the Poisson ratio, stating two independent material properties which can be measured by experimental testing.

In short,

$$\boxed{\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{E}}\,\underset{\sim}{\varepsilon}}$$

with the elasticity matrix $\underset{\approx}{\mathbf{E}}$.

The inverse relation can be given as,

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix} = \underbrace{\frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & \underset{\approx}{\mathbf{0}} & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & 0 & 0 \\ & \underset{\approx}{\mathbf{0}} & & 0 & 2(1+\nu) & 0 \\ & & & 0 & 0 & 2(1+\nu) \end{pmatrix}}_{\underset{\approx}{\mathbf{C}}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{pmatrix}$$

or

$$\boxed{\underset{\sim}{\varepsilon} = \underset{\approx}{\mathbf{C}}\, \underset{\sim}{\sigma}}$$

with the *Compliance Matrix* $\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{E}}^{-1}$.

2.4 Strain Energy and Elastic Potential

By use of the above conventions, the strain energy density can be expressed as,

$$\hat{U} = \frac{1}{2}\, \sigma_{ij}\, \varepsilon_{ij} = \frac{1}{2}\, \underset{\sim}{\sigma}^{\mathrm{T}}\, \underset{\sim}{\varepsilon}$$

from which follows the strain energy

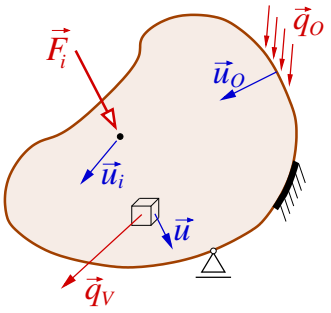
$$U = \int_{Vol} \hat{U}\, \mathrm{d}Vol$$

in a elastic body with volume Vol.

With the constitutive equation, $\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{E}}\underset{\sim}{\varepsilon}$, the symmetry of the elasticity matrix, $\underset{\approx}{\mathbf{E}} = \underset{\approx}{\mathbf{E}}^{\mathrm{T}}$, and the linear algebra rule, $(\underset{\approx}{\mathbf{A}}\underset{\sim}{\mathbf{b}})^{\mathrm{T}} = \underset{\sim}{\mathbf{b}}^{\mathrm{T}}\underset{\approx}{\mathbf{A}}^{\mathrm{T}}$, follows,

$$U = \int_{Vol} \left(\frac{1}{2}\, \underset{\sim}{\varepsilon}^{\mathrm{T}} \underset{\approx}{\mathbf{E}}\, \underset{\sim}{\varepsilon} \right) \mathrm{d}Vol$$

The total potential, V, of an elastic body is the sum of the potential of the internal forces, U, (the strain energy) and the potential of the external forces, W.



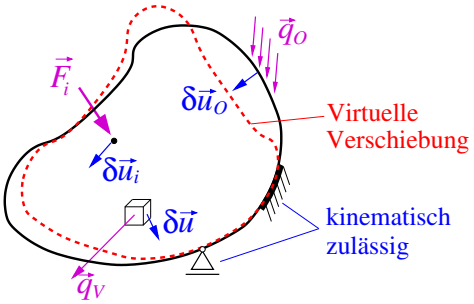
$$V = U + W$$
$$W = -\sum_i \vec{F}_i \cdot \vec{u}_i - \int_O \vec{q}_O \cdot \vec{u}_O \, \mathrm{d}O - \int_{Vol} \vec{q}_V \cdot \vec{u} \, \mathrm{d}Vol$$

The potential of the external forces comprise the effects of concentrated forces, $\vec{F}_i$, surface force densities, $\vec{q}_O$, and volume force densities, $\vec{q}_V$.

3 Variational Principles

One way to find equilibrium is the application of variational principles. Here, in the context of "displacement formulated Finite Elements" the principle of virtual displacements and the principle of minimum total potential energy will be shown.
Alternatively, the method of the weighted residuals can be applied in the context of the Finite Element Method (which is not followed here).

3.1 The Principle of Virtual Displacements



The virtual displacement is

- infinitesimally small
- kinematic admissible

A body is in equilibrium, if the virtual work is zero,

$$\boxed{\delta A = \delta A^{(\mathrm{a})} + \delta A^{(\mathrm{i})} = 0}$$

with the contribution of the external (a) and internal (i) work caused by virtual displace-

ments given by,

$$\delta A^{(\mathrm{a})} = \sum_i \vec{F}_i \cdot \delta\vec{u}_i + \int_O \vec{q}_O \cdot \delta\vec{u}_O \,\mathrm{d}O + \int_{Vol} \vec{q}_V \cdot \delta\vec{u} \,\mathrm{d}Vol$$

$$\delta A^{(\mathrm{i})} = -\int_{Vol} \sigma_{ij}\, \delta\varepsilon_{ij} \,\mathrm{d}Vol = -\int_{Vol} \underset{\sim}{\sigma}^{\mathrm{T}}\, \delta\underset{\sim}{\varepsilon} \,\mathrm{d}Vol \qquad \delta\underset{\sim}{\varepsilon} = \mathfrak{F}(\delta\vec{u})$$

Note: In the Finite Element Method all surface and volume force densities are converted to "consistent" nodal forces. Consequently,

$$\delta A^{(\mathrm{a})} = \underset{\sim}{F}^{\mathrm{T}}\, \delta\underset{\sim}{U}$$

with $\underset{\sim}{F}$ ... vector of nodal forces
$\underset{\sim}{U}$ ... vector nodal displacements

Then the principle of virtual displacements reads,

$$\int_{Vol} \underset{\sim}{\sigma}^{\mathrm{T}} \delta\underset{\sim}{\varepsilon} \mathrm{d}Vol = \underset{\sim}{F}^{\mathrm{T}}\, \delta\underset{\sim}{U}$$

3.2 Principle of Minimum Total Potential Energy

The principle of minimum total potential energy can be applied for conservative systems where all forces have a potential.

Since the potential is the negative work,

$$\delta A^{(\mathrm{a})} = -\delta W, \qquad \delta A^{(\mathrm{i})} = -\delta U$$

the principle of virtual displacement, $\delta A^{(\mathrm{a})} + \delta A^{(\mathrm{i})} = 0$, can be written as $\delta W + \delta U = 0$, and with $V = U + W$ equilibrium is expressed as

$$\boxed{\delta V = 0}$$

The condition for a minimum (i.e. stable equilibrium) follows from the second variation,

$$\delta^2 V > 0$$

V in general is a *functional* of the displacements $\vec{u}(x, y, z)$ and their derivatives.

If a Ritz' Ansatz (see later) is used, the problem simplifies to find the stationary value of a *function* of the displacements.

4 The RITZ' Approach (for Static Problems)

For the displacement field $\vec{u}(x,y,z)$, i.e. for their components $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ a Ritz' Ansatz is introduced of the form (e.g. for the x-component)

$$u(x,y,z) \rightarrow u^* = \sum_{j=1}^{N} a_{u,j}\, \varphi_{u,j} \overset{Einstein}{=} a_{u,j}\, \varphi_{u,j} = \underset{\sim}{\varphi}_u^{\mathrm{T}}\, \underset{\sim}{a}_u$$

and for the complete displacement vector

$$\vec{u}^* = \underset{\approx}{\boldsymbol{\varphi}}^{\mathrm{T}} \underset{\sim}{a}$$

which results in an *approximation of the displacement field* $\vec{u}^*(x,y,z)$. Here, $\varphi_j(x,y,z)$ are predefined *interpolation functions* which have to satisfy the kinematic, also called essential or Dirichlet boundary conditions (among other conditions). The *generalized coordinates* a_j are free coefficients. With such a Ritz' Ansatz an approximation of the potential V^* is obtained which is a *function* of the generalized coordinates.

$$V^* = V^*(a_1, a_2, \dots)$$

Applying the equilibrium condition

$$\delta V^* = 0 \Leftrightarrow \frac{\partial V^*}{\partial a_k}\, \delta a_k = 0$$

gives rise to

$$\frac{\partial V^*}{\partial a_1} = 0,\ \frac{\partial V^*}{\partial a_2} = 0 \dots \frac{\partial V^*}{\partial a_j} = 0 \quad \forall\, j$$

5 The Finite Element Method (as Ritz' Method)

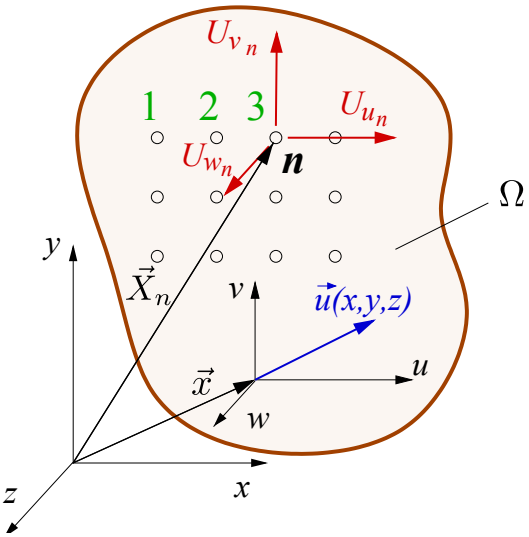
5.1 A Special Ritz' Ansatz

Before, the Ansatz was of the form

$$u(x,y,z) = a_{u_j}\,\varphi_{u_j}(x,y,z) = \underset{\sim}{\varphi}_u^{\mathrm{T}}\,\underset{\sim}{a}_u$$

where the coefficients a_j did not carry a mechanical "meaning".

Now the Ansatz is chosen such, that the coefficients are equal to the displacements of given points — the Finite Element *nodes* — in the domain. Each discrete node with number n is associated a set of interpolation functions $\varphi_{u_n}(x,y,z)$, $\varphi_{v_n}(x,y,z)$, $\varphi_{w_n}(x,y,z)$ valid over the entire domain Ω. Now, at any location $(x,y,z) \in \Omega$ the displacement vector can be expressed as



$$u = \sum_i U_{u_i}\varphi_{u_i}(x,y,z)\,; \qquad v, w \text{ likewise;}$$

consequently

$$u(x,y,z) = \underset{\sim}{\varphi}_u^{\mathrm{T}}(x,y,z)\,\underset{\sim}{U}_u$$

$$v(x,y,z) = \underset{\sim}{\varphi}_v^{\mathrm{T}}(x,y,z)\,\underset{\sim}{U}_v$$

$$w(x,y,z) = \underset{\sim}{\varphi}_w^{\mathrm{T}}(x,y,z)\,\underset{\sim}{U}_w$$

in short

$$\Rightarrow \boxed{\vec{u}(x,y,z) = \underset{\approx}{\boldsymbol{\varphi}}^{\mathrm{T}}(x,y,z)\,\underset{\sim}{U}}$$

with

$$\underset{\approx}{\boldsymbol{\varphi}}^{\mathrm{T}} = \begin{pmatrix} \underset{\sim}{\varphi_u}^{\mathrm{T}} & \underset{\sim}{0}^{\mathrm{T}} & \underset{\sim}{0}^{\mathrm{T}} \\ \underset{\sim}{0}^{\mathrm{T}} & \underset{\sim}{\varphi_v}^{\mathrm{T}} & \underset{\sim}{0}^{\mathrm{T}} \\ \underset{\sim}{0}^{\mathrm{T}} & \underset{\sim}{0}^{\mathrm{T}} & \underset{\sim}{\varphi_w}^{\mathrm{T}} \end{pmatrix}, \qquad \underset{\sim}{U} = \begin{pmatrix} \underset{\sim}{U_u} \\ \underset{\sim}{U_v} \\ \underset{\sim}{U_w} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ v_1 \\ v_2 \\ \vdots \\ w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

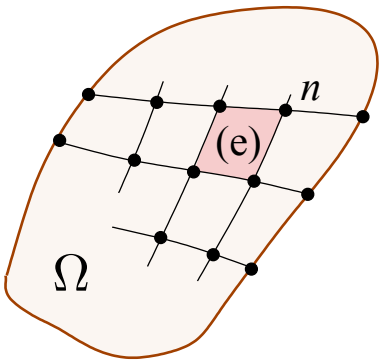
with u_1,u_2,$u_3 \ldots u_n$ being the x–components of the displacement vector of the nodes 1,2,3...n; likewise for v_i and w_i.

The displacement in x–direction of any point $(x, y, z) \in \Omega$ can be given as

$$\begin{aligned} u(x,y,z) &= u_1\varphi_{u_1}(x,y,z) + \ \ldots u_n\varphi_{u_n}(x,y,z)\cdots = \\ &= U_{u_1}\varphi_{u_1}(x,y,z) + \ldots U_{u_n}\varphi_{u_n}(x,y,z)\ldots \end{aligned}$$

For general case it is difficult (impossible) to find appropriate interpolation functions for the entire domain.

5.2 Ansatz for Small Sub-Domains — "Finite Elements"



The domain is divided into small sub-domains, the *Finite Elements* (e) and their boundaries form the *mesh* which is anchored at the *nodes* n.

The definition of the interpolation functions is given for the sub-domains, only, but no longer for the entire domain Ω (obeying some compatibility conditions, see later). Consequently, the displacement field at any point inside the Finite Element $(x, y, z) \in$ (e) reads

$$\boxed{\vec{u}(x,y,z) = \underset{\approx}{\boldsymbol{\varphi}}^{(\mathrm{e})^{\mathrm{T}}}(x,y,z)\,\underset{\sim}{U}^{(\mathrm{e})}}$$

e.g. for the x–component

$$u(x,y,z) = \sum_{i=1}^{N^{(\mathrm{e})}} \varphi_{u_i}^{(\mathrm{e})}(x,y,z)\,U_{u_i}^{(\mathrm{e})} = \underset{\sim}{\varphi}_u^{(\mathrm{e})^{\mathrm{T}}}\,\underset{\sim}{U}_u^{(\mathrm{e})}$$

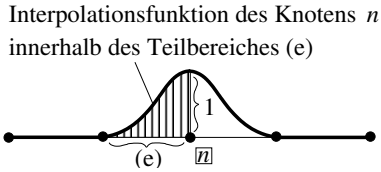
where $i = 1, 2....N^{(\mathrm{e})}$ denote the local node numbers of the element (e). From this, it gets obvious that the interpolation function $\varphi_i^{(\mathrm{e})}$ is associated to the node i and that it interpolates the contribution of node i over the element. The value at the node must be "one", generally written as,

$$\varphi_{k_j}^{(e)}\left(\vec{X}_i^{(e)}\right) = \delta_{ij}, \qquad k = u, v, w, \qquad i, j \ldots \text{node numbers}$$

and "zero" at the location of all other nodes in the element.

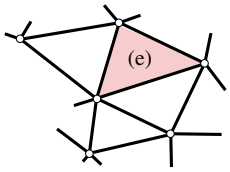
The following advantages arise;

* substantially simpler interpolation functions φ



* much easier to satisfy the essential boundary conditions

* topological equivalent elements have interpolation functions of the same type, e.g. triangular elements

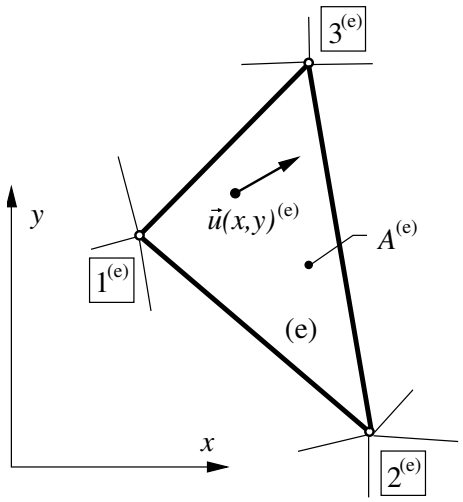


* complex shaped geometries can be modeled, i.e. the domain boundary

* suitable for automation (computer implementation)

5.3 Example — Constant Strain Triangle

As example a plane problem is shown where nodes have only two components of coordinates and displacement, respectively. For a triangular element the interpolation of the displacements and the strains will be shown.



$$\underset{\sim}{U}^{(\mathrm{e})} = \begin{pmatrix} u_1^{(\mathrm{e})} \\ u_2^{(\mathrm{e})} \\ u_3^{(\mathrm{e})} \\ v_1^{(\mathrm{e})} \\ v_2^{(\mathrm{e})} \\ v_3^{(\mathrm{e})} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{U}_u^{(\mathrm{e})} \\ \underset{\sim}{U}_v^{(\mathrm{e})} \end{pmatrix}$$

$$\vec{u}(x,y)^{(\mathrm{e})} = \underset{\approx}{\boldsymbol{\varphi}}^{(\mathrm{e})\mathrm{T}} \underset{\sim}{U}^{(\mathrm{e})}$$

$$u^{(\mathrm{e})} = \underset{\sim}{\varphi}_u^{(\mathrm{e})\mathrm{T}} \underset{\sim}{U}_u^{(\mathrm{e})}$$

$$v^{(\mathrm{e})} = \underset{\sim}{\varphi}_v^{(\mathrm{e})\mathrm{T}} \underset{\sim}{U}_v^{(\mathrm{e})}$$

$$\underset{\sim}{\varphi}_u^{(\mathrm{e})} = \frac{1}{2A^{(\mathrm{e})}} \begin{pmatrix} X_{x_2} X_{y_3} - X_{x_3} X_{y_2} + (X_{y_2} - X_{y_3})x + (X_{x_3} - X_{x_2})y \\ X_{x_3} X_{y_1} - X_{x_1} X_{y_3} + (X_{y_3} - X_{y_1})x + (X_{x_1} - X_{x_3})y \\ X_{x_1} X_{y_2} - X_{x_2} X_{y_1} + (X_{y_1} - X_{y_2})x + (X_{x_2} - X_{x_1})y \end{pmatrix}^{(\mathrm{e})}$$

$$\underset{\sim}{\varphi}_v^{(\mathrm{e})}(x,y) = \underset{\sim}{\varphi}_u^{(\mathrm{e})}(x,y)$$

with $A^{(\mathrm{e})}$ being the area of the element which is computed as,

$$2A^{(\mathrm{e})} = X_{x_1}(X_{y_2} - X_{y_3}) + X_{x_2}(X_{y_3} - X_{y_1}) + X_{x_3}(X_{y_1} - X_{y_2})$$

Consideration of the interpolation function at node $1^{(\mathrm{e})}$

$$\underset{\sim}{\varphi}_u^{(\mathrm{e})}(x_1,y_1) = \frac{1}{2A^{(\mathrm{e})}} \begin{pmatrix} X_{x_2} X_{y_3} - X_{x_3} X_{y_2} + X_{y_2} X_{x_1} - X_{y_3} X_{x_1} + X_{x_3} X_{y_1} - X_{x_2} X_{y_1} \\ X_{x_3} X_{y_1} - X_{x_1} X_{y_3} + X_{y_3} X_{x_1} - X_{y_1} X_{x_1} + X_{x_1} X_{y_1} - X_{x_3} X_{y_1} \\ X_{x_1} X_{y_2} - X_{x_2} X_{y_1} + X_{y_1} X_{x_1} - X_{y_2} X_{x_1} + X_{x_2} X_{y_1} - X_{x_1} X_{y_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

shows that it is "1" at the associated node and "0" at the others.
Likewise for $\underset{\sim}{\varphi}_v^{(\mathrm{e})}$ and the other nodes.

The strain component ε_{xx} follows form

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad u(x,y,z)^{(\mathrm{e})} = \underset{\sim}{\varphi}_u^{(\mathrm{e})^{\mathrm{T}}} \underset{\sim}{U}_u^{(\mathrm{e})} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\underset{\sim}{\varphi}_u^{(\mathrm{e})^{\mathrm{T}}})\, \underset{\sim}{U}_u^{(\mathrm{e})} =$$

$$= \frac{1}{2A^{(\mathrm{e})}} \begin{pmatrix} X_{y_2} - X_{y_3} \\ X_{y_3} - X_{y_1} \\ X_{y_1} - X_{y_2} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} u_1^{(\mathrm{e})} \\ u_2^{(\mathrm{e})} \\ u_3^{(\mathrm{e})} \end{pmatrix} \quad \Rightarrow \quad \varepsilon_{xx}(x,y)^{(\mathrm{e})} = \text{const.}^{(\mathrm{e})}$$

The result shows, that the strain ε_{xx} (as well as ε_{yy} and ε_{xy}) are independent of $(x,y)^{(\mathrm{e})}$, i.e. it is constant in (e), which is the reason for the name *Constant Strain Triangle*.

Note: Even if the displacements are continuous at the element boundaries, the strains (and stresses) are dis-continuous. The local balance equations are not fulfilled locally, however, the nodal forces and displacements are in equilibrium. Consequently, the total energy balance in the element is fulfilled which is called the "weak form" of the solution.

6 Derivation of Stiffness Matrices

Next the element stiffness matrix is sought for which relates the nodal forces to the nodal displacements. Here, the case is treated in which the element coordinates are given in the global coordinate system (e.g. corresponding to the Constant Strain Triangle in the previous section). A more general framework based on local element considerations will be presented later.

Element Stiffness Matrix from the Potential

The total potential energy reads

$$V = \sum_{(\mathrm{e})} V^{(\mathrm{e})} \qquad , \qquad V^{(\mathrm{e})} = U^{(\mathrm{e})} + W^{(\mathrm{e})},$$

with its contributions given at element level (e).

The strain energy of the element can be expressed as,

$$U^{(\mathrm{e})} = \frac{1}{2} \int_{Vol^{(\mathrm{e})}} \sigma_{ij} \varepsilon_{ij} \mathrm{d}Vol = \frac{1}{2} \int_{Vol^{(\mathrm{e})}} \underset{\sim}{\sigma}^{\mathrm{T}} \underset{\sim}{\varepsilon}\, \mathrm{d}Vol$$

with

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{E}}^{(\mathrm{e})}\underset{\sim}{\varepsilon} \quad , \quad \underset{\sim}{\sigma}^{\mathrm{T}} = \underset{\sim}{\varepsilon}^{\mathrm{T}}\underset{\approx}{\mathbf{E}}^{(\mathrm{e})^{\mathrm{T}}} \quad , \quad \underset{\approx}{\mathbf{E}}^{(\mathrm{e})^{\mathrm{T}}} = \underset{\approx}{\mathbf{E}}^{(\mathrm{e})}$$

and the definition

$$\underset{\sim}{\varepsilon} = \underset{\approx}{\mathbf{D}}^{(\mathrm{e})}\underset{\sim}{U}^{(\mathrm{e})} \qquad \text{from} \quad \underset{\sim}{\varepsilon} = \underset{\approx}{\mathbf{d}}\vec{u} = \underset{\approx}{\mathbf{d}}\underset{\approx}{\boldsymbol{\varphi}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\sim}{U}^{(\mathrm{e})}$$

follows

$$U^{(\mathrm{e})} = \frac{1}{2}\int_{Vol^{(\mathrm{e})}} \underset{\sim}{U}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{E}}^{(\mathrm{e})}\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}\underset{\sim}{U}^{(\mathrm{e})}\mathrm{d}Vol = \frac{1}{2}\underset{\sim}{U}^{(\mathrm{e})^{\mathrm{T}}}\int_{Vol^{(\mathrm{e})}} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{E}}\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}\,\mathrm{d}Vol\,\underset{\sim}{U}^{(\mathrm{e})}$$

From solid mechanics, it is known that the energy is a quadratic form of the displacements with the stiffness

$$U^{(\mathrm{e})} = \frac{1}{2}\underset{\sim}{U}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{K}}^{(\mathrm{e})}\underset{\sim}{U}^{(\mathrm{e})}$$

which renders the element stiffness matrix

$$\underset{\approx}{\mathbf{K}}^{(\mathrm{e})} = \int_{Vol^{(\mathrm{e})}} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{E}}^{(\mathrm{e})}\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}\mathrm{d}Vol$$

Assembly of the Total Stiffness Matrix

Since the assembly of all elements has to solved for equilibrium, the total stiffness matrix has to be built from all element stiffness matrices.

The coincidence transformation matrix, $\widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})}$, relates the total displacement vector $\underset{\sim}{U}$ to the element displacements as,

$$\underset{\sim}{U}^{(\mathrm{e})} = \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})}\underset{\sim}{U}$$

Then the strain energy of the element (e) reads

$$U^{(\mathrm{e})} = \frac{1}{2}\underset{\sim}{U}^{\mathrm{T}}\widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{K}}^{(\mathrm{e})}\widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})}\underset{\sim}{U},$$

from which the transformation of the element stiffness matrix follows

$$\widetilde{\underset{\approx}{\mathbf{K}}}^{(\mathrm{e})} = \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}}\underset{\approx}{\mathbf{K}}^{(\mathrm{e})}\widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})}$$

with $\widetilde{\underset{\approx}{\mathbf{K}}}^{(\mathrm{e})}$ being the element stiffness in global (total) consideration.

The potential of the external forces read

$$W^{(\mathrm{e})} = \sum_{i=1,N^{(\mathrm{e})}} -F_i^{(\mathrm{e})} U_i^{(\mathrm{e})} = -\underset{\sim}{F}^{(\mathrm{e})^{\mathrm{T}}} \underset{\sim}{U}^{(\mathrm{e})} = -\underset{\sim}{F}^{(\mathrm{e})^{\mathrm{T}}} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})} \underset{\sim}{U}$$

and the total potential can be given as the sum over all elements

$$V = \sum_{(\mathrm{e})} \left(\frac{1}{2} \underset{\sim}{U}^{\mathrm{T}} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{K}}^{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})} \underset{\sim}{U} - \underset{\sim}{F}^{(\mathrm{e})^{\mathrm{T}}} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})} \underset{\sim}{U} \right)$$

Applying the principle of minimum total potential energy, i.e. $\frac{\partial V}{\partial U_i} = 0 \; \forall i$, gives

$$\sum_{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{K}}^{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})} \underset{\sim}{U} - \sum_{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\sim}{F}^{(\mathrm{e})} = \underset{\sim}{0},$$

which can be written as

$$\boxed{\underset{\approx}{\mathbf{K}} \underset{\sim}{U} = \underset{\sim}{F}} \quad .$$

The expression,

$$\underset{\approx}{\mathbf{K}} = \sum_{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{K}}^{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})} = \sum_{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{K}}}^{(\mathrm{e})}$$

is the total stiffness matrix, and

$$\underset{\sim}{F} = \sum_{(\mathrm{e})} \widetilde{\underset{\approx}{\mathbf{T}}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\sim}{F}^{(\mathrm{e})} = \sum_{(\mathrm{e})} \widetilde{\underset{\sim}{F}}^{(\mathrm{e})} = \sum_{(\mathrm{e})} \widetilde{\underset{\sim}{F}}_{(a)}^{(\mathrm{e})} + \underbrace{\sum_{(\mathrm{e})} \widetilde{\underset{\sim}{F}}_{(i)}^{(\mathrm{e})}}_{\underset{\sim}{0}} = \sum_{(\mathrm{e})} \widetilde{\underset{\sim}{F}}_{(a)}^{(\mathrm{e})}$$

is the total load vector.

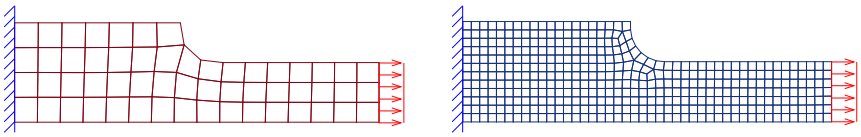
The sum over the internal forces $\underset{\sim}{F}_{(i)}^{(\mathrm{e})}$ is zero, since they are self equilibrated (if no external force is applied).

Consistent Nodal Loads

Since only nodal forces appear in $\underset{\approx}{\mathbf{K}}\,\underset{\sim}{U} = \underset{\sim}{F}$ which take action at the nodes, surface force and volume force densities must be converted to nodal forces. This must be done element wise, such that the work by the nodal loads is the same as the work by the distributed force densities, i.e. in an energetically consistent way.

7 Discretization and Accuracy

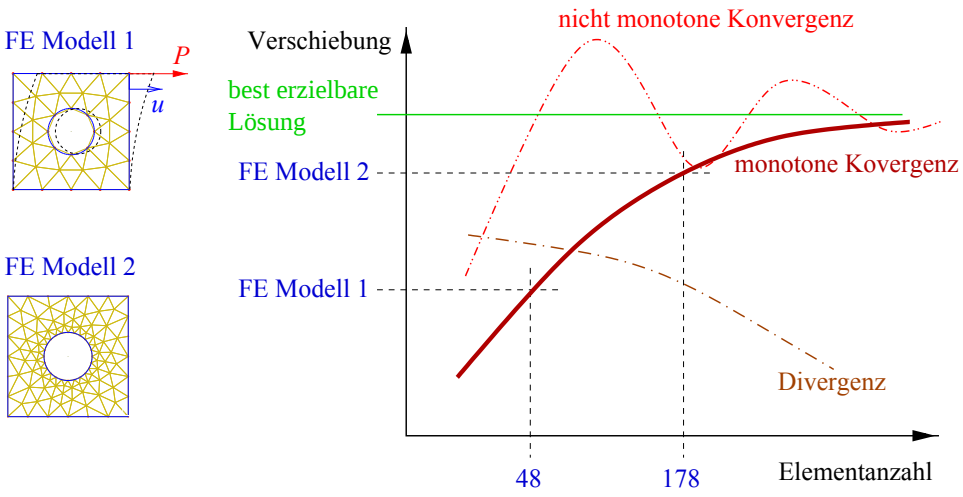
The accuracy of the solution depends (amount others) on the element formulation and on the mesh density.



The Finite Element Method also allows for selective mesh refinements in regions of interest or where high accuracy is desired.

Convergence

Monotonous convergence means, that with systematic increase of the mesh density the solution tends to the “true” response.

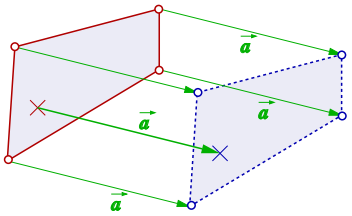


Completeness and Compatibility

Interpolation functions are required to be *complete*, i.e. to be able to describe rigid body movements and to be able describe states of constant strains.

This implies,

$$\sum_{i=1}^{N^{(e)}} \varphi_i^{(e)}(x,y,z) \equiv 1 \qquad \forall (x,y,z) \in (e)$$



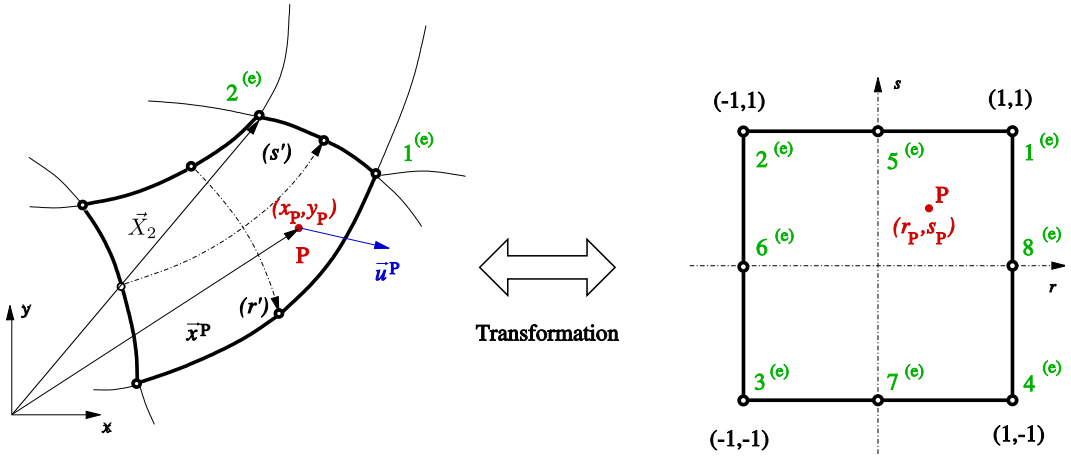
„Compatibility“ means, that the displacement field at element boundaries must be continuous to prevent overlapping and formation of gaps inside a continuum.

If adjacent elements have different interpolation functions, special measures must be take to achieve compatibility.

8 Isoparametric Finite Elements

An important step towards easy automation and efficiency has been the introduction of local reference elements.

8.1 Interpolation of Geometry and Displacement Field



Elements in the global domain can be transformed to a local reference element of the same type. For all elements of the same type, many pre-computations can be performed for the reference element once.

In the following, this is shown for a two-dimensional 8-noded quadrilateral element (with bi-quadratic interpolation functions).

$$x_{\mathrm{P}} = \sum_{j=1}^{n} f_j^{(\mathrm{e})}(r_{\mathrm{P}}, s_{\mathrm{P}})\, X_{x_j} \qquad x = \underset{\sim}{f}^{(\mathrm{e})^{\mathrm{T}}}(r,s)\, \underset{\sim}{X}_x{}^{(\mathrm{e})}$$

$$y_{\mathrm{P}} = \sum_{j=1}^{n} f_j^{(\mathrm{e})}(r_{\mathrm{P}}, s_{\mathrm{P}})\, X_{y_j} \qquad y = \underset{\sim}{f}^{(\mathrm{e})^{\mathrm{T}}}(r,s)\, \underset{\sim}{X}_y{}^{(\mathrm{e})}$$

$$\boxed{\vec{x} = \underset{\approx}{\mathbf{f}}^{(\mathrm{e})^{\mathrm{T}}}(r,s)\underset{\sim}{X}^{(\mathrm{e})}}$$

$(x_\mathrm{P}, y_\mathrm{P})$	coordinates of point $P \in$ (e)
(X_{x_j}, X_{y_j})	coordinates of node j
$\underset{\sim}{X}_x{}^{(\mathrm{e})}$	x-coordinates of the nodes $(X_{x_1}, X_{x_2}, \ldots, X_{x_n})^\mathrm{T}$
$\underset{\sim}{X}_y{}^{(\mathrm{e})}$	y-coordinates of the nodes $(X_{y_1}, X_{y_2}, \ldots, X_{y_n})^\mathrm{T}$
$\vec{x} = (x, y)^\mathrm{T}$	$\underset{\approx}{X}^{(\mathrm{e})} = (\underset{\sim}{X}_x{}^{(\mathrm{e})^\mathrm{T}}, \underset{\sim}{X}_y{}^{(\mathrm{e})^\mathrm{T}})^\mathrm{T}$

The interpolation of the displacements are given by

$$u^\mathrm{P} = \sum_{j=1}^{n} N_j^{(\mathrm{e})}(r_\mathrm{P}, s_\mathrm{P})\, U_{uj}^{(\mathrm{e})} \qquad u = \underset{\sim}{N}^{(\mathrm{e})^\mathrm{T}}(r,s)\, \underset{\sim}{U}_u{}^{(\mathrm{e})} \qquad \underset{\sim}{\varphi}_u{}^\mathrm{T} \longrightarrow \underset{\sim}{N}_u{}^\mathrm{T} \longrightarrow \underset{\sim}{N}^\mathrm{T}$$

$$v^\mathrm{P} = \sum_{j=1}^{n} N_j^{(\mathrm{e})}(r_\mathrm{P}, s_\mathrm{P})\, U_{vj}^{(\mathrm{e})} \qquad v = \underset{\sim}{N}^{(\mathrm{e})^\mathrm{T}}(r,s)\, \underset{\sim}{U}_v{}^{(\mathrm{e})} \qquad \underset{\sim}{N}_v = \underset{\sim}{N}_u = \underset{\sim}{N}$$

Note, that the interpolation functions $f_j^{(\mathrm{e})}(r,s)$ and $N_j{}^{(\mathrm{e})}(r,s)$, are defined in the local (r,s) space. All elements of the same type have identical interpolation functions, and the superscript (e) can be omitted, giving,

$$\boxed{\vec{u} = \underset{\approx}{\mathbf{N}}{}^\mathrm{T}(r,s)\, \underset{\sim}{U}^{(\mathrm{e})}}$$

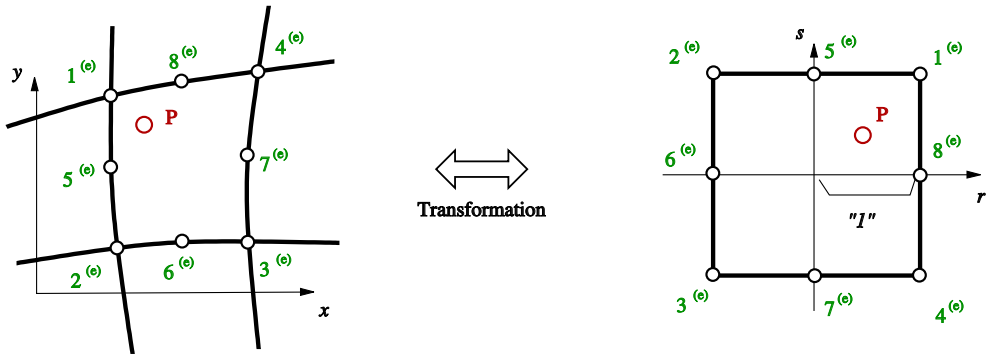
For the case of isoparametric elements $f_j(r,s) \equiv N_j(r,s) \qquad j = 1, 2 \ldots n$

Again, completeness is desired;

$$\begin{aligned} f_j\,(r_i, s_i, (t_i)) &= \delta_{ij} \\ N_j\,(r_i, s_i, (t_i)) &= \delta_{ij} \end{aligned} \qquad i, j \ldots \text{node number at reference elements}$$

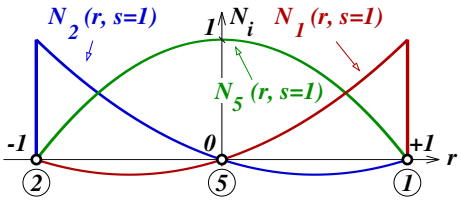
Examples

Isoparametric quadrilateral element with 8 nodes and bi-quadratic interpolation



$$
\begin{aligned}
N_i(r,s) &= \frac{1}{4}(1+rr_i)(1+ss_i)(rr_i+ss_i-1) \quad \text{for } i=1,2,3,4 \\
N_i(r,s) &= \frac{1}{2}(1-r^2)(1+ss_i) \quad \text{for } i=5,7 \\
N_i(r,s) &= \frac{1}{2}(1+rr_i)(1-s^2) \quad \text{for } i=6,8
\end{aligned}
$$

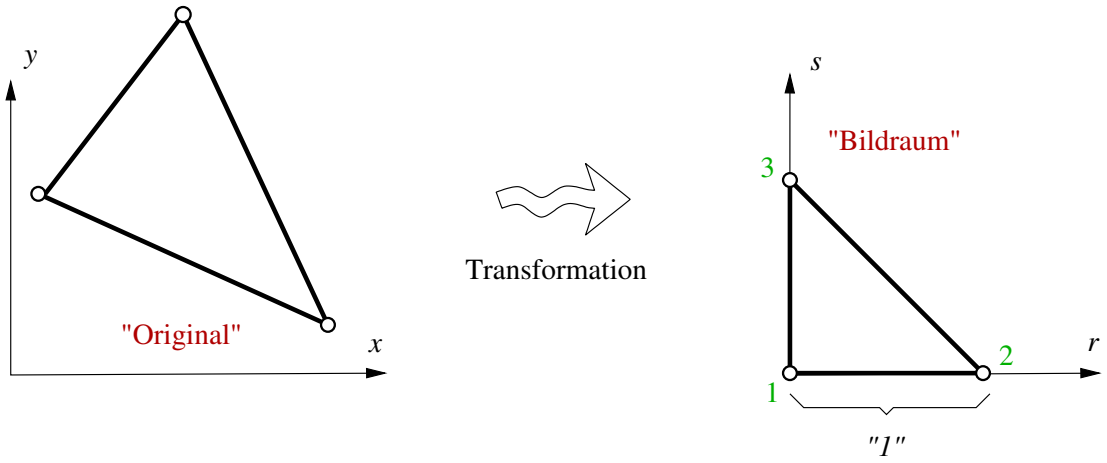
discussion for $s=1$



check for completeness, e.g. rigid body translation

$$\sum_{i=1}^{8} N_i(r,s) \equiv 1$$

Isoparametric Constant Strain Triangle



$$\underset{\sim}{N}(r,s) = \begin{pmatrix} 1-r-s \\ r \\ s \end{pmatrix}$$

8.2 The "Jacobi Matrix" of the Element

The computation of the strains requires the partial derivatives,

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial v}{\partial x}, \ldots,$$
$$\text{z.B.} \quad \varepsilon_{xx} = \frac{\partial u}{\partial x}, \ldots ,$$

but they cannot be given straightforward since, $\vec{u} = \underset{\approx}{\mathbf{N}}^{\mathrm{T}}(r,s,t)\,\underset{\sim}{U}^{(\mathrm{e})}$.

For two-dimensional cases we can write,

$$\left.\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial}{\partial s} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\frac{\partial y}{\partial s} \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} = \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}(r,s) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\boxed{\underset{\sim}{\partial}_{rs} = \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}(r,s)\ \underset{\sim}{\partial}_{xy}}$$

with $\underset{\approx}{\mathbf{J}}^{(\mathrm{e})}$ being the "Jacobi Matrix" of element (e).

Inversion gives the desired derivatives,

$$\boxed{\underset{\sim}{\partial}_{xy} = \underset{\approx}{\mathbf{J}}^{(\mathrm{e})^{-1}}(r,s)\ \underset{\sim}{\partial}_{rs}}$$

$$\begin{pmatrix} \frac{\partial(\,)}{\partial x} \\ \frac{\partial(\,)}{\partial y} \end{pmatrix} = \frac{1}{\det \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}} \begin{pmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{pmatrix} \begin{pmatrix} \frac{\partial(\,)}{\partial r} \\ \frac{\partial(\,)}{\partial s} \end{pmatrix}$$

Note that, because of $\vec{x} = \underset{\approx}{\mathbf{N}}^{\mathrm{T}}(r,s,t)\,\underset{\sim}{X}$, the Jacobi Matrix $\underset{\approx}{\mathbf{J}}^{(\mathrm{e})}$ depends only on the global coordinates $\underset{\sim}{X}^{(\mathrm{e})}$ of element (e).

Inversion of the Jacobi matrix is possible only if it is non-singular, i.e. $\underset{\approx}{\mathbf{J}}^{(\mathrm{e})}(r,s,t) \neq 0\ \forall(r,s,t)$. This gives restrictions to element shapes.

The volume of the element is given by $\mathrm{d}Vol = \mathrm{d}x\mathrm{d}y\mathrm{d}z = \det\underset{\approx}{\mathbf{J}}^{(\mathrm{e})}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t$.

8.3 The Element Stiffness Matrix of Isoparametric Elements

The element stiffness matrix is computed as

$$\underset{\approx}{\mathbf{K}}^{(\mathrm{e})} = \int_{Vol^{(\mathrm{e})}} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{E}}^{(\mathrm{e})} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})} \mathrm{d}Vol \qquad \text{with} \quad \underset{\approx}{\mathbf{D}}^{(\mathrm{e})} = \underset{\approx}{\mathbf{d}}^{(\mathrm{e})} \underset{\approx}{\mathbf{N}}^{\mathrm{T}} \quad .$$

For a two-dimensional element (for example) this reads,

$$\underset{\approx}{\mathbf{D}}^{(\mathrm{e})} = \begin{pmatrix} \frac{\partial}{\partial x}\underset{\sim}{N}^{\mathrm{T}} & \underset{\sim}{0}^{\mathrm{T}} \\ \underset{\sim}{0}^{\mathrm{T}} & \frac{\partial}{\partial y}\underset{\sim}{N}^{\mathrm{T}} \\ \frac{\partial}{\partial y}\underset{\sim}{N}^{\mathrm{T}} & \frac{\partial}{\partial x}\underset{\sim}{N}^{\mathrm{T}} \end{pmatrix}$$

The (1,1)-element of $\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}$ can be expressed as

$$\frac{\partial}{\partial x}\underset{\sim}{N}^{\mathrm{T}} = \frac{1}{\det\underset{\approx}{\mathbf{J}}^{(\mathrm{e})}} \left(\frac{\partial y}{\partial s}\frac{\partial}{\partial r}\underset{\sim}{N}^{\mathrm{T}} - \frac{\partial y}{\partial r}\frac{\partial}{\partial s}\underset{\sim}{N}^{\mathrm{T}} \right)$$

Introducing the short hand notation, $(\quad)' := \frac{\partial(\,)}{\partial r}$, $(\quad\dot{}\,) := \frac{\partial(\,)}{\partial s}$ the $\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}$ matrix can be written as

$$\underset{\approx}{\mathbf{D}}^{(\mathrm{e})} = \frac{1}{\det \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}} \underbrace{\begin{pmatrix} \dot{y}(\underset{\sim}{N}^{\mathrm{T}})' - y'(\dot{\underset{\sim}{N}}^{\mathrm{T}}) & \underset{\sim}{0}^{\mathrm{T}} \\ \underset{\sim}{0}^{\mathrm{T}} & -\dot{x}(\underset{\sim}{N}^{\mathrm{T}})' + x'(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \\ -\dot{x}(\underset{\sim}{N}^{\mathrm{T}})' + x'(\dot{\underset{\sim}{N}}^{\mathrm{T}}) & \dot{y}(\underset{\sim}{N}^{\mathrm{T}})' - y'(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \end{pmatrix}}_{\underset{\approx}{\mathbf{Q}}^{(\mathrm{e})}} = \frac{1}{\det \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}} \underset{\approx}{\mathbf{Q}}^{(\mathrm{e})} \quad .$$

For isoparametric elements further holds,

$$x = \underset{\sim}{N}^{\mathrm{T}} \underset{\sim}{X_x}^{(\mathrm{e})} \quad , \quad y = \underset{\sim}{N}^{\mathrm{T}} \underset{\sim}{X_y}^{(\mathrm{e})}$$

$$\Longrightarrow x' = \frac{\partial x}{\partial r} = \frac{\partial}{\partial r} \underset{\sim}{N}^{\mathrm{T}} \underset{\sim}{X_x}^{(\mathrm{e})} = \left(\underset{\sim}{N}^{\mathrm{T}} \right)' \underset{\sim}{X_x}^{(\mathrm{e})},$$

$$\dot{y} = \frac{\partial y}{\partial s} = \frac{\partial}{\partial s} \underset{\sim}{N}^{\mathrm{T}} \underset{\sim}{X_y}^{(\mathrm{e})} = \left(\dot{\underset{\sim}{N}}^{\mathrm{T}} \right) \underset{\sim}{X_y}^{(\mathrm{e})}$$

$$(\text{ likewise } \dot{x}, y' \,)$$

Derivatives are required only for $\underset{\sim}{N}^{(\mathrm{e})^{\mathrm{T}}}$, which are equal for all elements of the same type. The global coordinates of the elements, $\underset{\sim}{X}^{(\mathrm{e})}$, appear as simple coefficients.

Applying the short hand notation gives,

$$\underset{\approx}{\mathbf{Q}}^{(\mathrm{e})} = \begin{pmatrix} [(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \underset{\sim}{X_y}](\underset{\sim}{N}^{\mathrm{T}})' - [(\underset{\sim}{N}^{\mathrm{T}})' \underset{\sim}{X_y}](\dot{\underset{\sim}{N}}^{\mathrm{T}}) & \underset{\sim}{0}^{\mathrm{T}} \\ \underset{\sim}{0}^{\mathrm{T}} & [(\underset{\sim}{N}^{\mathrm{T}})' \underset{\sim}{X_x}](\dot{\underset{\sim}{N}}^{\mathrm{T}}) - [(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \underset{\sim}{X_x}](\underset{\sim}{N}^{\mathrm{T}})' \\ [(\underset{\sim}{N}^{\mathrm{T}})' \underset{\sim}{X_x}](\dot{\underset{\sim}{N}}^{\mathrm{T}}) - [(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \underset{\sim}{X_x}](\underset{\sim}{N}^{\mathrm{T}})' & [(\dot{\underset{\sim}{N}}^{\mathrm{T}}) \underset{\sim}{X_y}](\underset{\sim}{N}^{\mathrm{T}})' - [(\underset{\sim}{N}^{\mathrm{T}})' \underset{\sim}{X_y}](\dot{\underset{\sim}{N}}^{\mathrm{T}}) \end{pmatrix}$$

and the $\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}$ matrix is a function of the local coordinates, $\underset{\approx}{\mathbf{D}}^{(\mathrm{e})}(r,s)$, of the reference element (and the coefficients $\underset{\sim}{X}^{(\mathrm{e})}$).

Now the element stiffness matrix can be written as,

$$\begin{aligned} \int_{Vol^{(\mathrm{e})}} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{E}}^{(\mathrm{e})} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})} \mathrm{d}Vol \quad &\rightarrow \quad \int_{-1}^{+1} \int_{-1}^{+1} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})^{\mathrm{T}}}(r,s) \underset{\approx}{\mathbf{E}}^{(\mathrm{e})} \underset{\approx}{\mathbf{D}}^{(\mathrm{e})}(r,s) h \det \underset{\approx}{\mathbf{J}}^{(\mathrm{e})} \mathrm{d}r\mathrm{d}s \\ &= \quad \int_{-1}^{+1} \int_{-1}^{+1} \frac{1}{\det \underset{\approx}{\mathbf{J}}^{(\mathrm{e})}} \underset{\approx}{\mathbf{Q}}^{(\mathrm{e})^{\mathrm{T}}} \underset{\approx}{\mathbf{E}}^{(\mathrm{e})} \underset{\approx}{\mathbf{Q}}^{(\mathrm{e})} h \mathrm{d}r\mathrm{d}s \end{aligned}$$

with the element thickness h.

The integration is typically performed by Gauss quadrature, such as

$$\int_{\Omega} \Psi(r,s,t)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t = \sum_{j=1}^{\mathcal{O}} \Psi(r_j,s_j,t_j)\bar{w}_j$$

A further advantage of isoparametric elements is, that the integration involves the function $\Psi(r,s,t)$, given in local coordinates. Also this integration needs to be computed only once for the reference element.

9 Closing Remarks

Various extensions are readily developed and are widely used for Finite Element Method simulations. Among them are

- Transient problems with time-varying loads and responses
- "Dynamic" problems for which first and second time derivatives of the displacements enter the equations
- Eigenvalue problems to study buckling and resonance
- Non-linear formulations of the strain–displacement relation, the material response, etc.
- Dissipative and load history dependent mechanisms
- Multi-field and multi-physics coupled problems
- ...

Advantages of the Finite Element Method from the point of view of an Finite Element textbook as listed in [2].

- It can deal simply with nonhomogeneous and anisotropic situations (particularly when the direction of anisotropy is variable).
- The elements can be graded in shape and size to follow arbitrary boundaries and to allow for regions of rapid variation of the function sought, thus controlling the errors in a most efficient way ...
- Specified gradient or "radiation" boundary conditions are introduced naturally and with a better accuracy than in standard finite difference procedures.
- Higher order elements ... can be readily used to improve accuracy without complicating boundary conditions — a difficulty always arising with finite difference approximations of a higher order.
- Finally, but of considerable importance in the computer age, standard programs may be used for assembly and solution.

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[2] O. C. Zienkiewicz, R. L. Taylor, J. Zhu, The Finite Element Method: Its Basis and Fundamentals, Butterworth-Heinemann, 2013.

[3] K.-J. Bathe, Finite element procedures, Klaus-Jurgen Bathe, 2006.

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