Finite Element Methods: Introducion & Biomechanics

VU 317.039 & VU 317.526

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*Additional content for VU 317.526, Numbers in brackets indicate LVA units

Part I

Introduction

Overview Part I

1. **Motivation**

2. **Mathematical Background**
   - Notations
   - Coordinates
   - Tensors, Vectors, Matrices
   - Algebra
   - Analysis

Outline

1. **Motivation**

2. **Mathematical Background**
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A Basic Question

- Why should I learn the finite element theory if I like to apply the method only?
- Counter question: Why do you have to know how a car works if you only like to drive it?

Why Finite Elements Analysis (FEA)?

Experiments . . .

- are very expensive
- only certain parameters measurable

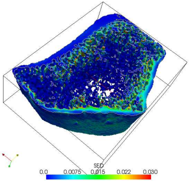
FE method allows . . .

- Simulation of a physical systems (loading, damage, flow, . . .)
- Unique insight - uncovers the invisible!

Prerequisite . . .

- Model is validated (correct model parameters)
- Validation needs experiments

Pro- and Cons of the FE Method



Advantages:

- Detailed insight into physical behavior
- Saving of experimental tests

Drawbacks:

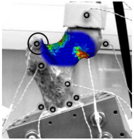
- Often models not (sufficiently) validated
- Colored pictures (results) fastly generated - true or false!
- Execution and interpretation needs FE-experts

General: The problem is not the FE method but the user!

Fields of Application

FE original used for solid mechanics, today widely used:

- Automotive & aerospace, engineering
- Mechanical & civil engineering
- Biomedical engineering



Especially in field of:

- Static problems (strength, stability, . . .)
- Dynamic problems (modal analysis, crash
- Acoustic
- Fluid dynamic
- Conduction/diffusion problems

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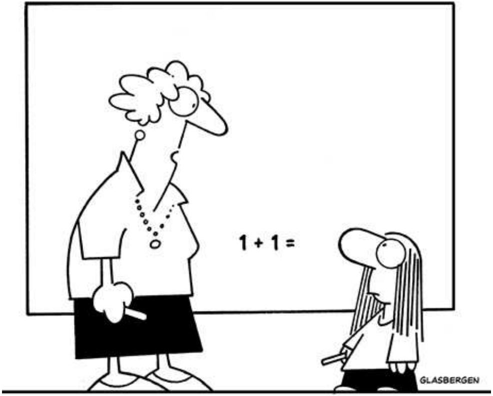
Why Maths?



"Yes, this will be useful to you later in life."
FEA

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Notations

There are three relevant notations

- Intrinsic
- Explicit
- Index

Intrinsic Notation

Scalar, vector, matrix:

$$a, \quad \underline{\boldsymbol{a}}, \quad \underline{\underline{\boldsymbol{A}}}$$

- Vectors: bold, small, 1 underline
- Matrix: bold, capital, 2 underlines

Explicit Notation

Displays all components in rows and columns

$$\underline{\boldsymbol{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \qquad \underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{32} & A_{33} \end{bmatrix}$$

Index Notation

Only for physical quantities:

$$\underline{\boldsymbol{a}} \leftrightarrow a_i \qquad \underline{\underline{\boldsymbol{A}}} \leftrightarrow A_{ij}$$

Conventions for Subscripts:

- subscript occurs once per term: free index f (e.g. space direction):

$$B_{fm}\, a_m + c_f = d_f$$

- subscript occurs twice: dummy/mute index m e.g.:

$$B_{fm}\, a_m = B_{f1}\, a_1 + B_{f2}\, a_2 + B_{f3}\, a_3$$

- subscript occurs more than twice: MISTAKE! (e.g. $a_j\, b_j\, c_j$)

Notation of Variables

Possible indices of variable C

$$^{c}C_{a}^{b}$$

- a. . . coordinate directions (1,2,3 or x, y, z)
- b. . . operators (square C^2, transpose $\underline{\underline{\boldsymbol{C}}}^{\mathrm{T}}$)
- c. . . additional notation(s) of variables e.g. nodes k and element numbers $\langle e \rangle$

Examples: ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$, ${}^{\langle e \rangle k}C$, X_1^2, σ_{12}

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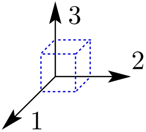
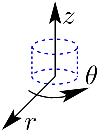
Coordinate System

Parametrization of point in space by three numbers

- Cartesian (x_1, x_2, x_3)
- Cylindrical (r, θ, z)
- Spherical (r, θ, ϕ)



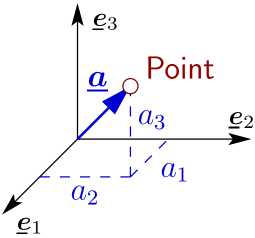


Example: Point in Space, Cartesian CS

Position of point $\underline{\boldsymbol{a}}$ is given by:

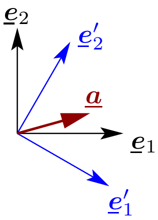
$$\underline{\boldsymbol{a}} = a_1\,\underline{\boldsymbol{e}}_1 + a_2\,\underline{\boldsymbol{e}}_2 + a_3\,\underline{\boldsymbol{e}}_3 = a_m\,\underline{\boldsymbol{e}}_m = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

with 3 basis vectors $\underline{\boldsymbol{e}}_1$, $\underline{\boldsymbol{e}}_2$, $\underline{\boldsymbol{e}}_3$



Change of Coordinate System

Task: $\underline{a}$ given in $\underline{e}_i$, transformed into new $\underline{e}'_i$ system



With transformation matrix $\underline{\underline{A}}$ from 1,2,3 to 1',2',3'

$$\underline{a}' = \underline{\underline{A}}\,\underline{a} \qquad \leftrightarrow \qquad a'_i = A_{im}\,a_m$$

where

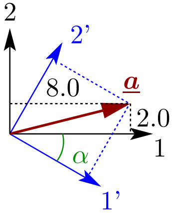
$$A_{im} = \cos\left(\measuredangle(\underline{e}'_i, \underline{e}_m)\right) = \underline{e}'_i \cdot \underline{e}_m$$

Example: Change of Coordinate System 2D

Given: $\alpha = 30°$

$$\underline{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \underline{\boldsymbol{a}} = \begin{bmatrix} 8.0 \\ 2.0 \end{bmatrix}$$

$$\underline{\boldsymbol{e}}'_1 = \begin{bmatrix} \cos(\alpha) \\ -\sin(\alpha) \end{bmatrix} \qquad \underline{\boldsymbol{e}}'_2 = \begin{bmatrix} \sin(\alpha) \\ \cos(\alpha) \end{bmatrix}$$



Solution:

$$\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \qquad \underline{\boldsymbol{a}}' = \underline{\underline{\boldsymbol{A}}}\,\underline{\boldsymbol{a}} = \begin{bmatrix} 5.928 \\ 5.732 \end{bmatrix}$$

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Physical Tensors

Examples

- Position $\underline{\boldsymbol{x}}$, displacement $\underline{\boldsymbol{u}}$, velocity $\underline{\boldsymbol{v}}$
- In textbooks also written as $\vec{x}$, $\vec{u}$, $\vec{v}$
- Stress $\underline{\underline{\boldsymbol{\sigma}}}$, strain $\underline{\underline{\boldsymbol{\varepsilon}}}$, stiffness $C_{ijkl}\ldots$

Characteristics

- $i, j = 1, 2, 3$ in 3D, dimension 3×1, 3×3
- Coordinate transformation: $\underline{\boldsymbol{a}}' = \underline{\underline{\boldsymbol{A}}}\,\underline{\boldsymbol{a}}$
- All below arithmetic rules allowed!

Physical Tensors of Order 0, 1, 2

- Tensor of order 0 = scalar, is invariant:

$$a' = a$$

- Tensor of order 1 = vector, 1 index, transformation law:

$$\underline{\boldsymbol{a}}' = \underline{\underline{\boldsymbol{A}}}\,\underline{\boldsymbol{a}} \quad \leftrightarrow \quad a_i' = A_{im}\,a_m$$

- Tensor of order 2 = matrix, 2 indices, transformation law:

$$Q_{ij}' = A_{im}\,A_{jn}\,Q_{mn}$$

Note: Number of transformation matrices = order of Tensor

Numerical Vectors/Matrices

Example:

- n measured displacements written as list/vector

$$\underline{\boldsymbol{U}} = [\,{}^1U \;\; {}^2U \;\; \ldots \; {}^nU]^{\mathrm{T}}$$

- Stiffness matrix $\underline{\underline{\boldsymbol{C}}}$ from stiffness tensor C_{ijkl}

Characteristics

- arbitrary dimensions $n \times 1$, $m \times n$
- Coordinate transformation needs “special” matrices
- Not all of below arithmetic rules allowed!

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Operation on Vectors (P+N)

Sum of vectors

$$\underline{\boldsymbol{s}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} \qquad s_f = a_f + b_f \qquad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

Multiplication by a scalar

$$\underline{\boldsymbol{s}} = \alpha\, \underline{\boldsymbol{a}} \qquad s_f = \alpha\, a_f \qquad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \alpha\, a_1 \\ \alpha\, a_2 \\ \alpha\, a_3 \end{bmatrix}$$

Operation on Vectors (P+N)

Scalar product (dot product, simple contraction)

$$\alpha = \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} \qquad \alpha = a_m \, b_m \qquad \alpha = a_1 \, b_1 + a_2 \, b_2 + a_3 \, b_3$$

can be also written as:

$$\alpha = \underline{\boldsymbol{a}}^{\mathrm{T}} \underline{\boldsymbol{b}} \qquad \alpha = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Euclidean norm of vectors = length

$$\|\underline{\boldsymbol{a}}\| = \sqrt{\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}}} \qquad \|\underline{\boldsymbol{a}}\| = \sqrt{a_m \, a_m} \qquad \|\underline{\boldsymbol{a}}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

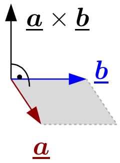
Operation on Vectors (P)

Cross product

$$\underline{\boldsymbol{s}} = \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} \qquad s_i = \epsilon_{imn}\, a_m\, b_n \qquad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{bmatrix}$$

where ϵ_{imn} is the (Levi-Civita) permutation tensor

$$\epsilon_{imn} = \begin{cases} \mathrm{sign}(i, m, n) & i \neq m,\, m \neq n,\, i \neq n \\ 0 & \text{in all other cases} \end{cases}$$



for example: $\epsilon_{123} = 1,\ \epsilon_{113} = 0,\ \epsilon_{213} = -1,\ \epsilon_{231} = 1,\ \ldots$

Vector is normal to surface, right hand rule, magnitude=area

Operation on Vectors (P+N)

Transformation of a vector (see above)

$$\underline{\boldsymbol{y}} = \underline{\underline{\boldsymbol{A}}}\,\underline{\boldsymbol{x}} \quad y_f = A_{fm}\,x_m \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

e.g. for numerical vectors (sum of row times column)

$$\underline{\boldsymbol{b}} = \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{a}} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} B_{11}\,a_1 + B_{12}\,a_2 \\ B_{21}\,a_1 + B_{22}\,a_2 \\ B_{31}\,a_1 + B_{32}\,a_2 \end{bmatrix}$$

dimension of new matrix: $n \times m \times m \times 1 = n \times 1$

Operation on Vectors (P)

Dyadic product

$$\underline{\underline{\boldsymbol{S}}} = \underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} \qquad S_{ij} = a_i \, b_j$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} a_1\,b_1 & a_1\,b_2 & a_1\,b_3 \\ a_2\,b_1 & a_2\,b_2 & a_2\,b_3 \\ a_3\,b_1 & a_3\,b_2 & a_3\,b_3 \end{bmatrix}$$

can be also written as

$$\underline{\underline{\boldsymbol{S}}} = \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}^{\mathrm{T}} \qquad \underline{\underline{\boldsymbol{S}}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

e.g. used to build up tensors

$$\underline{\underline{\boldsymbol{A}}} = A_{mn}\, \underline{\boldsymbol{e}}_m \otimes \underline{\boldsymbol{e}}_n$$

Operation on Matrices (P+N)

Addition

$$\underline{\underline{\boldsymbol{S}}} = \underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{B}}} \qquad S_{ij} = A_{ij} + B_{ij}$$

Multiplication

$$\underline{\underline{\boldsymbol{P}}} = \alpha\, \underline{\underline{\boldsymbol{A}}} \qquad P_{ij} = \alpha\, A_{ij}$$

Scalar product (double contraction)

$$\tau = \underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}} \qquad \tau = A_{ij}\, B_{ij}$$

Norm

$$\|\underline{\underline{\boldsymbol{A}}}\| = \sqrt{\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{A}}}} \qquad \|\underline{\underline{\boldsymbol{A}}}\| = \sqrt{A_{ij}\, A_{ij}}$$

Operation on Matrices (P+N)

Transposition

$$\underline{\underline{\boldsymbol{C}}}^{\mathrm{T}} \qquad C^{\mathrm{T}}_{ij} = C_{ji}$$

Composition (simple contraction)

$$\underline{\underline{\boldsymbol{C}}} = \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}} \qquad \underline{\underline{\boldsymbol{C}}} = \underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{B}}} \qquad C_{ij} = A_{im}\, B_{mj}$$

Identity or unit matrix (Kronecker delta)

$$\underline{\underline{\boldsymbol{I}}} \qquad I_{ij} = \delta_{ij} \qquad \underline{\underline{\boldsymbol{C}}}\,\underline{\underline{\boldsymbol{I}}} = \underline{\underline{\boldsymbol{I}}}\,\underline{\underline{\boldsymbol{C}}} = \underline{\underline{\boldsymbol{C}}}$$

Trace

$$\mathrm{tr}(\underline{\underline{\boldsymbol{S}}}) = S_{11} + S_{22} + \cdots + S_{nn}$$

Operation on Matrices (P+N)

Determinant

$$\det \underline{\underline{S}} = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} = |\underline{\underline{S}}| = s_{11}\, s_{22} - s_{12}\, s_{21}$$

Inverse matrix $(\det \underline{\underline{S}} \neq 0)$

$$\underline{\underline{S}} \cdot \underline{\underline{S}}^{-1} = \underline{\underline{I}} \qquad \longleftrightarrow \qquad \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \cdot \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\underline{\underline{I}}$. . . unit matrix

Similar Notations

Simple contractions

- Scalar product $\underline{a} \cdot \underline{b}$ $\quad a_m \, b_m$ $\quad \underline{a}^{\mathrm{T}} \, \underline{b}$
- Transformation $\underline{\underline{A}} \cdot \underline{b}$ $\quad A_{fm} \, b_m$ $\quad \underline{\underline{A}} \, \underline{b}$
- Composition $\underline{\underline{A}} \cdot \underline{\underline{B}}$ $\quad A_{im} \, B_{mj}$ $\quad \underline{\underline{A}} \, \underline{\underline{B}}$

Dyadic product

$\underline{a} \otimes \underline{b}$ $\quad a_i \, b_j$ $\quad \underline{a} \, \underline{b}^{\mathrm{T}}$

Example: Solving of Vector-Matrix Equation

$$
\begin{aligned}
\underline{\underline{S}} \cdot \underline{a} &= \underline{b} \qquad\qquad \Big| \qquad \underline{\underline{S}}^{-1} \cdot \\
\underline{\underline{S}}^{-1} \cdot \underline{\underline{S}} \cdot \underline{a} &= \underline{\underline{S}}^{-1} \cdot \underline{b} \\
\underline{\underline{I}} \cdot \underline{a} &= \underline{\underline{S}}^{-1} \cdot \underline{b} \\
\underline{a} &= \underline{\underline{S}}^{-1} \cdot \underline{b}
\end{aligned}
$$

Note: multiplication from left, application of identity

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Fields

Physical variables which are distributed over space $\underline{\boldsymbol{x}}$ called fields.

Scalar field:

- Each position $\underline{\boldsymbol{x}}$ corresponds with scalar $\phi(x_1, x_2, x_3) = \phi(\underline{\boldsymbol{x}})$
- Examples: density $\rho(\underline{\boldsymbol{x}})$ and pressure distribution $p(\underline{\boldsymbol{x}})$

Vector field:

- Each position $\underline{\boldsymbol{x}}$ corresponds with vector $\underline{\boldsymbol{f}}(x_1, x_2, x_3)$
- Examples: gravitational $\underline{\boldsymbol{g}}(\underline{\boldsymbol{x}})$ and velocity field $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}})$

Time:

- Fields can depend on time t: $\underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}) \to \underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}, t)$
- Arguments are sometimes omitted: $\underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}, t) \leftrightarrow \underline{\boldsymbol{f}}$

Derivatives

Partial derivative

$$\frac{(\cdot)}{\partial x_i} = \partial_i(\cdot) = (\cdot)_{,i}$$

e.g. in space for vector field $\underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}) = \underline{\boldsymbol{f}}$

$$\frac{\partial \underline{\boldsymbol{f}}}{\partial x_i} = \partial_i \underline{\boldsymbol{f}} = \underline{\boldsymbol{f}}_{,i} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

Chain rule

$$\frac{\partial(\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})}{\partial x_i} = \frac{\partial \underline{\boldsymbol{a}}}{\partial x_i} \cdot \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \cdot \frac{\partial \underline{\boldsymbol{b}}}{\partial x_i}$$

$$\frac{\partial(\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}})}{\partial x_i} = \frac{\partial \underline{\boldsymbol{a}}}{\partial x_i} \times \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \times \frac{\partial \underline{\boldsymbol{b}}}{\partial x_i}$$

Differential

Total differential $d(\cdot)$ of functions

$$\phi(\underline{\boldsymbol{x}}, t) = \phi(x_1, x_2, x_3, t) = \phi$$

gives

$$d\phi = \frac{\partial \phi}{\partial t} \cdot dt + \frac{\partial \phi}{\partial x_1} \cdot dx_1 + \frac{\partial \phi}{\partial x_2} \cdot dx_2 + \frac{\partial \phi}{\partial x_3} \cdot dx_3$$

Division by dt with $\frac{dx_i}{dt} = \dot{x}_i = v_i$ (velocity)

$$\frac{d\,\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial \phi}{\partial t} + v_i\,\frac{\partial \phi}{\partial x_i}$$

Differential

Also known as **material/particle derivative**

$$\frac{d\,(.)}{dt} = \frac{\partial(.)}{\partial t} + \underbrace{(\underline{\boldsymbol{v}} \cdot \nabla)}_{\text{Operator}}\,(.) = \frac{\partial(.)}{\partial t} + v_m \frac{\partial(.)}{\partial x_m}$$

where $(.)$ can be a scalar or vector field e.g.:

$$T(\underline{\boldsymbol{x}}, t),\ \underline{\boldsymbol{v}}(\underline{\boldsymbol{x}}, t)$$

Gradient of Scalar Field

With Nabla operator

$$\nabla(\cdot) = \underline{\boldsymbol{e}}_i \frac{\partial(\cdot)}{\partial x_i}$$

one gets the gradient which is a vector

$$\underline{\boldsymbol{g}} = \operatorname{Grad}\phi := \nabla\phi \qquad g_i = \phi_{,i} \qquad \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \phi = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{bmatrix}$$

Example: Potential flow with velocity potential $\phi(\underline{\boldsymbol{x}})$

- $\nabla\phi = \underline{\boldsymbol{v}}$ gives the velocity field

Gradient of a Vector Field

Gradient: Nabla operator with dyadic product

$$\nabla \underline{\boldsymbol{f}} := \left(\nabla \otimes \underline{\boldsymbol{f}}\right)^{\mathrm{T}} \qquad f_{i,j} \qquad \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

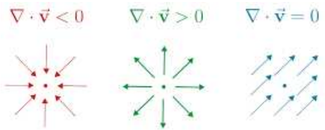
Divergence of a Vector Field

Divergence: Nabla Operator with dot product

$$\mathrm{Div}\,\underline{\boldsymbol{f}} := \nabla \cdot \underline{\boldsymbol{f}} \qquad f_{i,i} \qquad \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

Example: Flow field $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}})$

- $\mathrm{Div}\,\underline{\boldsymbol{v}} = \nabla \cdot \underline{\boldsymbol{v}}$ is magnitude of local source or sink flow



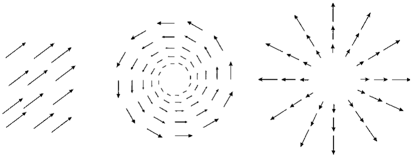
Rotation of a Vector Field

Rotation: Nabla operator with cross product

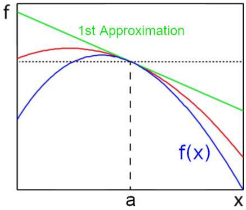
$$\mathrm{Curl}\, \underline{\boldsymbol{f}} := \nabla \times \underline{\boldsymbol{f}} \qquad \epsilon_{imn}\left(\frac{\partial f_m}{\partial x_n} - \frac{\partial f_n}{\partial x_m}\right) \qquad \begin{bmatrix} \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \end{bmatrix}$$

Example: Flow field $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}})$

- $\mathrm{Curl}\, \underline{\boldsymbol{v}} = \nabla \times \underline{\boldsymbol{v}}$ is the local angular velocity (middle)
- if $\nabla \times \underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \ \forall\ \underline{\boldsymbol{x}}$, there is no vortex in the field (left or right)



Taylor Expansion for 2 Variables



Taylor expansion for variable x_1, x_2 around a_1, a_2:

$$f(x_1, x_2) = f(a_1, a_2) + \frac{\partial f(a_1, a_2)}{\partial x_1}(x_1 - a_1) + \frac{\partial f(a_1, a_2)}{\partial x_2}(x_2 - a_2) + \mathcal{O}^2$$

or short (for more variables):

$$f(\underline{\boldsymbol{x}}) = f(\underline{\boldsymbol{a}}) + \nabla_x f(\underline{\boldsymbol{x}}) \cdot (\underline{\boldsymbol{x}} - \underline{\boldsymbol{a}}) + \mathcal{O}^2$$

For more Details . . .

Jesper Ferkingho-Borg
Introduction to vector and tensor analysis.
http://server.elektro.dtu.dk/personal/jfb/

List of Questions

- Which vector/matrix notations do you know?
- What are the conventions/rules for the index notation?
- How can we transform vector/matrices into different CS?
- How is a vector matrix equation solved?
- What are fields - examples?
- What can Taylor expansion be used for?

Part II

Continuum Mechanics

Overview Part II

3. **Continuums Mechanics**
   - Introduction
   - Concept of Stress

4. **Deformable Solid Mechanics**
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Solids, Liquids, Gases

Solids:

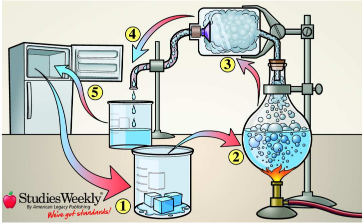
- high resistance to shear forces
- low compressibility

Liquids:

- low resistance to shear forces
- low compressibility

Gases:

- low resistance to shear forces
- high compressibility



The Basic Postulates of a Continua

Fundamental Balance Relations

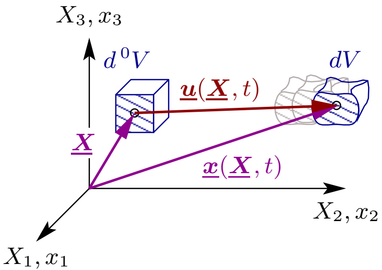
- Balance of mass (conservation of mass)
- Balance of linear momentum (1. or 2. Newton's law)
- Balance of energy (1st law of thermodynamics)

additionally to restrict/formulate e.g. constitutive equations:

- Thermodynamical state equations (constitutive equations, 2nd law of TD)
- Balance of angular momentum (symmetry of stress tensor)

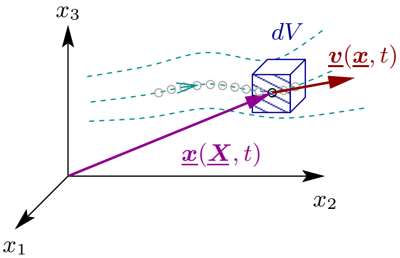
Motion: Lagrangian and Eulerian Approach

Lagrangian (Solid)



- dV moving/deforming in space
- dV contains same material
- i.e. observer fixed at material

Eulerian (Fluid)



- dV fixed in space
- material in dV changing
- i.e. observer fixed in space

Typical Field Variables of a Continua

Deformable body e.g.:

- Stress $\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{X}}, t)$
- Strain $\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\boldsymbol{X}}, t)$
- Displacement $\underline{\boldsymbol{u}}(\underline{\boldsymbol{X}}, t)$

Fluids e.g.:

- Density $\rho(\underline{\boldsymbol{x}}, t)$
- Pressure $p(\underline{\boldsymbol{x}}, t)$
- Velocity $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}}, t)$
- Temperature $T(\underline{\boldsymbol{x}}, t)$

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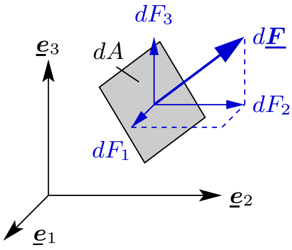
Concept of Stress - Stress Vector

Stresses are defined by infinitesimal small forces dF_i acting on area dA_i

$$t_1 = \lim_{dA \to 0} \frac{dF_1}{dA} \qquad (1)$$

$$t_2 = \lim_{dA \to 0} \frac{dF_2}{dA} \qquad (2)$$

$$t_3 = \lim_{dA \to 0} \frac{dF_3}{dA} \qquad (3)$$



which gives the definition of the stress vector (traction vector):

$$\underline{\boldsymbol{t}} = t_1\,\underline{\boldsymbol{e}}_1 + t_2\,\underline{\boldsymbol{e}}_2 + t_3\,\underline{\boldsymbol{e}}_3 = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \qquad [t_i] = \frac{N}{mm^2}$$

Stress → Force

Assuming a stress distribution within the area dA gives:

$$dF_1 = \int_{dA} t_1(\underline{\boldsymbol{x}})\, dA \tag{4}$$

$$dF_2 = \int_{dA} t_2(\underline{\boldsymbol{x}})\, dA \tag{5}$$

$$dF_3 = \int_{dA} t_3(\underline{\boldsymbol{x}})\, dA \tag{6}$$

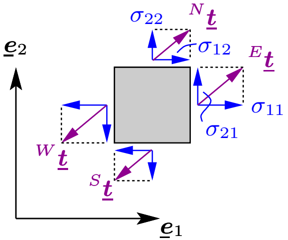
where stress t_i (unit e.g. N/mm^2) is a force density!

Stress Tensor

The Cauchy stress tensor (true stress)

- 1,2,3 stress components of traction vectors $\underline{\boldsymbol{t}}$
- Index: $\sigma_{(\text{direction})(\text{face})}$
- E, N, W, S... East, Nord, West, South surface

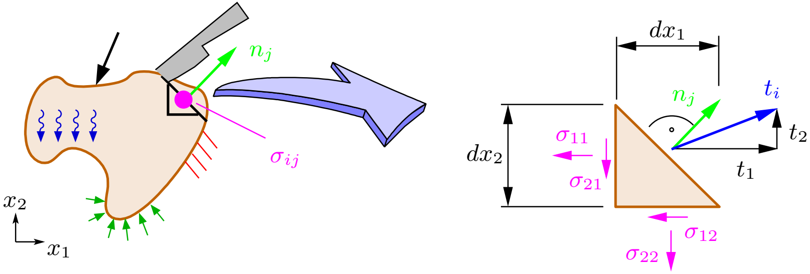
$$\underline{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}}, t) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Note: obeys tensor transformation law under a change of CS!

Traction Vector from Stress Tensor

- Cutting a part it follows in cutting plane
- at a point with stress tensor σ_{ij} and normal vector n_j
- acts the traction vector t_i



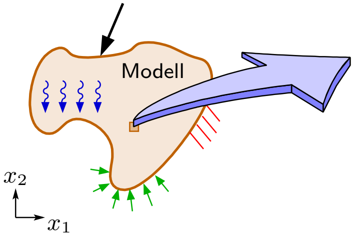
$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11}\, n_1 + \sigma_{12}\, n_2 \\ \sigma_{21}\, n_1 + \sigma_{22}\, n_2 \end{pmatrix}$$

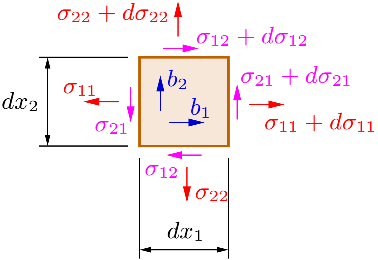
or shortly:

$$t_i = \sigma_{ij}\, n_j$$

Equilibrum of a 2D Volume Element

From Equilibrium for an infinitesimal small volume:





it follows ($dx_3 = h$) the static equilibrium condition:

$$
\begin{aligned}
\sum F_1 &= 0 : (\sigma_{11} + d\sigma_{11})\, dx_2\, h + (\sigma_{12} + d\sigma_{12})\, dx_1\, h - \sigma_{12}\, dx_1\, h - \sigma_{11}\, dx_2\, h + b_1\, dx_1\, dx_2\, h = 0 \\
\sum F_2 &= 0 : (\sigma_{22} + d\sigma_{22})\, dx_1\, h + (\sigma_{21} + d\sigma_{21})\, dx_2\, h - \sigma_{22}\, dx_1\, h - \sigma_{21}\, dx_2\, h + b_2\, dx_1\, dx_2\, h = 0 \\
\sum M &= 0 : (\sigma_{12} + d\sigma_{12})\, dx_1\, h\, \frac{dx_2}{2} - (\sigma_{21} + d\sigma_{21})\, dx_2\, h\, \frac{dx_1}{2} \ldots \\
\ldots &+ \sigma_{12}\, dx_1\, h\, \frac{dx_2}{2} - \sigma_{21}\, dx_2\, h\, \frac{dx_1}{2} = 0
\end{aligned}
$$

Static Equilibrium - Linear Momentum

and simplification of first two terms:

$$\begin{aligned} d\sigma_{11}\, dx_2\, h + d\sigma_{12}\, dx_1\, h + b_1\, dx_1\, dx_2\, h = 0 \quad &\Big| \quad : dx_1\, dx_2\, h \\ d\sigma_{22}\, dx_1\, h + d\sigma_{21}\, dx_2\, h + b_2\, dx_1\, dx_2\, h = 0 \quad &\Big| \quad : dx_1\, dx_2\, h \end{aligned}$$

furthermore, approximation of stress changes $d\sigma_{ij}$ via Taylor expansion:

$$d\sigma_{11} = \frac{\partial \sigma_{11}}{\partial x_1}\, dx_1 \quad \Rightarrow \quad \frac{d\sigma_{11}}{dx_1} = \frac{\partial \sigma_{11}}{\partial x_1} \qquad \ldots$$

gives static equilibrium equations in 2D

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + b_1 &= 0 \\ \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{21}}{\partial x_1} + b_2 &= 0 \end{aligned} \qquad (7)$$

Static Equilibrium - Angular Momentum

a simplification of the moment equilibrium:

$$2\underbrace{(\sigma_{12} - \sigma_{21})}_{=0} + \underbrace{(d\sigma_{12} - d\sigma_{21})}_{=0} = 0$$

requires a symmetric stress tensor (corresponding shear stresses):

$$\sigma_{21} = \sigma_{12} \qquad d\sigma_{21} = d\sigma_{12}$$

Static Equilibrium in 3D

In index notation it reads as:

$$\sigma_{ij,j} + b_i = 0 \qquad i,j = 1,2,3 \tag{8}$$

with $\sigma_{ij} = \sigma_{ji}$ (in 2D $i,j = 1,2$)

This relation can be also written tensor notation as:

Static Equilibrium

$$\underline{\underline{\boldsymbol{L}}}^T \, \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{0}} \qquad \forall x_i \tag{9}$$

Additional Definitions of Stress for FEM

For FEM stress rewritten as vector of stress components:

$$\text{2D:} \quad \underline{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \qquad \text{3D:} \quad \underline{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} \tag{10}$$

Note that it holds:

$$\sigma_{ij} = \sigma_{ji}$$

Principle Stresses

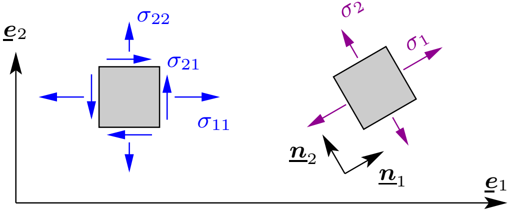
Problem: Orientation of a surface element such that only normal stresses appear?

$$\underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{n}} = \lambda \underline{\boldsymbol{n}}$$

Solution: via eigenvalue problem with

$$\underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{n}} = \lambda \underline{\boldsymbol{n}} \quad \rightarrow \quad (\underline{\underline{\boldsymbol{\sigma}}} - \lambda \underline{\underline{\boldsymbol{I}}}) \cdot \underline{\boldsymbol{n}} = \underline{\boldsymbol{0}}$$

- $\lambda = \sigma_1,\, \sigma_2,\, \sigma_3$ ... eigenvalues = principle stresses
- $\underline{\boldsymbol{n}} = \underline{\boldsymbol{n}}_1,\, \underline{\boldsymbol{n}}_2,\, \underline{\boldsymbol{n}}_3$ ... eigenvectors = orientation



Decomposition of Stress Tensor

Stress tensor can be decomposed into:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underbrace{{}^{\mathrm{h}}\underline{\underline{\boldsymbol{\sigma}}}}_{\text{hydrostatic}} + \underbrace{{}^{\mathrm{d}}\underline{\underline{\boldsymbol{\sigma}}}}_{\text{deviatoric}}$$

The associated hydrostatic stress tensor is:

$${}^{\mathrm{h}}\underline{\underline{\boldsymbol{\sigma}}} = \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}})\,\underline{\underline{\boldsymbol{I}}} = \begin{bmatrix} \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) & 0 & 0 \\ 0 & \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) & 0 \\ 0 & 0 & \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) \end{bmatrix}$$

with the mean normal stress:

$$\frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Hydrostatic Stress, Pressure

The negative hydrostatic stress = pressure

$$p = -\frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Deviatoric Stresses

The deviatoric stress tensor can be written as:

$$ {}^{\mathrm{d}}\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}} - {}^{\mathrm{h}}\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}} - (-p\,\underline{\underline{\boldsymbol{I}}}) $$

$$ {}^{\mathrm{d}}\underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{11} - \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \frac{1}{3}(\mathrm{tr}\underline{\underline{\boldsymbol{\sigma}}}) \end{bmatrix} $$

Equivalent Stresses

Tresca $(\sigma_1 > \sigma_2 > \sigma_3)$

$$^{\text{Tresca}}\sigma = \sigma_1 - \sigma_3$$

Von Mises

$$^{\text{vM}}\sigma = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

or with deviatoric stress

$$^{\text{vM}}\sigma = \sqrt{\frac{3}{2}\mathrm{tr}(\,^{\mathrm{d}}\underline{\underline{\boldsymbol{\sigma}}} : \,^{\mathrm{d}}\underline{\underline{\boldsymbol{\sigma}}})}$$

Note: These are scalar values!

Linear Stresses

Cauchy stresses are related to current configuration $\underline{\boldsymbol{x}}$:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{x}})$$

IF stresses are related to reference configuration $\underline{\boldsymbol{x}} \sim \underline{\boldsymbol{X}}$ for small displacements:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{X}})$$

we obtain **linearized stresses.**

For the **local equilibrium** conditions it holds:

$$\frac{\partial \sigma_{ij}(\underline{\boldsymbol{x}})}{\partial x_i} + b_i(\underline{\boldsymbol{x}}) = 0 \quad \rightarrow \quad \frac{\partial \sigma_{ij}(\underline{\boldsymbol{X}})}{\partial X_i} + b_i(\underline{\boldsymbol{X}}) = 0 \qquad (11)$$

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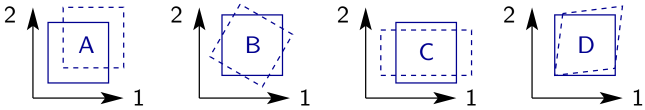
Kinematics: Motion of Solid Element



Different type of motion:

- **A** rigid body translation
- **B** rigid body rotation
- **C** extension
- **D** pure shear

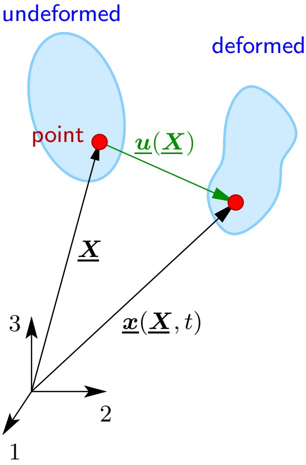
Point: Position and Displacement Field

The position of a material particle is given by:

$$\underline{\boldsymbol{X}},\ \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t)$$

Which gives the displacement field by:

$$\underline{\boldsymbol{u}}(\underline{\boldsymbol{X}}, t) = \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t) - \underline{\boldsymbol{X}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

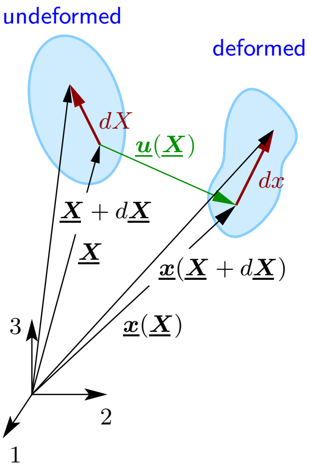


Line: Gradient of Deformation

Transformation of line element

$$d\underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t) = \underline{\underline{\boldsymbol{F}}}(\underline{\boldsymbol{X}}, t) \cdot d\underline{\boldsymbol{X}}$$

- $\underline{\underline{\boldsymbol{F}}}$ . . . deformation gradient



Derivation: Gradient of Deformation

A Taylor development around $\underline{\boldsymbol{X}}$ gives:

$$\underline{\boldsymbol{x}}(\underline{\boldsymbol{X}} + d\underline{\boldsymbol{X}}) = \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}) + \nabla_X \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}) \cdot d\underline{\boldsymbol{X}} + \mathcal{O}^2$$

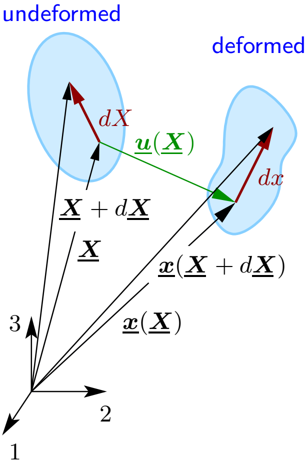
canceling higher order terms and rearranging

$$\underline{\boldsymbol{x}}(\underline{\boldsymbol{X}} + d\underline{\boldsymbol{X}}) - \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}) = d\underline{\boldsymbol{x}} = \nabla_X \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}) \cdot d\underline{\boldsymbol{X}}$$

gives:

Gradient of Deformation

$$\underline{\boldsymbol{F}}(\underline{\boldsymbol{X}}, t) = \nabla_X \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t)$$



Derivation: Gradient of Deformation

Note: instead of scalar field f, we have a vector field $d\underline{\boldsymbol{x}}$:

$$d\underline{\boldsymbol{x}} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}}_{\underline{\underline{\boldsymbol{F}}}} \cdot \underbrace{\begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}}_{d\underline{\boldsymbol{X}}}$$

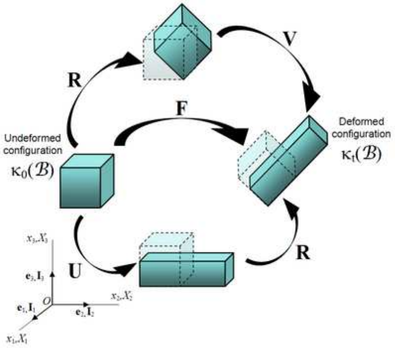
i.e. one equation for each space direction.

Gradient of Deformation: Polar Decomposition

Polar decomposition of deformation gradient

$$\underline{\underline{\boldsymbol{F}}} = \underline{\underline{\boldsymbol{R}}} \cdot \underline{\underline{\boldsymbol{U}}} = \underline{\underline{\boldsymbol{V}}} \cdot \underline{\underline{\boldsymbol{R}}}$$

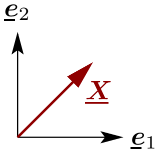
- $\underline{\underline{\boldsymbol{U}}}$ . . . right stretch tensor
- $\underline{\underline{\boldsymbol{V}}}$ . . . left stretch tensor
- $\underline{\underline{\boldsymbol{R}}}$ . . . rotation tensor

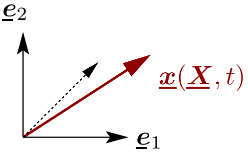


Example: Gradient of Deformation

Given: are the homogeneous deformation of line element:

$$\underline{\boldsymbol{X}} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t) = \begin{bmatrix} 1.7\, X_1 \\ 1.1\, X_2 \end{bmatrix}$$





Unknown: The gradient of deformation

$$\underline{\underline{\boldsymbol{F}}}(\underline{\boldsymbol{X}}, t) = \begin{bmatrix} \frac{\partial x_1}{\partial X1} & \frac{\partial x_1}{\partial X2} \\ \frac{\partial x_2}{\partial X1} & \frac{\partial x_2}{\partial X2} \end{bmatrix} = \begin{bmatrix} 1.7 & 0 \\ 0 & 1.1 \end{bmatrix}$$

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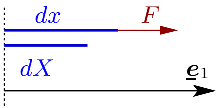
Concept of Strain: 1D

A (small) strain is defined in 1D as:



$$\varepsilon = \frac{dx - dX}{dX}$$

- ε ... strain (linear or nominal strain)
- dX ... initial length of a line element
- dx... stretched length of a line element

with $du = dx - dX$ we simple get:

$$\varepsilon = \frac{du}{dX} \quad \xrightarrow[\text{Taylor}]{} \quad \varepsilon = \frac{\partial u}{\partial X}$$

Note: Analogue for: $dX \rightarrow d\underline{\boldsymbol{X}}$, $dx \rightarrow d\underline{\boldsymbol{x}}$, $du \rightarrow d\underline{\boldsymbol{u}}$

Right Cauchy Green Deformation Tensor

Problem: How do we get 3D relations?

Solution: deformation gradient + rotational independent stretch measure

$$\underline{\underline{\boldsymbol{F}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{F}}} = \underline{\underline{\boldsymbol{U}}}^{T} \cdot \underbrace{\underline{\underline{\boldsymbol{R}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{R}}}}_{=\underline{\underline{\boldsymbol{I}}}} \cdot \underline{\underline{\boldsymbol{U}}} = \underbrace{\underline{\underline{\boldsymbol{U}}}^{2}}_{\underline{\underline{\boldsymbol{U}}}=\underline{\underline{\boldsymbol{U}}}^{\mathrm{T}}}$$

Alternativ Definition: Green Lagrange strain tensor

Alternativ Definition: Green Lagrange strain tensor

$$^{\mathrm{GL}}\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{I}}})$$

inserting the definition of displacement ($\underline{\boldsymbol{u}} = \underline{\boldsymbol{x}} - \underline{\boldsymbol{X}}$) gives

$$\begin{aligned}
^{\mathrm{GL}}\underline{\underline{\varepsilon}} &= \frac{1}{2}\big((\nabla_X(\underline{\boldsymbol{u}} - \underline{\boldsymbol{X}}))^{\mathrm{T}} \cdot (\nabla_X(\underline{\boldsymbol{u}} - \underline{\boldsymbol{X}})) - \underline{\underline{\boldsymbol{I}}}\big) \\
&= \frac{1}{2}\big((\nabla_X \underline{\boldsymbol{u}}^{\mathrm{T}} + \nabla_X \underline{\boldsymbol{u}} + \nabla_X \underline{\boldsymbol{u}}^{\mathrm{T}} \cdot \nabla_X \underline{\boldsymbol{u}}\big) \\
&= \frac{1}{2}\big(\underbrace{\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}}_{\text{linear}} + \frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j}\big)
\end{aligned}$$

Linearized Strains

For small strains it holds (see above):

Linear Strain

$$\underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{X}}, t) = \frac{1}{2}(\nabla_X \underline{\boldsymbol{u}} + \nabla_X \underline{\boldsymbol{u}}^{\mathrm{T}}) \qquad \varepsilon_{ij}(\underline{\boldsymbol{X}}, t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{12}$$

or written explicitly

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\right) = \frac{\partial u_1}{\partial X_1} \\ \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) \\ &\dots \end{aligned}$$

Note: The volume changes is given by the trace:

$$\frac{dV}{d\,^0V} \cong \mathrm{tr}(\underline{\boldsymbol{\varepsilon}}) = (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Vector of Strain Components

vector of strain components ($\neq$ strain tensor!):

$$\text{2D:} \quad \underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} \qquad \text{3D:} \quad \underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} \tag{13}$$

with engineering strains $\gamma_{ij} = 2\,\varepsilon_{ij}$ one can write:

Displacement-Strain Relations

$$\underline{\boldsymbol{\varepsilon}} = \underline{\underline{\boldsymbol{L}}}\,\underline{\boldsymbol{u}} \tag{14}$$

with displacement vector (3D)

$$\underline{\boldsymbol{u}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{15}$$

Differential Operator Matrix

and differential operator matrix $\underline{\underline{L}}$:

$$\underline{\underline{L}} = \begin{pmatrix} \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} \\ 0 & \frac{\partial}{\partial X_3} & \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} & 0 & \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} & 0 \end{pmatrix} . \tag{16}$$

Remarks: linear strains:

- only for small displacements/strains valid ($\varepsilon < 1\%$)
- in case of big displacements/strains $\rightarrow$ non/linear relations necessary!

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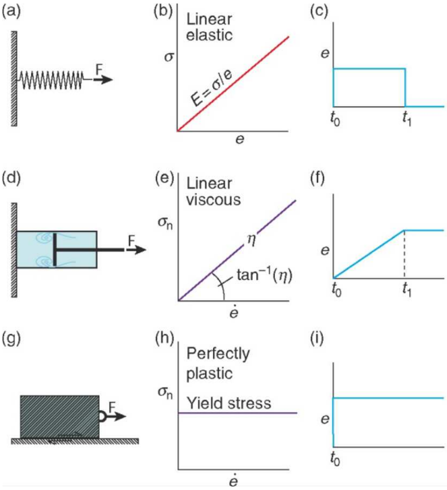
Material Behavior

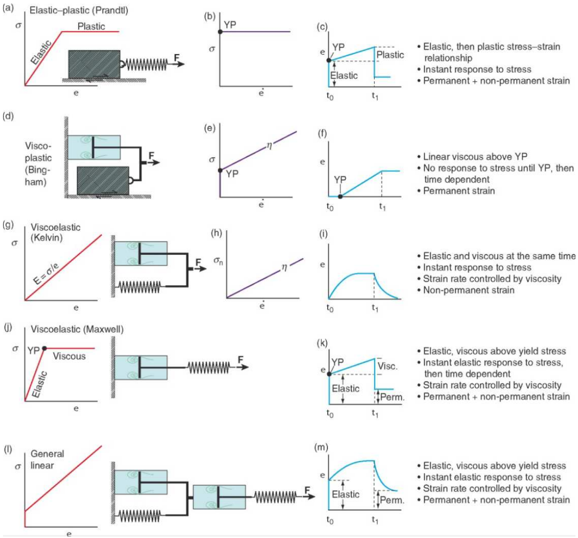


- Linear stress–strain relationship
- Instant response to stress
- Non-permanent strain

- Linear stress–strain rate relationship
- Stress depends on strain rate
- Delayed response to stress (the more time, the more strain)
- Permanent strain

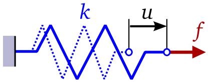
- Deforms at constant stress once the yield stress is achieved
- Constant stress regardless of strain rate
- Permanent strain

Combined Material Behavior



Hooke's Law

Spring law: $f = k\ u$



Analogue: Simplest material law of linear elastic material:

Hooke's Law

$$\underline{\boldsymbol{\sigma}} = \underline{\underline{C}}\ \underline{\boldsymbol{\varepsilon}} \tag{17}$$

- $\underline{\underline{C}}$ ...stiffness matrix
- **Note:** Material law = relation between stresses and strains!
- in books the following relation is often used (index notation):

$$\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl}$$

2D Material Law

In 2D the isotropic (=independent of direction) isothermal material law:

$$\underbrace{\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}}_{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)(1-2\,\nu)} \begin{pmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\,\nu)}{2} \end{pmatrix}}_{\underline{\underline{C}}} \cdot \underbrace{\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}}_{\underline{\varepsilon}}$$

with

- E ... Young's modul (e.g. steel 210 GPa)
- ν ... Poission's ratio (e.g. steel 0.3)

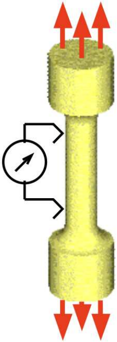
alternatively shear module can be used:

$$G = \frac{E}{2\,(1+\nu)}$$

2D Material Law

Easier to remember is inverse relation:

$$\underbrace{\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}}_{\underline{\varepsilon}} = \underbrace{\begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{pmatrix}}_{\underline{\underline{C}}^{-1}} \cdot \underbrace{\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}}_{\underline{\sigma}}$$



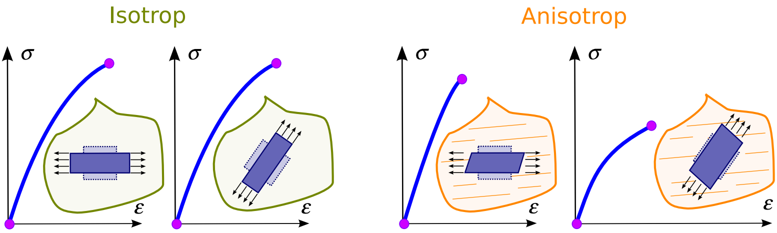
Example: uni-axial tension test i.e.

$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ 0 \\ 0 \end{pmatrix}$$

the strain follow directly as:

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\sigma_{11} \quad \Longrightarrow \quad \sigma_{11} = E\,\varepsilon_{11} \\ \varepsilon_{22} &= -\frac{\nu}{E}\sigma_{11} \quad \Longrightarrow \quad \textbf{lateral contraction effect} \end{aligned}$$

Isotropic and Anisotropic Behavior



- Isotropic material: material behavior same in all directions
- Anisotropic Material: material behavior depend on direction

Special Case: Orthotropic Material Behavior

Bones show orthotropic material behavior (=special case of anisotropy):

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}$$

mit

- E_1, E_2, E_3 ... Young's (longitudinal-) module
- ν_{ij} ... Poisson ratio
- G_{23}, G_{13}, G_{12} ... shear module

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Balance of Mass

The balance of mass is given by:

Balance of Mass

$$\rho(\underline{\boldsymbol{x}})\ \det(\underline{\underline{\boldsymbol{F}}}) - \rho(\underline{\boldsymbol{X}}) = 0 \tag{18}$$

where $\underline{\underline{\boldsymbol{F}}} = \nabla_X \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, t)$ is the deformation gradient

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Static Equilibrium

Starting point is Eq. 11 for small displacements:

$$\frac{\partial \sigma_{ij}(\underline{\boldsymbol{X}})}{\partial X_i} + b_i(\underline{\boldsymbol{X}}) = 0 \tag{19}$$

with $\sigma_{ij} = \sigma_{ji}$

This realation can be also written tensor notation as:

Static Equilibrium

$$\underline{\underline{\boldsymbol{L}}}^T \; \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{0}} \qquad\qquad \nabla_x \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{0}} \tag{20}$$

Note: Vector of stress components $\underline{\boldsymbol{\sigma}}$ and stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$!

Static Equilibrium - Displacements based

Using strain-displacement relation (6×)

$$\underline{\varepsilon} = \underline{\underline{L}}\, \underline{u}$$

and material law (6×)

$$\underline{\sigma} = \underline{\underline{C}}\, \underline{\varepsilon}$$

and inserting in static equilibrium (3×)

$$\underline{\underline{L}}^T\, \underline{\sigma} + \underline{b} = \underline{0}$$

it follows the static equilibrium (Navier space equation):

$$\underline{\underline{L}}^T\, \underline{\underline{C}}\, \underline{\underline{L}}\, \underline{u} + \underline{b} = \underline{0}$$

Thus we have 15 equations for 15 unknowns:

$$\underline{u}(\underline{X}),\ \underline{\sigma}(\underline{X}),\ \underline{\varepsilon}(\underline{X})$$

List of Questions

- What are the differences between solid and fluid mechanics
- What means Lagrangian and what Eulerian consideration?
- Which field variables in solids/fluids do you know?
- What is a stress tensor, stress vector, vector of stress components?
- Which fundamental balance equations do you know?
- Which different stress definitions du you know?
- What is a deformation gradient, what does it describe?
- What is a strain, what definition is it used in FEM?

List of Questions

- **What is the Hook Law?**
- **What means local equilibrium in solid mechanics?**

Part III

FE Introduction, Simple 1D Example

Overview Part III

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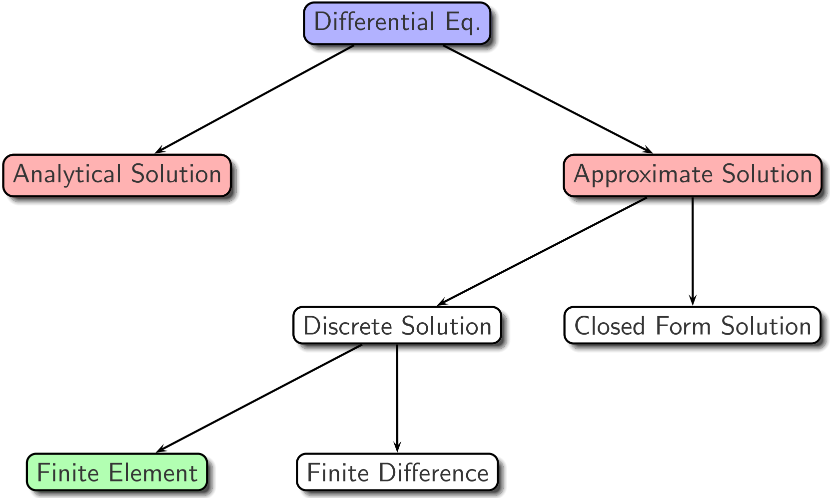
Solution Procedure for Different



Examples: 2nd order Differential Equations

Thermal Conduction

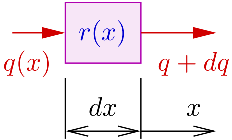
$$\frac{d^2\Theta(x)}{dx^2} + \frac{r(x)}{k} = 0$$

with

energy balance: $-\frac{dq(x)}{dx} + r(x) = 0$

Fourier law: $q(x) = -k\frac{d\Theta(x)}{dx}$

$\Theta(x)$	...	temperature
$q(x)$	...	heat flux
$r(x)$	...	heat source
k	...	heat transfer coef.



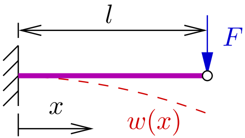
Beam Bending

$$\frac{d^2w(x)}{dx^2} = -\frac{M(x)}{E\,I}$$

with

equilib.: $M(x) = F\,(l - x)$

$w(x)$	...	bending displ.
$M(x)$	...	moment
$E\,I$	...	bending stiffness
l	...	beam length



Scope of this Lecture

Scope of this Lecture

"Linear Static Finite Element Method in Solid Mechanics"

The world is non-linear!

- Nevertheless: linear static FEM often a good approximation

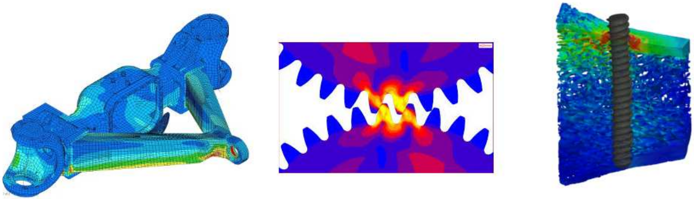


Figure: Typical Examples: Engineering structures (left, middle) and bone screw (right)

What means linear?

Linear: means

- small displacements/strains (otherwise geometric nonlinearity)
- linear elastic material (otherwise material non-linearity)
- no contact/friction

Non-linear problems:

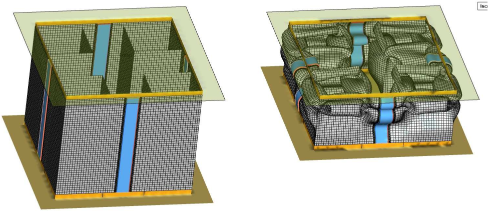


Figure: Full non-linearity: crash box: Initial (left) and final deformation (right). Large displacements/strains, plasticity, contact (ILSB, Grohs, 2005).

What means static?

Static: Means solution is time independent

- dynamic problems: e.g. crash (inertia and damping important!)
- free vibrations: eigenfrequencies, eigenmodes

Figure: Dynamic crash simulation (left) and vibration of a bridge (right).

What means Finite Element Method?

Finite Element Method - FEM = partition of interested domain in small regions (=finite elements)

Alternative methods are e.g.:

- particle method (PFEM, meshfree method)
- boundary element method
- discrete element method

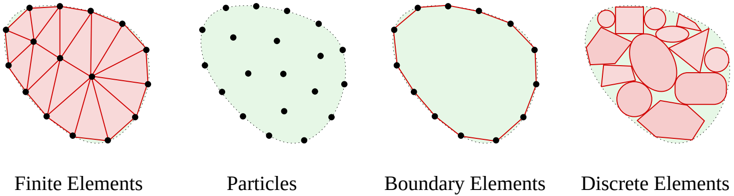


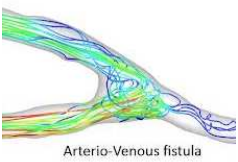
Figure: Numerical methods: finite element, particle, boundary element, discrete element method

What means Solid Mechanics?

Solid mechanic: branch of physics/mathematics concerning solid, deformable matter.

Other branches:

- fluid mechanic (flow of liquids, gases)
- continuum mechanic (fluid + solid)
- electrostatic, electrodynamic
- rigid body mechanic (kinematic, dynamic)
- thermal problems (heat transfer, heat conduction)



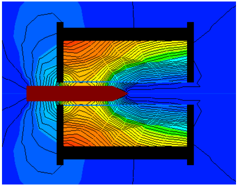


Figure: Fluid mechanic problem (left), electromagnetic problem (right)

Lagrangian or Eulerian Approach

Lagrangian Approach

- material based consideration
- element has always same volume
- e.g. solid mechanics problems → classical FEM

Eulerian Approach

- volume based consideration
- element fixed in space - continuum flows through
- e.g. fluid flow, forming simulation

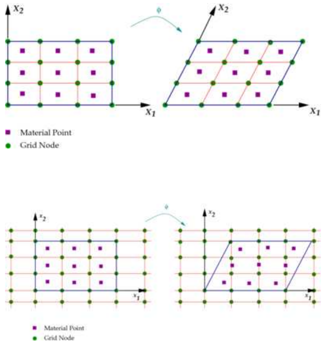


Figure: Lagrangian (top) and Eulerian mesh (bottom)

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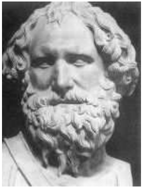
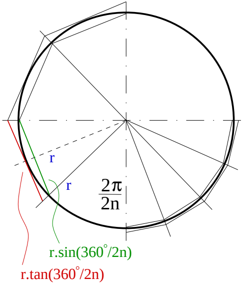
First Usage of Discretization

Archimedes (287 - 212 v. Chr.)





- Approximation for circumference

$$U = 2r\pi$$

- Consideration of inner/outer polygon gives:

$$3\,\frac{1}{7} > \pi > 3\,\frac{10}{71}$$

- Finite number of chord pieces

History of FEM


Ritz


Galerkin


Courant


Argyris


Clough


Zienkiewicz

1851 Schellbach: Variational calculus for minimum surface problems

1908 Ritz: New method for solution of variational problems - Ritz Ansatz

1915 Galerkin: Method of weighted residuals

1943 Courant: Variational method using element wise trial functions

1954 Argyris: Matrix formulation of problem equations

1960 Clough: Usage of FEM notation

1967 Zienkiewicz: Pioneer and first FEM text book

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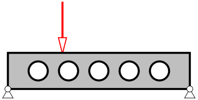
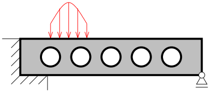
Motivation - Why FEM?

Classical beam problem: $\Rightarrow$ analytical solution possible

But: most mechanical components show complex:



- geometry



- boundary conditions
- material behavior

Motivation - Why FEM?

- local stress state are of interest

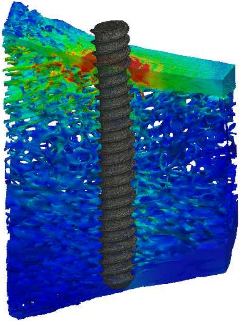


Figure: Stress concentration around bone screw

Motivation - Why FEM?

- nonlinearities present (material, geometry, contact)

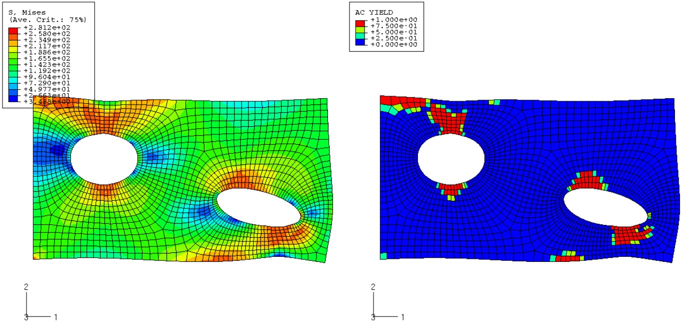


Figure: Plastic material behavior: Von Mises stress distribution (left) and yield flag (right)

- ⇒ solution only with finite element method possible!

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Spring System

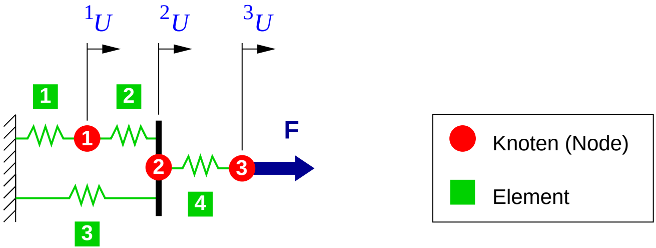

Figure: Spring system

- connected springs with load
- points at which elements are connected = nodes
- springs = elements

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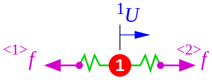
Equations using Nodal Equilibrium

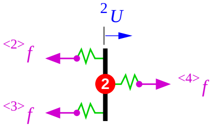
Solution method 1:

Node 1:



$$\begin{aligned} \sum F_x &= 0 \\ -\,^{\langle 1 \rangle}f + \,^{\langle 2 \rangle}f &= 0 \\ -\,^{\langle 1 \rangle}K\,^{1}U + \,^{\langle 2 \rangle}K(\,^{2}U - \,^{1}U) &= 0 \quad (21) \end{aligned}$$

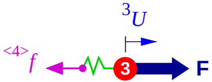
Node 2:



$$\begin{aligned} \sum F_x &= 0 \\ -\,^{\langle 2 \rangle}f - \,^{\langle 3 \rangle}f + \,^{\langle 4 \rangle}f &= 0 \\ -\,^{\langle 2 \rangle}K(\,^{2}U - \,^{1}U) - \,^{\langle 3 \rangle}K\,^{2}U + & \\ + \,^{\langle 4 \rangle}K(\,^{3}U - \,^{2}U) &= 0 \quad (22) \end{aligned}$$

Equations using Nodal Equilibrium

Node 3:



$$\begin{aligned} \sum F_x &= 0 \\ -\,{}^{\langle 4\rangle}f + F &= 0 \\ -\,{}^{\langle 4\rangle}K\,({}^{3}U - {}^{2}U) + F &= 0 \end{aligned} \qquad (23)$$

Assembling (21)–(23) gives:

$$\begin{aligned} -\,{}^{\langle 1\rangle}K\,{}^{1}U + {}^{\langle 2\rangle}K\,({}^{2}U - {}^{1}U) &= 0 \\ {}^{\langle 2\rangle}K\,({}^{2}U - {}^{1}U) - {}^{\langle 3\rangle}K\,{}^{2}U + {}^{\langle 4\rangle}K\,({}^{3}U - {}^{2}U) &= 0 \\ -\,{}^{\langle 4\rangle}K\,({}^{3}U - {}^{2}U) + F &= 0 \end{aligned}$$

Setup of Governing Equations

and in matrix form:

$$\begin{pmatrix} ({}^{\langle 1\rangle}K + {}^{\langle 2\rangle}K) & -{}^{\langle 2\rangle}K & 0 \\ -{}^{\langle 2\rangle}K & ({}^{\langle 2\rangle}K + {}^{\langle 3\rangle}K + {}^{\langle 4\rangle}K) & -{}^{\langle 4\rangle}K \\ 0 & -{}^{\langle 4\rangle}K & {}^{\langle 4\rangle}K \end{pmatrix} \begin{pmatrix} {}^{1}U \\ {}^{2}U \\ {}^{3}U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \tag{24}$$

or short:

$$\underline{\underline{\boldsymbol{K}}}\,\underline{\boldsymbol{U}} = \underline{\boldsymbol{F}} \tag{25}$$

with

- $\underline{\underline{\boldsymbol{K}}}$ . . . global stiffness matrix
- $\underline{\boldsymbol{U}}$ . . . (nodal) displacement vector
- $\underline{\boldsymbol{F}}$ . . . (nodal) force vector

= discrete governing equation for solid mechanic problems!

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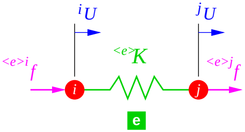
One-Dimensional Spring Element

Solution method 2:

- equation (24) comes from nodal equilibrium
- alternative: finite element → FEM formulation



Typical **spring element** - notations:

- element number: $\langle e \rangle$
- node number: i, j
- Nodal displacements = degrees of freedom (DOF): ^{i}U, ^{j}U
- nodal forces: $^{\langle e \rangle i}f$, $^{\langle e \rangle j}f$
- element stiffness (=spring stiffness): $^{\langle e \rangle}K$

Problem Statements

Problem 1: How can the stiffness matrix of spring element $^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$ (=element stiffness matrix) be obtained?

Problem 2: How can this "element" be used to get the response of the whole system?

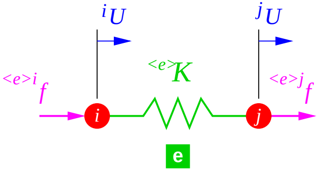
Final goal: Obtain equation (24).

Derivation of Element Stiffness Matrix

Problem 1: How does the element stiffness matrix $^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}}$ look like?

$$\underbrace{\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}}_{^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}}} \begin{pmatrix} {}^{i}U \\ {}^{j}U \end{pmatrix} = \begin{pmatrix} {}^{\langle e\rangle i}f \\ {}^{\langle e\rangle j}f \end{pmatrix} \qquad (26)$$

Solution: If node j is fixed $({}^{j}U = 0)$ and node i is moved by "1" $({}^{i}U = 1)$ it follows:



$$\begin{aligned} {}^{\langle e\rangle i}f &= {}^{\langle e\rangle}K\ {}^{i}U \\ {}^{\langle e\rangle j}f &= -{}^{\langle e\rangle}K\ {}^{i}U \end{aligned} \qquad (27)$$

Derivation of Element Stiffness Matrix

Insertion in equation (26) gives:

$$\begin{pmatrix} {}^{\langle e \rangle}K & ? \\ -{}^{\langle e \rangle}K & ? \end{pmatrix} \begin{pmatrix} {}^{i}U \\ {}^{j}U \end{pmatrix} = \begin{pmatrix} {}^{\langle e \rangle i}f \\ {}^{\langle e \rangle j}f \end{pmatrix}$$

in same way with ${}^{i}U = 0$ and ${}^{j}U = 1$ element stiffness matrix follows:

$$\underbrace{\begin{pmatrix} {}^{\langle e \rangle}K & -{}^{\langle e \rangle}K \\ -{}^{\langle e \rangle}K & {}^{\langle e \rangle}K \end{pmatrix}}_{{}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}} \begin{pmatrix} {}^{i}U \\ {}^{j}U \end{pmatrix} = \begin{pmatrix} {}^{\langle e \rangle i}f \\ {}^{\langle e \rangle j}f \end{pmatrix} \qquad (28)$$

Note:

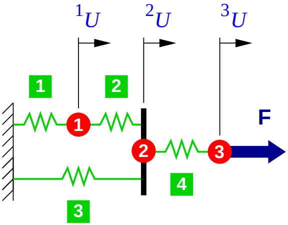
- element stiffness matrix ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$ is symmetric!
- provides relation between nodal displacements and nodal forces

Derivation of Global Stiffness Matrix

Problem 2: How can this "element" be used to get response of whole system?

Solution: Relationship between element node numbering (i, j) and global numbering (1, 2, 3)

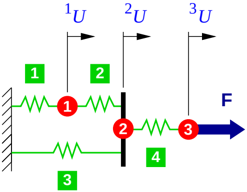
Element 2: With $i = 1$ and $j = 2$



$$\begin{pmatrix} ^{\langle 2 \rangle}K & -\,^{\langle 2 \rangle}K & 0 \\ -\,^{\langle 2 \rangle}K & ^{\langle 2 \rangle}K & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} ^1U \\ ^2U \\ ^3U \end{pmatrix} = \begin{pmatrix} ^{\langle 2 \rangle 1}f \\ ^{\langle 2 \rangle 2}f \\ 0 \end{pmatrix} \tag{29}$$

Derivation of Global Stiffness Matrix

Element 1: $i =?$ has no global DOF $\Rightarrow$ only one contribution



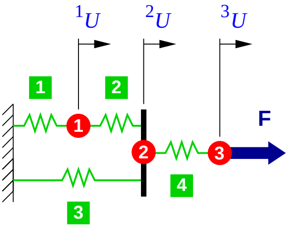
$$\begin{pmatrix} ^{\langle 1\rangle}K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} ^1U \\ ^2U \\ ^3U \end{pmatrix} = \begin{pmatrix} ^{\langle 1\rangle 1}f \\ 0 \\ 0 \end{pmatrix}$$

Element 3: Same as element 1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & ^{\langle 3\rangle}K & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} ^1U \\ ^2U \\ ^3U \end{pmatrix} = \begin{pmatrix} 0 \\ ^{\langle 3\rangle 2}f \\ 0 \end{pmatrix}$$

Derivation of Global Stiffness Matrix

Element 4: Same as element 2



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & {}^{\langle 4\rangle}K & -{}^{\langle 4\rangle}K \\ 0 & -{}^{\langle 4\rangle}K & {}^{\langle 4\rangle}K \end{pmatrix} \begin{pmatrix} {}^1U \\ {}^2U \\ {}^3U \end{pmatrix} = \begin{pmatrix} 0 \\ {}^{\langle 4\rangle 2}f \\ {}^{\langle 4\rangle 3}f \end{pmatrix}$$

Note: Assembling of global matrix can be easily automated for computer use

Assembly of the Global Stiffness Matrix

Adding up equations gives:

$$\begin{pmatrix} ({}^{\langle 1\rangle}K + {}^{\langle 2\rangle}K) & -{}^{\langle 2\rangle}K & 0 \\ -{}^{\langle 2\rangle}K & ({}^{\langle 2\rangle}K + {}^{\langle 3\rangle}K + {}^{\langle 4\rangle}K) & -{}^{\langle 4\rangle}K \\ 0 & -{}^{\langle 4\rangle}K & {}^{\langle 4\rangle}K \end{pmatrix} \begin{pmatrix} {}^{1}U \\ {}^{2}U \\ {}^{3}U \end{pmatrix} = \dots \tag{30}$$

$$\dots = \begin{pmatrix} {}^{\langle 1\rangle 1}f + {}^{\langle 2\rangle 1}f \\ {}^{\langle 2\rangle 2}f + {}^{\langle 3\rangle 2}f + {}^{\langle 4\rangle 2}f \\ {}^{\langle 4\rangle 3}f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \tag{31}$$

- right hand side follows from the equilibrium at nodes ($\sum F_x = 0$)
- same results as equation (24) (equation using nodal equilibrium)

Note

For global stiffness matrix "assembly" step necessary!

Solution Using Gauss Elimination

Assume values: Inserting ${}^{\langle 1\rangle}K = 500$, ${}^{\langle 2\rangle}K = 250$, ${}^{\langle 3\rangle}K = 2000$, ${}^{\langle 4\rangle}K = 1000$ and $F = 1000$ it follows:

$$\begin{pmatrix} 750 & -250 & 0 \\ -250 & 3250 & -1000 \\ 0 & -1000 & 1000 \end{pmatrix} \begin{pmatrix} {}^{1}U \\ {}^{2}U \\ {}^{3}U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \end{pmatrix}$$

Question: How can a solution for ${}^{i}U$ be obtained?

Answer: z.B. Gauss elimination

Note:

Solution of system is numerically expensive!

Solution Using Gauss Elimination

Procedure:

- normalize row 1 (divide by 750):

$$\begin{pmatrix} 1 & -0.33 & 0 \\ -250 & 3250 & -1000 \\ 0 & -1000 & 1000 \end{pmatrix} \begin{pmatrix} {}^1U \\ {}^2U \\ {}^3U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \end{pmatrix}$$

- add "250" times the first row to the second row

$$\begin{pmatrix} 1 & -0.33 & 0 \\ 0 & 3166.7 & -1000 \\ 0 & -1000 & 1000 \end{pmatrix} \begin{pmatrix} {}^1U \\ {}^2U \\ {}^3U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \end{pmatrix}$$

Solution Using Gauss Elimination

- repeat normalization and addition to get a triangular matrix:

$$\begin{pmatrix} 1 & -0.33 & 0 \\ 0 & 1 & -0.316 \\ 0 & 0 & 684.2 \end{pmatrix} \begin{pmatrix} {}^1U \\ {}^2U \\ {}^3U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \end{pmatrix}$$

- solve the system of equations from bottom to top e.g.:

$$\begin{aligned} 684.2\,{}^3U &= 1000 \Longrightarrow {}^3U \\ {}^2U - 0.316\,{}^3U &= 0 \Longrightarrow {}^2U \\ &\vdots \\ \begin{pmatrix} {}^1U \\ {}^2U \\ {}^3U \end{pmatrix} &= \begin{pmatrix} 0.1538 \\ 0.4616 \\ 1.4615 \end{pmatrix} \end{aligned}$$

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Model of Spring System
Equations Using Nodal Equilibrium
Equations Using Spring Element

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Summary

Steps of a finite element calculation:

1. partition of the domain in elements
2. computation of element stiffness matrix ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$
3. assembly of global stiffness matrix $\underline{\underline{\boldsymbol{K}}}$
4. solving of governing equations
5. (post-processing: computation of ${}^{\langle e \rangle}f$ with equation (28)).

Note:

Advantage of finite element modeling: can be automated

List of Questions

- **What are the limits of a linear static Finite Element Analysis?**
- **What's the use of the Finite Element Method?**
- **What are the basic steps of an FE analysis?**

Recommendable Literature

Bathe, K.-J. (1996).
Finite Element Procedures.
Prentice-Hall, Inc., New Jersey, USA.

Part IV

Weighted Residuals, Ritz Ansatz

Overview Part IV

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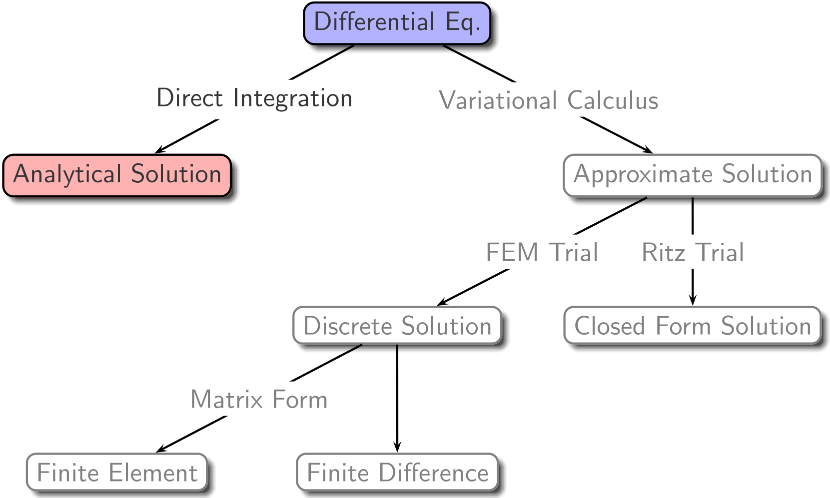
Solution Procedure - Direct Integration



Example: Uniform Bar with Body and Tip Load

Task: Derivation of the exact solution for:

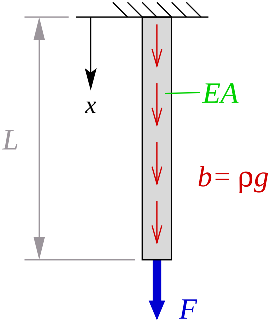


Figure: Uniform bar with body load $b = \rho\, g$ and tip load F

$\Rightarrow$ Demonstrate how weighted residuals work using an example

Derivation of the Differential Equation

Starting point is the local static equilibrium (see later):

$$\frac{d\sigma(x)}{dx} + b \quad = \quad 0 \qquad 0 \leq x \leq L \tag{32}$$

the constitutive equation

$$\sigma(x) = E\,\varepsilon(x) \tag{33}$$

and the displacement-strain relation

$$\varepsilon(x) = \frac{du(x)}{dx} \tag{34}$$

lead to the differential equation for this problem:

$$E\frac{d^2u(x)}{dx^2} + b \quad = \quad 0 \qquad 0 \leq x \leq L \tag{35}$$

Solution of the Differential Equation

The differential equation for this problem is:

$$E\frac{d^2u(x)}{dx^2} + b \quad = \quad 0 \qquad 0 \le x \le L \tag{36}$$

with boundary conditions (BC):

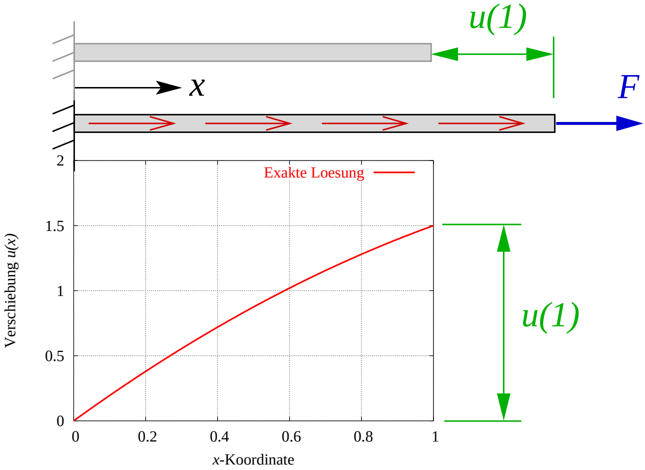
$$u|_{x=0} \quad = \quad 0 \qquad \text{disp. BC} \tag{37}$$

$$E\,A\frac{du(x)}{dx}|_{x=L} \quad = \quad F \qquad \text{force BC} \tag{38}$$

Integration and considering of BCs gives:
Exact solution for bar problem

$$\boxed{u(x) = \frac{1}{E}\Big((\frac{F}{A} + b\,L)\,x - \frac{b}{2}\,x^2\Big)} \tag{39}$$

Graphical Representation of the Solution



Practical Problems

- Complex geometry, etc. $\Rightarrow$ exact solution (eq. (36)) not available
- $\Rightarrow$ Approximation necessary i.e. exact solution u becomes approximation $\bar{u}$:

$$\boxed{u(x) \rightarrow \bar{u}(x)} \tag{40}$$

- **Problem**: Approximation do usually not satisfy differential equation (e.g. Eq. (36)):

$$E\frac{d^2\bar{u}(x)}{dx^2} + b \neq 0 \tag{41}$$

- **Work around**: alternative formulation for eq. (36) necessary

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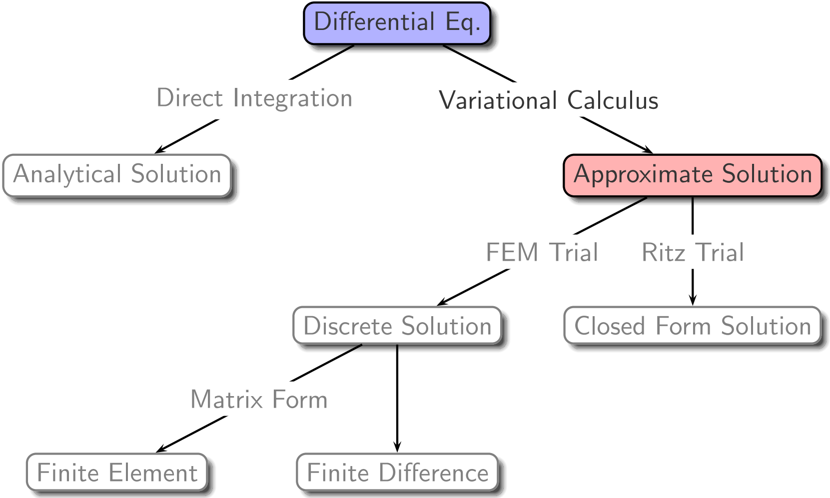
Solution Procedure - Variational Calculus



Overview Approximated Solution Methods

Goal

New approximated equation instead of exact differential equation!

Alternative methods to obtain approximated governing equation:

- Variational principles (energy principles):
  - Hamilton's principle
  - Principle of virtual work $\delta A = 0$ (displ., force)
  - Galerkin method
  - Minimum total potential energy principle $\delta V = 0$
  - ...
- Weighted residual method
  - simple but powerful mathematical tool
  - some of above principle follow from this principle

Weighted Residual Method

Example: Uniform bar with body and tip load (see eq. (36)-(38))

Mathematical problem differential equation (DE) with BC reads as:

$$\begin{aligned} E\frac{d^2u(x)}{dx^2} + b &= 0 && \text{differential equation} \\ u(x)|_{x=0} &= 0 && \text{displacement BC} \\ E\,A\frac{du(x)}{dx}|_{x=L} &= F && \text{force BC} \end{aligned}$$

$u(x)$. . . unknown

Weighted Residual Method

If in DE an approximate solution $\bar{u}(x)$ is used instead of $u(x)$:

$$R(x) = E\frac{d^2\bar{u}(x)}{dx^2} + b \neq 0 \qquad\qquad R(x)\ldots \quad \text{error or residual}$$

In a second step this error is weighted:

$$w(x)\ R(x) \neq 0 \qquad\qquad w(x)\ldots \quad \text{weighting function}$$

Finally the error is summed (i.e. integrated) over the volume

$$\int_V w(x)R(x)dV = A\int_0^L w(x)R(x)dx = 0$$

and it is required that averaged error is zero!

Weighted Residual Method

Instead of (36) following equation is obtained:

Weighted Residual Method

$$A \int_0^L w(x) \Big(\underbrace{E \frac{d^2 \bar{u}(x)}{dx^2} + b}_{R(x)} \Big) dx = 0 \tag{42}$$

- Weighting function: $w(x)$, arbitrary function
- Approximation of displacements $\bar{u}(x)$
- Residual (error): $R(x) \neq 0$ because $u \neq \bar{u}$

Note:

- Instead of $R(x) = E\frac{d^2 u(x)}{dx^2} + b = 0$ (exact solution)
  $A \int w(x) R(x) dx = 0$ is required!
- Equation (42) is known as strong form (details later)

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Motivation, Trial Function

Problem: Find an approximate solution for the bar problem using weighted residuals (Eq. (42)**)**

- Assumption: Solution (Eq. (39)) is not known.
- Proposed solution for Eq. (36) = displacement trial (ansatz) function:

$$\bar{u}(x) = \frac{F}{E\,A}x + a_1 \sin\left(\frac{\pi}{2\,L}x\right) \tag{43}$$

- **Notes** with respect to trial function
  - principle shape of solution is given $\Rightarrow$ from experience
  - Task = find unknown coefficient a_1

Boundary Conditions

Requirement for the solution: BC have to be fulfilled:

$$\begin{aligned} \bar{u}|_{x=0} &= 0 \qquad \rightarrow \text{trivial} \\ E\,A\,\frac{d\bar{u}}{dx}|_{x=L} &= F \end{aligned}$$

Check of the 2. BC:

$$E\,A\,\frac{d\bar{u}}{dx}|_{x=L} \quad = \quad E\,A\left(\frac{F}{E\,A} + \underbrace{a_1\,\frac{\pi}{2\,L}\,\cos\left(\frac{\pi}{2\,L}x\right)}_{=0 \ \text{for} x=L}\right)|_{x=L} = F$$

Compution of the Residual

If trial function Eq. (43) is inserted in Eq. (42) with $A, E, F, L, b = 1$:

$$R(x) = -a_1 \frac{\pi^2}{4} \sin\left(\frac{\pi}{2} x\right) + 1 \neq 0 \tag{44}$$

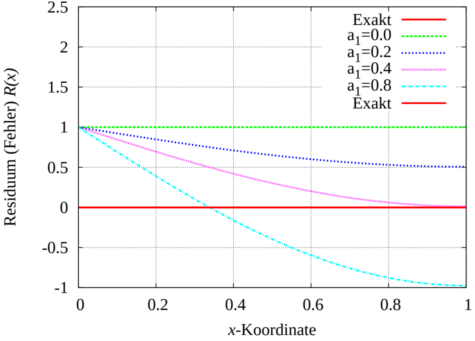

Figure: Residuals for different a_1 values over body

Calculation of a_1 - Subdomain Method

Question: How can the coefficients a_1 be calculated?

Answer: Appropriate choice of weighting function!

Subdomain method $w(x) = 1$:

- Average error is zero
- i.e. area below and above the "0" line in Fig. 11 are equal!
- Evaluation of Eq. (42) gives:

$$
\begin{aligned}
\int_0^L 1\, R(x)\, dx &= 0 \qquad (45)\\
\int_0^L 1 \left(-\, a_1 \frac{\pi^2}{4} \sin\left(\frac{\pi}{2}\, x\right) + 1 \right) dx &= 0\\
\Rightarrow \underline{a_1 = \frac{2}{\pi}} &= 0.637
\end{aligned}
$$

Calculation of a_1 - Galerkin Method

Galerkin method:

$$w(x) = d_1 \, sin(\frac{\pi}{2\,L} x)$$

- Weighting function is (part of) displacement trial function (d_1... arbitrary coefficient)
- Interpretation: Error is weighted more in areas of bigger displacements
- Evaluation of Eq. (42) gives (note d_1 cancels out):

$$\int_0^L \underbrace{d_1 \, sin(\frac{\pi}{2\,L} x)}_{w(x)} \underbrace{\Big(- a_1 \frac{\pi^2}{4} \sin\big(\frac{\pi}{2} x\big) + 1 \Big)}_{R(x)} dx \quad = \quad 0 \qquad (46)$$

$$\Rightarrow \underline{a_1 = \frac{16}{\pi^3}} \quad = \quad 0.516$$

Solution for the Bar Problem

Approximated solution for the bar problem using Galerkin (strong form)

$$\boxed{\bar{u}(x) = \frac{F}{E\,A}x + \frac{16}{\pi^3}\,\sin\left(\frac{\pi}{2\,L}x\right)} \tag{47}$$

Comparison of Different Solutions

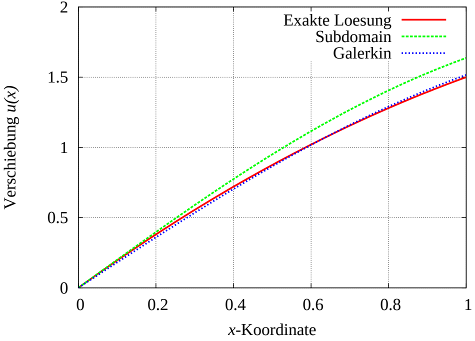

Figure: Solution for bar problem: Exact, Subdomain and Galerkin solution

- Galerkin most accurate solution
- **Note:** More terms in trial function $\Rightarrow$ higher accuracy

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Review: Strong Form

Approximative strong form of weighted residual method (=Eq. (42), bar problem)

$$
\begin{aligned}
A\int_0^L w\left(E\frac{d^2\bar{u}}{dx^2}+b\right)dx &= 0 \\
\bar{u}|_{x=0} &= 0 \qquad \text{displacement BC} \\
E\,A\,\frac{d\bar{u}}{dx}|_{x=L} &= F \qquad \text{force BC}
\end{aligned}
$$

Advantages (compared to exact DE):

- often no exact solution available
- approximative solution can be used for displacements $u \to \bar{u}$ e.g. (Eq. (43)):

$$\bar{u} = \frac{F}{E\,A}x + a_1\,\sin\left(\frac{\pi}{2\,L}x\right)$$

Transition: Strong $\rightarrow$ Weak Form

Drawbacks of strong form:

- displacement trial $\bar{u}$ has to fulfill force BC $\Rightarrow$ complex trial functions
- displacement trial $\bar{u}$ has to be at least $2\times$ continuously differentiable

Workaround: Reformulate strong form (=Eq. (42))

- mathematical trick (partial differentation):

$$\frac{d}{dx}\left(w(x)\,\frac{d\bar{u}(x)}{dx}\right) = \frac{dw}{dx}\frac{d\bar{u}}{dx} + w\,\frac{d^2\bar{u}}{dx^2}$$

- inserting in Eq. (42) gives:

$$\underbrace{E\,A\int_0^L \underbrace{\frac{d}{dx}\left(w\,\frac{d\bar{u}}{dx}\right)}_{f'(x)} dx}_{f(x)|_0^L} - E\,A\int_0^L \frac{dw}{dx}\frac{d\bar{u}}{dx}dx + A\int_0^L w\,b dx = 0$$

Weak Form of Weighted Residuals

- further simplifications + insertion of force BC

$$\underbrace{E\,A\left(w\,\frac{d\bar{u}}{dx}\right)|_L}_{=w(L)\;F} - \underbrace{E\,A\left(w\,\frac{d\bar{u}}{dx}\right)|_0}_{=0 \quad \text{if} \quad w(0)=0} - E\,A\int_0^L \frac{dw}{dx}\frac{d\bar{u}}{dx}dx + A\int_0^L w\,b dx = 0$$

Assumption: Weighting function fulfills displacement BC!

- It follows:

Weak Form of Weighted Residuals

$$E\,A\int_0^L \frac{dw}{dx}\,\frac{d\bar{u}}{dx}dx = A\int_0^L w\,b\,dx + w(L)\,F \qquad (48)$$

- Remarkable points:
  - weak . . . trial function $1\times$ continuously differentiable
  - more solutions for the weak than strong form possible
  - trial function has not to fulfill force BC

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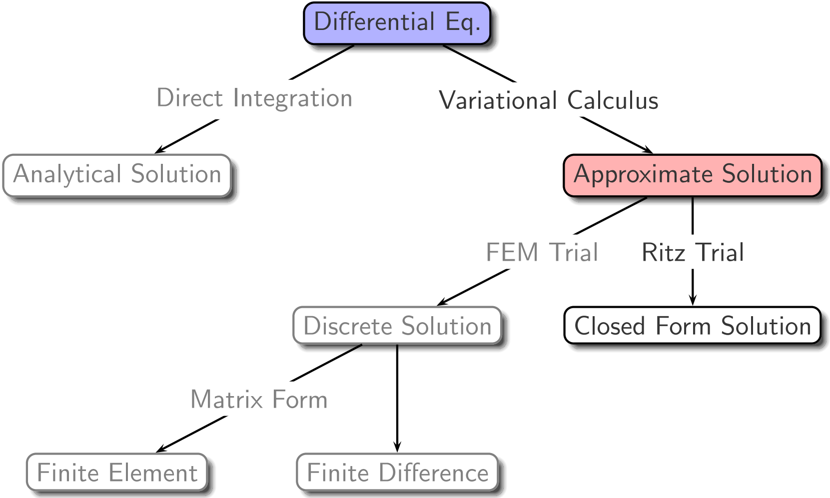
Solution Procedure - Ritz Trial



General Ritz Ansatz Function

The previous trial function can be written as:

$$\bar{u}(x) = \underbrace{\frac{F}{E\,A}x}_{\phi_0(x)} + \underbrace{a_1 \,\sin\left(\frac{\pi}{2\,L}x\right)}_{a_j\,\phi_j(x)}$$

It follows the general Ritz ansatz function:

$$\bar{u}(x) = \phi_0(x) + \sum_{j=1}^{n} a_j\,\phi_j(x) \tag{49}$$

- one term which fulfills displacement BC: $\phi_0(x)$
- given shape functions: $\phi_j(x)$ (location dependent)
- unkown coefficients: a_j
- $\Rightarrow$ solution linear combination of basis functions!

Graphical Explanation of Ritz Ansatz

- Example:

$$\bar{u}(x) = f(x) = \underbrace{0.7}_{a_1}\underbrace{\sin(5x)}_{\phi_1} + \underbrace{1.5}_{a_2}\underbrace{x}_{\phi_2} + \underbrace{\frac{1}{3}}_{a_3}\underbrace{\log(150x+1)}_{\phi_3}$$

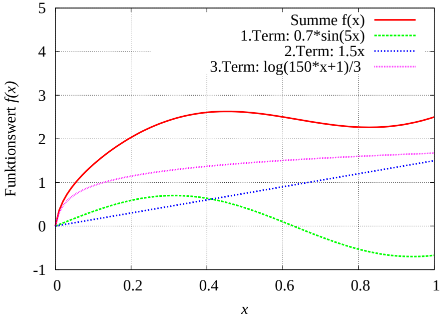


Figure: Ritz ansatz with 3 basis functions

Derivative of a Ritz Ansatz, Weighting Function

What happens if Ritz trial is inserted in Eq. (48)?

- Derivative of approximate solution:

$$\frac{d\bar{u}(x)}{dx} = \frac{d\phi_0(x)}{dx} + \sum_{j=1}^{n} a_j \frac{d\phi_j(x)}{dx} \tag{50}$$

- Derivative of Galerkin weighting function:

$$w(x) = \sum_{i=1}^{n} d_i \, \phi_i(x) \qquad \frac{dw(x)}{dx} = \sum_{i=1}^{n} d_i \frac{d\phi_i(x)}{dx} \tag{51}$$

Usage of Ritz Ansatz in Weak Form Solution

Substituting Ritz ansatz in Eq. (48) gives:

$$E\,A\int_0^L\left(\sum_{i=1}^n d_i\,\frac{d\phi_i}{dx}\right)\left(\frac{d\phi_0}{dx}+\sum_{j=1}^n a_j\,\frac{d\phi_j}{dx}\right)dx = A\int_0^L\left(\sum_{i=1}^n d_i\,\phi_i\right)b\,dx+\left(\sum_{i=1}^n d_i\,\phi_i(L)\right)F \qquad (52)$$

rearranging:

$$\sum_{i=1}^n d_i\,\underbrace{\left[E\,A\int_0^L\left(\frac{d\phi_i}{dx}\right)\left(\frac{d\phi_0}{dx}+\sum_{j=1}^n a_j\,\frac{d\phi_j}{dx}\right)dx - A\int_0^L \phi_i\,b\,dx-\phi_i(L)\,F\right]}_{=0} = 0 \qquad (53)$$

Because equation has to be fullfilled for arbitrary d_i it holds:

$$E\,A\int_0^L\left(\frac{d\phi_i}{dx}\right)\left(\frac{d\phi_0}{dx}+\sum_{j=1}^n a_j\,\frac{d\phi_j}{dx}\right)dx - A\int_0^L \phi_i\,b\,dx-\phi_i(L)\,F = 0 \qquad (54)$$

System of Equations

Moving known terms to the right hand side:

$$E\,A \int_0^L \left(\frac{d\phi_i}{dx}\right) \left(\sum_{j=1}^{n} a_j \,\frac{d\phi_j}{dx}\right) dx =$$

$$\underbrace{A \int_0^L \phi_i\, b\, dx}_{\text{volume forces}} - \underbrace{E\,A \int_0^L \left(\frac{d\phi_i}{dx}\right)\left(\frac{d\phi_0}{dx}\right) dx}_{\text{boundary displacements}} + \underbrace{\phi_i(L)\,F}_{\text{boundary forces}} \qquad i = 1, n$$

gives n equations $(i = 1, \ldots, n\ !)$ for n unknown values a_j.

We can expand this equation to see individual terms:

$$\begin{pmatrix} E\,A \int_0^L \left(\frac{d\phi_1}{dx}\right)\left(\frac{d\phi_1}{dx}\right) dx & \cdots & E\,A \int_0^L \left(\frac{d\phi_1}{dx}\right)\left(\frac{d\phi_n}{dx}\right) dx \\ E\,A \int_0^L \left(\frac{d\phi_2}{dx}\right)\left(\frac{d\phi_1}{dx}\right) dx & \cdots & E\,A \int_0^L \left(\frac{d\phi_2}{dx}\right)\left(\frac{d\phi_n}{dx}\right) dx \\ \vdots & \vdots & \vdots \\ E\,A \int_0^L \left(\frac{d\phi_n}{dx}\right)\left(\frac{d\phi_1}{dx}\right) dx & \cdots & E\,A \int_0^L \left(\frac{d\phi_n}{dx}\right)\left(\frac{d\phi_n}{dx}\right) dx \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \qquad (55)$$

Force Vector, Equations for Unknowns

using the known force vector f_i:

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} A \int_0^L \phi_1 \, b \, dx - E\,A \int_0^L \left(\frac{d\phi_1}{dx}\right)\left(\frac{d\phi_0}{dx}\right) dx + \phi_1(L)\,F \\ A \int_0^L \phi_2 \, b \, dx - E\,A \int_0^L \left(\frac{d\phi_2}{dx}\right)\left(\frac{d\phi_0}{dx}\right) dx + \phi_2(L)\,F \\ \vdots \\ A \int_0^L \phi_n \, b \, dx - E\,A \int_0^L \left(\frac{d\phi_n}{dx}\right)\left(\frac{d\phi_0}{dx}\right) dx + \phi_n(L)\,F \end{pmatrix} \tag{56}$$

it follows:

Equations for unknown coefficients

$$\underline{\underline{\boldsymbol{\Phi}}} \cdot \underline{\boldsymbol{a}} = \underline{\boldsymbol{f}} \tag{57}$$

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Trial Function and Derivatives

Problem: Find an approximate solution for the bar problem using a single term Ritz ansatz for $\bar{u}$.

- Trial function for Eq. (36) in form of (compare with Eq. (43)):

$$\bar{u}(x) = \sum_{j=1}^{1} a_j \, \phi_j(x) \qquad \text{with} \qquad \phi_1(x) = \sin\left(\frac{\pi}{2\,L} x\right) \qquad 0 \leq x \leq L \tag{58}$$

Remark: $\phi_0 = 0$ because $\bar{u}(x = 0) = 0$ (=given displacement BC)

- Using the derivative of the function

$$\frac{d\bar{u}}{dx} = a_1 \frac{\pi}{2\,L} \cos\left(\frac{\pi}{2\,L} x\right)$$

Evaluation and Solution

- and inserting in Eq. (48) gives with $w(x) = \bar{u}(x)$ (Galerkin) and $A, E, L, F, b = 1$:

$$\left[\frac{\pi^2}{4}\int_0^L \left(\cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)\right)dx\right]a_1 = \int_0^L \sin\left(\frac{\pi}{2}x\right)dx + 1$$

- Evaluating the integrals yields:

$$a_1 = 1.326$$

- **Solution: Approximated, closed form displacements:**

$$\boxed{\bar{u}(x) = 1.326\,\sin\left(\frac{\pi}{2\,L}x\right) \qquad 0 \leq x \leq L} \tag{59}$$

Comparison of Strong and Weak Form Solution

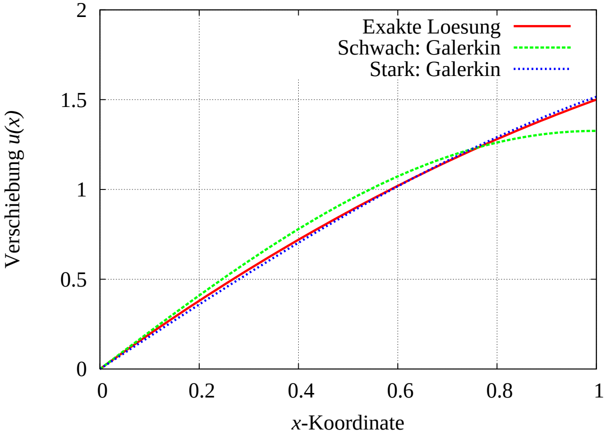

Figure: Bar problem: Exact, weak and strong Galerkin solution

Comparison of Strong - Weak Form Solution

- weak form solution less accurate (if similar trial functions used)
- but: trial function simpler as for strong form
- strong form trial function usable for weak form - but not vice versa!

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Summary

- Overview of approximated solution methods
- Usage of weighted residual method to get solution of DE (strong form)

$$A \int_0^L w(x) \underbrace{\left(E \frac{d^2 \bar{u}(x)}{dx^2} + b \right)}_{R(x)} dx = 0$$

  - introduction of displacement trial function ⇒ approximation
  - fulfills displacement and force boundary conditions
  - solution = calculation of unknown coefficient a_1 for $\bar{u}$

$$\bar{u}(x) = \frac{F}{E\,A} x + a_1 \sin\left(\frac{\pi}{2\,L} x\right)$$

  - weighting function: subdomain method $w(x) = 1$, Galerkin method $w(x) \sim \bar{u}(x)$

Summary

- Transition from the strong to the weak form:

$$E\,A \int_0^L \frac{dw}{dx}\,\frac{d\bar{u}}{dx} dx = A \int_0^L w\,b\,dx + w(L)\,F$$

  - trial function have to fulfill displacement BC
  - simpler trial functions possible

- Application of Ritz ansatz in weak form equation
  - trial and weighting function:

$$\bar{u}(x) = \phi_0 + \sum_{j=1}^{n} a_j\,\phi_j \qquad w(x) = \sum_{i=1}^{n} d_i\,\phi_i$$

  - it follows: equation for unknowns:

$$\underline{\underline{\boldsymbol{\Phi}}} \cdot \underline{\boldsymbol{a}} = \underline{\boldsymbol{f}}$$

List of Questions

- What is the use of the weighted residual method?
- What is the basic idea behind the weighted residual method?
- What means strong/weak form?
- What are the advantages/drawbacks of the weak form?
- What is a general Ritz Ansatz (general trial function)?
- Why is such a trial function needed?

Part V

Shape and Trial Functions

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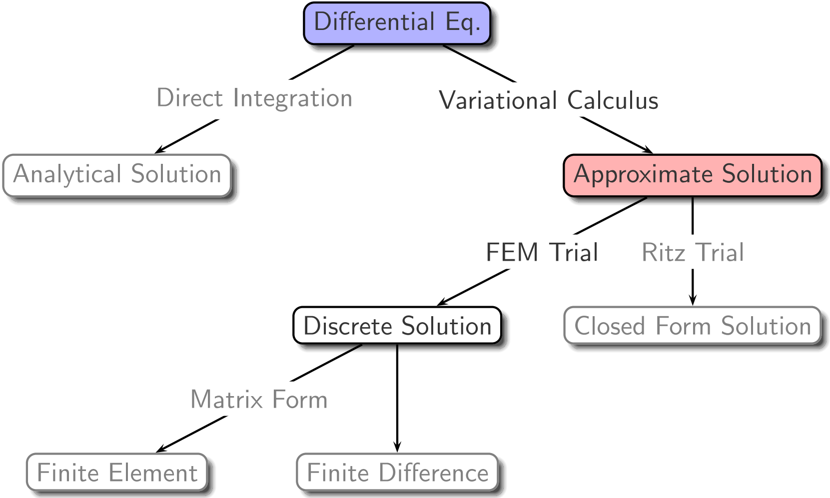
Solution Procedure - FEM Trial Function



Motivation

- Currently trial function over whole domain (Eq. (43), (58)) i.e.:

$$\bar{u}(x) = a_1 \underbrace{\sin\left(\frac{\pi}{2\,L}x\right)}_{\phi(x)} \qquad 0 \leq x \leq L$$

a_1 ... unknown coefficient, $\phi(x)$ ... basis, shape or interpolation function

- Weaknesses of this approach:
  - coefficients a_1 pure mathematical, no physical meaning
  - hard to find basis functions in case of complex geometry (because $\phi(x)$ in $0 \leq x \leq L$)

- **Workaround:** Partitioning of the domain $\Rightarrow$ $\boxed{\text{Finite Element Method}}$

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Linear 1D Element

- **Problem:** Derivation of linear trial function

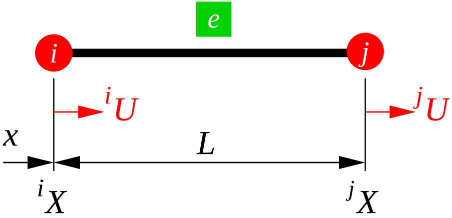

Figure: Simple linear 1D element

- Assume that displacement in element e is linear:

$$^{\langle e \rangle}\bar{u}(x) = c_1 + c_2\, x \tag{60}$$

- Displacements $^{i,j}U$ at nodes i, j using nodal coordinates $^{i,j}X$ are:

$$^{i}U = c_1 + c_2\ ^{i}X \qquad ^{j}U = c_1 + c_2\ ^{j}X$$

- **Note:** Nodal values are written in capital letters (^{i}X, ^{i}U)

Linear 1D Element

Assembling in matrix form gives:

$$\begin{pmatrix} 1 & {}^{i}X \\ 1 & {}^{j}X \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} {}^{i}U \\ {}^{j}U \end{pmatrix}$$

and solving with respect to c_1, c_2

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{{}^{j}X - {}^{i}X} \begin{pmatrix} {}^{j}X & -{}^{i}X \\ -1 & 1 \end{pmatrix} \begin{pmatrix} {}^{i}U \\ {}^{j}U \end{pmatrix} .$$

Reinserting c_1, c_2 into Eq. (60) gives the trial function:

$$^{\langle e \rangle}\bar{u}(x) = \left(\frac{{}^{j}X - x}{{}^{j}X - {}^{i}X} \right) {}^{i}U + \left(\frac{x - {}^{i}X}{{}^{j}X - {}^{i}X} \right) {}^{j}U \qquad (61)$$

with shape or interpolation function $^{\langle e \rangle i}N$**,** $^{\langle e \rangle j}N$**:**

$$\boxed{^{\langle e \rangle}\bar{u}(x) = \left({}^{\langle e \rangle i}N(x) \right) {}^{i}U + \left({}^{\langle e \rangle j}N(x) \right) {}^{j}U} \qquad (62)$$

Remarks on Shape and Trial Function

- Shape and trial functions are not the same!
- Shape (=interpolation) functions ${}^{\langle e \rangle i}N$ are given functions
- Trial functions ${}^{\langle e \rangle}\bar{u}$ are unknown functions, obtained from FEM solution
- Trial function = Sum of shape functions times nodal displacements (DOFs)

$$
{}^{\langle e \rangle}\bar{u}(x) = \left({}^{\langle e \rangle i}N(x) \right) {}^{i}U + \left({}^{\langle e \rangle j}N(x) \right) {}^{j}U
$$

Characteristics of Shape Functions

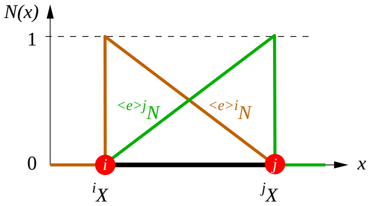

Figure: Shape function of linear 1D element

Characteristics of Shape Functions

- Only defined within an element $\langle e \rangle$
- $^{\langle e \rangle i}N$ is "'1"' at node ^{i}X
- $^{\langle e \rangle i}N$ is "'0"' at all other nodes
- Sum of shape functions is "'1"' ($\sum_{i=1}^{n_{\mathrm{kn}}} {}^{\langle e \rangle i}N(x) = 1 \quad \forall x$)

Characteristics of Shape Functions

Notes:

- In contrast to that general Ritz Ansatz defined over whole domain.
- Shape function is only zero at nodes - not inbetween!
- $\sum_{i=1}^{n_{\mathrm{kn}}} {}^{\langle e \rangle i}N = 1$ is known as completeness condition.
- Completeness: Trial functions allow rigid body movements!

Example: Rigid body movement U i.e.:

$$U = {}^{i}U = {}^{j}U = \bar{u}(x) \quad \forall x$$

hold only if:

$$\bar{u}(x) = {}^{i}N(x)\,{}^{i}U + {}^{j}N(x)\,{}^{j}U = \left({}^{i}N(x) + {}^{j}N(x)\right) U = \underbrace{\sum_{k=1}^{n_{\mathrm{kn}}} {}^{k}N(x)}_{=1\ \forall x}\, U$$

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Natural Coordinates

- Shape function in real (actual) element coordinates

$$^{\langle e \rangle i}N(x) = \frac{^{j}X - x}{^{j}X - {^{i}X}} \qquad ^{\langle e \rangle j}N(x) = \frac{x - {^{i}X}}{^{j}X - {^{i}X}} \tag{63}$$

- Introduction of natural coordinates $^{i}\xi = -1,\ ^{j}\xi = 1$

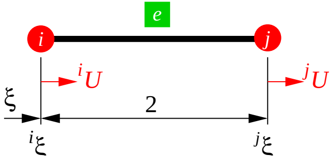

Figure: Linear element in natural coordinates

- Shape functions for natural coordinates ($x \longrightarrow \xi$):

$$^{\langle e \rangle i}N(\xi) = \frac{1 - \xi}{2} \qquad ^{\langle e \rangle j}N(\xi) = \frac{\xi + 1}{2} \tag{64}$$

Advantages of Natural Coordinates

- Shape functions (Eq. (64)) do not depend on real coordinates ${}^{j}X$, x!
- Same shape function for same element type - independent of element size!
- Same numerical integration for same element types (see later)

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Quadratic Line Element

Quadratic interpolation requires 3 nodes:

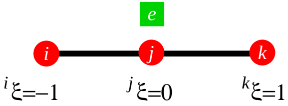

Figure: Quadratic element in natural coordinates

The trial function for element $\langle e \rangle$ reads as:

$$^{\langle e \rangle}\bar{u}(\xi) = {}^{\langle e \rangle i}N(\xi)\ {}^{i}U + {}^{\langle e \rangle j}N(\xi)\ {}^{j}U + {}^{\langle e \rangle k}N(\xi)\ {}^{k}U = \sum_{k=1}^{^{\langle e \rangle}n_{\mathrm{kn}}} {}^{\langle e \rangle k}N(\xi)\ {}^{k}U \qquad (65)$$

- $^{k}U\ldots$ nodal displacements
- $^{\langle e \rangle k}N(\xi)\ldots$ shape functions
- $^{\langle e \rangle}n_{\mathrm{kn}}\ldots$ number of nodes in element

Quadratic Trial Functions - Lagrange Polynomal

Question: How are shape functions derived generally?

Answer: Lagrange Polynomal

Quadratc Trial Functions - Lagrange Polynomal

- First define a function that is zero at all nodes except the ith node:

$$^{\langle e\rangle i}N(\xi) = {}^{i}c\,(\xi - {}^{1}\xi)\,(\xi - {}^{2}\xi)\ldots(\xi - {}^{i-1}\xi)\,(\xi - {}^{i+1}\xi)\ldots(\xi - {}^{n}\xi) \qquad (66)$$

  Note: Term $(\xi - {}^{i}\xi)$ is missing! ${}^{i}c$ is a constant

- Force this function to equal 1 at node i ${}^{\langle e\rangle 1}N = 1$ $(\xi = {}^{i}\xi)$:

$$1 = {}^{i}c\,({}^{i}\xi - {}^{1}\xi)\,({}^{i}\xi - {}^{2}\xi)\ldots({}^{i}\xi - {}^{i-1}\xi)\,({}^{i}\xi - {}^{i+1}\xi)\ldots({}^{i}\xi - {}^{n}\xi) \qquad (67)$$

- Solve equation for ${}^{i}c$ and substitute into Eq. (66) to get Lagrange Polynomal:

$$^{\langle e\rangle i}N(\xi) = \frac{(\xi - {}^{1}\xi)\,(\xi - {}^{2}\xi)\ldots(\xi - {}^{i-1}\xi)\,(\xi - {}^{i+1}\xi)\ldots(\xi - {}^{n}\xi)}{({}^{i}\xi - {}^{1}\xi)\,({}^{i}\xi - {}^{2}\xi)\ldots({}^{i}\xi - {}^{i-1}\xi)\,({}^{i}\xi - {}^{i+1}\xi)\ldots({}^{i}\xi - {}^{n}\xi)} \qquad (68)$$

Quadratic Trial Function from Lagrange Polynomal

- Evaluation of Eq. (68) for $n = 3$ (quadratic element):

$$
{}^{\langle e \rangle 1}N(\xi) = \frac{(\xi - {}^{2}\xi)\,(\xi - {}^{3}\xi)}{({}^{1}\xi - {}^{2}\xi)\,({}^{1}\xi - {}^{3}\xi)} = \frac{(\xi - 0)\,(\xi - 1)}{(-1 - 0)\,(-1 - 1)} = \frac{\xi}{2}(\xi - 1)
$$

$$
{}^{\langle e \rangle 2}N(\xi) = \frac{(\xi - {}^{1}\xi)\,(\xi - {}^{3}\xi)}{({}^{2}\xi - {}^{1}\xi)\,({}^{2}\xi - {}^{3}\xi)} = \frac{(\xi + 1)\,(\xi - 1)}{(0 - (-1))\,(0 - 1)} = (1 - \xi^2)
$$

$$
{}^{\langle e \rangle 3}N(\xi) = \cdots = \cdots = \frac{\xi}{2}(\xi + 1)
$$

- yields the quadratic trail function:

$$
{}^{\langle e \rangle}\bar{u}(\xi) = \frac{\xi}{2}(\xi - 1)\ {}^{i}U + (1 - \xi^2)\ {}^{j}U + \frac{\xi}{2}(\xi + 1)\ {}^{k}U
$$

Graphical Representation of Shape Functions

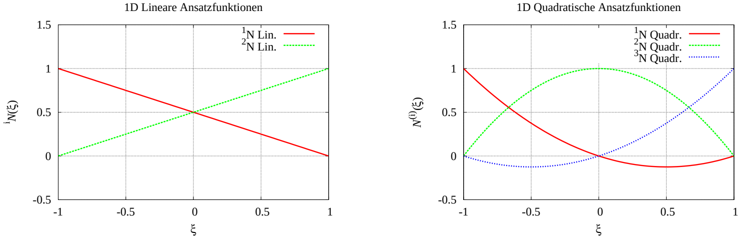

Figure: Linear (left) and quadratic 1D shape functions in natural coordinates

Note:

- It holds: ${}^{\langle e \rangle i}N({}^{j}\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- Completeness (for rigid body movements) $\sum_{\langle e \rangle n_{\mathrm{kn}}} {}^{\langle e \rangle i}N = 1$

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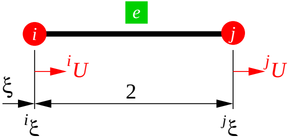
Motivation

Revisiting linear 1D trial function:

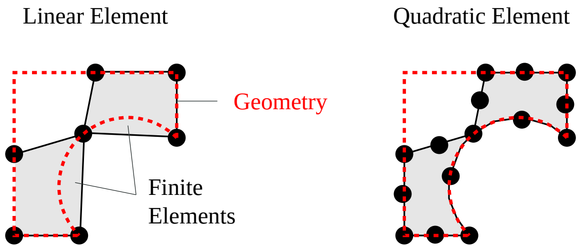
$$^{\langle e \rangle}\bar{u}(\xi) = {}^{\langle e \rangle i}N(\xi)\ {}^{i}U + {}^{\langle e \rangle j}N(\xi)\ {}^{j}U$$



- Shape functions $^{\langle e \rangle i}N(\xi)$, $^{\langle e \rangle j}N(\xi)$ are given (depend on location)
- Nodal displacements ^{i}U, ^{j}U has to be calculated (fixed location)
- Solution $^{\langle e \rangle}\bar{u}(\xi)$ follows from discrete FEM solution!
- i.e. overall solution follows from solution at nodes ^{i}U, ^{j}U and given shape functions $^{\langle e \rangle i}N(\xi)$, $^{\langle e \rangle j}N(\xi)$!
- **Note:** Solutions comes in natural coordinates $^{\langle e \rangle}\bar{u}(\xi)$ (not $^{\langle e \rangle}\bar{u}(x)$)

Interpolation of Geometry

- **Question 1:** How do I compute ${}^{\langle e \rangle}\bar{u}(x)$ $(x = x(\xi))$?
- **Question 2:** How do I define e.g. holes $x = x(\xi)$?



- **Answer:** Interpolation of geometry with nodal coordinates ${}^{i,j}X$ + position shape functions ${}^{\langle e \rangle i}\hat{N}(\xi)$ e.g.:

$$\bar{x}(\xi) = {}^{\langle e \rangle i}\hat{N}(\xi)\ {}^{i}X + {}^{\langle e \rangle j}\hat{N}(\xi)\ {}^{j}X$$

Definition of Isoparametric Elements

- If position shape function ${}^{\langle e\rangle j}\hat{N}(\xi)$ and displacement shape function ${}^{\langle e\rangle i}N(\xi)$ are the same $\rightarrow$ isoparametric elements
- Remark: Geometry is not the exact geometry!
- Errors of coarse meshes around holes using linear elements

Isoparametric Elements

Interpolation of geometry with displacement trial functions ${}^{\langle e\rangle k}N(\xi)$:

$$ {}^{\langle e\rangle}\bar{x}(\xi) = \sum_{k=1}^{n_{\mathrm{kn}}} {}^{\langle e\rangle k}N(\xi)\ {}^{k}X $$

Note:

- Nodal position ${}^{k}X$ in real space
- $n_{\mathrm{kn}}\ldots$ number of nodes in element

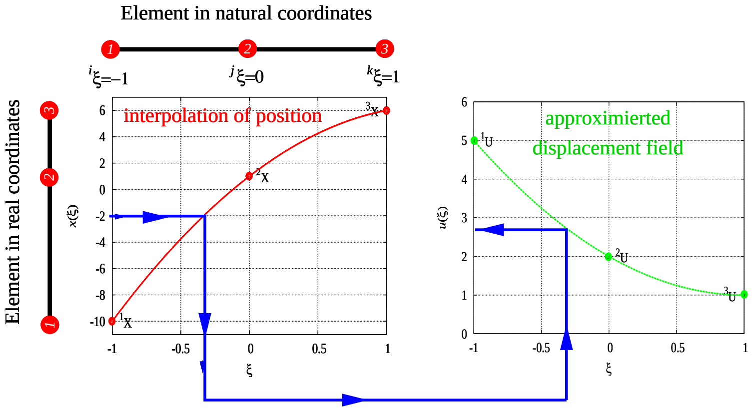
Example

Exercise: Calculate and sketch for an 1D element with three nodes ($^1X = -10,\ ^2X = 1,\ ^3X = 6$) which has given displacements at nodes $^1U = 5,\ ^2U = 2,\ ^3U = 1$:

- approximated geometry $\bar{x}(\xi)$
- approximate displacement field $\bar{u}(\xi)$
- approximate displacement at $x = -2$ ($\bar{u}(x = -2)$)

Example - Solution

- approximated geometry $\bar{x}(\xi)$
- approximate displacement field $\bar{u}(\xi)$
- approximate displacement at $x = -2$ ($\bar{u}(x = -2)$)



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Shape Functions

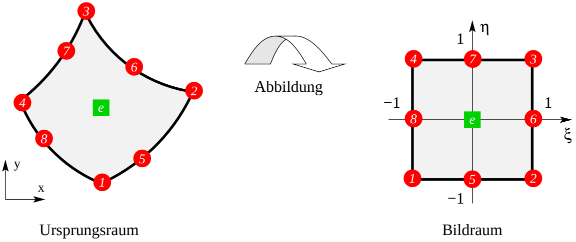

Figure: Quadratic 2D element in real (left) and natural coordinates (right)

Shape functions for such an element ($\langle e \rangle$ not written) are:

$$^{1}N(\xi, \eta) \quad = \quad -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1)$$

Shape Functions

and for the remaining nodes:

$$
\begin{aligned}
{}^{2}N(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\
{}^{3}N(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\
{}^{4}N(\xi,\eta) &= -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1) \\
{}^{5}N(\xi,\eta) &= \frac{1}{2}(1-\xi^2)(1-\eta) \\
{}^{6}N(\xi,\eta) &= \frac{1}{2}(1-\eta^2)(1+\xi) \\
{}^{7}N(\xi,\eta) &= \frac{1}{2}(1-\xi^2)(1+\eta) \\
{}^{8}N(\xi,\eta) &= \frac{1}{2}(1-\eta^2)(1-\xi)
\end{aligned} \tag{69}
$$

Sketch of Shape Functions

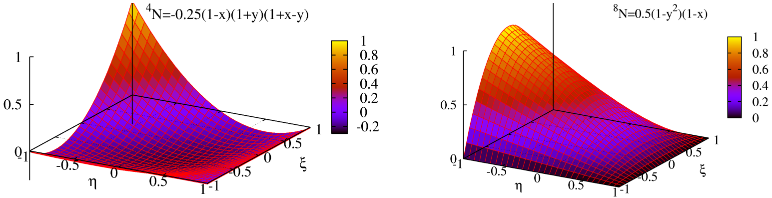

Figure: Shape function for corner node (left) und mid node (right) of a quadratic 8-noded element

- **Note:** Shape function is "1" at considered node and "0" at other nodes!

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Definition of Trial Functions

Starting point is Eq. (65). Introduction of two natural coordinates gives:

$$\begin{aligned} \bar{u}_1(\xi,\eta) &= \sum_{k=1}^{8} {}^k N_1(\xi,\eta)\ {}^k U_1 \\ \bar{u}_2(\xi,\eta) &= \sum_{k=1}^{8} {}^k N_2(\xi,\eta)\ {}^k U_2 \end{aligned} \qquad (70)$$

- $i = 1, 2$ direction in space, along ξ and η
- $\bar{u}_i(\xi,\eta)$ displacement field in ξ and η in direction i
- ${}^k U_i$ nodel displacement of node k in direction i

Matrix of Element Trial Functions

2D Element Trial Functions

$$ {}^{\langle e\rangle}\underline{\bar{\boldsymbol{u}}}(\xi,\eta) = {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}(\xi,\eta) \cdot {}^{\langle e\rangle}\underline{\boldsymbol{U}} \tag{71} $$

- ${}^{\langle e\rangle}\underline{\bar{\boldsymbol{u}}}(\xi,\eta)$ approximated displacement field
- ${}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}(\xi,\eta)$ matrix of shape functions
- ${}^{\langle e\rangle}\underline{\boldsymbol{U}}$ vector of unknown nodal displacements of the element

written in matrix form:

$$ \begin{pmatrix} \bar{u}_1(\xi,\eta) \\ \bar{u}_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} {}^1N(\xi,\eta) & 0 & {}^2N(\xi,\eta) & \dots & {}^8N(\xi,\eta) & 0 \\ 0 & {}^1N(\xi,\eta) & 0 & {}^2N(\xi,\eta) & \dots & {}^8N(\xi,\eta) \end{pmatrix} \cdot \begin{pmatrix} {}^1U_1 \\ {}^1U_2 \\ {}^2U_1 \\ {}^2U_2 \\ \vdots \\ {}^8U_1 \\ {}^8U_2 \end{pmatrix} $$

Matrix of Element Trial Functions

Notes:

- vector ${}^{\langle e\rangle}\underline{\boldsymbol{U}}$ contains DOFs in x, y (nodal displacements)
- same interpolation in both space directions used!

$$ {}^{k}N_1 = {}^{k}N_2 = {}^{k}N $$

(Advandage of natural coordinates)

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Derivations in Real and Natural Coordinates

- In chapter 9 its shown that derivatives of trial functions are needed:

$$\frac{\partial}{\partial x}\left({}^{\langle e \rangle}\underline{\bar{\boldsymbol{u}}}(\xi,\eta)\right) = \frac{\partial}{\partial x}\left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}(\xi,\eta)\right) \cdot {}^{\langle e \rangle}\underline{\boldsymbol{U}}$$

(**Note:** Nodal displacements ${}^{\langle e \rangle}\underline{\boldsymbol{U}}$ do not depend on ξ, η!)

- i.e. one needs derivation of shape function (in node k):

$$\frac{\partial^k N(\xi,\eta)}{\partial x} \tag{72}$$

- But: ${}^k N(\xi,\eta)$ is not a direct function of x
- Workaround: Usage of chain rule

$$\begin{aligned}
\frac{\partial^k N}{\partial \xi} &= \frac{\partial^k N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial^k N}{\partial y}\frac{\partial y}{\partial \xi} \\
\frac{\partial^k N}{\partial \eta} &= \frac{\partial^k N}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial^k N}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned}$$

Mapping Function: Jacobian Operator

In matrix notation:

$$\begin{pmatrix} \frac{\partial^k N}{\partial \xi} \\ \frac{\partial^k N}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^k N}{\partial x} \\ \frac{\partial^k N}{\partial y} \end{pmatrix}$$

with

Jacobian Operator of a 2D element

$$\underline{\underline{\boldsymbol{J}}} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \tag{73}$$

= relation between natural and real coordinate derivatives

Evaluation of Jacobian Operator

Derivatives $\frac{\partial x}{\partial \xi}$ follow from interpolation of geometry, e.g. 2D:

$$\bar{x}(\xi,\eta) = {}^1N(\xi,\eta)\ {}^1X + {}^2N(\xi,\eta)\ {}^2X + \cdots + {}^8N(\xi,\eta)\ {}^8X$$

and

$$\frac{\partial x}{\partial \xi} = \frac{\partial\, {}^1N}{\partial \xi}\ {}^1X + \frac{\partial\, {}^2N}{\partial \xi}\ {}^2X + \cdots + \frac{\partial\, {}^8N}{\partial \xi}\ {}^8X$$

Note: Nodal coordinates ${}^kX$ do not depend on ξ, η!

For Eq. (72) an inverse relation is required:

$$\begin{pmatrix} \frac{\partial^k N}{\partial x} \\ \frac{\partial^k N}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial^k N}{\partial \xi} \\ \frac{\partial^k N}{\partial \eta} \end{pmatrix} = \frac{1}{\det \underline{\underline{\boldsymbol{J}}}} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^k N}{\partial \xi} \\ \frac{\partial^k N}{\partial \eta} \end{pmatrix}$$

with inverse Jacobian $\underline{\underline{\boldsymbol{J}}}^{-1}$.

Explanation Determinant of Jacobian

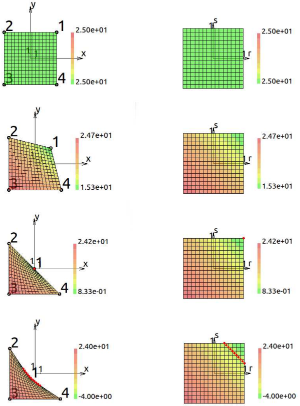

Figure: Element side length 10×10. Left orginal, right natural coordinates. From top to bottom movement of node 1. Red points $\det \underline{\underline{J}} = 0$.

Note: $\det \underline{\underline{J}}$ gives the area change (2D) or volume change (3D).

Inverse Jacobian

- **Note:** Inversion only possible if $\underline{\underline{J}}$ not singular i.e. $\det \underline{\underline{J}} \neq 0$!
- Attention in the case of some element geometries!

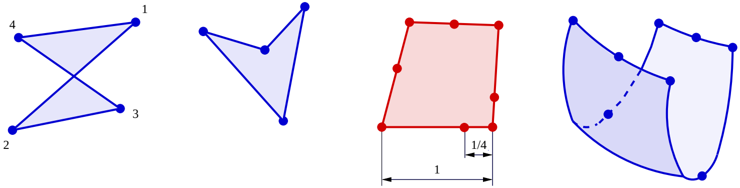

Figure: Critical element geometries which show singular Jacobian. Wrong node numbering, distorted element, crack tip element, folded element (from left to right)

- Utilization of singularity: crack tip element

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Integration in Natural Coordinaten

- Example: One like do integrate $u(x,y)$ over x,y:

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} u(x,y)\,dxdy$$

- Known is only $u(\xi,\eta)$. In that case it holds:

Integration in natural coordinates

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} u(x,y)\,dxdy = \int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} \underbrace{u\big(x(\xi,\eta),y(\xi,\eta)\big)}_{u(\xi,\eta)}\,\det \underline{\boldsymbol{J}}(\xi,\eta)\,d\xi d\eta \tag{74}$$

with determinant of Jacobian operator $\det \underline{\boldsymbol{J}}(\xi,\eta)$

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4-Noded Element

Linear 4-noded element, displacement of node 1 and 4

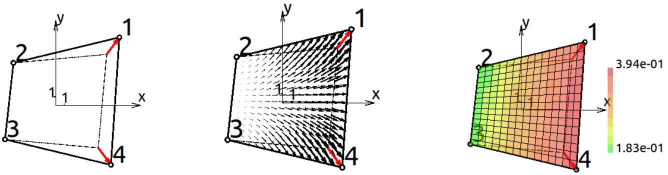


Figure: Nodal displacements $\underline{\boldsymbol{U}}$ (left), displacement field $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})$ (middle), σ_{11} stresses (right).

4-Noded vs. 8-Noded Element

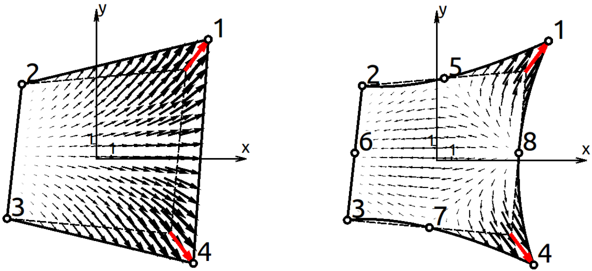
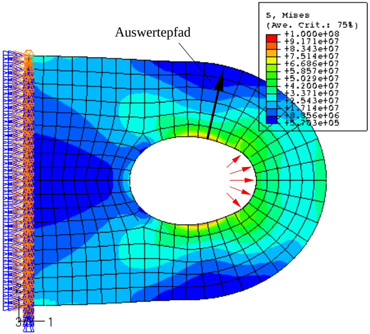


Figure: Displacement fields $\underline{u}(\underline{x})$ of a 4-noded (left) and 8-noded (right) element by using the same nodal displacements.

Hole with Internal Pressure: Stresses

- Linear and quadratic hexahedral elements, linear static analyses
- Stress distribution along hole in radial direction



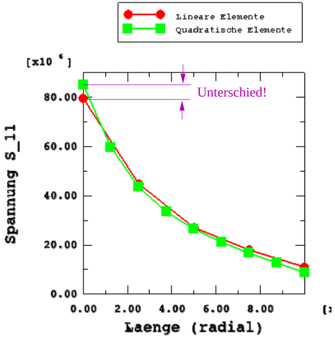


Figure: σ_{11} stress (left) and distribution of σ_{11} for linear (red) and quadratic elements (green, right)

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Shape Functions, Trial Functions

- Shape (interpolation) functions ${}^{\langle e \rangle k}N(\xi, \eta)$ in natural coordinates ξ, η
  - Defined only within elemente $\langle e \rangle$
  - ${}^{\langle e \rangle k}N(\xi, \eta)$ is "1" at considered node k
  - ${}^{\langle e \rangle k}N(\xi, \eta)$ is "0" on all other nodes
  - Sum of shape functions is "1" ($\sum_{k=1}^{n_{\text{kn}}} {}^{\langle e \rangle k}N(\xi, \eta) = 1 \quad \forall \xi, \eta$)

- Introduction of 2D element trial functions (${}^{\langle e \rangle}\underline{\boldsymbol{U}} \ldots$ nodal displacement)

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = {}^{\langle e \rangle}\underline{\bar{\boldsymbol{u}}}(\xi, \eta) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}(\xi, \eta) \cdot {}^{\langle e \rangle}\underline{\boldsymbol{U}}$$

Isoparametric Elements, Jacobian Operator

- Definition of isoparametric 2D elements : ${}^{\langle e \rangle k}N(\xi, \eta)$

$$ {}^{\langle e \rangle}\underline{\bar{\boldsymbol{x}}}(\xi, \eta) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}(\xi, \eta) \cdot {}^{\langle e \rangle}\underline{\boldsymbol{X}} $$

(Interpolation of Geometry with displacement trial function)

- Discussion of Jacobian operator

$$ \underline{\underline{\boldsymbol{J}}} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} $$

Necessary when using natural (ξ, η) instead of real (x, y) coordinates

List of Questions

- What is a shape (interpolation) function (sketch)?
- What are the properties of shape functions?
- What is a trial function?
- How are trial and shape functions related?
- What is the difference between general (Ritz) and FEM trial function?
- What are natural coordinates?
- What are advantages/drawbacks of natural coordinates?
- What are isoparametric elements?
- What is the Jacobian operator?
- When do the Jacobian become singular?

Part VI

Governing FEM Equations

Overview Part VI

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- Derivation of Trial and Weighting Functions
- Derivation of Global Governing FEM Equations
- Discussion of Terms in Governing FEM Equation

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- Principle of Virtual Work, Governing Equation
- Example: 2D Element Stiffness Matrix

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- Pre-processing, Solution, Post-processing
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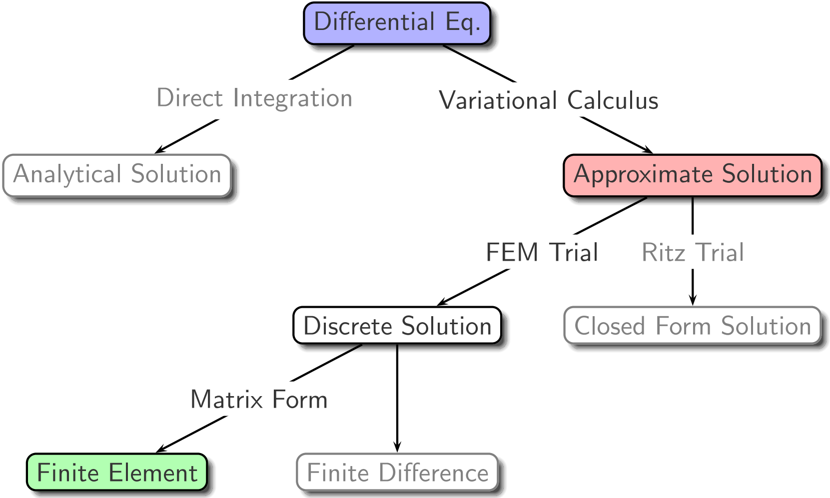
Solution Procedure - Matrix Formulation



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Element Trial Function

Review:

- Element trial function 1D (compare with Eq. (71))

$$^{\langle e\rangle}\bar{u}(\xi) = {}^{\langle e\rangle 1}N(\xi)\ {}^{\langle e\rangle 1}U + \cdots + {}^{\langle e\rangle n_{\text{kn}}}N(\xi)\ {}^{\langle e\rangle n_{\text{kn}}}U = \sum_{k=1}^{n_{\text{kn}}} {}^{\langle e\rangle k}N(\xi)\ {}^{\langle e\rangle k}U$$

with

- approximation within the element $^{\langle e\rangle}\bar{u}(\xi)$
- given shape function $^{\langle e\rangle k}N(\xi)$
- unknown nodal displacements $^{\langle e\rangle k}U$
- n_{kn} . . . number of nodes in element

Graphical Representation of Element Trial Function

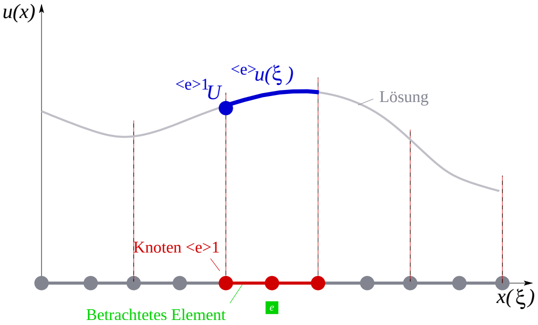


Figure: Element trial function (blue)

- element trial function is defined within the element
- global (overall) trial function = sum of element trial functions

Introduction of connectivity matrix

- **Note:** Assembly of Eq. (29) and (30) requires:
- Element stiffness matrices have to be placed on the right place in global stiffness matrix
- done by connectivity matrix (=transformation matrix)

$$^{\langle e \rangle}\underline{\boldsymbol{U}} = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{T}}} \cdot \underline{\boldsymbol{U}}$$

- $^{\langle e \rangle}\underline{\boldsymbol{U}}$ ... element nodal displacement
- $^{\langle e \rangle}\underline{\underline{\boldsymbol{T}}}$ ... connectivity matrix
- $\underline{\boldsymbol{U}}$ ... nodal point displacement of total assembly

Example: Connectivity Matrix

Example: Compute connectivity matrix for following problem:

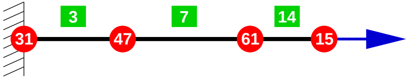


Figure: Simple model containing 3 linear 1D elements

for element 7 it holds:

$$^{\langle 7 \rangle}\underline{U} = \begin{pmatrix} ^{47}U \\ ^{61}U \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{^{\langle 7 \rangle}\underline{\underline{T}}} \begin{pmatrix} ^{31}U \\ ^{47}U \\ ^{61}U \\ ^{15}U \end{pmatrix}$$

Example: Connectivity Matrix

Computation of remaining matrices: element 3

$$^{\langle 3 \rangle}\underline{\underline{\boldsymbol{T}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and element 14

$$^{\langle 14 \rangle}\underline{\underline{\boldsymbol{T}}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to be continued . . .

Globale Shape function

- Shape function over whole domain follows from:

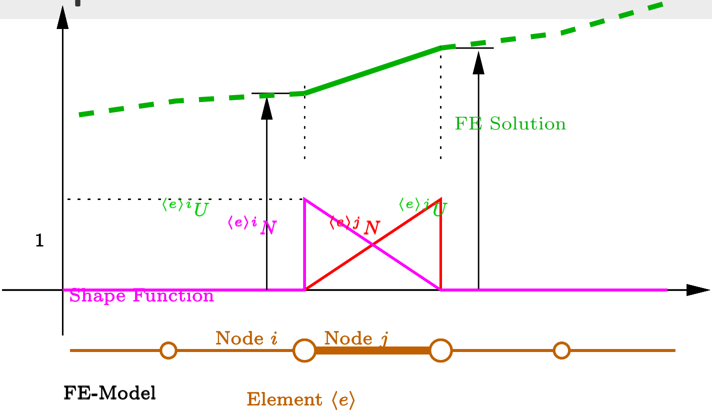
$$
\begin{aligned}
\bar{u} &= \sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}\; {}^{\langle e\rangle}\underline{\boldsymbol{U}} \\
&= \sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}\; {}^{\langle e\rangle}\underline{\underline{\boldsymbol{T}}} \cdot \underline{\boldsymbol{U}} \\
&= \left(\sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}\; {}^{\langle e\rangle}\underline{\underline{\boldsymbol{T}}} \right) \cdot \underline{\boldsymbol{U}} \\
\bar{u} &= \underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{U}}
\end{aligned}
$$

Global Trial Function

$$
\bar{u} = \sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}\; {}^{\langle e\rangle}\underline{\boldsymbol{U}} = \underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{U}} \tag{75}
$$

$\underline{\underline{\boldsymbol{N}}}$ . . . global shape function, $\underline{\boldsymbol{U}}$ nodal displacement

Sketch of Shape and Trial Function



Note:

- Shape function $^{\langle e \rangle i}N$, $^{\langle e \rangle j}N$ defined only within an element $\neq 0$
- nodal displacement (unknown) - scales shape function
- finally trial function is obtained ($=$ FE solution)

Example: Globale Shape Function

Example continuation: Compute $\underline{\underline{\boldsymbol{N}}} = \sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}\; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{T}}}$

Matrix of element shape function:

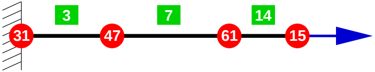
$$
\begin{aligned}
{}^{\langle 3 \rangle}\underline{\underline{\boldsymbol{N}}} &= \begin{pmatrix} {}^{\langle 3 \rangle 31}N & {}^{\langle 3 \rangle 47}N \end{pmatrix} \\
{}^{\langle 7 \rangle}\underline{\underline{\boldsymbol{N}}} &= \begin{pmatrix} {}^{\langle 7 \rangle 47}N & {}^{\langle 7 \rangle 61}N \end{pmatrix} \\
{}^{\langle 14 \rangle}\underline{\underline{\boldsymbol{N}}} &= \begin{pmatrix} {}^{\langle 14 \rangle 61}N & {}^{\langle 14 \rangle 15}N \end{pmatrix}
\end{aligned}
$$

(calculation of element shape function in Eq. (64)**)**

Example: Globale Shape Function

Assembly of global shape function:

$$
\begin{aligned}
\underline{N} &= {}^{\langle 3\rangle}\underline{\underline{N}}\;{}^{\langle 3\rangle}\underline{\underline{T}} + {}^{\langle 7\rangle}\underline{\underline{N}}\;{}^{\langle 7\rangle}\underline{\underline{T}} + {}^{\langle 14\rangle}\underline{\underline{N}}\;{}^{\langle 14\rangle}\underline{\underline{T}} = \\
&= \underbrace{\begin{pmatrix} {}^{\langle 3\rangle 31}N & {}^{\langle 3\rangle 47}N \end{pmatrix}}_{1\times 2} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{2\times 4} + \dots \\
&= \begin{pmatrix} {}^{\langle 3\rangle 31}N & {}^{\langle 3\rangle 47}N & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & {}^{\langle 7\rangle 47}N & {}^{\langle 7\rangle 61}N & 0 \end{pmatrix} + \dots \\
&\dots \begin{pmatrix} 0 & 0 & {}^{\langle 14\rangle 61}N & {}^{\langle 14\rangle 15}N \end{pmatrix} \\
&= \begin{pmatrix} {}^{\langle 3\rangle 31}N & {}^{\langle 3\rangle 47}N + {}^{\langle 7\rangle 47}N & {}^{\langle 14\rangle 61}N + {}^{\langle 7\rangle 61}N & {}^{\langle 14\rangle 15}N \end{pmatrix}
\end{aligned}
$$



Note: Element shape functions are summed in nodes!

Partition of Trial Function

Additional step:

- Partition of trial function in known and unknown parts

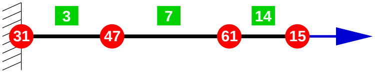
Partition of Trial Function

$$\underline{\bar{u}} = \underline{\underline{N}} \cdot \underline{U} = {}^{0}\underline{\underline{N}} \cdot {}^{0}\underline{U} + {}^{?}\underline{\underline{N}} \cdot {}^{?}\underline{U} \tag{76}$$

${}^{0}\underline{U}$ ... known (given) nodal displacements
${}^{?}\underline{U}$ ... unknown (calculated) nodal displacements

Example: Partition of Trial Function

Example continuation: Compute partition of trial function



- **known and unknown displacements:**

$$ {}^{0}\underline{U} = {}^{31}U \qquad {}^{?}\underline{U} = \begin{pmatrix} {}^{47}U \\ {}^{61}U \\ {}^{15}U \end{pmatrix} $$

- **global trial function follows as ($ {}^{0}\underline{\underline{N}} \cdot {}^{0}\underline{U} + {}^{?}\underline{\underline{N}} \cdot {}^{?}\underline{U} $):**

$$ \bar{u} = \underbrace{{}^{\langle 3\rangle 31}N \; {}^{31}U}_{\text{known}} + \underbrace{\left({}^{\langle 3\rangle 47}N + {}^{\langle 7\rangle 47}N \quad {}^{\langle 14\rangle 61}N + {}^{\langle 7\rangle 61}N \quad {}^{\langle 14\rangle 15}N \right)}_{{}^{?}\underline{\underline{N}}} \cdot \underbrace{\begin{pmatrix} {}^{47}U \\ {}^{61}U \\ {}^{15}U \end{pmatrix}}_{{}^{?}\underline{U}} $$

Matrix Dimension of Individual Terms

- Global Trial Function:

$$\underbrace{\underline{\bar{u}}}_{n_\mathrm{d}\times 1} = \sum_{e=1}^{n_\mathrm{el}} \underbrace{{}^{\langle e\rangle}\underline{\underline{N}}}_{n_\mathrm{d}\times n_\mathrm{ke}} \cdot \underbrace{{}^{\langle e\rangle}\underline{\underline{T}}}_{n_\mathrm{ke}\times n_\mathrm{kn}} \cdot \underbrace{\underline{U}}_{n_\mathrm{kn}\times 1} = \underbrace{\underline{\underline{N}}}_{n_\mathrm{d}\times n_\mathrm{kn}} \cdot \underbrace{\underline{U}}_{n_\mathrm{kn}\times 1} =$$

$$= \underbrace{{}^{0}\underline{\underline{N}}}_{n_\mathrm{d}\times n_\mathrm{k0}} \cdot \underbrace{{}^{0}\underline{U}}_{n_\mathrm{k0}\times 1} + \underbrace{{}^{?}\underline{\underline{N}}}_{n_\mathrm{d}\times n_\mathrm{k?}} \cdot \underbrace{{}^{?}\underline{U}}_{n_\mathrm{k?}\times 1}$$

with

- n_d ... dimensionality of the problem (1,2,3)
- n_ke ... number of degrees of freedom (DOF) within an element
- n_k0 ... number of known DOFs of total assemblage
- $n_\mathrm{k?}$ ... number of unknown DOFs of total assemblage
- n_kn ... total number of DOFs $n_\mathrm{kn} = n_\mathrm{k0} + n_\mathrm{k?}$

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Derivation of Trial Function 1D

For Eq. (48) (weak form) derivatives of displacements are needed

- derivatives of displacements:

$$\frac{d\bar{u}}{dx} = \frac{d\,{}^{0}\underline{\underline{\boldsymbol{N}}}}{dx} \cdot {}^{0}\underline{\boldsymbol{U}} + \frac{d\,{}^{?}\underline{\underline{\boldsymbol{N}}}}{dx} \cdot {}^{?}\underline{\boldsymbol{U}}$$

- definition of derivatives of shape functions ${}^{0}\underline{\underline{\boldsymbol{B}}}$, ${}^{?}\underline{\underline{\boldsymbol{B}}}$ gives:

$$\frac{d\bar{u}}{dx} = {}^{0}\underline{\underline{\boldsymbol{B}}} \cdot {}^{0}\underline{\boldsymbol{U}} + {}^{?}\underline{\underline{\boldsymbol{B}}} \cdot {}^{?}\underline{\boldsymbol{U}}$$

Remark: A similar Eq. follows for the 3D case $\bar{u} \rightarrow \underline{\bar{\boldsymbol{u}}}$ (see later)

Derivation of Weighting Function 1D

- with weighting function similar to Eq. (51) (Galerkin)

$$w = {}^{?}\underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{d}} = \underline{\boldsymbol{d}}^{\mathrm{T}} \cdot {}^{?}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}$$

$\underline{\boldsymbol{d}}$ ... arbitrary coefficients

- follows derivation of weighting function (1D)

$$\frac{dw}{dx} = \underline{\boldsymbol{d}}^{\mathrm{T}} \cdot {}^{?}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}$$

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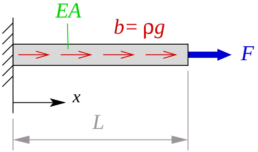
Review: Weak form of weighted residual method

Example: Bar with body and tip load:



- Starting point is weak form of weighted residual method (Eq. (48)):

$$E\,A \int_0^L \frac{dw}{dx}\,\frac{d\bar{u}}{dx} dx = A \int_0^L w\,b\,dx + w(L)\,F$$

Governing FEM Equations 1D

- inserting of trial function, weighting function and their derivatives

$$E\,A \int_0^L \underline{d}^{\mathrm{T}} \cdot {}^{?}\underline{\underline{B}}^{\mathrm{T}} \left({}^{0}B\ {}^{0}U + {}^{?}\underline{\underline{B}} \cdot {}^{?}\underline{U} \right) dx = A \int_0^L \underline{d}^{\mathrm{T}} \cdot {}^{?}\underline{\underline{N}}^{\mathrm{T}}\, b\, dx + \underline{d}^{\mathrm{T}} \cdot {}^{?}\underline{\underline{N}}(L)^{\mathrm{T}}\, F$$

- rearrange with respect to $\underline{d}^{\mathrm{T}}$ gives

$$\underline{d}^{\mathrm{T}} \cdot \underbrace{\left[E\,A \int_0^L {}^{?}\underline{\underline{B}}^{\mathrm{T}} \left({}^{0}B\ {}^{0}U + {}^{?}\underline{\underline{B}} \cdot {}^{?}\underline{U} \right) dx - A \int_0^L {}^{?}\underline{\underline{N}}^{\mathrm{T}}\, b\, dx - {}^{?}\underline{\underline{N}}(L)^{\mathrm{T}}\, F \right]}_{=0 \text{ da } \underline{d}^{\mathrm{T}} \text{ arbitrary}} = 0$$

- **Note:** Here one see why $\underline{d}^{\mathrm{T}}$ is an arbitrary constant!

Governing FEM Equations 1D

- further rearrangement with respect to nodal displacements

Governing FEM Equation 1D

$$\underbrace{\left(E\,A \int_0^L {}^{?}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\;{}^{?}\underline{\underline{\boldsymbol{B}}}\,dx\right)}_{\underline{\underline{\boldsymbol{K}}}}\;{}^{?}\underline{\boldsymbol{U}} = \underbrace{A \int_0^L {}^{?}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\, b\, dx + {}^{?}\underline{\underline{\boldsymbol{N}}}(L)^{\mathrm{T}}\, F - E\,A \int_0^L {}^{?}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\;{}^{0}B\;{}^{0}U\, dx}_{\underline{\boldsymbol{F}}} \tag{77}$$

- it follows the well known equation:

$$\underline{\underline{\boldsymbol{K}}} \cdot \underline{\boldsymbol{U}} = \underline{\boldsymbol{F}}$$

Alternative Form of Governing FEM Equation

An alternative equation is obtained if:

- global trial function is used in form of:

$$\bar{u} = \left(\sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}\ {}^{\langle e\rangle}\underline{\underline{\boldsymbol{T}}}\right)\cdot\underline{\boldsymbol{U}} = \left(\sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle ?}\underline{\underline{\boldsymbol{N}}}\ {}^{\langle e\rangle ?}\underline{\underline{\boldsymbol{T}}}\right)\cdot {}^{?}\underline{\boldsymbol{U}} + \left(\sum_{e=1}^{n_{\text{el}}} {}^{\langle e\rangle 0}\underline{\underline{\boldsymbol{N}}}\ {}^{\langle e\rangle 0}\underline{\underline{\boldsymbol{T}}}\right)\cdot {}^{0}\underline{\boldsymbol{U}}$$

- with ${}^{\langle e\rangle 0}\underline{\underline{\boldsymbol{T}}}$ and ${}^{\langle e\rangle ?}\underline{\underline{\boldsymbol{T}}}$ as new transformation matrices
- finally the sum and integration in Eq. (77) can be interchanged i.e.

$$\int \sum \cdots = \sum \int \ldots$$

because of

$$\int \sum_{i=1}^{2} a_i = \int (a_1 + a_2) = \int a_1 + \int a_2 = \sum_{i=1}^{2} \int a_i$$

Alternative Form of Governing FEM Equation

It follows (details are in Appendix)

Governing FEM Equation (simplified)

$$\overset{\text{Assemble}}{\underset{n_{\text{el}}}{\sum}} \underbrace{\left\{ E\,A \int_0^{L_e} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx \right\}}_{\text{element stiffness matrix}} {}^{?}\underline{\boldsymbol{U}} = \overset{\text{Assemble}}{\underset{n_{\text{el}}}{\sum}} \underbrace{\left\{ A \int_0^{L_e} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\, b\, dx \right\}}_{\text{volume force}} + \ldots$$

$$\ldots \overset{\text{Assemble}}{\underset{n_{\text{el}}}{\sum}} \underbrace{\left\{ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}(L)^{\mathrm{T}}\, F \right\}}_{\text{surface force}} - \overset{\text{Assemble}}{\underset{n_{\text{el}}}{\sum}} \underbrace{\left\{ E\,A \int_0^{L_e} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx\, {}^{0}U \right\}}_{\text{force from known displacement}} \qquad (78)$$

with the assembly function (see Appendix)

$$\overset{\text{Assemble}}{\underset{n_{\text{el}}}{\sum}} \left\{ \ldots \right\}$$

Remarks: Alternative Form of Governing FEM Equation

- **Assembly:** with assembly function $\sum_{n_{\rm el}}^{\text{Assemble}} \left\{ \ldots \right\}$ (put element stiffness matrices on right place in global matrix (compare with Eq. (29) or (30))
- **Governing FEM Equation:** finite number of unknowns $(= n_{\rm k?})$
- **Unkowns:** not given DOF $^{?}\underline{\boldsymbol{U}}$
- **Integration** of terms at element level $\int_0^{L_{\rm e}} \ldots$

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Element Stiffness Matrix

Element stiffness matrix from Eq. (78)

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \int_0^{L_\mathrm{e}} \left(^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^\mathrm{T} \; ^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx$$

the shape function (see Eq. (61)) reads as:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}} = \begin{pmatrix} \frac{^jX - x}{^jX - {}^iX} & \frac{x - {}^iX}{^jX - {}^iX} \end{pmatrix}$$

derivations of this function are:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \frac{-1}{L_\mathrm{e}} & \frac{1}{L_\mathrm{e}} \end{pmatrix} \qquad \text{with} \qquad L_\mathrm{e} = {}^jX - {}^iX$$

inserting in element stiffness matrix $^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$ gives:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \int_0^{L_\mathrm{e}} \begin{pmatrix} \frac{-1}{L_\mathrm{e}} \\ \frac{1}{L_\mathrm{e}} \end{pmatrix} \cdot \begin{pmatrix} \frac{-1}{L_\mathrm{e}} & \frac{1}{L_\mathrm{e}} \end{pmatrix} dx$$

Element Stiffness Matrix

simplified

$$^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \int_0^{L_\mathrm{e}} \begin{pmatrix} \frac{1}{L_\mathrm{e}^2} & -\frac{1}{L_\mathrm{e}^2} \\ -\frac{1}{L_\mathrm{e}^2} & \frac{1}{L_\mathrm{e}^2} \end{pmatrix} dx = \frac{E\,A}{L_\mathrm{e}^2} \int_0^{L_\mathrm{e}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} dx$$

and integrated

$$^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}} = \frac{E\,A}{L_\mathrm{e}^2} \begin{pmatrix} L_\mathrm{e} & -L_\mathrm{e} \\ -L_\mathrm{e} & L_\mathrm{e} \end{pmatrix} = \frac{E\,A}{L_\mathrm{e}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

yields our old friend (cp. Eq. (28))

Consistent Nodal Forces

Volume forces: (see Eq. (78))

$$
{}^{\mathrm{b}}\underline{\boldsymbol{F}} = A \int_0^{L_\mathrm{e}} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \, b \, dx
$$

Inserting the shape function

$$
{}^{\mathrm{b}}\underline{\boldsymbol{F}} = A \int_0^{L_\mathrm{e}} \begin{pmatrix} \frac{{}^{j}X - x}{{}^{j}X - {}^{i}X} \\ \frac{x - {}^{i}X}{{}^{j}X - {}^{i}X} \end{pmatrix} b \, dx = A \int_0^{L_\mathrm{e}} \frac{b}{L_\mathrm{e}} \begin{pmatrix} {}^{j}X - x \\ x - {}^{i}X \end{pmatrix} dx
$$

and integration gives:

$$
{}^{\mathrm{b}}\underline{\boldsymbol{F}} = A \, b \, L_\mathrm{e} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
$$

- i.e. a constant volume force is distributed at two nodes
- **Note:** consistent nodal forces give same virtual work as real force

Sketch Consistent Nodal Forces

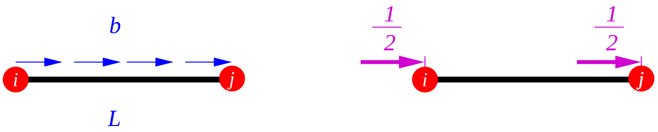

Figure: Consistent nodal force (right) for a 2 noded bar element with constant volume force (left). $A, b, L = 1$

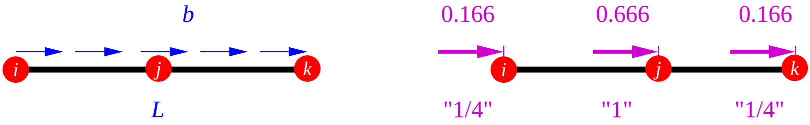

Figure: Consistent nodal force (right) for a 3 noded bar element with constant volume force (left). $A, b, L = 1$

Consistent Nodal Forces: Example

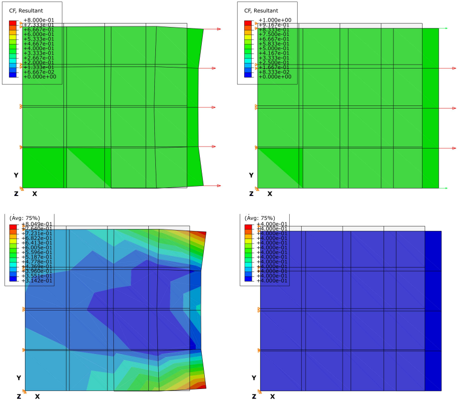

Figure: Nodal forces (top) and von Mises stress (bottom). Averaged (left) and consistent (right).

Surface Force

Surface force: (see Eq. (78))

$$ {}^{\mathrm{O}}\underline{\boldsymbol{F}} = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\, F $$

- Note: ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}$ is here a vector!
- in 1D only a single force
- higher dimensions - similar to volume force

Force from Given Displacement

Force from given displacement: (see Eq. (78))

$$ {}^{\mathrm{U}}\underline{\boldsymbol{F}} = E\,A \int_0^{L_{\mathrm{e}}} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx\, {}^{0}U $$

- Same as element stiffness matrix, same treatment

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Overview

- usage of Principle of Virtual Work (see Appendix)
- (allows transition from 1D to 2D/3D equations)
- application of the same formalism as in previous chapter (cp. Eq. (78))
- gives matrix form equation

Matrix Form Equation of Static Equilibrium

Governing FEM Equation for Statics

$$\overset{\text{Assemble}}{\underset{n_{\mathrm{el}}}{\sum}} \underbrace{\left\{ \int_{\langle e \rangle V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \, dV \right\}}_{\text{element stiffness matrix } {}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}} \underline{\boldsymbol{U}} = \overset{\text{Assemble}}{\underset{n_{\mathrm{el}}}{\sum}} \underbrace{\left\{ \int_{\langle e \rangle V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{b}} \, dV \right\}}_{\text{consistent volume forces} F_{\mathrm{V}}} + \dots$$

$$\dots + \overset{\text{Assemble}}{\underset{n_{\mathrm{el}}}{\sum}} \underbrace{\left\{ \int_{\langle e \rangle S} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{t}} \, dS \right\}}_{\text{consistent surface forces} F_{\mathrm{O}}} + \underbrace{\underline{\boldsymbol{F}}_{\mathrm{K}}}_{\text{nodal forces}} \tag{79}$$

or simplified:

Governing FEM Equation for Statics

$$\underline{\underline{\boldsymbol{K}}} \, \underline{\boldsymbol{U}} = \underline{\boldsymbol{F}}_{\mathrm{V}} + \underline{\boldsymbol{F}}_{\mathrm{O}} + \underline{\boldsymbol{F}}_{\mathrm{K}}$$

Matrix Form Equation of Static Equilibrium - Remarks

- external forces are: volume force vector ${}^{\langle e\rangle}\underline{\boldsymbol{b}}$, surface force vector ${}^{\langle e\rangle}\underline{\boldsymbol{t}}$, and nodal force vector $\underline{\boldsymbol{F}}_{\mathrm{K}}$
- integration is applied at element level (${}^{\langle e\rangle}V$, ${}^{\langle e\rangle}S$)
- known and unknown displacements on the right hand side ($\bar{\underline{\boldsymbol{u}}} = \underline{\underline{\boldsymbol{N}}}\,\underline{\boldsymbol{U}} = {}^{0}\underline{\underline{\boldsymbol{N}}}\,{}^{0}\underline{\boldsymbol{U}} + {}^{?}\underline{\underline{\boldsymbol{N}}}\,{}^{?}\underline{\boldsymbol{U}}$), in contrast to Eq. (78)
- known nodal force $\underline{\boldsymbol{F}}_{\mathrm{K}}$ are inserted directly

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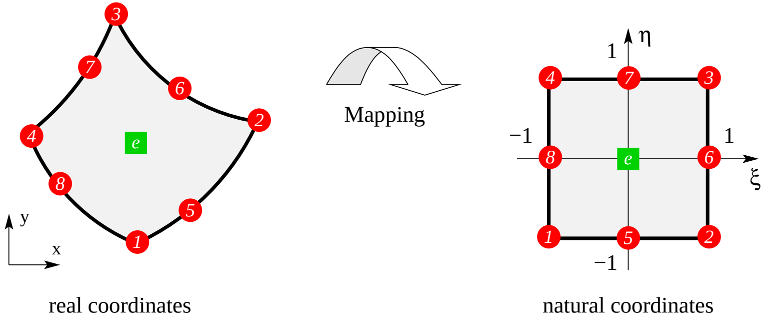
Element Stiffness Matrix 8-noded 2D Element

Element (see Fig. 20):



element stiffness matrix for this element:

$$ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = \int_{{}^{\langle e \rangle}V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \, dV $$

Analysis Steps

Following steps are necessary:

- usage of element trial function

$$^{\langle e \rangle}\underline{\bar{\boldsymbol{u}}} = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}\ {}^{\langle e \rangle}\underline{\boldsymbol{U}}$$

- computation of element strains

$$^{\langle e \rangle}\underline{\boldsymbol{\varepsilon}} = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}\ {}^{\langle e \rangle}\underline{\boldsymbol{U}}$$

- computation of terms of ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}$ matrix
- utilization of material stiffness matrix ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}}$ (with $\underline{\boldsymbol{\sigma}} = \underline{\underline{\boldsymbol{C}}} \cdot \underline{\boldsymbol{\varepsilon}}$)
- computation of element stiffness matrix ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$

Element Trial Function

trial functions are summarized in section 13:

$$\underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{^{\langle e \rangle}\underline{\bar{\boldsymbol{u}}}} = \underbrace{\begin{pmatrix} ^1N & 0 & ^2N & \dots & ^8N & 0 \\ 0 & ^1N & 0 & ^2N & \dots & ^8N \end{pmatrix}}_{^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}} \cdot \underbrace{\begin{pmatrix} ^1U_1 \\ ^1U_2 \\ ^2U_1 \\ ^2U_2 \\ \vdots \\ ^8U_1 \\ ^8U_2 \end{pmatrix}}_{^{\langle e \rangle}\underline{\boldsymbol{U}}}$$

with shape functions:

$$\begin{aligned} ^1N(\xi,\eta) &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1) \\ &\dots \\ ^8N(\xi,\eta) &= \frac{1}{2}(1-\eta^2)(1-\xi) \end{aligned}$$

Element Strains

displacement field $\bar{u}(\xi,\eta)$ within element reads:

$$\begin{aligned}\bar{u}_1 &= {}^1N\ {}^1U_1 + {}^2N\ {}^2U_1 + \cdots + {}^8N\ {}^8U_1 \\ \bar{u}_2 &= {}^1N\ {}^1U_2 + {}^2N\ {}^2U_2 + \cdots + {}^8N\ {}^8U_2\end{aligned}$$

which gives the strain ($x_1 = x$, $x_2 = y$):

$$\varepsilon_{11} = \frac{\partial \bar{u}_1}{\partial x} = \frac{\partial\,{}^1N}{\partial x}\ {}^1U_1 + \frac{\partial\,{}^2N}{\partial x}\ {}^2U_1 + \cdots + \frac{\partial\,{}^8N}{\partial x}\ {}^8U_1$$

analog for ε_{22}

$$\varepsilon_{22} = \frac{\partial \bar{u}_2}{\partial y} = \frac{\partial\,{}^1N}{\partial y}\ {}^1U_2 + \frac{\partial\,{}^2N}{\partial y}\ {}^2U_2 + \cdots + \frac{\partial\,{}^8N}{\partial y}\ {}^8U_2$$

and shear strain γ_{12}

$$\gamma_{12} = \frac{\partial \bar{u}_1}{\partial y} + \frac{\partial \bar{u}_2}{\partial x} = \frac{\partial\,{}^1N}{\partial y}\ {}^1U_1 + \frac{\partial\,{}^1N}{\partial x}\ {}^1U_2 + \cdots + \frac{\partial\,{}^2N}{\partial y}\ {}^8U_1 + \frac{\partial\,{}^8N}{\partial x}\ {}^8U_2$$

Computation of ${}^{\langle e \rangle}\underline{\underline{B}}$ Matrix

Rewriting these three equations in matrix form yields:

$$\underbrace{\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}}_{{}^{\langle e \rangle}\underline{\varepsilon}} = \underbrace{\begin{pmatrix} \frac{\partial\,{}^{1}N}{\partial x} & 0 & \frac{\partial\,{}^{2}N}{\partial x} & 0 & \dots & \frac{\partial\,{}^{8}N}{\partial x} & 0 \\ 0 & \frac{\partial\,{}^{1}N}{\partial y} & 0 & \frac{\partial\,{}^{2}N}{\partial y} & \dots & 0 & \frac{\partial\,{}^{8}N}{\partial y} \\ \frac{\partial\,{}^{1}N}{\partial y} & \frac{\partial\,{}^{1}N}{\partial x} & \frac{\partial\,{}^{2}N}{\partial y} & \frac{\partial\,{}^{2}N}{\partial x} & \dots & \frac{\partial\,{}^{8}N}{\partial y} & \frac{\partial\,{}^{8}N}{\partial x} \end{pmatrix}}_{{}^{\langle e \rangle}\underline{\underline{B}}} \cdot \underbrace{\begin{pmatrix} {}^{1}U_1 \\ {}^{1}U_2 \\ {}^{2}U_1 \\ {}^{2}U_2 \\ \vdots \\ {}^{8}U_1 \\ {}^{8}U_2 \end{pmatrix}}_{{}^{\langle e \rangle}\underline{U}}$$

it follows the strain and the unknown ${}^{\langle e \rangle}\underline{\underline{B}}$ matrix:

$${}^{\langle e \rangle}\underline{\varepsilon} = {}^{\langle e \rangle}\underline{\underline{B}}\ {}^{\langle e \rangle}\underline{U}$$

Remark: Equation could be directly derived from:

$${}^{\langle e \rangle}\underline{\varepsilon} = \left[\underline{\underline{L}}\ {}^{\langle e \rangle}\underline{\underline{N}}\right]\ {}^{\langle e \rangle}\underline{U}$$

Usage of Material Stiffness Matrix ${}^{\langle e \rangle}\underline{\underline{C}}$

material stiffness matrix ${}^{\langle e \rangle}\underline{\underline{C}}$ follows from material law:

$$\underbrace{\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}}_{\underline{\sigma}} \underbrace{= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{pmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\,\nu)}{2} \end{pmatrix}}_{\underline{\underline{C}}} \cdot \underbrace{\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}}_{\underline{\varepsilon}}$$

which can be evaluated if E, ν are known

Computation of Element Stiffness Matrix ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}$

Inserting of previous equations gives element stiffness matrix:

$$
{}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = \int_{{}^{\langle e \rangle}V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \, dV
$$

Practical relevance of this example:

- **element type:** have to be provided (here quadratic)
- **derivatives of displacements** are needed ⇒ constant shape functions not allowed (at least linear)
- **Jacobi matrix:** Inverse have to be calculated ⇒ interpret errors correctly
- **material stiffness matrix:** material parameter necessary

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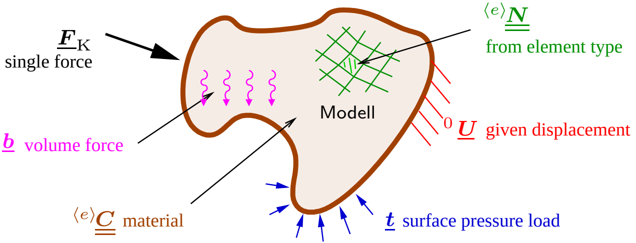
Model Generation

(1) Pre-Processing:

- conceptual model (simplifications, geometry, ...)
- material definition (${}^{\langle e \rangle}\underline{\underline{C}}$)
- FE-mesh generation (elements n_{el}, element type ${}^{\langle e \rangle}\underline{\underline{N}}$)
- application of displacement BCs (${}^{0}\underline{U}$)
- application of force BCs ($\underline{t}$, $\underline{b}$, $\underline{F}_{\text{K}}$)



Computation of Unkown Nodal Displacements

(2) Solving: Displacements

- computation of element stiffness matrices ${}^{\langle e \rangle}\underline{\underline{K}}$
- computation of global stiffness matrices $\underline{\underline{K}}$ (assembly)
- computation of consistent nodal forces ${}^{0}\underline{F}$
- splitting of system of equations $\underline{F} = \underline{\underline{K}} \cdot \underline{U}$ in:

$$\begin{pmatrix} {}^{0}\underline{F} \\ {}^{?}\underline{F} \end{pmatrix} = \begin{pmatrix} {}^{0?}\underline{\underline{K}} & {}^{00}\underline{\underline{K}} \\ {}^{??}\underline{\underline{K}} & {}^{?0}\underline{\underline{K}} \end{pmatrix} \cdot \begin{pmatrix} {}^{?}\underline{U} \\ {}^{0}\underline{U} \end{pmatrix}$$

  known ${}^{0}\underline{U}$ and unknown ${}^{?}\underline{U}$ nodal displacements
- computation of unknown nodal displacements ${}^{?}\underline{U}$ from:

$${}^{0?}\underline{\underline{K}} \cdot {}^{?}\underline{U} = {}^{0}\underline{F} - {}^{00}\underline{\underline{K}} \cdot {}^{0}\underline{U}$$

Support Forces, Stresses, and Strains

(3) Solving: Additional Results

- computation of support forces (nodes where ${}^{0}\underline{U}$ given)

$$ {}^{?}\underline{F} = {}^{??}\underline{\underline{K}} \cdot {}^{?}\underline{U} + {}^{?0}\underline{\underline{K}} \cdot {}^{0}\underline{U} $$

- computation of strains

$$ {}^{\langle e\rangle}\underline{\varepsilon} = {}^{\langle e\rangle}\underline{\underline{B}} \; {}^{\langle e\rangle}\underline{U} $$

- computation of stresses

$$ {}^{\langle e\rangle}\underline{\sigma} = {}^{\langle e\rangle}\underline{\underline{C}} \; {}^{\langle e\rangle}\underline{\varepsilon} $$

Graphical Interpretations

(4) Post-Processing: Ploting Analysis Results

- deformation (true or scaled)
- stresses (von Mises, principle stresses, . . .)
- further results: line plots, vector plots, etc.

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Model, Geometry, and Material Definitions

Connecting lug consists of:

- steel lug (orange) and brass bearing bush (yellow)
- linear elastic material

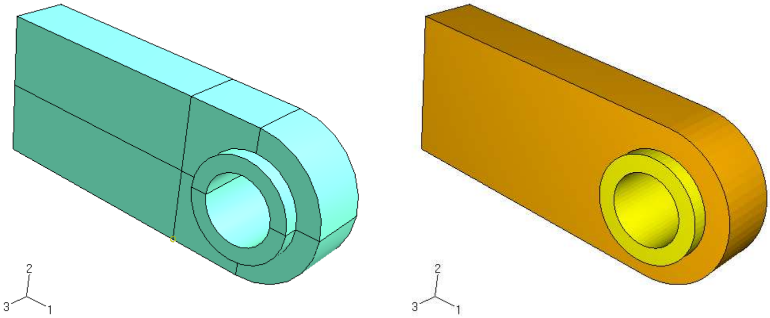

Figure: Geometry (left) and material definition (right)

Loading and Finite Element Mesh

Connecting lug is:

- clamped at rear end, pressure load in bush (see Fig. 348, left)
- mesh consists of hexahedral elements

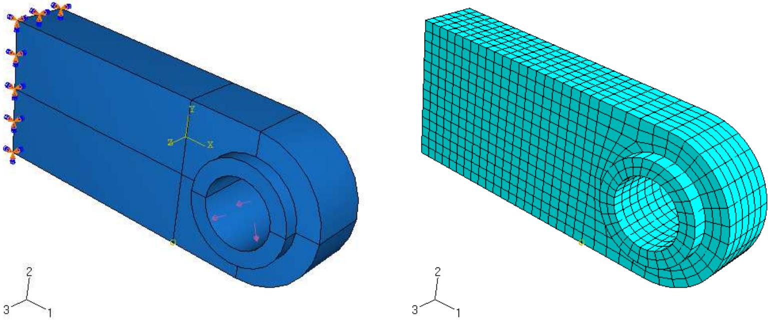

Figure: Load definitions (left) and finite element mesh (right)

Results: Deformations and Stresses

- linear static analysis
- displacement magnification = $40\times$

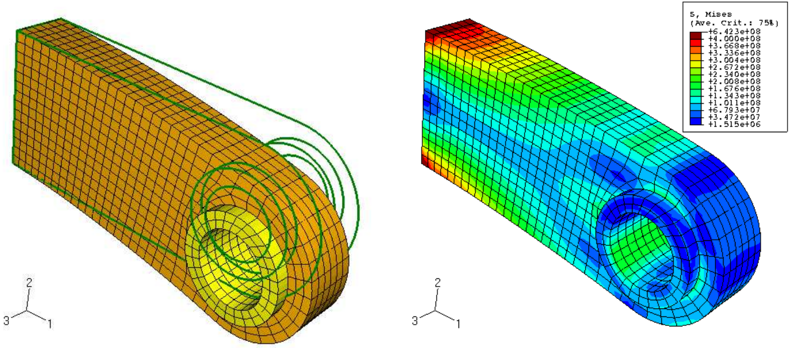

Figure: Deformation (left) and von Mises stress (right)

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Global Trial Function, Equilibrium

Introduction of global trial function (Eq. (75))

$$\underline{\bar{u}} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e \rangle}\underline{\underline{N}}\ {}^{\langle e \rangle}\underline{U} = \underline{\underline{N}} \cdot \underline{U}$$

- $\underline{\bar{u}}$ . . . approximated solution = displacement
- $\underline{\underline{N}}$ . . . global shape functions = known
- $\underline{U}$ . . . nodal displacements (DOFs) = unknown

Derivation of static equilibrium (3D)

$$\underline{\underline{L}}^T \underline{\sigma} + \underline{b} = \underline{0} \qquad \longleftrightarrow \qquad \sigma_{ij,j}(\boldsymbol{x}) + b_i(\boldsymbol{x}) = 0 \qquad i = 1, 2, 3$$

- $\underline{\sigma}$ . . . vector of stress components = unknown
- $\underline{\underline{L}}$ . . . differential operator matrix ($\partial/\partial x_i$)
- $\underline{b}$ . . . volume force vector (e.g. gravity) = known

Material Law, Governing FEM Equation

Introduction of material law (isothermal) using displacements

$$\underline{\boldsymbol{\sigma}} = \underline{\underline{\boldsymbol{C}}}\,\underline{\boldsymbol{\varepsilon}} = \underline{\underline{\boldsymbol{C}}}\,\underline{\underline{\boldsymbol{L}}}\,\underline{\bar{\boldsymbol{u}}} = \underline{\underline{\boldsymbol{C}}}\,\underline{\underline{\boldsymbol{L}}}\left[\underline{\underline{\boldsymbol{N}}}\,\underline{\boldsymbol{U}}\right]$$

- $\underline{\underline{\boldsymbol{C}}}$ . . . material stiffness matrix
- $\underline{\boldsymbol{\varepsilon}}$ . . . vector of strain components

Finally governing FEM equation

$$\overset{\text{Assemble}}{\sum_{n_{\text{el}}}} \underbrace{\left\{\int_{\langle e\rangle V} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\;{}^{\langle e\rangle}\underline{\underline{\boldsymbol{C}}}\;{}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}}\,dV\right\}}_{\text{element stiffness matrix } {}^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}}} \underline{\boldsymbol{U}} = \dots$$

$$\dots \overset{\text{Assemble}}{\sum_{n_{\text{el}}}} \underbrace{\left\{\int_{\langle e\rangle V} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\;{}^{\langle e\rangle}\underline{\boldsymbol{b}}\,dV\right\}}_{\text{consistent volume forces}} + \overset{\text{Assemble}}{\sum_{n_{\text{el}}}} \underbrace{\left\{\int_{\langle e\rangle S} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\;{}^{\langle e\rangle}\underline{\boldsymbol{t}}\,dS\right\}}_{\text{consistent surface forces}} + \underline{\boldsymbol{F}}_{\mathrm{K}}$$

List of Questions

- **What is the need of a connectivity matrix?**
- **What is the difference between weak form of static equilibrium and governing FEM equation?**
- **What is the difference between element and global trial function?**
- **Look at the governing FEM equation of the bar problem. What is the meaning of the individual terms?**

List of Questions

- **Look at the governing FEM equation in 2D and try to find out which parameters are needed to compute the individual terms.**
- **What steps are performed in a linear static FEA (use equations)?**

Part VII

Numerical Integration

Overview Part VII

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Integration of Matrix Form FE Equations

- equations contain integrals:
  - $^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = \int_{^{\langle e \rangle}V} {^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}} \; {^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}}} \; {^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}} \, dV$ ... element stiffness matrix
  - $^{\langle e \rangle \mathrm{b}}\underline{\boldsymbol{F}} = \int_{^{\langle e \rangle}V} {^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}} \; {^{\langle e \rangle}\underline{\boldsymbol{b}}} \, dV$ ... consistent volume force vector
  - $^{\langle e \rangle \mathrm{O}}\underline{\boldsymbol{F}} = \int_{^{\langle e \rangle}S} {^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}} \; {^{\langle e \rangle}\underline{\boldsymbol{t}}} \, dS$ ... consistent surface force vector
- integrals are in real space ($dV = dx\,dy\,dz$)
- but: shape function in natural coordinates ξ, η, ζ
- mathematically it holds in case of coordinate transforms (1D):

$$\int_{x_a}^{x_b} f(x)dx = \int_{\xi_a}^{\xi_b} f\Big(x(\xi)\Big) \left[\frac{d\Big(x(\xi)\Big)}{d\xi}\right] d\xi$$

Example: Integration of Element Stiffness Matrix

Integration of element stiffness matrix of 2-noded 1D element:

$$^{\langle e\rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \int_0^{L_\mathrm{e}} \left({}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}}^\mathrm{T}\ {}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}} \right) dx = E\,A \int_{-1}^{+1} {}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}}(\xi)^\mathrm{T}\ {}^{\langle e\rangle}\underline{\underline{\boldsymbol{B}}}(\xi)\, \frac{dx(\xi)}{d\xi} d\xi$$

using interpolation of x (isoparametric element)

$$x(\xi) = {}^1N(\xi)\ {}^1X + {}^2N(\xi)\ {}^2X$$

gives the derivative (= Jacobi matrix (73))

$$\begin{aligned}
\frac{dx(\xi)}{d\xi} &= \frac{d}{d\xi}\Big(\underbrace{\frac{1}{2}(1-\xi)}_{{}^1N(\xi)}\Big)\ {}^1X + \frac{d}{d\xi}\Big(\underbrace{\frac{1}{2}(1+\xi)}_{{}^2N(\xi)}\Big)\ {}^2X \\
&= \left(\frac{1}{2}(-1)\right)\ {}^1X + \left(\frac{1}{2}(1)\right)\ {}^2X \\
&= \frac{L_\mathrm{e}}{2}
\end{aligned}$$

Example: Integration of Element Stiffness Matrix

Derivatives of shape function follow from:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}} = \begin{pmatrix} {}^{1}N & {}^{2}N \end{pmatrix} \Longrightarrow {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \frac{d\,{}^{1}N}{dx} & \frac{d\,{}^{2}N}{dx} \end{pmatrix}$$

it holds for e.g. ${}^{1}N$

$$\begin{aligned} \frac{d\,{}^{1}N(\xi)}{d\xi} &= \frac{d\,{}^{1}N(\xi)}{dx}\frac{dx}{d\xi} \\ \left(\frac{1}{2}(-1)\right) &= \frac{d\,{}^{1}N(\xi)}{dx}\frac{L_{\mathrm{e}}}{2} \end{aligned}$$

which gives

$$\frac{d\,{}^{1}N(\xi)}{dx} = \left(-\frac{1}{L_{\mathrm{e}}}\right)$$

Example: Integration of Element Stiffness Matrix

The first term of element stiffness matrix follows from:

$$E\,A \int_{-1}^{+1} {}^{\langle e \rangle 1}B(\xi)\; {}^{\langle e \rangle 1}B(\xi)\, \frac{dx(\xi)}{d\xi} d\xi = E\,A \int_{-1}^{+1} \Big(-\frac{1}{L_{\mathrm{e}}}\Big)\Big(-\frac{1}{L_{\mathrm{e}}}\Big) \frac{L_{\mathrm{e}}}{2} d\xi = \frac{E\,A}{L_{\mathrm{e}}}$$

similar result as before!

Noteable points:

- integral is solved analytically
- **Question:** How can complex integral be solved in general?
- **Answer:** Numerical integration.

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Introduction

- Suggestion for the integration of a function:

$$\int_{-1}^{+1} f(\xi)\, d\xi \approx \sum_{i=1}^{{}^{\mathrm{IP}}n} {}^{i}w\, f({}^{i}\xi)$$

with

- ${}^{\mathrm{IP}}n$ ... number of integration points (IP) = Gauss points
- ${}^{i}w$ ... integration weight IP i (given)
- ${}^{i}\xi$ ... location of IP i in natural coordinates (given)
- $f({}^{i}\xi)$ ... function value at IP i

- integral is replaced by a sum
- in general: Sum is not exact
- if specific IP and weights are chosen → integral can be integrated exactly

Numerical Integration of a Linear Function

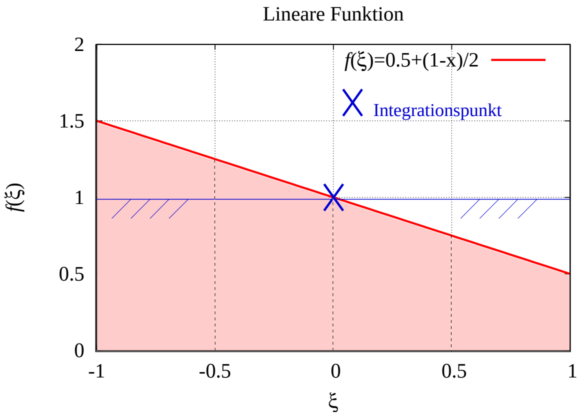

Figure: Numerical integration of a linear function

Numerical Integration of a Linear Function

Function can be evaluated exactly using:

$$\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{i=1}^{1} {}^{i}w\, f({}^{i}\xi) = 2\, f(0)$$

Remarks:

- one integration point
- evaluation of function at ${}^{1}\xi = 0$
- integration weight ${}^{1}w = 2$ ("support")

Numerical Integration of a Quadratic Function

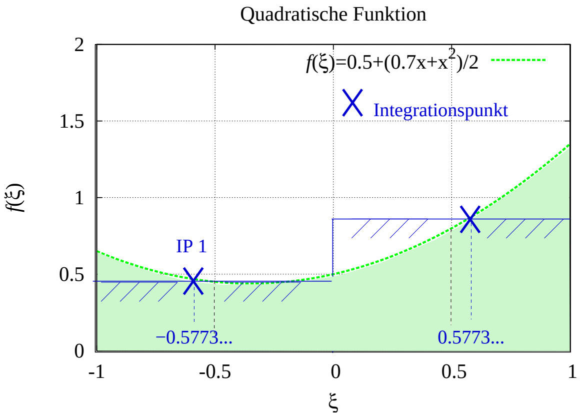

Figure: Numerical integration of a quadratic function

Numerical Integration of a Quadratic Function

Function can be evaluated exactly using:

$$\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{i=1}^{2} {}^{i}w\, f({}^{i}\xi) = 1\, f(-\frac{1}{\sqrt{3}}) + 1\, f(\frac{1}{\sqrt{3}})$$

Remarks:

- two integration points
- evaluation of function at ${}^{1}\xi = -\frac{1}{\sqrt{3}}$, ${}^{2}\xi = \frac{1}{\sqrt{3}}$
- integration weight ${}^{1}w = {}^{2}w = 1$

Higher Order Gauss-Legendre Integration

Question: Order of polynom (N) known. How many integration points ${}^{\mathrm{IP}}n$ are needed to integrate the function exactly?

Answer: In general it holds:

Number of Integration Points for exact Integration of Polynom

$$N = 2\,{}^{\mathrm{IP}}n - 1 \tag{80}$$

Location and weights of integration points (see Bathe):

${}^{\mathrm{IP}}n$	Location ${}^{i}\xi$	Weight ${}^{i}w$	Sketch 1D
1	0.0	2.0	
2	±0.5773502692...	1.0	
3	±0.7745966692... 0.0	0.5555555555... 0.8888888888...	

Example of Integration

Function $f(\xi)$	Integral $\int_{-1}^{+1} \ldots d\xi$	Evaluation	Value
ξ	$\left(\frac{\xi^2}{2}\right)\Big\vert_{-1}^{+1}$	$\frac{1}{2} - \frac{1}{2}$	$= 0$
$\xi - \xi^2$	$\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)\Big\vert_{-1}^{+1}$	$0 - \left(\frac{1}{3} - \frac{-1}{3}\right)$	$=$ -$\frac{2}{3}$
$\xi - \xi^2 + \xi^3$	$\left(\frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4}\right)\Big\vert_{-1}^{+1}$	$0 - \frac{2}{3} + \frac{1}{4} - \frac{1}{4}$	$=$ -$\frac{2}{3}$
$\xi - \xi^2 + \xi^3 - \xi^4$	$\left(\frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} - \frac{\xi^5}{5}\right)\Big\vert_{-1}^{+1}$	$0 - \frac{2}{3} - \left(\frac{1}{5} - \frac{-1}{5}\right)$	$=$ -$\frac{2}{3}$ - $\frac{2}{5}$

Integration with 2 integration points ${}^{1,2}\xi = \pm 1/\sqrt{3}$, ${}^{1,2}w = 1$

Function $f(\xi)$	Integral $\sum_{i=1}^{2} {}^{i}w\, f({}^{i}\xi)$	Value
ξ	$-1\frac{1}{\sqrt{(3)}} + 1\frac{1}{\sqrt{(3)}}$	$= 0$
$\xi - \xi^2$	$0 - \left(1\frac{1}{3} + 1\frac{1}{3}\right)$	$=$ -$\frac{2}{3}$
$\xi - \xi^2 + \xi^3$	$0 - \frac{2}{3} + \left(-1\frac{1}{\sqrt{3}^3} + \frac{1}{\sqrt{3}^3}\right)$	$=$ -$\frac{2}{3}$
$\xi - \xi^2 + \xi^3 - \xi^4$	$0 - \frac{2}{3} - \left(\frac{1}{9} + \frac{1}{9}\right)$	$= -\frac{2}{3} - \frac{2}{9}$ not exact!

Findings and Remarks for this Example

Findings:

- function up to 3rd order is integrated exactly
- to many integration points: no problem - but inefficient
- to few integration points: integral is not integrated exactly!
- i.e. beside approximation of solution, additional approximation due to integration!

Remark:

- due to transformation in case of isoparametric elements
- functions are not polynoms anymore
- that means: Integration is always approximative!

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From Analytical to Numerical Integration

In introduction element stiffness matrix was integrated analytically

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \int_0^{L_\mathrm{e}} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^\mathrm{T}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx = E\,A \int_{-1}^{+1} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}(\xi)^\mathrm{T}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}(\xi)\, \frac{dx(\xi)}{d\xi} d\xi$$

Using numerical integration gives:

Numerical Integration of Element Stiffness Matrix 1D

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \sum_{i=1}^{^{\mathrm{IP}}n} {}^{i}w \left[{}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}({}^{i}\xi)^\mathrm{T}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}({}^{i}\xi)\, \frac{dx(\xi)}{d\xi}\Big|_{\xi = {}^{i}\xi} \right]$$

Example: Integration of Element Stiffness Matrix

Derivation of shape function

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \frac{d\,^1N}{dx} & \frac{d\,^2N}{dx} \end{pmatrix} = \begin{pmatrix} -\frac{1}{L_\mathrm{e}} & \frac{1}{L_\mathrm{e}} \end{pmatrix}$$

together with Jacobi Matrix

$$\frac{dx(\xi)}{d\xi} = \frac{L_\mathrm{e}}{2}$$

give:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = E\,A \sum_{i=1}^{^{\mathrm{IP}}n} {}^{i}w \left[\begin{pmatrix} -\frac{1}{L_\mathrm{e}} \\ \frac{1}{L_\mathrm{e}} \end{pmatrix} \begin{pmatrix} -\frac{1}{L_\mathrm{e}} & \frac{1}{L_\mathrm{e}} \end{pmatrix} \frac{L_\mathrm{e}}{2} \right]$$

Example: Integration of Element Stiffness Matrix

or simplified:

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = \frac{E\,A}{L_\mathrm{e}} \sum_{i=1}^{^{\mathrm{IP}}n} {}^{i}w \left[\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \right]$$

For constant function (order 0) one integration point sufficient

$$^{\mathrm{IP}}n = 1 \qquad {}^{1}w = 2 \qquad {}^{1}\xi = 0$$

and it follows our old friend

$$^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}} = \frac{E\,A}{L_\mathrm{e}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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Governing Equation of Static Equilibrium

In chapter 16 the matrix formulation of FEM Eq. given as:

$$\sum_{n_{\text{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{^{\langle e \rangle}V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \, dV \right\}}_{\text{Element Stiffness Matrix } {}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}} \underline{\boldsymbol{U}} = \dots$$

$$\dots \sum_{n_{\text{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{^{\langle e \rangle}V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{b}} \, dV \right\}}_{\text{consistent volume forces}} + \sum_{n_{\text{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{^{\langle e \rangle}S} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{t}} \, dS \right\}}_{\text{consistent surface forces}}$$

Integrals are similar to previous ones:

$$\int_{y_a}^{y_b} \int_{x_a}^{x_b} f(x,y) dx dy = \int_{\eta_a}^{\eta_b} \int_{\xi_a}^{\xi_b} f\Big(x(\xi,\eta), y(\xi,\eta)\Big) \det \underline{\underline{\boldsymbol{J}}} \, d\xi \, d\eta$$

with determinant of Jacobi Matrix $\det \underline{\underline{\boldsymbol{J}}}$

Numerical Integration of Element Stiffness Matrix

in line with previous element stiffness matrix it follows:

Numerical Integration of Element Stiffness Matrix 2D

$$^{\langle e \rangle}\underline{\underline{K}} = t \sum_{i=1}^{^{\mathrm{IP}}n_\xi} \sum_{j=1}^{^{\mathrm{IP}}n_\eta} \left[{}^{\langle e \rangle}\underline{\underline{B}}({}^i\xi, {}^j\eta)^{\mathrm{T}}\, \underline{\underline{C}}\; {}^{\langle e \rangle}\underline{\underline{B}}({}^i\xi, {}^j\eta) \det \underline{\underline{J}}({}^i\xi, {}^j\eta)\; {}^i w\; {}^j w \right]$$

with

- $^{\mathrm{IP}}n_\xi$, $^{\mathrm{IP}}n_\eta$ ...number of integration points in ξ and η
- t ...element thickness (is dz in 3D case)
- $^i w\; ^j w$ ...integration weights
- $\underline{\underline{C}}$ ...2D material stiffness matrix

Location of Integration Points

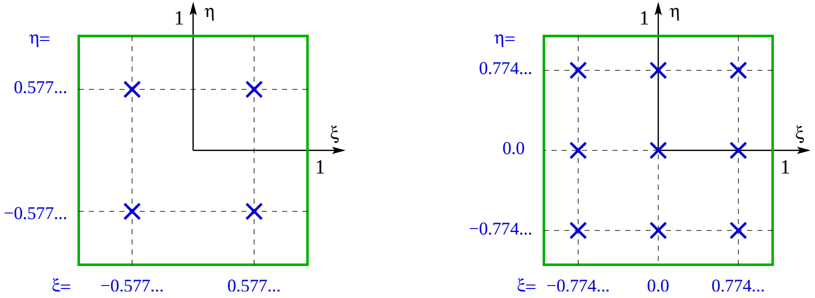

Figure: Integration points of 2D element. 2×2 integration (left), 3×3 integration (right)

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Fully Integrated Elements

Full integration

- previous integrals for polynoms are solved exactly
- number of required integration points from Eq. (80)
- full integration in case of 2D element is shown in Fig. 38

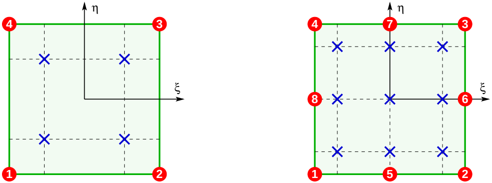

Figure: Fully integrated 4-noded (left) and 8-noded element

Reduced Integrated Elements

Reduced integration

- previous integrals are solved approximatly
- number of applied integration points is less than required
- reduced integration in case of 2D element is shown in Fig. 39

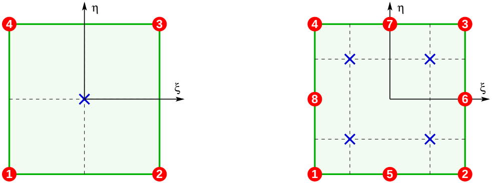

Figure: Reduced integrated 4-noded (left) and 8-noded element

Example: Cantilever Beam

Effect of integration (and discretization):

- cantilever beam under point load (Fig. 40 left)
- 4 different meshes (Fig. 40 right)
- usage of linear and quadratic quad elements

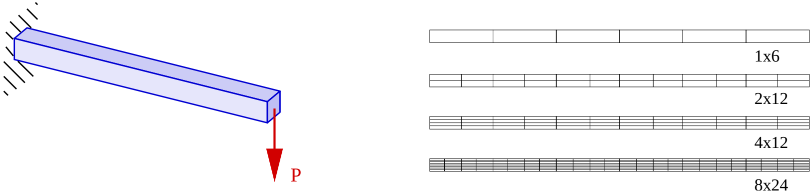

Figure: Cantilever beam (left) and meshes of 2D model

Example: Cantilever Beam

Table: Normalized beam deflections with fully integrated 2D elements

Element	1×6	2×12	4×12	8×24
linear 4-noded	0.074	0.242	0.242	0.561
quadr. 8-noded	0.994	1.000	1.000	1.000

- linear elements are to stiff in bending
- quadratic elements are more effective

Reason for the problems of linear elements:

- in general: elements approximate displacement field not very well
- additionally in case of bending: shear locking

What is Shear Locking?

Explanation of **shear locking effect**

- considering a material element under pure bending (Fig. 41)

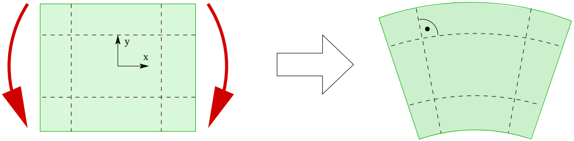

Figure: Deformation of material under pure bending

- material lines are plotted on element at undeformed state (dashed)
- lines show constant curvature in deformed state
- angle between horizontal and vertical lines remains 90°

What is Shear Locking?

- 8-noded element (Fig 42) shows same deformation

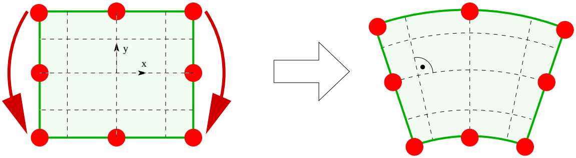

Figure: Pure bending (left) and deformation (right) of quadratic element with 8 nodes

- reason why this element converges fast (see Table 1)

What is Shear Locking?

- 4-noded element can not resolve this deformation (Fig. 43)

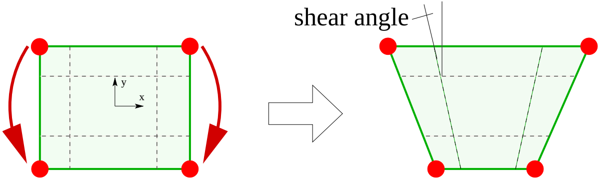

Figure: Pure bending (left) and deformation (right) of linear element with 4 nodes

- because of linear trial function only straight lines possible
- dashed lines show shear deformation $\rightarrow$ parasitic shear stresses
- $\Rightarrow$ reason why stiffness is to high (see Table 1)
- shear deformation makes element stiffer = shear locking

Workaround against Shear Locking - Reduced Integration

Table: Normalized beam deflections with reduced integrated 2D elements

Element	1×6	2×12	4×12	8×24
linear 4-noded	20.3 (Hourglassing)	1.308	1.051	1.012
quadrat. 8-noded	1.000	1.000	1.000	1.000

- linear elements converge much faster
- in case of coarse mesh there is problem: Hourglassing

Reduced Integration - Hourglassing

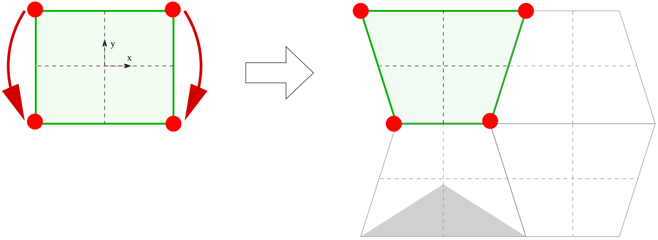

Figure: Hourglassing (zero energy mode) in case of reduced integrated linear 4-noded elements

- deformation pattern looks like hourglasses
- material lines (dashed) show no deformation
- i.e. no energy (zero energy) is needed for this deformation pattern

Shear Locking versus Hourglassing

Shear locking versus hourglassing

- appears mainly in linear QUAD oder BRICK elements under bending
- if one is avoided - the other appears!
- it can be seen how important a deep understanding is!
- if you are unsure - compare both integration schemes!

Other possibility: **Incompatible mode elements**

- overcome shear locking in linear fully integrated elements
- uses additional degrees of freedoms
- not always implemented in FE software

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Motivation

Questions: During reading solver manual

- Why do I get stesses at integration points (and nodal points)?
- Why do I get displacements only in nodal points?
- What is the difference between integration and nodal point values?

Calculation of Nodal Point Values

During solving of the equation:

$$\sum_{n_{\text{el}}}^{\text{Assemble}} \left\{ t \sum_{i=1}^{{}^{\text{IP}}n_\xi} \sum_{j=1}^{{}^{\text{IP}}n_\eta} \left[{}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}({}^{i}\xi, {}^{j}\eta)^{\text{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}({}^{i}\xi, {}^{j}\eta) \det \underline{\underline{\boldsymbol{J}}}({}^{i}\xi, {}^{j}\eta) \; {}^{i}w \; {}^{j}w \right] \right\} \underline{\boldsymbol{U}} = \dots$$

$$\dots \sum_{n_{\text{el}}}^{\text{Assemble}} \left\{ t \sum_{i=1}^{{}^{\text{IP}}n_\xi} \sum_{j=1}^{{}^{\text{IP}}n_\eta} \left[{}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}({}^{i}\xi, {}^{j}\eta)^{\text{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{b}} \det \underline{\underline{\boldsymbol{J}}}({}^{i}\xi, {}^{j}\eta) \; {}^{i}w \; {}^{j}w \right] \right\} \dots$$

following values are computed element by element:

- ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}({}^{i}\xi, {}^{j}\eta)$ derivation of shape function in IP ${}^{i}\xi, {}^{j}\eta$
- ${}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}({}^{i}\xi, {}^{j}\eta)$ shape function in IP ${}^{i}\xi, {}^{j}\eta$

and finally displacement $\underline{\boldsymbol{U}}$ is obtained in nodal points

It's clear why displacements are obtained in nodal points!

Calculation of Stresses at Integration Points

The question arises why stresses are obtained at IPs.

- strains follow directly from ${}^{\langle e \rangle}\underline{\boldsymbol{U}}$ (part of $\underline{\boldsymbol{U}}$) as:

$$ {}^{\langle e \rangle}\underline{\boldsymbol{\varepsilon}}\left({}^{i}\xi,\, {}^{j}\eta\right) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}\left({}^{i}\xi,\, {}^{j}\eta\right)\; {}^{\langle e \rangle}\underline{\boldsymbol{U}} $$

  i.e. strains are computed at integration point level

- stresses follow from the material law:

$$ {}^{\langle e \rangle}\underline{\boldsymbol{\sigma}}\left({}^{i}\xi,\, {}^{j}\eta\right) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}}\; {}^{\langle e \rangle}\underline{\boldsymbol{\varepsilon}}\left({}^{i}\xi,\, {}^{j}\eta\right) $$

  i.e. stresses are also computed at integration point level!

Calculation of Stresses at Nodal Points

Stresses at nodal points

- for this purpose strains are computed at nodal points:

$$^{\langle e \rangle}\underline{\boldsymbol{\varepsilon}}(\pm 1, \pm 1) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}(\pm 1, \pm 1)\ {}^{\langle e \rangle}\underline{\boldsymbol{U}}$$

  and it follows for stresses:

$$^{\langle e \rangle}\underline{\boldsymbol{\sigma}}(\pm 1, \pm 1) = {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}}\ {}^{\langle e \rangle}\underline{\boldsymbol{\varepsilon}}(\pm 1, \pm 1)$$

- **Note:** There are stresses from each element around a node!

Explanation of Stress Jumps

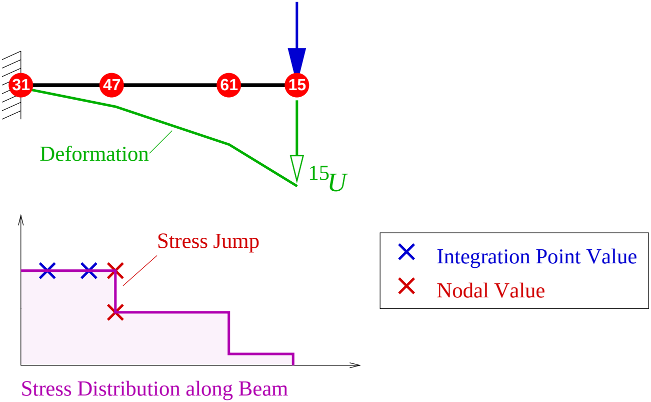

Figure: Deformation (top) and stress distribution of cantilever beam (bottom).

Nodal Stress Values

Remarks on Fig. 45

- displacement is not smooth from element to element
- stress is proportional to the first derivate of $\underline{\bar{\boldsymbol{u}}}$
- $\Rightarrow$ this is reason why stresses show jumps
- more than one stress value at node

Nodal Stress Values

Possible stress outputs:

- at integration point
- at node (not averaged, some values)
- averaged at node
- at element centroid

Remarks:

- inaccurate in case of big stress jumps this means high stress gradients
- averaging not allowed over different materials or element properties (element thickness)
- stress jumps can be used as error estimates

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Numerical Integration 1D

Matrix FEM equations are intergrated numerically with Gauss-Legendre integration:

$$\int_{-1}^{+1} f(\xi)\, d\xi \approx \sum_{i=1}^{{}^{\mathrm{IP}}n} {}^{i}w\, f({}^{i}\xi)$$

- ${}^{\mathrm{IP}}n$ ... number of integration points IP (Gauss points)
- ${}^{i}w$ ... integration weight IP i (given)
- ${}^{i}\xi$ ... location of IP i in natural coordinates (given)
- $f({}^{i}\xi)$ ... function value at IP i

Number of integration points for exact integration of polynom:

$$N = 2\,{}^{\mathrm{IP}}n - 1$$

- N ... order of polynom which is integrated

Notations & Output

Notations with respect to numerical integration:

- **fully integrated:** matrix FEM equation (mostly) integrated exactly
- **reduced integrated:** matrix FEM equation integrated approximately
- **shear locking:** linear fully integrated elements give stiffer response because of artifical shear deformations
- **hourglassing:** linear reduced integrated elements show large deformations with zero energy

Output at nodes or integration points:

- **nodes:** displacement, forces
- **integration points:** stresses, strains

List of Questions

- Why are governing FEM equations integrated numerically?
- What is the formula of numerical integration of a function with one variable?
- What are integration points (IPs) and integration weights?
- How can a function be integrated exactly?
- Where are the IP located in the case of a linear full integrated quad element?
- What means shear locking and how can it be prevented?
- What means reduced/fully integrated (problems)?
- In which case are governing equations approximated although the elements are fully integrated?
- Which variables are obtained at nodes, which at IP - why?
- Where and why do stresses (strains) show jumps?
- Which problems appear if IP stresses are transformed to nodal stresses?

Part VIII

Discretization Hints

Overview Part VIII

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What means Convergence?

- Monotone convergence means FE solution approximates exact solution if mesh density is increased

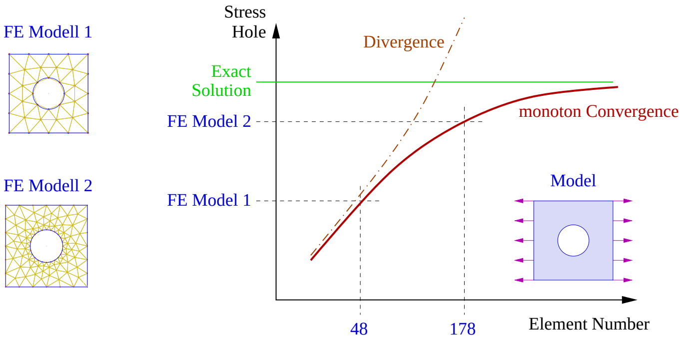

Figure: Monotone convergence for hole problem

- Challenge: To know if FE solution is close enough to exact solution.

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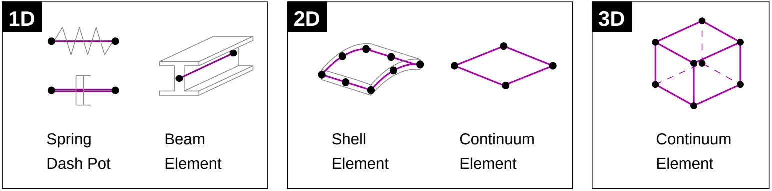
Characterizing Elements

- **Familiy:** continuum, structural, springs, dashpots, . . .



- **DOFs:** degrees of freedom (translation, rotation, temperature, . . .)
- **Number of nodes:** order of interpolation (linear, quadratic)
- **Formulation:** Lagrangian oder Eulerian material description
- **Integration:** fully, reduced

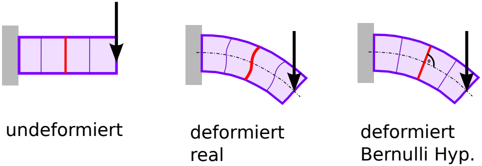
Element Family

Most commonly used:

- continuum elements (1/D, 2/D and 3/D solids, axial symmetric elements)
- structural elements (beam, truss, shell elements, . . .)

Difference of **structural elements**:

- part of displacement field follows from kinematic condition
- kinematic condition: Bernoulli-/Timoshenko beam theory or Kirchhoff-/Mindlin-Reissner shell theory



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Structural Element: 1D - Truss

- forces only along truss axis

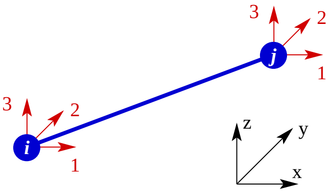

Figure: 1D element: truss

- 6 DOF total
- 3 DOF per node (translational)

Structural Element: 1D - Beam

- 3D structure replaced by line!
- rotational DOFs introduced in nodes (for bending stiffness)

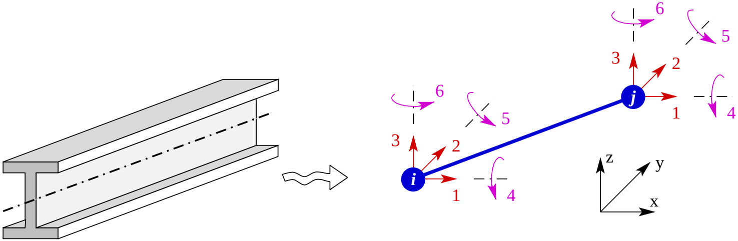

Figure: Beam element: real structure (left) and beam element (right)

- 12 DOF total
- 6 DOF per node (3 translational, 3 rotational)

Structural Element: 1D - Beam

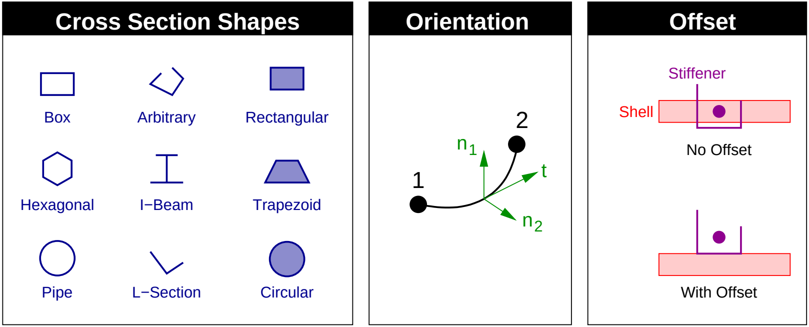

Figure: Beam element: Predefined cross sections (left), orientations of element tangents and cross section axes (middle), and correct connection between beam and shell elements (right)

Structural Element: 2D - Shell

- same as beam but in 2D

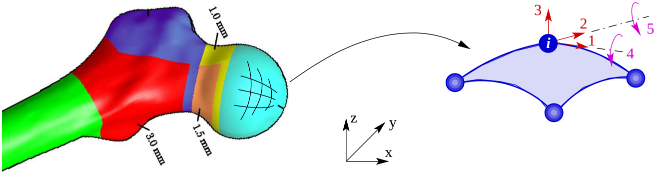

Figure: Shell element: real structure (left) and shell element (right)

- 5 DOF per node (3 translational, 2 rotational)
- no stiffness normal to shell (would be 6th DOF)!

Structural Element: 2D - Shell

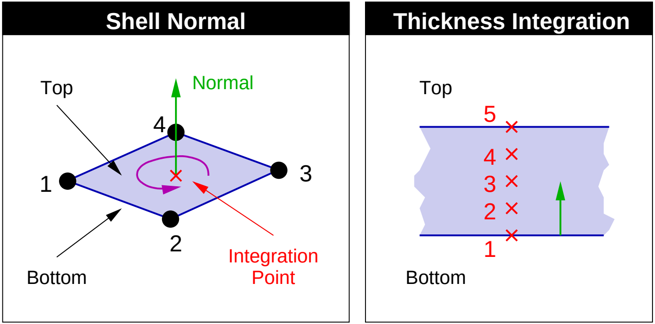

Figure: Shell Elements: Definition of shell normal (left) and position of integration points over shell thickness (right)

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Continuum Element 2D - Quad, Tria

- plane (2D) Solid Element - **Note:** not shell elements!
- quadrilateral (quad) and triangular (tria) elements possible

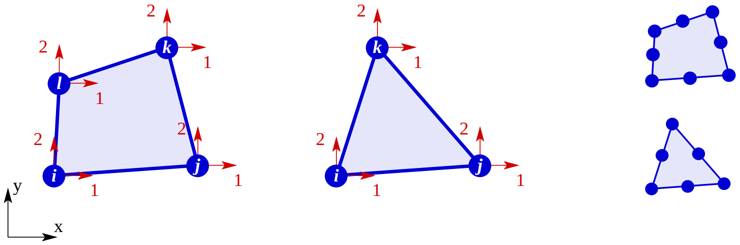

Figure: 2D elements: linear quad element (left), linear tria element (middle), and corresponding quadratic elements (right)

- 2 DOF per node (2 translational)

Continuum Element 2D - Plane Stress

- for e.g. strips

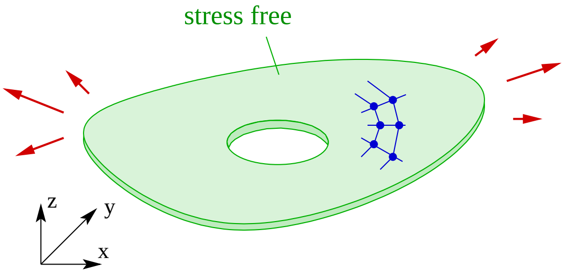

Figure: Part with plane stress state

- 2 DOF per node (2 translational)
- $\sigma_{zz}=\sigma_{xz}=\sigma_{yz}=0,\ \varepsilon_{zz} \neq 0$

Continuum Element 2D - Plane Strain

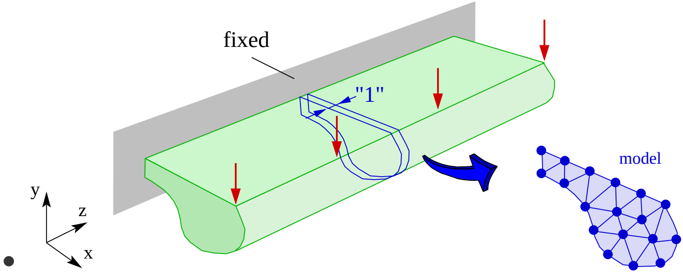

Figure: Part with plane strain state

- 2 DOF per node (2 translational)
- ε_{zz}=ε_{xz}=ε_{yz}=0, $\sigma_{zz} \neq 0$

Continuum Element 2D - Axial Symmetric

- **Note:** Geometry, BC, material, and solution are symmetric!

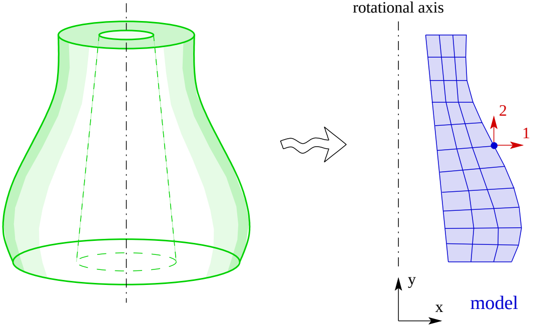

Figure: Axial symmetic part (left) and FE-model (right)

- reduction of modelling and computing costs
- not applicable in case of bending and/or torsion!

Continuum Element 3D - Hexahedron, Tetrahedron

- also known as 3D solid elements

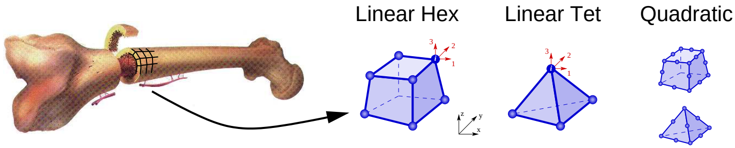

Figure: 3D elements: linear hexa (links), linear tetra (middle), and corresponding quadratic elements (right)

- 3 DOF per node (3 translational)
- linear tetrahedrons are bad elements $\Rightarrow$ avoid usage!

Combination of Different Element Types

- in principle for all element types possible

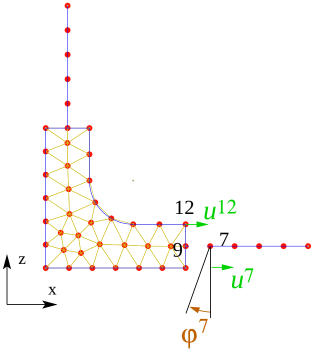

Figure: Combination of different element types

- consistency condition necessary: e.g. $u^{12} = u^7 + z\,\varphi^7$ (with MPC)
- usage of transition/coupling elements

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General Remarks

Motivation:

- many elements: high computing costs, long waiting time, slow post-processing
- few elements: fast results but solution (e.g. around holes) not accurate

Workaround:

- local mesh refinement where it is necessary
- three possibilities (combinations) available

Possible Mesh Refinements:

- h-refinement
- p-refinement
- d-refinement
- adaptive mesh refinement

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Change of Element Size: h-Refinement

- change of element size

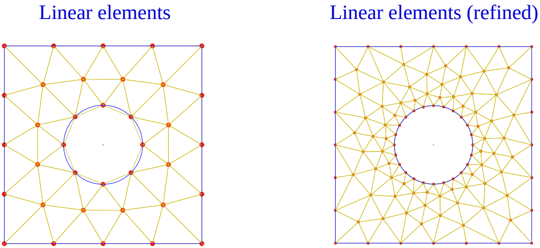

Figure: h-refinement: inital mesh (left) and refined mesh (right)

- element edge length is denoted as "h" $\Rightarrow$ h-refinement
- refinement: stress gradient better resolved around hole

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Change of Interpolation Function: p-Refinement

- same element size but higher order of interpolation functions

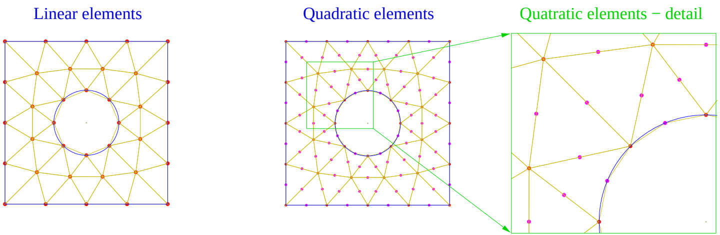

Figure: p-refinement: inital mesh (left), refined mesh (middle), and mesh detail (right)

- order of interpolation function is denoted as"p" $\Rightarrow$ p-refinement
- more efficient as h-refinement

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Change of Dimensionality: d-Refinement

- resolving of details (notches, supports,...) due to element of higher dimensionality

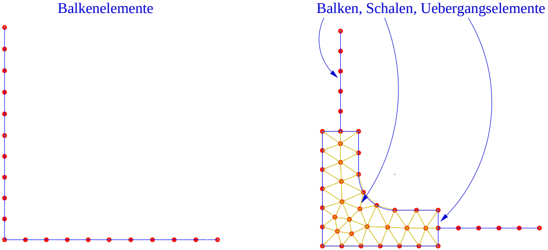

Figure: d-refinement: inital mesh (left) and refined mesh (right)

- dimensionality denoted as $d \Rightarrow$ d-refinement
- less DOFs but same model size

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Adaptive Mesh Refinement

- self-controlled mesh refinement using error estimators
- error estimators: e.g. stress jump or element energy density

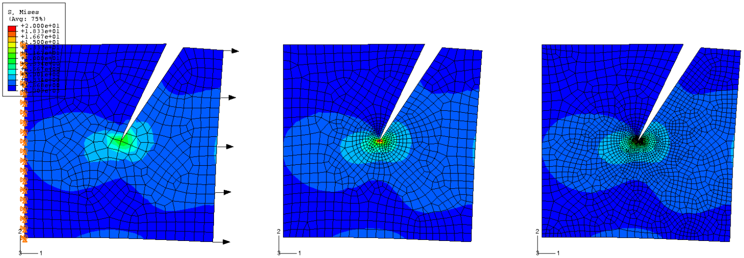

Figure: Adaptive mesh refinement at crack tip: inital mesh (left), refined meshes (middle, right)

Adaptive Mesh Refinement - Algorithm

Algorithm steps:

1. compute stress in first (start) model
2. estimate error
3. refine in regions of high error
4. generate new model and go back to step 1
5. done if error is acceptable

Remark: can lead to meaningless results (e.g. introduction of point loads)

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Coordinate System

- choose a proper coordinate system!
- easier application of loads and BCs

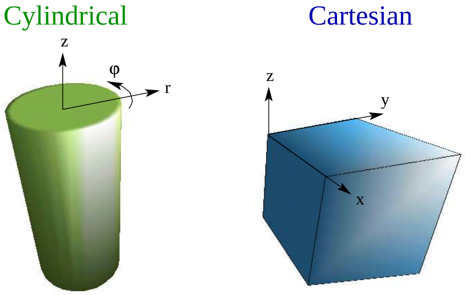

Figure: Different coordinate systems: cylindrical (left) cartesian (right)

Compatibility of Meshes

- gaps or overlaps during deformation

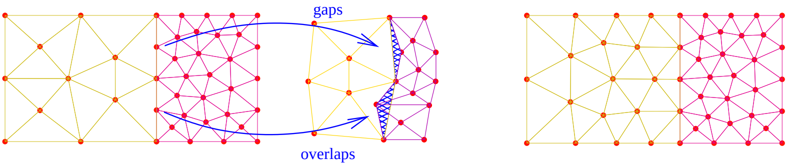

Figure: Compatibility: not compatible mesh (left), gaps and overlaps (middle), compatible mesh (right)

- harder to dedect: superposed nodes, removal of second node = equivalence
- compatibility problems → jumps in displacement field

Mesh Density

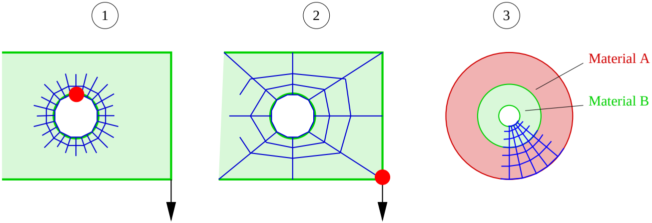

Figure: Meshes for stress analysis around hole (1), for displacement analysis at load introduction (2), for material interfaces (3)

- Stress analysis (1): Refinement in interested regions (around hole)
- Deformation analysis (2): Coarser mesh sufficient (e.g. displacement of load introduction)
- Material boundaries (3) are also element boundaries

Edge Aspect Ratio and Element Warpage

- within reasonable limits

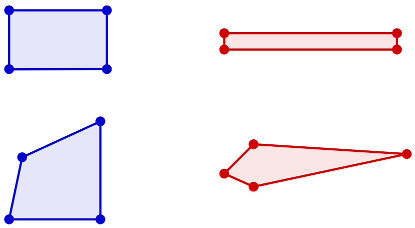

Figure: Correct (left) and problematic (right) edge aspect ratio / element warpage

- Rule of thumb: edge aspect ratio max. 1:4, angle min. 30°
- **Note:** Rule of thumb important if solution is unknown!

Exploitation of Symmetry

Problem shows symmetry if:

- **Geometry** is symmetric
- **Load** and BCs are symmetric
- **Material** is symmetric (e.g. isotropic)
- **Solution** is symmetric (attention in case of stability problems!)

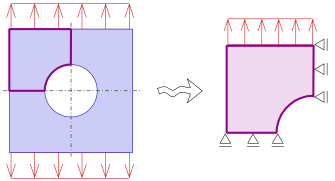

Figure: Symmetry: Whole model (left) and reduced model (right)

- symmetry conditions can considerable reduce computing costs!
- computing time is proportional to DOF^2

List of Questions

- What means convergence of a solution?
- How are elements characterized?
- What is the difference between structural and continuum elements?
- Describe typical structural elements.
- Describe typical continuum elements.
- What kind of mesh refinements are available?
- What is the most efficient mesh refinement method in a practical sense?
- What means meshes has to be compatible?
- What has to be symmetric in order to exploit symmetry?

Part IX

Musculoskeletal Biomechanics

Overview Part IX

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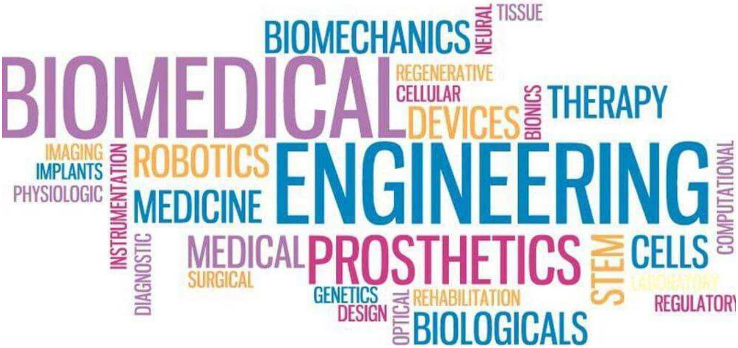
Biomedical Engineering / Biomechanics



Musculoskeletal System

Components

- Bone, cartilage, ligaments
- Muscles, tendons

Function of Bone

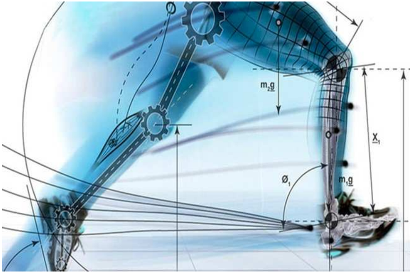
- Build up the human skeleton
- Supportive structure for locomotion
- Protects organs

Also. . .

- Nutrient storage
- Red blood cell production

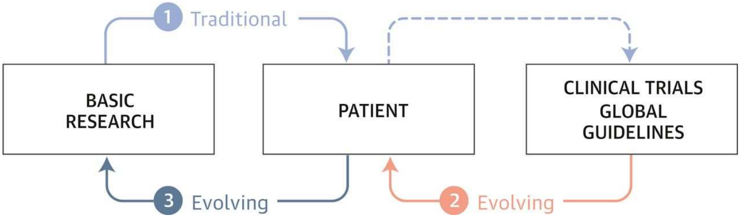
Musculoskeletal Biomechanics

- evaluate musculoskeletal pathologies, injuries and treatments
- through the use of computer simulations and laboratory experiments
- of the whole body down to the organ level



Translational Biomechanics

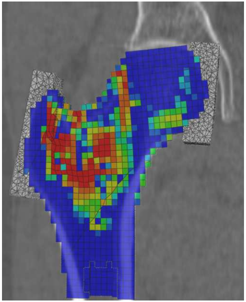
- develop new biomechanically related medical solutions
- in collaboration with clinics and research partners
- to improve human health



Finite Elements in Biomechanics

Key Points

- based on X-ray or 3D CT/MRI images
- in-silico (by computer) modelling
- uncover the invisible



Benefits

- individualization (patient)
- less (animal) experiments
- prediction possible

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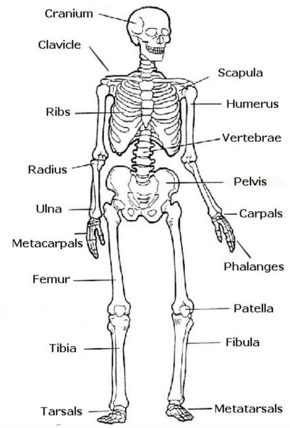
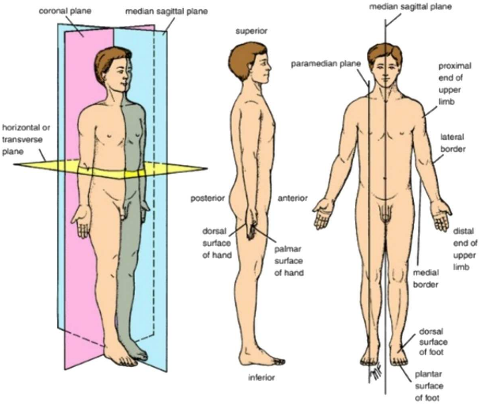
Human Skeletal System



English / German designations

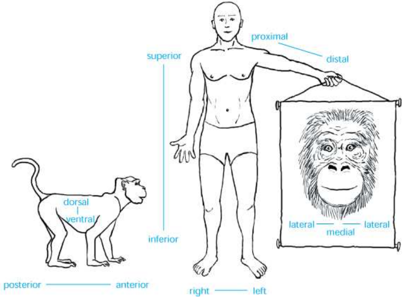
- Skull (Cranium+Mandible) . . . Kopf
- Radius/Ulna. . . Speiche/Elle
- Humerus . . . Oberarm
- Pelvis. . . Becken
- Femur. . . Oberschenkel
- Tibia. . . Schienbein
- Fibula. . . Wadenbein
- Spine. . . Wirbelsäule
- Vertebra. . . Wirbelkörper

Anatomical Planes



- Sagittal plane: divides body into left and right section
- Coronal plane: divides body into front (anterior) and back (posterior) section
- Transverse plane: divides body into an upper (superior) and lower (inferior) section

Anatomical Directions



- Superior: Higher
- Inferior: Lower
- Anterior: Toward the front
- Posterior: Toward the rear
- Proximal: Closer to the torso
- Distal: Farther from the torso
- Medial: Closer to the midline
- Lateral: Farther from midline
- Dorsal: Toward the back of the torso
- Ventral: Toward the front of the torso

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Bone Types

Differentiation of 2 types of bone:

- compacta (cortex or compact bone)
- cancellous bone or trabecular bone

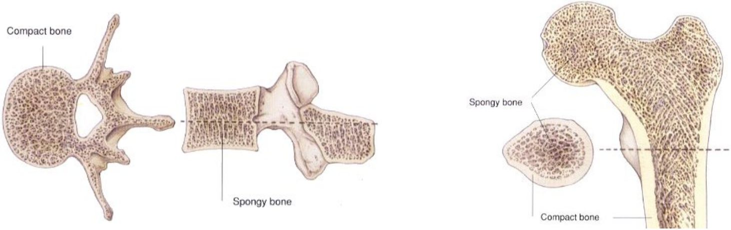


Figure: Vertebra (left), femur (right)

Structure of Cortical Bone

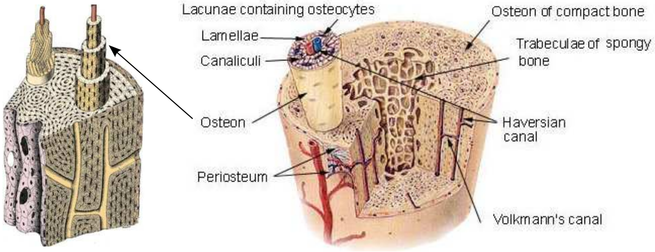
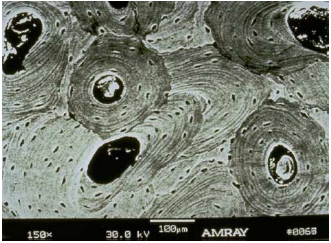


Figure: Cortical Bone

- Build up of osteons (Havers System)
- Each osteon consists of:
  - Lamellae (concentric layers of collagen / hydroxyapatite)
  - Havers canal with blood vessels, nerves, etc inside

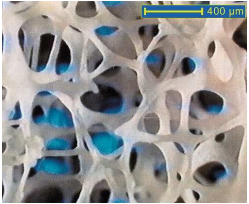
Structure of Cortical Bone

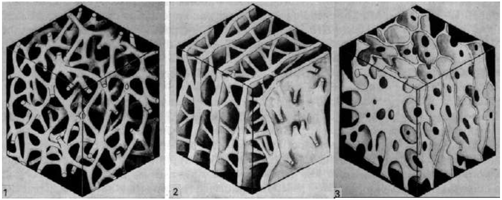


- 80 % of skeletal mass, 20% bone surface
- porosity 5-10%, density $\rho = 2000 \mathrm{kg/m}^3$
- low bone turnover rate 2-3%/year[1]

[1] Physiology of Bone Formation, Remodeling, and Metabolism, Usha Kini and B. N. Nandeesh

Structure of Trabecular Bone

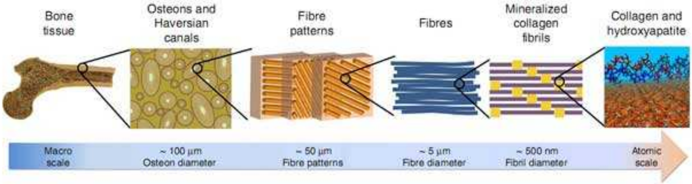




- 20 % of skeletal mass, 80% bone surface
- spongy, consits of beams and/or plates
- porosity 65-95%
- high bone turnover rate 5-10%/year[2]

[2] Physiology of Bone Formation, Remodeling, and Metabolism, Usha Kini and B. N. Nandeesh

Micro-Structure of Bone



Bone = multi hierarchical composite material

- complex micro-structure at several length scales
- hydroxylapatite (rigid) + collagen (soft) + proteins + water

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Mechanical Properties of Bone

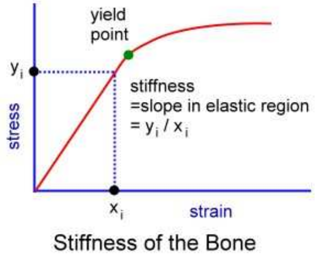
Properties depend on the material ...

- and geometry of the bone,
- composition of bone,
- properties of constituents,

and on the loading ...

- rate of applied load,
- direction of applied load.

Bone Stiffness



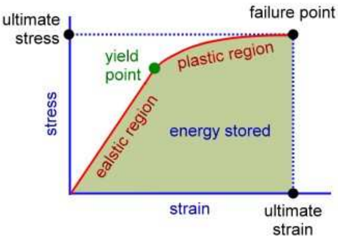
- elastic modulus E slope of stress-strain curve
- E: steel > titanium > bone > polymer

http://www.pt.ntu.edu.tw/hmchai/Biomechanics/BMmaterial/BMbone.htm

Measured Elastic Constants of Compact Bone

Autor	**Reilly&Burstein**	**Yoon&Katz**	**Knets et.al.**	**Ashman et.al.**
Ort	Femur	Femur	Tibia	Femur
Symmetrie	Trans. Ortho	Trans. Ortho	Ortho	Ortho
Technik	Mechanisch	Ultraschall	Mechanisch	Ultraschall
E_1 [GPa]	11.5	18.8	6.91	12.0
E_2 [GPa]	11.5	18.8	8.51	13.4
E_3 [GPa]	17.0	27.4	18.4	20.0
G_{23} [GPa]	3.3	8.71	4.91	6.23
G_{31} [GPa]	3.3	8.71	3.56	5.61
G_{12} [GPa]	3.6	7.17	2.41	4.53
ν_{23} [1]	0.31	0.193	0.14	0.235
ν_{31} [1]	0.31	0.193	0.12	0.222
ν_{12} [1]	0.58	0.312	0.49	0.376

Bone Yield & Strength



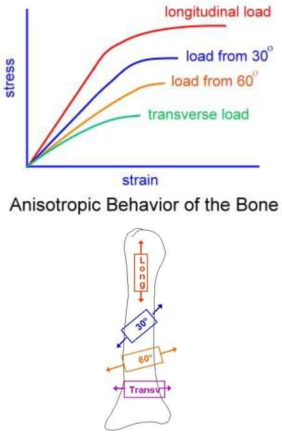
Stress-Strain Curve of the Bone

- elastic region (Hooke's law)
- yield point (first damage)
- plastic regions (accumulation of damage)
- ultimate stress = strength
- energy = area under the curve

Measured Strength of Compact Bone

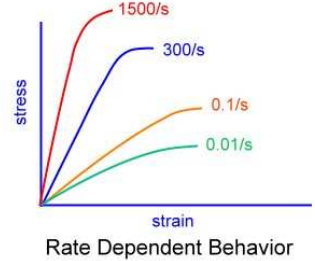
Variable	Wert
$R_{11}^{(+)}$	39 MPa
$R_{11}^{(-)}$	190 MPa
$R_{22}^{(+)}$	50 MPa
$R_{22}^{(-)}$	150 MPa
$R_{33}^{(+)}$	156 MPa
$R_{33}^{(-)}$	237 MPa
R_{21}	56 MPa
R_{31}	66 MPa
R_{23}	71 MPa

Anisotropic Behavior of Bone



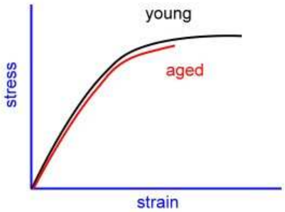
- anisotropy: directional dependency
- reason: bone is composite material
- stiffness and strength influenced

Rate Dependency



- properties change with rate of loading
- if loading rate ↑, stiffness+strength+energy ↑
- low-energy fracture: single crack
- high-energy fracture: severe destruction

Degenerative Changes of Bone



Effect of Degeneration

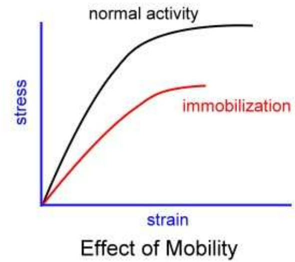
Morphological changes:

- loss of bone mass (osteoporosis)
- micro-structural changes

Mechanical changes:

- stiffness and strength $\sim$
- energy storage capacity $\downarrow$ (brittleness)

Adaptive Response



Response of bone on:

- internal changes z.B. calcium, hormone level, ...
- external mechanical loads (immobilization)

bed rest $\sim$ 1% of loss of bone mass per week

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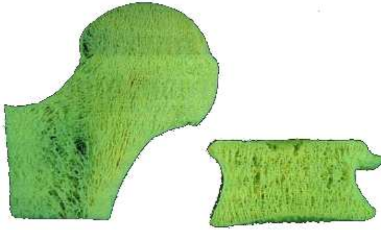
Trabecular Bone Density



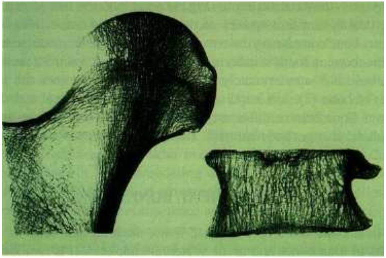
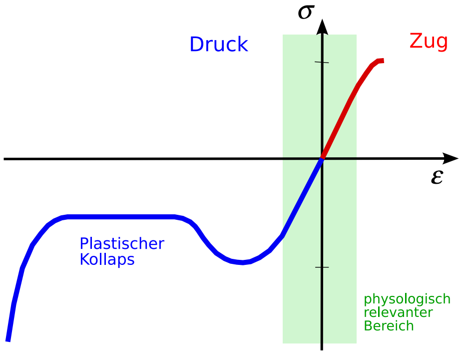


Figure: Femoral, vertebral body section (left) and associated bone density(right)

- Density varies greatly (depending on the location)
- $\Rightarrow$ inhomogeneous material behavior

Tension/Compression Behavior of Trabecular Bone



- different tension/compression behavior (foam like)

Determination of Material Behavior

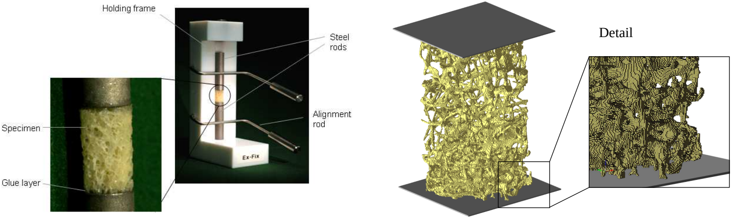


Figure: Material test (right) and FE model (left)

Possibilities of determining material parameters are:

- material tests
- analytical micromechanical models
- morphology analysis (power laws)
- Finite element analysis based on CT scans

Elastic Constants from Power Laws

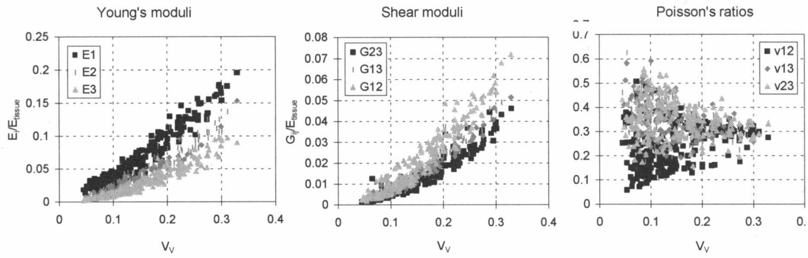


Figure: Orthotropic elastic constants from FE analyses (Cowin, 2001)

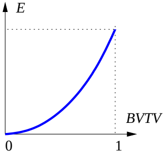
Material values show a power law like relation:

$$E \sim E_{\text{Tissue}}\ \rho^k \sim E_{\text{Tissue}}\ (BVTV)^k$$

Isotropic and Orthotropic Models

Isotropic "power law" model:

$$E = E_0\, \rho^k \qquad \frac{E}{\nu} = \frac{E_0}{\nu_0}\, \rho^k$$

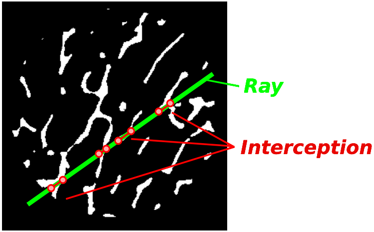


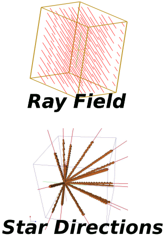
Orthotropic "fabric based" model (Zysset 1995)

$$E_i = E_0\, \rho^k\, (m_i^2)^l \qquad \frac{E_i}{\nu_{ij}} = \frac{E_0}{\nu_0}\, \rho^k\, (m_i\, m_j)^l \qquad G_{ij} = G_0\, \rho^k\, (m_i\, m_j)^l$$

- ρ ... density (=BVTV... bone volume to total volume)
- m_i ... fabric eigenvalues
- E_0, ν_0, G_0 ... tissue properties (from experiments/models)
- k, l ... power coefficients for density and fabric (from models)
- E_i, G_{ij}, ν_{ij} ... local unknown material properties

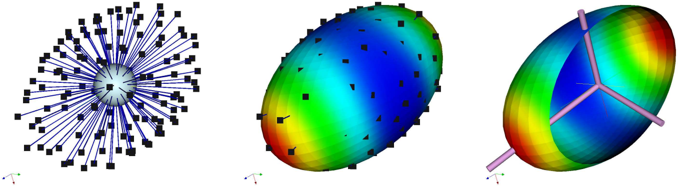
Fabric Tensor Computation





- Many parallel test beams (= ray field) are sent through model
- Many directions are considered (= star directions)
- Measurement of the Mean Intercept Length (MIL)

Fabric Tensor Computation



Original Distribution ***Fabric Tensor*** ***Eigenvectors (Orientation)***

- **Fabric** Tensor: Approximation of the MIL distribution with ellipsoid
- **Eigen vectors**: Orientation of material based on approximate distribution
- **Eigen values**: = Magnitude of Eigen vectors, anisotropy of the material

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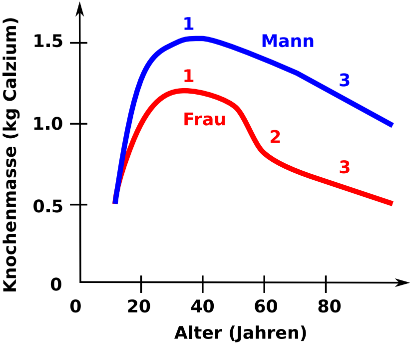
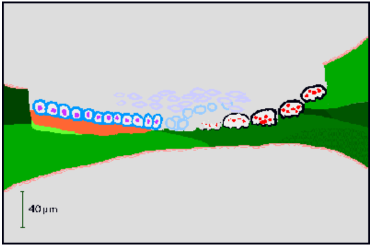
Bone Mass Changes over Life



- 1. . . peak bone mass
- 2. . . menopausal bone loss
- 3. . . bone loss during aging

Bone is Alive!



Bone is constantly being rebuilt (approx. 3-5%)

- old bone is removed: resorbtion
- new bone is built: formation

Bone remodelling based on cell activity

- **osteclasts:** resorb bone
- **osteblasts:** form bone

Osteoporosis

- From age 45, more bone is resorbed than built up
- As a result: loss of bone mass = osteoporosis

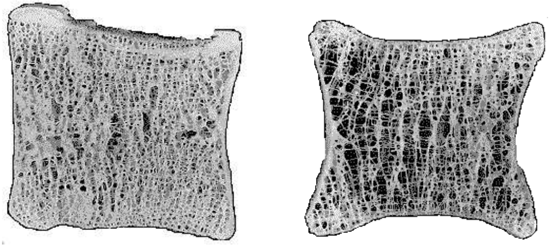


Figure: Osteoporosis in vertebral bodies - loss of bone mass: healthy (left), pathological (right).

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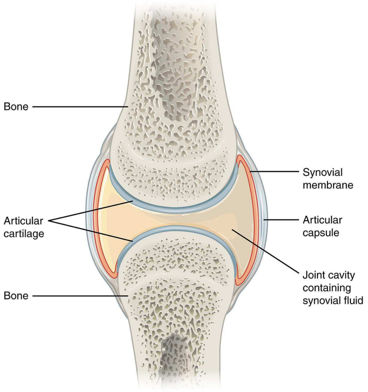
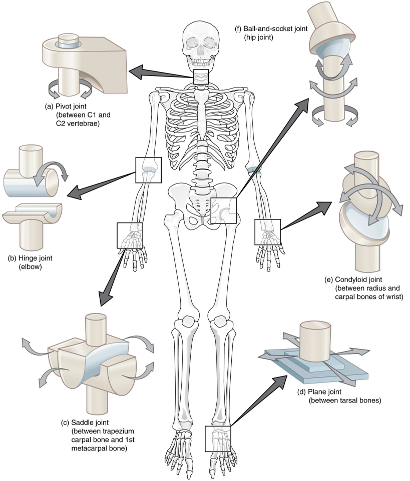
Structure of Synovial Joint

- Bone: hard tissue
- Articular cartilage: smooth, slippery surface
- Joint capsule: surrounds joint
- Joint cavity: filled with synovial fluid

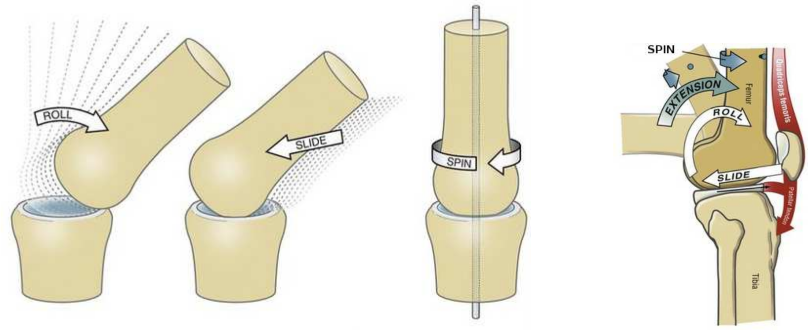


Types of Synovial Joints

- Pivot, hinge, saddle, plane, ball joints
- characterized via kinematics
- up to 3 translation and 3 rotations
- constraints lead to forces or moments



Movement between Joint Surfaces

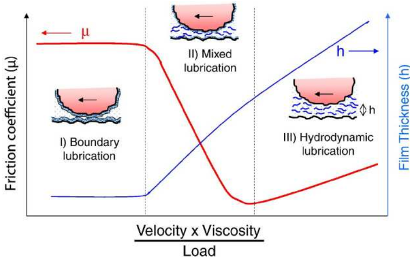


roll slide spin combined

Lubrication of Joints

Biotribology point of view

- articular cartilage forms the two bearing surfaces
- synovial fluid acts as the lubricant
- different types of lubrication - Stribeck curve



Challenge: joints experience large and variable joint reaction forces

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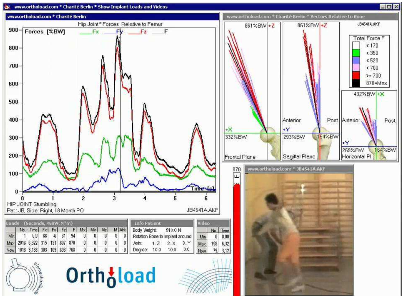
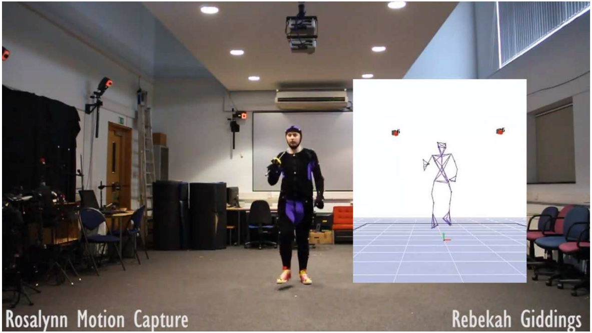
Measurement of Joint Forces via Implants



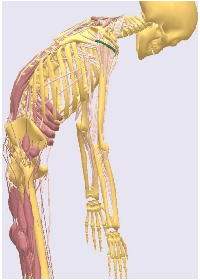
Calculation of Joint Forces



Step 1: Motion Capture

- record 3D movement of people based markers & cameras
- transfer information to skeletal computer model
- used in science, industry, movies

Calculation of Joint Forces



Step 2: Musculoskeletal Model

- add muscles & tendons to skeleton
- apply movements from motion capture
- minimize muscle activations in musculoskeletal model
- simulation result: muscle and joint forces

Part X

Modeling of Bone using FEA

Overview Part X

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- Homogenized Voxel Models
- Homogenized Smooth Models

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- Mapping Algorithm

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33 **Applications**

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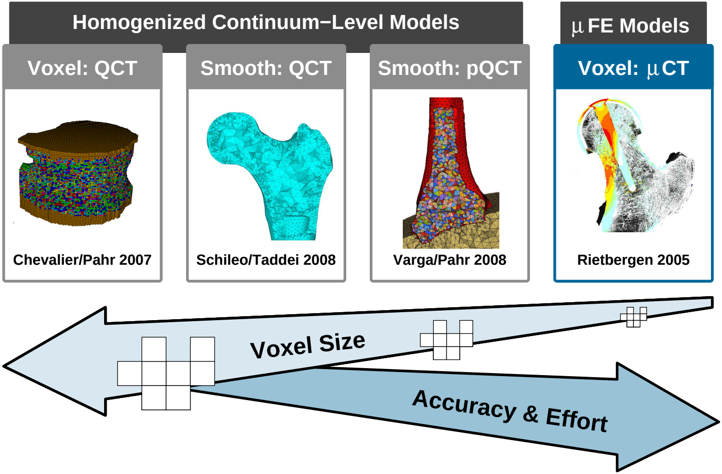
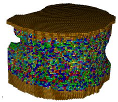
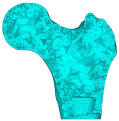
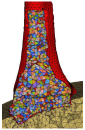
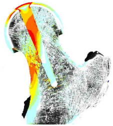
Possible FE Simulation Models



Pro and Con's of Different Models

	Voxel: QCT	Smooth: QCT	Smooth: pQCT	Voxel: μCT
				
Aquisition In−Vivo				
Modelling Effort				
Computational Effort				
Accuracy				

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μFE: Detailed Anatomical Bone Models

- Modeling of full bone (cortex + cancellous bone)

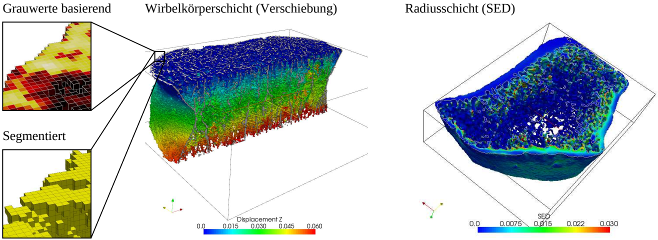

Figure: Full bone models

- 1 voxel = 1 hexahedral element, isotropic material
- resolution ~4-80 μm (many elements)
- numerically expensive (special soft- and hardware necessary)
- in-vitro: donor studies, in-vivo: distal regions (radius, tibia)

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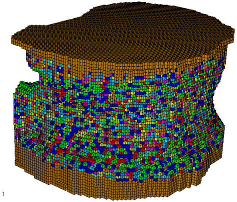
Homogenized QCT Bone Models



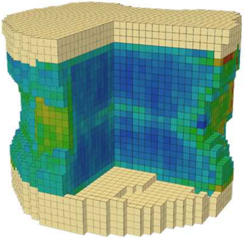


Figure: Embedded (left) and partly mask vertebral body (right)

- resolution ~mm (small amout of elements)
- usually isotropic, density dependend material→mapping!
- simple, fast generated but (usually) less accurate model
- in-vivo voxel data (from patients)

Smoothing of Voxel Type Models

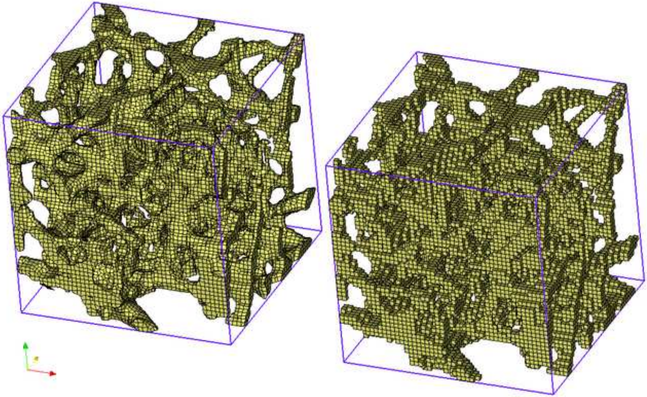


Figure: Smoothed (improved) model with Taubin's algorithm (left) and original model (right)

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Homogenized Smooth Models

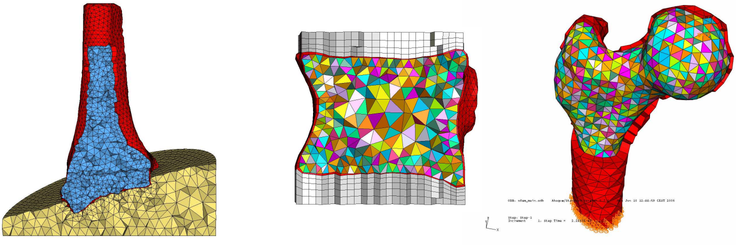

Figure: Smooth hFE models including cortex of radius, vertebral body, and femur (from left to right)

- homogenized tetrahedral/hexahedral elements→mapping
- challenging: iso-surface extraction + special cortex meshing
- but: numerical less expensive and similar accuracy as μFE

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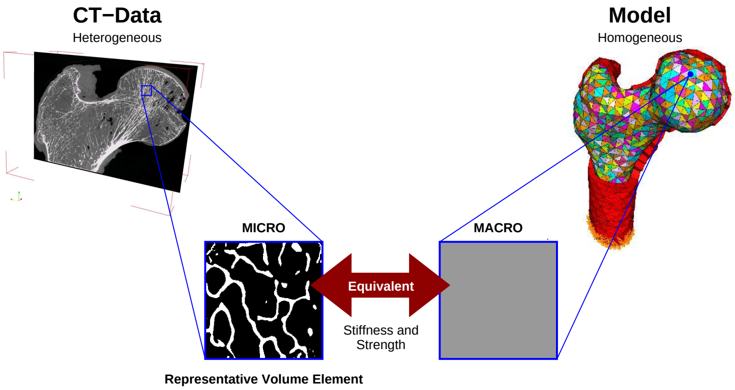
Modelling of a Hierarchical Structure

Homogenization example: Cancellous Bone

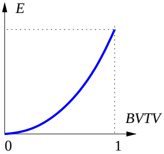


Question: Relation between Morphology $\leftrightarrow$ Stiffness

Isotropic and Orthotropic Models

Isotropic "power law" model:

$$E = E_0 \, \rho^k \qquad \frac{E}{\nu} = \frac{E_0}{\nu_0} \, \rho^k$$



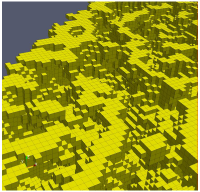
Orthotropic "fabric based" model (Zysset 1995)

$$E_i = E_0 \, \rho^k \, (m_i^2)^l \qquad \frac{E_i}{\nu_{ij}} = \frac{E_0}{\nu_0} \, \rho^k \, (m_i \, m_j)^l \qquad G_{ij} = G_0 \, \rho^k \, (m_i \, m_j)^l$$

Possible Ways to compute BVTV

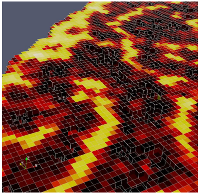
CT image → threshold → BVTV

- right choice of threshold is important
- usually calibration study necessary



CT image → BMD → BVTV

- calibration study necessary
- potentially more robust than thresholding



BMD. . . bone mineral density

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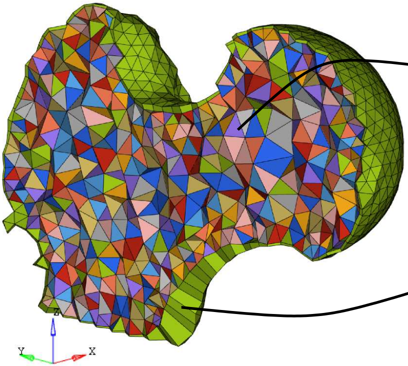
What kind of Material?



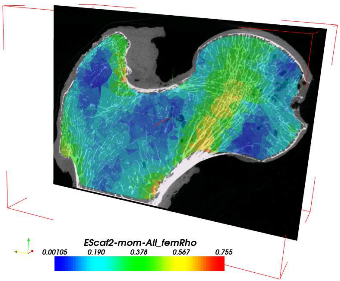
– isotropic (BVTV)
– orthotropic (Morphology)

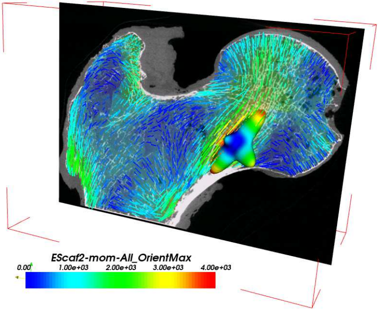
– isotropic

Orthotropic Material Mapping: Example Femur

Fabric based power law model (Zysset 1995)

$$E_i = \underbrace{E_0\,\rho^k}_{\text{power law}}\ \underbrace{(m_i^2)^l}_{\text{fabric}}$$





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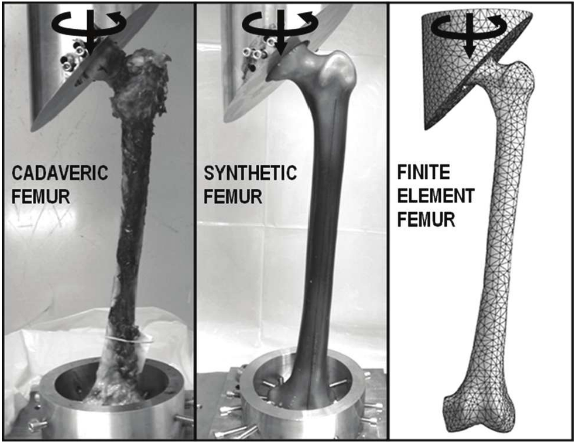
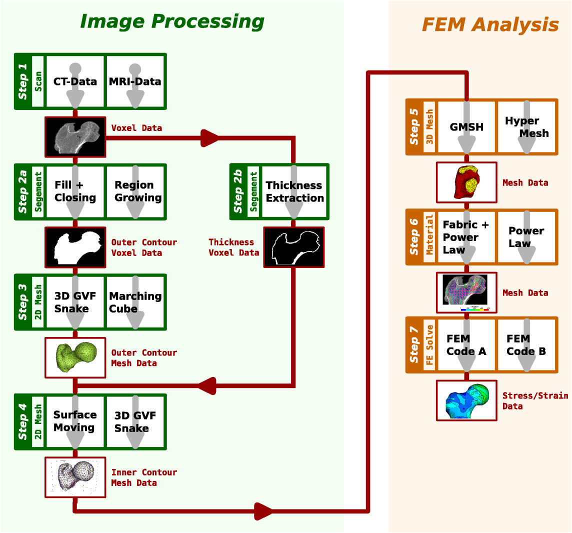
Possibilities to Assess Structural Behavior



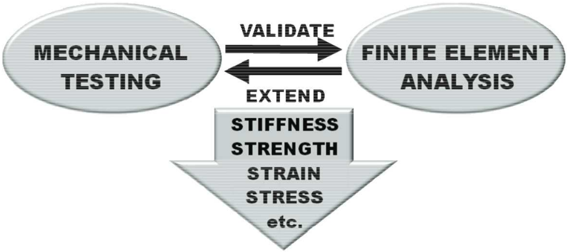
www.intechopen.com/books/finite-element-analysis

From CT Data to hFE Model



Pahr & Zysset, 2009

Biomechanics Research Diagram



Combination of mechanical testing and FEA:

- minimising time & cost for testing
- use testing for model validation (less testing in future)
- gain better in-sight (parameter variations)

Verification & Validation

Verification

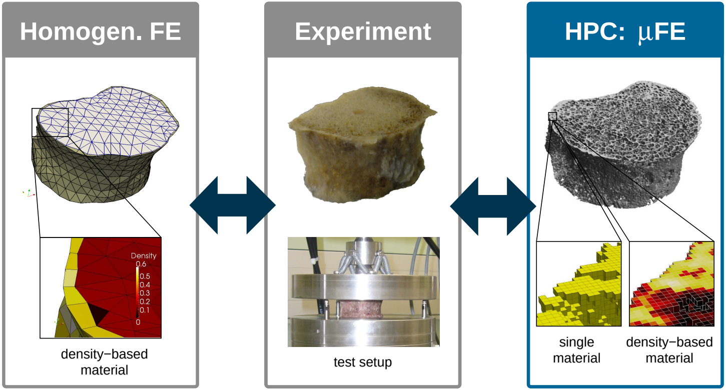
- Software & calculation verification
- Is the FE calculation correct?

Validation

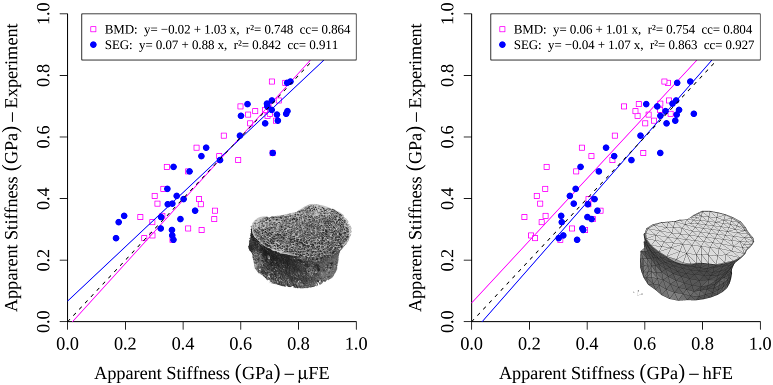
- Modeling validation
- Does the model agree with the reality?

Vertebrae: Homogenized FE vs. μFE Models

Goal: Experimental validation of FE models ($\sim$40 samples)



Vertebrae: Apparent Strength



(experiment and simulation → linear relationship)

List of Questions

- What are the main components of the human musculoskeletal system?
- Which bone types do you know and what are their main characteristics?
- Draw a typical stress-strain diagram of compact bone and label characteristic areas.
- How can you recognize the anisotropy of a material such as in the case of bone?
- What is the relationship between density and stiffness in trabecular bone?
- What are fabric tensors and what are they used for?
- Why can you get osteoporosis when you get older?

List of Questions

- **How are synovial joints constructed and what is meant by tribology in this context?**
- **What are the possibilities to determine joint forces in humans?**
- **What kind of FE models are possible with clinical CTs, what are pro- and cons?**
- **What is meant by material mapping in FEA models in biomechanics?**
- **What research option do you know of to study the structural behavior of bones?**

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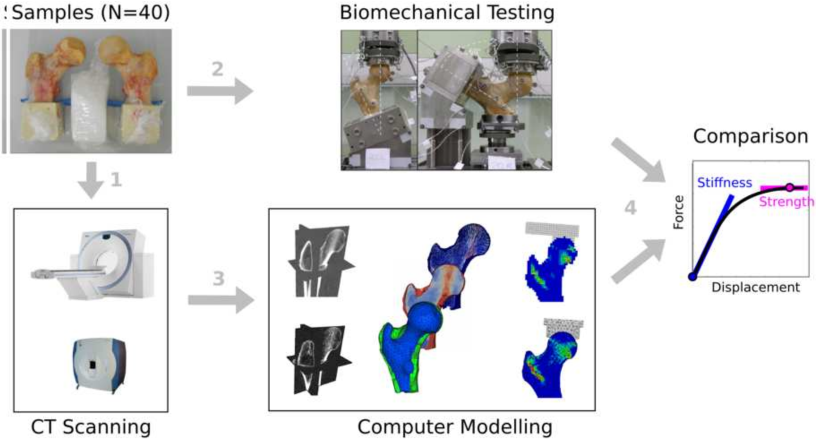
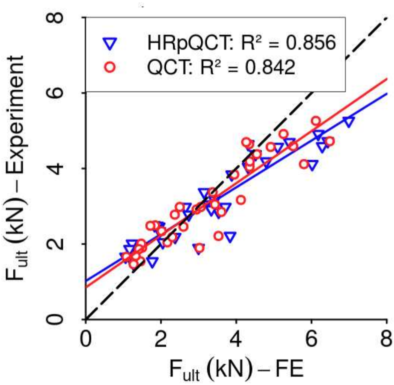
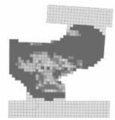
Biomechanical Study on Femoral Bone: Overview



Luisier, Dall'Ara, Pahr: J Mech Behav Biomed Mater, 2014, 32C, 287-299

Biomechanical Study on Femoral Bone: Validation

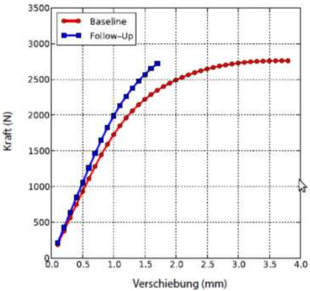


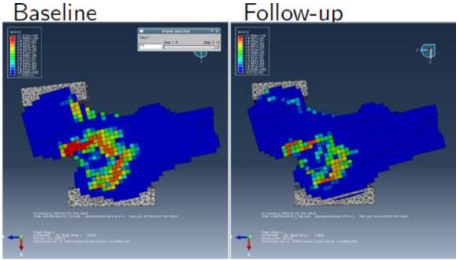


Fall: 85% of variability explained - model is **predictive**!

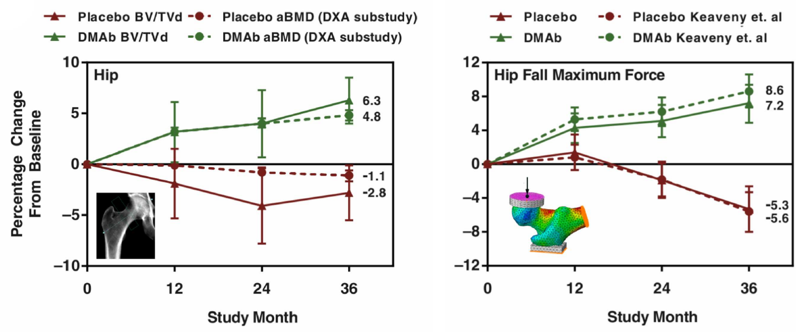
Osteoporosis Therapy: FEA Application

Osteoporosis Therapy: Follow-up Study





Osteoporosis Therapy: Freedom Trial



Zysset, Pahr, Engelke, Genant, McClung, Kendler, Recknor, Kinzl, Schwiedrzik, Museyko, Wang, Libanati: Bone, 2015, 81, 122-130

Vertebroplasty Study (N=40)

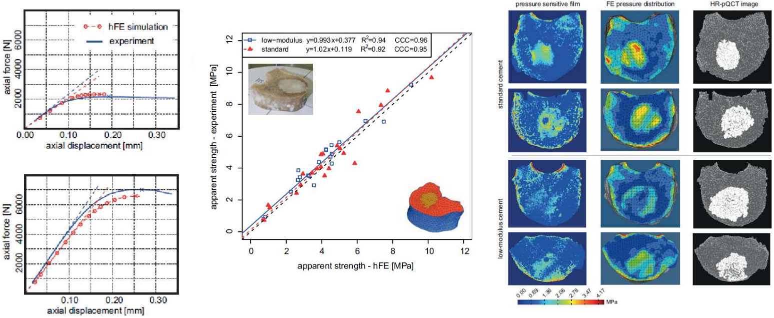
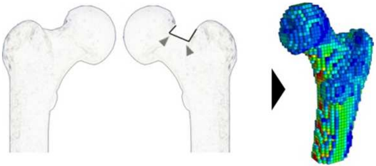


Figure: $F - u$ curves (left), 1:1 prediction (middle), pressure distributions (right)

Kinzl, Schwiedrzik, Zysset, Pahr: Clin Biomech, 2013, 28, 15-22

Pathologic Fractures: Metastatic Lesions



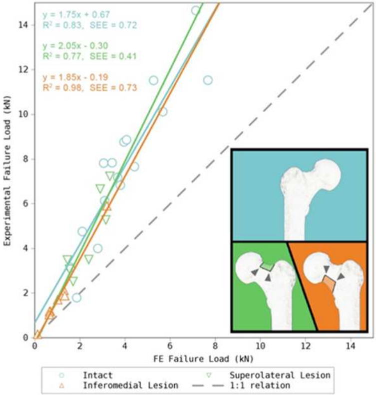


Figure: Intact bone, bone with lesion, FEA Model, comparison FEA vs. experiment (from left to right)

Benca, Reisinger, Patsch, Hirtler, Synek, Stenicka, Windhager, Mayr, Pahr. J Ortho Res, 2017, 35

Volar Plate Osteosynthesis

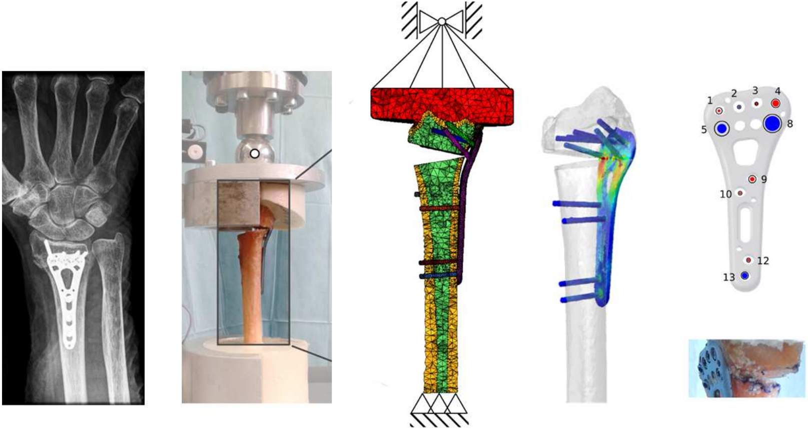


Figure: X-ray, experimental setup, FEA model, FEA results on plate, FEA screw loads (from left to right)

Synek, Chevalier, Baumbach, Pahr. J Biomech,. 2016, 49

Done!

All the best and have fun with the

Finite Element Method in Biomechanics!

Dieter H. Pahr

Part XI

Appendix

Overview Appendix

34 **Appendix**
Derivation of Assembly Function
General Governing FEM Equation

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Derivation of Assembly Function
General Governing FEM Equation

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Derivation of Assembly Function
General Governing FEM Equation

Global Trial Function

- assembly function is needed for Matrix Formulation of FE Equations
- for the derivation the global trial function is written as:

$$\underline{\bar{\boldsymbol{u}}} = \underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{U}} = \underbrace{\left(\sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}\, {}^{\langle e \rangle 0}\underline{\underline{\boldsymbol{T}}} \right)}_{{}^{0}\underline{\underline{\boldsymbol{N}}}} \cdot {}^{0}\underline{\boldsymbol{U}} + \underbrace{\left(\sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}\, {}^{\langle e \rangle ?}\underline{\underline{\boldsymbol{T}}} \right)}_{{}^{?}\underline{\underline{\boldsymbol{N}}}} \cdot {}^{?}\underline{\boldsymbol{U}} \quad (81)$$

- one gains the correct equation - but also the more complex one!

Governing FEM Equation

- inserting Eq. (81) gives:

$$ {}^{0}\underline{\underline{N}} = \sum_{e=1}^{n_{\text{el}}} {}^{\langle e \rangle}\underline{\underline{N}}\, {}^{\langle e \rangle 0}\underline{\underline{T}} \qquad {}^{?}\underline{\underline{N}} = \sum_{e=1}^{n_{\text{el}}} {}^{\langle e \rangle}\underline{\underline{N}}\, {}^{\langle e \rangle ?}\underline{\underline{T}} $$

$$ {}^{0}B = \sum_{e=1}^{n_{\text{el}}} {}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle 0}\underline{\underline{T}} \qquad {}^{?}\underline{\underline{B}} = \sum_{e=1}^{n_{\text{el}}} {}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle ?}\underline{\underline{T}} $$

- and it follows:

$$ \left(E\,A \int_0^L \sum_{e=1}^{n_{\text{el}}} ({}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle ?}\underline{\underline{T}})^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle ?}\underline{\underline{T}}\, dx\right) {}^{?}\underline{\underline{U}} = A \int_0^L \sum_{e=1}^{n_{\text{el}}} ({}^{\langle e \rangle}\underline{\underline{N}}\, {}^{\langle e \rangle ?}\underline{\underline{T}})^{\mathrm{T}}\, b\, dx \ldots $$

$$ \cdots + \sum_{e=1}^{n_{\text{el}}} ({}^{\langle e \rangle}\underline{\underline{N}}\, {}^{\langle e \rangle ?}\underline{\underline{T}})^{\mathrm{T}}\, F - E\,A \int_0^L \left(\sum_{e=1}^{n_{\text{el}}} ({}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle ?}\underline{\underline{T}})^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{B}}\, {}^{\langle e \rangle 0}\underline{\underline{T}}\, dx\right) {}^{0}U $$

Governing FEM Equation

Interchanging of the summation and integration and take into account that shape functions is only $\neq 0$ within an element:

Governing FEM Equation of Static Equilibrium

$$\left(\sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e\rangle ?}\underline{\underline{T}}^{\mathrm{T}} \underbrace{E\,A\int_0^{L_{\mathrm{e}}} \left({}^{\langle e\rangle}\underline{\underline{B}}^{\mathrm{T}}\;{}^{\langle e\rangle}\underline{\underline{B}}\right) dx}_{\text{Element Stiffness Matrix}}\;{}^{\langle e\rangle ?}\underline{\underline{T}}\right) {}^{?}\underline{U} = \sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e\rangle ?}\underline{\underline{T}}^{\mathrm{T}} \underbrace{A\int_0^{L_{\mathrm{e}}} {}^{\langle e\rangle}\underline{\underline{N}}^{\mathrm{T}}\, b\, dx}_{\text{Volume Force}} \dots$$

$$\dots + \sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e\rangle ?}\underline{\underline{T}}^{\mathrm{T}} \underbrace{{}^{\langle e\rangle}\underline{\underline{N}}(L)^{\mathrm{T}}\, F}_{\text{Surface Force}} - \left(\sum_{e=1}^{n_{\mathrm{el}}} {}^{\langle e\rangle ?}\underline{\underline{T}}^{\mathrm{T}} \underbrace{E\,A\int_0^{L_{\mathrm{e}}} \left({}^{\langle e\rangle}\underline{\underline{B}}^{\mathrm{T}}\;{}^{\langle e\rangle}\underline{\underline{B}}\right) dx\;{}^{0}U}_{\text{Force from known Displacement}}\;{}^{\langle e\rangle 0}\underline{\underline{T}}\right)$$

Note: Integration: $L \longrightarrow L_{\mathrm{e}}$!

Governing FEM Equation

This Eq. can be written in a simplified form as:

Governing FEM Equation of Static Equilibrium (simplified)

$$\sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ E\, A \int_0^{L_{\mathrm{e}}} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx \right\}}_{\text{Element Stiffness Matrix}} {}^{?}\underline{\boldsymbol{U}} = \sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ A \int_0^{L_{\mathrm{e}}} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\, b\, dx \right\}}_{\text{Volume Force}} + \dots$$

$$\dots \sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}(L)^{\mathrm{T}}\ F \right\}}_{\text{Surface Force}} - \sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ E\, A \int_0^{L_{\mathrm{e}}} \left({}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\ {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \right) dx\, {}^{0}U \right\}}_{\text{Force from known Displacement}}$$

with the assembly function

$$\sum_{n_{\mathrm{el}}}^{\text{Assemble}} \left\{ \dots \right\}$$

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Derivation of Assembly Function

General Governing FEM Equation

Static Equilibrium Condition

Starting point is the static equilibrium condition:

$$\sigma_{ij,j}(\boldsymbol{x}) + b_i(\boldsymbol{x}) = 0 \qquad i = 1, 2, 3$$

with following boundary conditions (see Fig. 80):

- natural (dynamic) BCs at surface $S_{f \neq 0}$ with traction $\hat{t}_j$:

$$\sigma_{ij}(\boldsymbol{x})\, n_i = \hat{t}_j \tag{82}$$

- essential (kinematic) BCs at surface S_u

$$u_i = \hat{u}_i \tag{83}$$

  with given displacements $\hat{u}_i$.

Sketch of Boundary Conditions

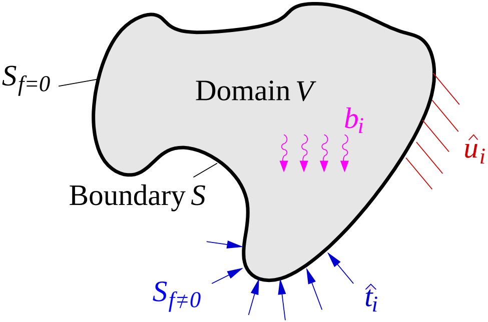

Figure: Sketch of Boundary Conditions

Residual, Virtual Displacement

Weighted residual method (see Eq. (42)) reads:

$$\int_V \underbrace{\underline{\boldsymbol{w}}^{\mathrm{T}}}_{\text{weighting}} \; \underbrace{\underline{\boldsymbol{R}}}_{\text{residual}} \; dV = 0 \qquad \longleftrightarrow \qquad \int_V w_i \, R_i \, dV = 0 \tag{84}$$

Residual follows from static equilibrium:

$$\underline{\boldsymbol{R}} = \underline{\underline{\boldsymbol{L}}}^T \, \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{b}} \qquad \longleftrightarrow \qquad R_i = \sigma_{ij,j} + b_i \tag{85}$$

Weighting function: e.g. usage of virtual displacement:

$$w_i = \delta u_i \tag{86}$$

What are Virtual Displacements?

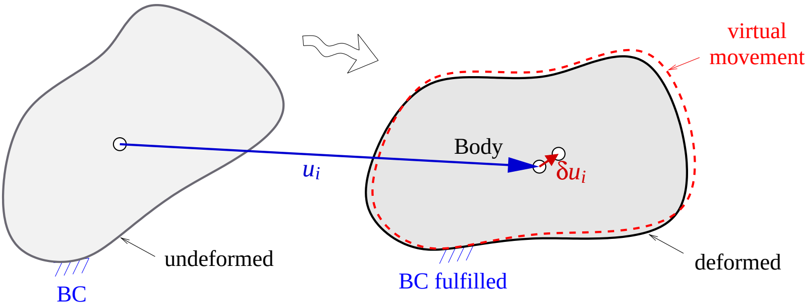

Figure: Sketch of virtual displacement

- virtual displacement is a very small arbitrary displacement
- But: It must fulfill the displ. BCs i.e. where $u_i = \hat{u}_i$ it holds $\delta u_i = 0$
- virtual displacement $\delta \underline{\boldsymbol{u}}$ corresponds to $\underline{\boldsymbol{d}}^{\mathrm{T}}\ \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}$!

Weak Form of Static Equilibrium

Integrating of Eq. (85) and the using virtual displacement δu_i:

$$\int_V (\sigma_{ij,j} + b_i)\delta u_i \, dV = 0\,, \tag{87}$$

(=weak form) and considering that:

$$(\sigma_{ij}\,\delta u_i)_{,j} = \sigma_{ij,j}\,\delta u_i + \sigma_{ij}\,\delta u_{i,j} \quad \rightarrow \quad \sigma_{ij,j}\,\delta u_i = -\sigma_{ij}\,\delta u_{i,j} + (\sigma_{ij}\,\delta u_i)_{,j} \tag{88}$$

it follows that

$$-\int_V \sigma_{ij}\,\delta u_{i,j}\,dV + \int_V (\sigma_{ij}\,\delta u_i)_{,j}\,dV + \int_V b_i\,\delta u_i\,dV = 0 \tag{89}$$

where the derivative of the virtual displacement

$$\delta u_{i,j}$$

is known as virtual strain

$$\delta u_{i,j} = \delta\varepsilon_{ij}$$

Weak Form of Static Equilibrium

using Gauss theorem (from volume integrals to surface integrals)

$$\int_V (\sigma_{ij}\,\delta u_i)_{,j}\,dV = \int_S \sigma_{ij}\,n_j\,\delta u_i\,dS$$

with normal vector n_j und traction vector t_i

$$t_i = \sigma_{ij}\,n_j$$

follows

Weak Form of Static Equilibrium = Principle of Virtual Work

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int_S t_i\,\delta u_i\,dS + \int_V b_i\,\delta u_i\,dV \tag{90}$$

Discussion

Discussion of Eq. (90)

- **surface integral:** $S = S_f$ is the surface where stresses (forces) are active i.e. $t_i = \hat{t}_i$
- **left side:** internal work of unknown stresses (displacements) and virtual strains
- **right side:** external work from external force and volume force
- equation says that internal and external virtual works are equal

Note: Principle of virtual work is derived from weighted residual method!

Derivation of Matrix Form Equation

Note: Eq. (90) can not be used for FEM because:

- not a discrete form (trial function $\underline{\bar{\boldsymbol{u}}}$ have to be introduced),
- stress is unknown in equation

1. Step: Inserting trial function

- global displacement trial function (Eq. (76)) reads as:

$$\begin{aligned} \underline{\bar{\boldsymbol{u}}} &= \underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{U}} \\ \delta\underline{\bar{\boldsymbol{u}}} &= \underline{\underline{\boldsymbol{N}}} \cdot \delta\underline{\boldsymbol{U}} \qquad \Longrightarrow \qquad \delta\underline{\bar{\boldsymbol{u}}}^{\mathrm{T}} = \delta\underline{\boldsymbol{U}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \end{aligned}$$

- for the strain (see Eq. (14)) it holds:

$$\begin{aligned} \underline{\boldsymbol{\varepsilon}} &= \underline{\underline{\boldsymbol{L}}} \cdot \left[\underline{\underline{\boldsymbol{N}}} \cdot \underline{\boldsymbol{U}} \right] = \underline{\underline{\boldsymbol{B}}} \cdot \underline{\boldsymbol{U}} \\ \delta\underline{\bar{\boldsymbol{\varepsilon}}} &= \underline{\underline{\boldsymbol{B}}} \cdot \delta\underline{\boldsymbol{U}} \qquad \Longrightarrow \qquad \delta\underline{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}} = \delta\underline{\boldsymbol{U}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \end{aligned}$$

Derivation of Matrix Form Equation

Inserting these relationships in Eq. (90) yields:

$$\int_V \delta\underline{\boldsymbol{U}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\,\underline{\boldsymbol{\sigma}}\,dV - \int_V \delta\underline{\boldsymbol{U}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{b}}\,dV - \int_S \delta\underline{\boldsymbol{U}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{t}}\,dS = 0$$

rearrangement with respect to $\delta\underline{\boldsymbol{U}}^{\mathrm{T}}$ gives:

$$\underbrace{\delta\underline{\boldsymbol{U}}^{\mathrm{T}}}_{\neq 0}\cdot\underbrace{\left(\int_V \underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\,\underline{\boldsymbol{\sigma}}\,dV - \int_V \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{b}}\,dV - \int_S \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{t}}\,dS\right)}_{=0} = 0$$

as shown with $\underline{\boldsymbol{d}}^{\mathrm{T}}$, $\delta\underline{\boldsymbol{U}}^{\mathrm{T}}$ is arbitrary i.e. $\delta\underline{\boldsymbol{U}}^{\mathrm{T}} \neq 0$ and it follows:

$$\int_V \underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\,\underline{\boldsymbol{\sigma}}\,dV = \int_V \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{b}}\,dV + \int_S \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{t}}\,dS$$

But: Unknown is still $\underline{\boldsymbol{\sigma}}$ (and not the displacement)!

Derivation of Matrix Form Equation

2. Step: Displacement as unknown (instead of stress)
Applying Hook's yields:

$$
\begin{aligned}
\underline{\boldsymbol{\sigma}} &= \underline{\underline{\boldsymbol{C}}}\,\underline{\boldsymbol{\varepsilon}} \\
&= \underline{\underline{\boldsymbol{C}}}\,(\underline{\underline{\boldsymbol{L}}}\,\underline{\boldsymbol{u}}) \\
&= \underline{\underline{\boldsymbol{C}}}\,(\underline{\underline{\boldsymbol{L}}}\,[\underline{\underline{\boldsymbol{N}}}\,\underline{\boldsymbol{U}}]) \\
\underline{\boldsymbol{\sigma}} &= \underline{\underline{\boldsymbol{C}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{U}}
\end{aligned}
$$

inserting this into the previous equations gives:

$$
\int_V \underline{\underline{\boldsymbol{B}}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{C}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{U}}\,dV = \int_V \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{b}}\,dV + \int_S \underline{\underline{\boldsymbol{N}}}^{\mathrm{T}}\,\underline{\boldsymbol{t}}\,dS
$$

Governing FEM Equation of Static Equilibrium

Finally integration and summation are interchanged (cp. Eq. (78)) and point loads (=nodal forces) $\underline{\boldsymbol{F}}_{\mathrm{K}}$ are added:

$$\sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{\langle e \rangle V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{C}}} \; {}^{\langle e \rangle}\underline{\underline{\boldsymbol{B}}} \, dV \right\}}_{\text{element stiffness matrix } {}^{\langle e \rangle}\underline{\underline{\boldsymbol{K}}}} \underline{\boldsymbol{U}} = \sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{\langle e \rangle V} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{b}} \, dV \right\}}_{\text{consistent volume forces } F_{\mathrm{V}}} + \dots$$

$$\dots + \sum_{n_{\mathrm{el}}}^{\text{Assemble}} \underbrace{\left\{ \int_{\langle e \rangle S} {}^{\langle e \rangle}\underline{\underline{\boldsymbol{N}}}^{\mathrm{T}} \; {}^{\langle e \rangle}\underline{\boldsymbol{t}} \, dS \right\}}_{\text{consistent surface forces } F_{\mathrm{O}}} + \underbrace{\underline{\boldsymbol{F}}_{\mathrm{K}}}_{\text{nodal forces}} \tag{91}$$

or simply:

Governing FEM Equation of Static Equilibrium

$$\underline{\underline{\boldsymbol{K}}}\,\underline{\boldsymbol{U}} = \underline{\boldsymbol{F}}_{\mathrm{V}} + \underline{\boldsymbol{F}}_{\mathrm{O}} + \underline{\boldsymbol{F}}_{\mathrm{K}}$$